Cosmology II

Problem sheet 1

Robertson-Walker metric and Friedmann equations

Exercise 1:

(a) Given the (isotropic) metric:

$$ds^2 = dt^2 - a(t)^2 \left(f[r]dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right) , \tag{1}$$

with r, θ, ϕ comoving coordinates, and $f[r]$ an arbitrary function of r, compute the scalar curvature ${}^3R$ for the 3-dimensional space. Show then that the homogeneity of space implies :

$$f[r] = \frac{1}{1 - kr^2} , \tag{2}$$

where $k = +1,\ 0,\ -1$ (Robertson-Walker metric).

Hint: the curvature cannot depend on the spatial position if the universe is homogeneous.

(b) Calculate the Ricci tensor and Ricci scalar for the Robertson-Walker metric. Calculate the 0-0 and $i - i$ components of the Einstein equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} , \tag{3}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of a perfect fluid

$$T^\mu_\nu = \mathrm{diag}(\rho, -p, -p, -p) \tag{4}$$

with $p = p(t)$ and $\rho = \rho(t)$.

(c) For the energy-momentum tensor (4) derive the $\mu = 0$ - component of the energy momentum conservation equation

$$T^{\mu\nu}{}_{;\nu} = 0 \tag{5}$$

(";" denotes the covariant derivative). Solve the resulting differential equation for $\rho = \rho(t)$ assuming an equation of state $p = w\rho$ with $w = \mathrm{const}$.