Cosmology II

Problem sheet 4

Power law inflation and tensor perturbations

Exercise 1:

Consider the inflaton potential

$$V(\phi) = V_0 \exp\left\{-\sqrt{\frac{16\pi}{\lambda}}\frac{\phi}{m_P}\right\}, \tag{1}$$

where V_0 and λ are a priori free parameters of the model. Assume that the universe is flat and that the energy density is dominated by the inflaton.

(a) Compute ρ, p, the equation of motion and the slow-roll equations. Do we need the slow-roll approximation?

(b) Compute solutions for $\phi(t)$, $w(t)$, $a(t)$ and the slow-roll parameters ϵ and η. When does inflation end?

(c) What conditions have to be satisfied for a "successful" inflation in this model? Why is it called power-law inflation?

(d) Use the expressions given in the lecture to compute the perturbations. What is the spectral index? How much gravitational waves are produced?

Exercise 2:

General relativity predicts the existence of gravitational waves, which represent a perturbation of the metric:

$$ds^2 = dt^2 - a^2(t)\,(\delta_{ij} + h_{ij})\,dx^i dx^j. \tag{2}$$

If we fix a coordinate system and assume that the gravitational waves propagate in the z - direction, i.e. with the wave vector $\vec{k} = k\vec{e}_z$, we can write

$$h_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The two (small) functions h_+ and $h_\times$ describe the two polarizations of the gravitational wave. In this exercise we now derive the equation of motion for the amplitude h ($= h_+$ or $h_\times$) in first order.

(a) Find the Christoffel symbols for the tensor perturbations.

(b) Calculate the Ricci tensor and show that:

1.
$$R_{00} = -3\frac{d^2a/dt^2}{a}, \tag{3}$$
i.e. tensor perturbations do not appear at first-order in R_{00}

2.
$$R_{ij} = g_{ij}\left(\frac{d^2a/dt^2}{a} + 2H^2\right) + \frac{3}{2}a^2 H h_{ij,0} + a^2\frac{h_{ij,00}}{2} + \frac{k^2}{2}h_{ij} \tag{4}$$
where $H = \frac{\dot{a}}{a}$ and "," means partial derivative, i.e. $h_{ij,0} = \partial_0 h_{ij} = \partial_t h_{ij}$

(c) Compute the Ricci scalar $\mathcal{R} = g^{00}R_{00} + g^{ij}R_{ij}$ and show that the tensor perturbations do not affect the Ricci scalar at first order, i.e. $\delta\mathcal{R} = 0$.

(d) Because at first-order $\delta\mathcal{R} = 0$, the first order Einstein tensor is $\delta G^i{}_j = \delta R^i{}_j$. Then we have
$$\delta G^i{}_j = \delta^{ik}\left[\frac{3}{2}a^2 H h_{ij,0} + a^2\frac{h_{ij,00}}{2} + \frac{k^2}{2}h_{ij}\right]. \tag{5}$$
Why?

(e) Consider h_+ and the combination $\delta G^1{}_1 - \delta G^2{}_2$. Show that
$$\delta G^1{}_1 - \delta G^2{}_2 = 3Hh_{+,0} + h_{+,00} + \frac{k^2 h_+}{a^2}. \tag{6}$$
Introduce conformal time and show that then
$$a^2[\delta G^1{}_1 - \delta G^2{}_2] = \ddot{h}_+ + 2\frac{\dot{a}}{a}\dot{h}_+ + k^2 h_+. \tag{7}$$
The "dot" denotes derivative with respect to conformal time.

(f) Use that the corresponding right hand side of the Einstein equations is zero, i.e. $T^1_1 - T^2_2 = 0$ (why?) such that finally
$$\ddot{h}_+ + 2\frac{\dot{a}}{a}\dot{h}_+ + k^2 h_+ = 0. \tag{8}$$
By using the ${}^1{}_2$ component of the Einstein equations one can also show that the same equation holds for $h_\times$.