LIAPUNOV SPECTRA FOR INFINITE CHAINS OF NONLINEAR OSCILLATORS

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Abstract. We argue that the spectrum of Liapunov exponents for long chains of nonlinear oscillators, at large energy per mode, may be well approximated by the Liapunov exponents of products of independent random matrices. If, in addition, statistical mechanics applies to the system, the elements of these random matrices have a distribution which may be calculated from the potential and the energy alone. Under a certain isotropy hypothesis (which is not always satisfied) we argue that the Liapunov exponents of these random matrix products can be obtained from the density of states of a typical random matrix. This construction uses an integral equation first derived by Newman. We then derive and discuss a method to compute the spectrum of a typical random matrix. Putting the pieces together we see that the Liapunov spectrum can be computed from the potential between the oscillators.

1. Introduction

In the paper [7], Livi *et al.* discuss a numerical experiment concerning Liapunov exponents for Hamiltonian systems with many degrees of freedom. They plot, for a variety of such systems, the graph of μ_i versus i/N, where μ_i is the i-th Liapunov exponent (ordered $\mu_1 \geq \cdots \geq \mu_N$) in a system with N degrees of freedom. They observe that the curves thus obtained seem to be independent of N for large N and, to a lesser extent, also independent of the system under analysis. Furthermore, they observe that the graphs form essentially straight lines.

We propose here an explanation of these findings. Our explanation is based on a twofold reduction:

– We first argue that in a Hamiltonian system describing a long chain of non-linear oscillators, in which *equipartition* of the energy holds, and which is ergodic, the tangent matrices to the Hamiltonian flow $\tau \to \Phi_\tau$ are symplectic matrices which are, for short times τ, well approximated by

$$\mathrm{d}\Phi_\tau \approx \mathbf{S}_\tau(\mathbf{\Omega}) \equiv \exp\left(\tau \begin{pmatrix} \mathbf{0} & -\mathbf{\Omega} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}\right) , \tag{1.1}$$

where $\mathbf{\Omega}$ is a *tri-diagonal* matrix of the form

$$\mathbf{\Omega} = \begin{pmatrix} \omega_1 & -\omega_1 & 0 & \dots & 0 \\ -\omega_1 & \omega_1+\omega_2 & -\omega_2 & \dots & 0 \\ 0 & -\omega_2 & \omega_2+\omega_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \omega_{N-1} \end{pmatrix} , \tag{1.2}$$

and the ω_i's are independent, identically distributed random variables with density $F(\omega)$. Because of equipartition, the density F depends only on the ergodic distribution of the coordinate of *one* oscillator, which in turn depends on the coupling potential. The importance and existence of an equipartition threshold has been discussed in Livi *et al.*[6], but our description will be more detailed and goes beyond previous studies.

– Assuming now that the matrices $\mathbf{\Omega}$ have a known distribution, the problem remains of finding the distribution of Liapunov exponents for the product

$$\prod_{j=0}^{n-1} \mathrm{d}\Phi_\tau(j\tau)$$

of the tangent maps $\mathrm{d}\Phi_\tau(j\tau)$ at times $t = j\tau$, for $j = 0, \ldots, n-1$. We shall show that at *large energy per oscillator* and sufficient anharmonicity, there is a choice of τ for which the approximation (1.1) is valid, and where, furthermore, successive matrices are statistically independent. We derive an integral equation which describes the connection between the density F of the ω_i's and the integrated density K of eigenvalues of $\mathbf{\Omega}$, and we show that it has a unique solution. The density of states can be read off this solution by a result of Simon and Taylor [12]. We then make the assumption that successive matrices of the form $\mathbf{S}_\tau(\mathbf{\Omega})$ tend to rotate tangent vectors isotropically in space. Numerical evidence, presented in Section 6, shows that this assumption may, or may not, be satisfied for a given system. For those systems which do satisfy the isotropy hypothesis, we can extend the theory of Newman [8], [9], in which the distribution of Liapunov exponents for products of random matrices is studied. We arrive at a relation between the expected integrated density of states K for large random matrices and the corresponding integrated density of Liapunov exponents for their (random) product.

These ideas lead to an explicit algorithm, described in Section 2, for determining the integrated density of Liapunov exponents for chains of oscillators. Applying this algorithm shows that the Liapunov spectrum is close to being a straight line. It depends on the probability distribution of the tangent matrices, and hence on the potential.

2. The Determination of the Liapunov Spectrum

In order to compute the density of Liapunov exponents, we shall show in Section 4 that we need to compute the density of states for a random tridiagonal matrix of the form

$$\mathbf{\Omega} = \begin{pmatrix} \omega_1 & -\omega_1 & 0 & \ldots & 0 \\ -\omega_1 & \omega_1+\omega_2 & -\omega_2 & \ldots & 0 \\ 0 & -\omega_2 & \omega_2+\omega_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \omega_{N-1} \end{pmatrix},$$

where the ω_i are independent, identically distributed random variables, with density $F(\omega)$.

We make the following assumptions on F:

F1. $\int dx F(x) = 1,\ F(x) \geq 0.$

F2. $\sup_\xi F(\xi)(1+\xi^2) < \infty.$

F3. The support of F lies on one side of 0.

Remark. The function F is, in the setting of Section 3, related to the second derivative of the interaction potential V. The conditions F1–F3 can then be easily reexpressed in terms of V, in particular, F3 follows from the convexity of the potential.

Under the assumptions F1–F3, the integrated density of Liapunov exponents $H(\mu)$ in the limit $N \to \infty$ is given by the following algorithm:

A1. For every $\lambda \in \mathbf{R}$, $\lambda \neq 0$, find a positive function u_λ such that

$$u_\lambda(t) \;=\; \int dt'\, F\Big(\frac{t'(t+\lambda)}{t'-t-\lambda}\Big)\Big(\frac{t'}{t'-t-\lambda}\Big)^2 u_\lambda(t')\ ,$$

and

$$\int_{-\infty}^{\infty} dt'\, u_\lambda(t') \;=\; 1\ .$$

A2. Determine $L(\lambda)$ by

$$L(\lambda) \;=\; 1 - \int_0^\infty dz \int_{-\infty}^{\infty} d\omega\, u_{-\lambda}(\omega(z-1))|\omega| F(\omega)\ .$$

(L is the integrated density of eigenvalues of matrices of the form of $\mathbf{\Omega}$. The condition F3 is only needed in this step.)

A3. Now compute $K(\lambda)$, the integrated density of eigenvalues of the absolute value of a matrix of the form (1.1), as follows: Given λ, we define $\gamma = \lambda^{1/2}$ and then $\lambda_-(\lambda)$ as the square root of the smaller of the two eigenvalues of the matrix

$$\begin{pmatrix} \cos^2(\tau\gamma) + \gamma^{-2}\sin^2(\tau\gamma) & (\gamma^{-1}-\gamma)\cos(\tau\gamma)\sin(\tau\gamma) \\ (\gamma^{-1}-\gamma)\cos(\tau\gamma)\sin(\tau\gamma) & \cos^2(\tau\gamma) + \gamma^2\sin^2(\tau\gamma) \end{pmatrix}\ .$$

Then we define

$$K(\lambda) = \begin{cases} 0, & \text{for } \lambda \leq 0, \\ L(\lambda_-^{-1}(\lambda)), & \text{for } 0 \leq \lambda \leq 1, \\ 1 - K(\lambda^{-1}), & \text{for } \lambda > 1. \end{cases} \tag{2.1}$$

Here, λ_-^{-1} denotes the inverse function of λ_-.

Remark. The numbers $\lambda_-(\lambda)$ and $\lambda_+(\lambda) = 1/\lambda_-(\lambda)$ are eigenvalues of $|\mathbf{S}|$ when λ is an eigenvalue of $\mathbf{\Omega}$. This follows by explicit calculation and observing that

$$\mathbf{S}_\tau(\mathbf{\Omega}) \;=\; \begin{pmatrix} \cos(\tau\mathbf{\Gamma}) & -\mathbf{\Gamma}\sin(\tau\mathbf{\Gamma}) \\ \mathbf{\Gamma}^{-1}\sin(\tau\mathbf{\Gamma}) & \cos(\tau\mathbf{\Gamma}) \end{pmatrix} , \tag{2.2}$$

with $\mathbf{\Gamma} = \mathbf{\Omega}^{1/2}$. Note that only integer powers of $\mathbf{\Omega}$ occur in $\mathbf{S}_\tau(\mathbf{\Omega})$.

A4. Determine $\mu_{\max}$ by

$$\mu_{\max} \;=\; \tfrac{1}{2}\log\Bigl(\int d\lambda\, \lambda^2 K'(\lambda)\Bigr) ,$$

and $\mu_{\min} = -\mu_{\max}$. Then, $H(\mu) = 0$ if $\mu \le \mu_{\min}$, $H(\mu) = 1$ if $\mu \ge \mu_{\max}$, and for all other μ, the function $H(\mu)$ is the nonzero solution of the equation

$$\int ds\, K'(s) \frac{s^2}{H(\mu)e^{2\mu} + (1 - H(\mu))s^2} \;=\; 1 . \tag{2.3}$$

In Section 5 we show the existence of solutions to the equation in A1 and the existence of the integral in A2 under the conditions F1–F2. In Section 6, we apply these formulas to a case similar to that considered by Livi *et al.*, and compare them with direct numerical studies.

3. The Random Matrix Approximation

In this section, we argue that for a long chain of coupled oscillators, at large energy per oscillator, and in a regime where equipartition of the energy holds, one can approximate the spectrum of Liapunov exponents by studying *random products* of *random matrices* of the form of those appearing in Section 2. Our arguments make precise some ideas which appear in Paladin and Vulpiani [10].

Given a system with Hamiltonian

$$\mathcal{H}(p, q) \;=\; \sum_{i=1}^{N} p_i^2/2m + \sum_{i=1}^{N-1} V(q_i - q_{i+1}) , \tag{3.1}$$

we let Φ_t be the associated Hamiltonian flow. The Liapunov exponents are then given by the large time behavior of $\mathrm{d}\Phi_t$. For example, Oseledec's theorem tells us that if the system has an ergodic invariant measure, then for almost every choice of the initial point $(\mathbf{p}^0, \mathbf{q}^0)$, (w.r.t. the invariant measure),* the largest Liapunov exponent is given by

$$\lambda_1 \;=\; \lim_{t\to\infty} \frac{1}{t} \log \|\mathrm{d}\Phi_t(\mathbf{p}^0, \mathbf{q}^0)\| \; . \tag{3.2}$$

Suppose $\tau > 0$ is some time interval. Then if we set

$$(\mathbf{p}^n, \mathbf{q}^n) \;=\; \Phi_{n\tau}(\mathbf{p}^0, \mathbf{q}^0) \;, \tag{3.3}$$

the chain rule of differentiation implies

$$\mathrm{d}\Phi_{n\tau}(\mathbf{p}^0, \mathbf{q}^0) \;=\; \prod_{j=0}^{n-1} \mathrm{d}\Phi_\tau(\mathbf{p}^j, \mathbf{q}^j) \; . \tag{3.4}$$

We now study (3.4) when τ is small. Then we obtain a good approximation to the solution of the equations of motion in the time interval $[\tau l, \tau(l+1)]$ by approximating the Hamiltonian by its Taylor series about the point $(\mathbf{p}^l, \mathbf{q}^l)$ to second order. This gives an effective Hamiltonian of the form

$$\begin{aligned}\mathcal{H}^{(2)}(\mathbf{p},\mathbf{q}) \;&=\; \tfrac{1}{2}\sum_{i=1}^{N}(p_i - p_i^l)^2 + \sum_{i=1}^{N}(p_i - p_i^l)p_i^l + \tfrac{1}{2}\sum_{i=1}^{N}(p_i^l)^2 \\ &+\mathcal{V}(\mathbf{q}^l) + \sum_{i=1}^{N}(q_i - q_i^l)\frac{\partial \mathcal{V}}{\partial q_i}(\mathbf{q}^l) + \tfrac{1}{2}\sum_{i,j=1}^{N}(q_i - q_i^l)(q_j - q_j^l)\frac{\partial^2 \mathcal{V}}{\partial q_i \partial q_j}(\mathbf{q}^l) \;,\end{aligned}$$

with $\mathcal{V}(\mathbf{q}) \;=\; \sum_{i=1}^{N-1} V(q_i - q_{i+1})$. We have set the masses m equal to 1. Note that the equations of motion for this reduced Hamiltonian are

$$\frac{d}{dt}\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix} \;=\; \mathbf{K}^l \;+\; \mathbf{M}^l\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix} \; .$$

Here, $\mathbf{K}^l$ is the column vector whose components are

$$(\mathbf{K}^l)_i = \begin{cases} -\dfrac{\partial \mathcal{V}}{\partial q_i} + \displaystyle\sum_{j=1}^{N} q_j^l \frac{\partial^2 \mathcal{V}}{\partial q_i \partial q_j}(\mathbf{q}^l) \;, & \text{for } i = 1, \ldots, N, \\ 0 \;, & \text{for } i = N+1, \ldots, 2N. \end{cases}$$

* If we suppose ergodicity, this is just the Liouville measure restricted to the energy shell, cf. e.g.[4]

and $\mathbf{M}^l$ is the matrix

$$\mathbf{M}^l = \begin{pmatrix} \mathbf{0} & -\frac{\partial^2 \mathcal{V}}{\partial q_i \partial q_j}(\mathbf{q}^l) \\ \mathbf{1} & \mathbf{0} \end{pmatrix} .$$

We can solve these equations explicitly, and if we denote the corresponding flow by Φ_t^l, we have

$$\Phi_t^l \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} = e^{\mathbf{M}^l t} \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} + e^{\mathbf{M}^l t} \int_0^t dt' \, e^{-\mathbf{M}^l t'} \mathbf{K}^l .$$

Hence,

$$\mathrm{d}\Phi_\tau(\mathbf{p}^l, \mathbf{q}^l) \approx \mathrm{d}\Phi_\tau^l = e^{\tau \mathbf{M}^l} , \tag{3.5}$$

when τ is sufficiently small. We then expect that *the Liapunov exponents of our Hamiltonian should be well approximated by those of the product of matrices*

$$\prod_{l=0}^{n-1} e^{\tau \mathbf{M}^l} . \tag{3.6}$$

Note that because of the short range nature of $\mathcal{V}$, we can compute the matrix

$$(\mathbf{A}(\mathbf{q}^l))_{ij} \equiv \frac{\partial^2 \mathcal{V}}{\partial q_i \partial q_j}(\mathbf{q}^l) ,$$

and we find

$$(\mathbf{A}(\mathbf{q}^l))_{ij} = \begin{cases} V''(q_i^l - q_{i+1}^l) + V''(q_{i-1}^l - q_i^l), & \text{if } i = j, \\ -V''(q_i^l - q_j^l), & \text{if } |i-j| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

(The terms at the boundary ($i = 1, N$ or $j = 1, N$) are different, but this is irrelevant for what follows.) Note that $\sum_j (\mathbf{A}(\mathbf{q}^l))_{ij} = 0$ and $\sum_j (\mathbf{A}(\mathbf{q}^l))_{ji} = 0$ for $i = 2, \ldots, N-1$. Note also that the matrices $\exp(\tau \mathbf{M}^l)$ are symplectic.

We now make a number of precise physical assumptions about the behavior of the system (3.1). They will show that

- *The numbers $V''(q_i^l - q_{i+1}^l)$ occurring in the matrices $\mathbf{A}(\mathbf{q}^l)$ can be viewed as random variables, ω_i.*

- *There is a range of τ for which the approximation (3.5) is valid <u>and</u> for which successive matrices $\mathbf{M}^l$ and $\mathbf{M}^{l+1}$ are statistically independent.*

Our assumptions are:

E1. The energy of the orbit is $E \cdot N$ and the energy per oscillator, E, is large.

E2. The system is in a state of equipartition of the energy. *

E3. For large q, the function V behaves like $V(q) \approx q^\alpha$, with $\alpha > 2$.

From equipartition and the virial theorem, we deduce that the energy in the system will be divided equally between the oscillators and between kinetic and potential energy. Using E1–E3, we see that typically the momenta and positions will have values

$$\bar{p} \approx E^{1/2} \ , \ \bar{q} \approx E^{1/\alpha} \ .$$

In the following paragraphs, we perform calculations of orders of magnitude. We therefore omit all indices, and assume that the statistical properties of q_i and of $q_i - q_{i+1}$ are the same. We also use interchangeably $\mathcal{V}$ and V to denote the potential. If we consider the equations of motion corresponding to the Hamiltonian (3.1), we see that the equations of motion are

$$\begin{aligned} \dot{p} &= -\mathrm{grad}\mathcal{V}(q) \ , \\ \dot{q} &= p \ . \end{aligned}$$

The solutions to these equations for times Δt may thus be approximated by

$$\begin{aligned} p(\Delta t) &\approx p(0) - \Delta t\, V'(q(0)) \ , \\ q(\Delta t) &\approx q(0) + \Delta t\, p(0) \ . \end{aligned} \tag{3.7}$$

We estimate the time Δt over which this approximation remains valid by comparing the above values of $p(\Delta t)$ and $q(\Delta t)$ to the values we would have obtained if we further subdivided the time interval. We find

$$\begin{aligned} p(\Delta t) &\approx p(\frac{\Delta t}{2}) - \frac{\Delta t}{2} V'(q(\frac{\Delta t}{2})) \\ &\approx p(0) - \frac{\Delta t}{2} V'(q(0)) - \frac{\Delta t}{2} V'(q(0) + \frac{\Delta t}{2}\, p(0)) \\ &\approx p(0) - \Delta t V'(q(0)) - (\frac{\Delta t}{2})^2 V''(q(0)) p(0) \ . \end{aligned}$$

* In [6], numerical evidence is presented for the existence of a finite equipartition threshold as $N \to \infty$.

Similarly,

$$\begin{aligned} q(\Delta t) &\approx q(\frac{\Delta t}{2}) + \frac{\Delta t}{2}\,p(\frac{\Delta t}{2}) \\ &\approx q(0) + \frac{\Delta t}{2}\,p(0) + \frac{\Delta t}{2}\left(p(0) - \frac{\Delta t}{2}\,V'(q(0))\right) \\ &\approx q(0) + \Delta t\,p(0) - (\frac{\Delta t}{2})^2 V'(q(0)) \ . \end{aligned}$$

Thus, the approximation Eq.(3.5) will be good if

$$|(\Delta t)^2 V''(q(0))p(0)| \ll |\Delta t\,V'(q(0))| \ , \tag{3.8}$$

$$|(\Delta t)^2 V'(q(0))| \ll |\Delta t\,p(0)| \ . \tag{3.9}$$

These are implied by

$$\begin{aligned} |\Delta t| &\ll \min\left(\left|\frac{V'(q(0))}{p(0)V''(q(0))}\right|, \left|\frac{p(0)}{V'(q(0))}\right|\right) \\ &\approx \min\left(\frac{E^{1-1/\alpha}}{E^{1/2}E^{1-2/\alpha}}, \frac{E^{1/2}}{E^{1-1/\alpha}}\right) \\ &\approx E^{(1/\alpha)-(1/2)} \ . \end{aligned} \tag{3.10}$$

Note that the flow Eq.(3.7) gives an expression for the tangent map which is

$$\mathrm{d}\Phi_{\Delta t}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \mathbf{1} & -\Delta t\,V''(q) \\ \Delta t\,\mathbf{1} & \mathbf{1} \end{pmatrix} \approx \exp\left(\Delta t \begin{pmatrix} \mathbf{0} & -V''(q) \\ \mathbf{1} & \mathbf{0} \end{pmatrix}\right) \ , \tag{3.11}$$

provided $|(\Delta t)^2\,V''(q)| \ll 1$, which is valid if

$$|\Delta t| \ll E^{1/\alpha - 1/2} \ . \tag{3.12}$$

Note that this is the same inequality as the one in (3.10).

Remark. We should also check that when we further subdivide the time interval we do not change the tangent map to the flow very much either. To see this, note that the tangent map to the flow computed with a time interval $\Delta t/2$ is just

$$\begin{pmatrix} \mathbf{1} - (\frac{\Delta t}{2})^2 V''(q(0)) & -\Delta t V''(q(0)) - (\frac{\Delta t}{2})^2 V'''(q(0))p(0) \\ \Delta t \mathbf{1} & \mathbf{1} - (\frac{\Delta t}{2})^2 V''(q(0)) \end{pmatrix} \ .$$

This is close to the approximation we previously derived, provided

$$\begin{aligned} |(\Delta t)^2 V''(q(0))| &\ll 1 \ , \\ |(\Delta t)^2 V'''(q(0))p(0)| &\ll |\Delta t V''(q(0))| \ . \end{aligned}$$

Both of these inequalities are satisfied if $|\Delta t| \ll E^{1/\alpha-1/2}$.

We now compute the amount by which the the components, V'', of the tangent matrix change during a time Δt. This change is of the order

$$\Delta t \frac{d}{dt} V'' \approx \Delta t\, V'''(q)\dot{q} \approx \Delta t\, V'''(q)p \approx \Delta t\, E^{\frac{1}{2}} E^{1-\frac{3}{\alpha}} .$$

Suppose now we assume $\Delta t \approx E^{1/\alpha-1/2}$, in accordance with (3.12). Then the change in the tangent matrix elements will be of order

$$E^{\frac{1}{\alpha}-\frac{1}{2}} E^{\frac{1}{2}} E^{1-\frac{3}{\alpha}} \approx E^{1-\frac{2}{\alpha}} .$$

This is of the same order of magnitude as the size of V''. Hence, under the assumptions E1–E3, we see that for $\Delta t \approx E^{2/\alpha-1}$ the elements in the tangent map $\mathrm{d}\Phi_{\Delta t}$ at $t = 0$ will be very different from those in $\mathrm{d}\Phi_{\Delta t}$, at $t = \Delta t$ and so we may choose them to be *randomly and independently distributed* with respect to their previous values and among each other.

Note that the distribution F of the elements of $\mathrm{d}\Phi_{\Delta t}$ depends on the potential V and on the invariant measure of the flow. We make the further assumption

E4. Statistical mechanics holds at large energies for systems with many degrees of freedom.

We can then go one step further and compute the density F at energy E per degree of freedom from a knowledge of the potential alone. First note that the quantities $V''(q_i^l - q_{i+1}^l)$ and $V''(q_{i-1}^l - q_i^l)$ are uncorrelated. Thus we can treat the various elements of the matrices $\mathbf{A}(\mathbf{q}^l)$ as independent random variables. Because of equipartition, the density F of these random variables can be determined numerically by sampling the coordinates of two oscillators. Furthermore, we can compute the density from a knowledge of the potential alone. The argument goes as follows: The integral of the function F up to ω is, according to what we have said above, the probability that $V''(q_i - q_{i+1})$ takes a value less than ω. We denote $G(\omega) = \int_{-\infty}^{\omega} d\omega' F(\omega')$. In the canonical ensemble, we first calculate

$$G_\beta(\omega) = \frac{\int_{E(\omega)} e^{-\beta \mathcal{H}_N(p,q)} dp\, dq}{\int e^{-\beta \mathcal{H}_N(p,q)} dp\, dq} , \tag{3.13}$$

where

$$E(\omega) = \Big\{ (p,q) \in \mathbf{R}^{2N} \,|\, V''(q_{N/2} - q_{N/2+1}) < \omega \Big\} . \tag{3.14}$$

We may integrate over the p's and we get

$$G_\beta(\omega) \;=\; \frac{\int_{E(\omega)} e^{-\beta \sum_{i=1}^{N-1} V(q_i - q_{i+1})}\, dq}{\int e^{-\beta \sum_{i=1}^{N-1} V(q_i - q_{i+1})}\, dq} \;, \tag{3.15}$$

which, by a change of variables to $q_i - q_{i+1}$, leads to

$$G_\beta(\omega) \;=\; \frac{\int_{V''(x)<\omega} e^{-\beta V(x)}\, dx}{\int e^{-\beta V(x)}\, dx} \;. \tag{3.16}$$

Observe that if V is convex, then the support of G_β is contained in $\{\omega > 0\}$, which is our condition F3. By the standard techniques of statistical mechanics, we have then

$$G(\omega) \;=\; G_{\beta^*}(\omega) \;,$$

where β^* is determined by the equation

$$E \cdot N \;=\; \frac{\int \mathcal{H}_N(p,q) e^{-\beta^* \mathcal{H}_N(p,q)} dp\, dq}{\int e^{-\beta^* \mathcal{H}_N(p,q)} dp\, dq} \;, \tag{3.17}$$

and E is the energy per oscillator. The simple set of Eqs (3.13)–(3.17) together with our earlier results show:

The density H of Liapunov exponents of a chain of non-linear oscillators at high energy per oscillator can be predicted through a series of integral transforms of the interaction potential, by solving first Eq.(3.17) to obtain the probability distribution of the elements of the matrices (3.11) and then applying the algorithm A1–A4 to the matrices (3.11).

4. Liapunov Exponents for Products of Symplectic Matrices

In the present section, we argue that one can apply a modification of the theory developed by Wachter [13] and Newman [9] to evaluate the distribution of Liapunov exponents of a random product of matrices in terms of the density of eigenvalues of a single matrix. We begin by studying symplectic matrices of size $2N \times 2N$ and derive one of the formulas in Newman for this class of matrices. We can then apply his theory to take the thermodynamic limit $N \to \infty$.

Let $\mathbf{S}_j(\Omega)$ be a set of independent, identically distributed random matrices of the form of Eq.(1.1); however, the index j denotes now different matrices, and not the size of a time interval τ, which we assume fixed. We also define

$$\mathbf{S}^n \equiv \prod_{j=1}^{n} \mathbf{S}_j(\Omega) \ .$$

If we assume that the Liapunov exponents of the product are ordered as $\mu_1 \geq \mu_2 \geq \ldots$, then for almost every choice of (linearly independent) normalized vectors $x_1, \ldots, x_k$ we have

$$\mu_1 + \cdots + \mu_k = \lim_{n\to\infty} \frac{1}{n} \log\left(\|\mathbf{S}^n x_1 \wedge \cdots \wedge \mathbf{S}^n x_k\|\right) \ .$$

Let $x_l(j) = \mathbf{S}^j x_l$, for $l = 1, \ldots, k$ and $j = 0, \ldots, n$. Then

$$\begin{aligned}\mu_1 + \cdots + \mu_k &= \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log\left(\frac{\|\mathbf{S}_j x_1(j-1) \wedge \cdots \wedge \mathbf{S}_j x_k(j-1)\|}{\|x_1(j-1) \wedge \cdots \wedge x_k(j-1)\|}\right) \\ &= \lim_{n\to\infty} \frac{1}{n} \left[\sum_{j=1}^{n} \log\left\|\frac{\mathbf{S}_j x_1(j-1)}{\|x_1(j-1)\|} \wedge \cdots \wedge \frac{\mathbf{S}_j x_k(j-1)}{\|x_k(j-1)\|}\right\|\right. \\ &\quad \left. - \sum_{j=1}^{n} \log\left\|\frac{x_1(j-1)}{\|x_1(j-1)\|} \wedge \cdots \wedge \frac{x_k(j-1)}{\|x_k(j-1)\|}\right\|\right] \ .\end{aligned} \tag{4.1}$$

It is a consequence of Oseledec's theorem that $\mu_1 + \cdots + \mu_k$ is almost surely independent of the choice of $x_1, \ldots, x_k$. Therefore, we treat $x_1, \ldots, x_k$ as independent random vectors, uniformly distributed on the unit sphere. Denoting the expectation with respect to these variables as E_x, we have

$$\mu_1 + \cdots + \mu_k = \lim_{n\to\infty} \frac{1}{n} E_x\left(\log\|\mathbf{S}^n x_1 \wedge \cdots \wedge \mathbf{S}^n x_k\|\right)$$

$$= \lim_{n\to\infty} \frac{1}{n} \left[\sum_{j=1}^{n} E_x \left(\log \left\| \frac{\mathbf{S}_j x_1(j-1)}{\|x_1(j-1)\|} \wedge \cdots \wedge \frac{\mathbf{S}_j x_k(j-1)}{\|x_k(j-1)\|} \right\| \right) \right.$$
$$\left. - \sum_{j=1}^{n} E_x \left(\log \left\| \frac{x_1(j-1)}{\|x_1(j-1)\|} \wedge \cdots \wedge \frac{x_k(j-1)}{\|x_k(j-1)\|} \right\| \right) \right] . \tag{4.2}$$

We now make the following hypothesis.

R1. The vectors $e_l(j) \equiv x_l(j-1)/\|x_l(j-1)\|$, $j = 1, 2, \ldots$, are, for fixed l, and almost every choice of x_l and $\mathbf{S}_1, \mathbf{S}_2, \ldots$, uniformly distributed on the unit sphere.

Remark. This hypothesis means that multiplication of a random vector by a random sequence of matrices of the form $\mathbf{S}_j$ does not tend to rotate that vector into any preferred directions. This hypothesis seems *not* to be satisfied in general for the systems under consideration. Numerical experiments rather suggest that the vectors are *localized* in the following sense: If one of the components of $e_l(j)$ is large (in modulus) then nearby components tend to be large, too. As j varies, the components at which these maxima occur move in an apparently random fashion. In the absence of a theory of these time-dependent localization problems, we continue our analysis under the hypothesis R1. In those cases where numerical evidence indicates that R1 is most nearly satisfied, we verify that we obtain good agreement between theory and numerical experiment, see Section 6.

Consider now the first term in the expression (4.2) in the light of this hypothesis. We first note that it can be rewritten as

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} E_x \left(\log \|\mathbf{S}_j e_1(j) \wedge \cdots \wedge \mathbf{S}_j e_k(j)\| \right) ,$$

where the $e_l(j)$ are random, uniformly distributed unit vectors in $\mathbf{R}^{2N}$. Note that $e_l(j)$ is independent of $e_{l'}(j)$ if $l \neq l'$, since these two vectors are the product of a sequence of invertible matrices and two independent random variables. Therefore, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} E_x \left(\log \|\mathbf{S}_j e_1(j) \wedge \cdots \wedge \mathbf{S}_j e_k(j)\| \right) = \mathcal{E} \left(\log \|\mathbf{S} e_1 \wedge \cdots \wedge \mathbf{S} e_k\| \right) ,$$

where $\mathcal{E}$ denotes the expectation with respect to both the random matrices $\mathbf{S}$ and the independent, uniformly distributed, unit vectors x_l. We have omitted the indices j over which these expectations are taken.

Let $X_l(j)$ be $2N$-dimensional vectors whose entries are Gaussian random variables of mean zero and variance 1. Note that the subspace spanned by $\{x_1(j),\ldots,x_k(j)\}$ is equidistributed with that spanned by $\{X_1(j),\ldots,X_k(j)\}$. Thus, we have

$$\mathcal{E}\left(\log\|\mathbf{S}e_1\wedge\cdots\wedge\mathbf{S}e_k\|\right) = \mathcal{E}\left(\log\|\mathbf{S}X_1\wedge\cdots\wedge\mathbf{S}X_k\|\right) - k\mathcal{E}\left(\log\|X\|\right) .$$

On the l.h.s., $\mathcal{E}$ denotes the expectation w.r.t. the probability distribution of the random matrices $\mathbf{S}$ and the random unit vectors x_l. On the r.h.s., it denotes expectation w.r.t. the random matrices $\mathbf{S}$ and the Gaussian random variables X_l. The term $k\mathcal{E}\left(\log\|X\|\right)$ comes from normalizing the X_l's.

Following Newman, we let $G(\sigma_1,\ldots,\sigma_{2N})$ be the $2N\times 2N$ matrices whose entries are independent mean zero Gaussian random variables, such that elements of the l-th row have variance σ_l^2. Then

$$\begin{aligned}\mathcal{E}\left(\log\|\mathbf{S}X_1\wedge\cdots\wedge\mathbf{S}X_k\|\right) &= \mathcal{E}\left(\log\|\mathbf{S}G(1,\ldots,1)v_1\wedge\cdots\wedge\mathbf{S}G(1,\ldots,1)v_k\|\right)\\ &= \tfrac{1}{2}\mathcal{E}\left(\log\ \det\ (P_kG^T(1,\ldots,1)\mathbf{S}^T\mathbf{S}G(1,\ldots,1)P_k)\right) .\end{aligned}$$

Here, $v_1,\ldots,v_{2N}$ are the natural basis vectors in $\mathbf{R}^{2N}$, and P_k is the projection onto the span of $v_1,\ldots,v_k$. Let $\mathbf{O}_j$ be the orthogonal matrix which diagonalizes $\mathbf{S}_j^T\mathbf{S}_j$, i.e., $\mathbf{O}_j\mathbf{S}_j^T\mathbf{S}_j\mathbf{O}_j^T = \mathbf{D}_j^2$. Then

$$\begin{aligned}&\tfrac{1}{2}\mathcal{E}\left(\log\ \det\ (P_kG^T(1,\ldots,1)\mathbf{S}^T\mathbf{S}G(1,\ldots,1)P_k)\right)\\ &\qquad = \tfrac{1}{2}\mathcal{E}\left(\log\ \det\ (P_kG^T(1,\ldots,1)\mathbf{O}^T\mathbf{D}^2\mathbf{O}G(1,\ldots,1)P_k)\right)\\ &\qquad = \tfrac{1}{2}\mathcal{E}\left(\log\ \det\ (P_kG^T(1,\ldots,1)\mathbf{D}^2G(1,\ldots,1)P_k)\right)\end{aligned}$$

because $\mathbf{O}G(1,\ldots,1)$ is equidistributed with $G(1,\ldots,1)$. If we let $\lambda_1(j),\ldots,\lambda_{2N}(j)$ be the eigenvalues of $\mathbf{D}_j$, i.e., the eigenvalues of $|\mathbf{S}_j|$, and we define

$$\Phi_k(\lambda_1,\ldots,\lambda_{2N}) = \tilde{\mathcal{E}}\left(\log\|G(\lambda_1,\ldots,\lambda_{2N})v_1\wedge\cdots\wedge G(\lambda_1,\ldots,\lambda_{2N})v_k\|\right) ,$$

where $\tilde{\mathcal{E}}$ denotes an average over the entries of G with $\lambda_1,\ldots,\lambda_{2N}$ fixed, we obtain

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\log\left(\|\mathbf{S}_je_1(j)\wedge\cdots\wedge\mathbf{S}_je_k(j)\|\right)\\ &\qquad = \mathcal{E}\left(\log\|\mathbf{D}G(1,\ldots,1)v_1\wedge\cdots\wedge\mathbf{D}G(1,\ldots,1)v_k\|\right) - k\mathcal{E}\left(\log\|X\|\right)\\ &\qquad = \mathcal{E}\left(\log\|G(\lambda_1,\ldots,\lambda_{2N})v_1\wedge\cdots\wedge G(\lambda_1,\ldots,\lambda_{2N})v_k\|\right) - k\mathcal{E}\left(\log\|X\|\right)\\ &\qquad = \hat{\mathcal{E}}(\Phi_k(\lambda_1,\ldots,\lambda_{2N})) - k\mathcal{E}\left(\log\|X\|\right) .\end{aligned}$$

Here, $\hat{\mathcal{E}}$ denotes an average over the random eigenvalues $\lambda_1, \ldots, \lambda_{2N}$. A similar argument shows that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log \left\| \frac{x_1(j-1)}{\|x_1(j-1)\|} \wedge \cdots \wedge \frac{x_k(j-1)}{\|x_k(j-1)\|} \right\| = \Phi_k(1,\ldots,1) - k\mathcal{E}(\log \|X\|) \ .$$

Thus, we obtain

$$\mu_1 + \cdots + \mu_k = \hat{\mathcal{E}}(\Phi_k(\lambda_1,\ldots,\lambda_{2N})) - \Phi_k(1,\ldots,1) \ , \tag{4.3}$$

which is precisely Eq.(1.12) of [9].

This allows us to apply the powerful result of Newman [9, Theorem 2.11]. His work uses an interesting study of the density of states of random matrices due to Wachter [13]. We repeat Newman's result in its original form (adapted to our notation), and state then which assumptions are changed in our application.

Theorem 4.1. *Let $\mu_1^N \geq \cdots \geq \mu_N^N$ denote the Liapunov exponents for the product, $\mathbf{M}_t \cdots \mathbf{M}_1$ of $N \times N$ real i.i.d. random matrices and let H_N denote the empirical distribution of the μ_i^N's,*

$$H_N(\mu) = N^{-1} \cdot (\text{number of } i \text{ with } \mu_i^N \leq \mu \ ).$$

Suppose $\mathbf{M}_1^T\mathbf{M}_1$ has, for each N, a rotation invariant distribution. Let K_N denote the (random) empirical distribution of the eigenvalues λ_i^N of $|\mathbf{M}_1|$. Suppose that there exist nonrandom $\underline{\lambda} > 0$ and $\overline{\lambda} < \infty$ such that, with probability one, $\underline{\lambda} \leq \lambda_i^N \leq \overline{\lambda}$ for all i and N. Suppose further that there is a nonrandom distribution function K such that with probability one, $K_N \to K$ (at all continuity points of K). Then $H_N(\mu) \to H(\mu)$ for all μ, where H is a continuous distribution function satisfying:

$$H(\mu) = 0, \text{ for } \mu \leq -\tfrac{1}{2}\log\Bigl(\int \lambda^{-2}\, dK(\lambda)\Bigr) \ ,$$

$$H(\mu) = 1, \text{ for } \mu \geq \tfrac{1}{2}\log\Bigl(\int \lambda^{2}\, dK(\lambda)\Bigr) \ .$$

$H(\mu)$ is, for all other μ, the unique solution in $(0,1)$ of

$$\int dK(\lambda) \frac{\lambda^2}{H\, e^{2\mu} + (1-H)\lambda^2} = 1.$$

The discussion leading to (4.3) shows how the assumption on rotational invariance is replaced by the weaker assumption R1. The assumption that $|\mathbf{S}|$ has eigenvalues uniformly bounded away from 0 and ∞ is obviously fulfilled in our case, since $\mathbf{S}$ is symplectic and bounded, i.e., its eigenvalues come in pairs $\lambda \leftrightarrow \lambda^{-1}$.

5. Random Tridiagonal Matrices

In this section we justify steps A1 and A2 of the algorithm of Section 2. We want to study the distribution of eigenvalues of the matrix (1.2), and therefore we consider first the eigenvalue problem for such matrices. Recall that

$$\mathbf{\Omega} = \begin{pmatrix} \omega_1 & -\omega_1 & 0 & \dots & 0 \\ -\omega_1 & \omega_1+\omega_2 & -\omega_2 & \dots & 0 \\ 0 & -\omega_2 & \omega_2+\omega_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \omega_{N-1} \end{pmatrix} .$$

The eigenvalue problem $\mathbf{\Omega x} = \lambda \mathbf{x}$ can be rewritten as the system of equations in the unknowns $x_1, \dots, x_N$,

$$-\omega_{n-1}x_{n-1} \ + \ (\omega_{n-1}+\omega_n)x_n \ - \ w_n x_{n+1} \ = \ \lambda x_n \ . \tag{5.1}$$

for $n = 2, \dots, N-1$, and with obvious modifications for $n = 1$ and $n = N$. We rewrite the problem, defining $z_n = x_{n+1}/x_n$, as a system of transformations

$$z_n \ = \ \frac{1}{\omega_n}\left(\omega_{n-1}+\omega_n-\lambda-\frac{\omega_{n-1}}{z_{n-1}}\right) \ . \tag{5.2}$$

If we denote the distribution of z_n by G_n and that of the random variables ω_n by F, and assume that we are given G_1, then we can write

$$\begin{aligned} &G_{n+1}(z_{n+1}) \\ &\quad = \int \prod_{j=1}^{n+1} F(\omega_j)d\omega_j \prod_{j=1}^{n} dz_j G_1(z_1) \prod_{j=2}^{n+1} \delta\big(z_j - \frac{1}{\omega_j}(\omega_{j-1}+\omega_j-\lambda-\frac{\omega_{j-1}}{z_{j-1}})\big) \ . \end{aligned} \tag{5.3}$$

Suppose we define inductively a sequence of functions of *two* variables by

$$B_1(\omega, z) \ = \ G_1(z) \ , \tag{5.4}$$

$$B_{j+1}(\omega, z) \ = \ \int F(\omega')B_j(\omega', z')\delta\big(z - \frac{1}{\omega}(\omega'+\omega-\lambda-\frac{\omega'}{z'})\big)d\omega'\,dz' \ . \tag{5.5}$$

Clearly, we find

$$\begin{aligned} G_2(z_2) \ &= \ \int F(\omega_2)F(\omega_1)G_1(z_1)\delta\big(z_2 - \frac{1}{\omega_2}(\omega_1+\omega_2-\lambda-\frac{\omega_1}{z_1})\big)\,d\omega_1\,d\omega_2\,dz_1 \\ &= \ \int F(\omega_2)B_2(\omega_2, z_2)\,d\omega_2 \ , \end{aligned} \tag{5.6}$$

and, by induction

$$G_n(z_n) \;=\; \int F(\omega_n) B_n(\omega_n, z_n)\, d\omega_n \;, \tag{5.7}$$

for all $n \geq 2$. Instead of studying G_n, for which we want to show the existence of a limit as $n \to \infty$, we study the functions B_n. We rewrite Eq.(5.5) as

$$B_{j+1}(\omega, z) \;=\; \int F(\omega') B_j(\omega', z') \delta\big(\omega(z-1) + \omega'(\frac{1}{z'} - 1) + \lambda)\big) |\omega|\, d\omega'\, dz' \;, \tag{5.8}$$

and we see that $|\omega|^{-1} B_{j+1}(\omega, z)$ is a function of the combination $\omega(z-1)$ alone. We thus define

$$U_k(\omega(z-1)) \;=\; B_k(\omega, z)|\omega|^{-1} \text{ for } k = 1, 2, \ldots \;, \tag{5.9}$$

and we set

$$\begin{aligned} t \;&=\; \omega(z-1) \;, \\ t' \;&=\; \omega'(z'-1) \;. \end{aligned} \tag{5.10}$$

Note that the Jacobian of the transformation $(\omega', z') \to (\omega', t')$ is $|\omega'|$ and that $z' = 1 + t'/\omega'$, and

$$\omega'(\frac{1}{z'} - 1) \;=\; \omega' \frac{-t'/\omega'}{1 + t'/\omega'} \;=\; -\frac{t'\omega'}{t' + \omega'} \;. \tag{5.11}$$

Therefore, we get

$$U_{j+1}(t) \;=\; \int dt'\, d\omega'\, F(\omega') U_j(t') \delta\big(t - \frac{t'\omega'}{t' + \omega'} + \lambda\big) \;. \tag{5.12}$$

We want to eliminate ω' by integrating over the δ-function. Note that the δ-function has support on

$$\omega' \;=\; -\frac{t'(t+\lambda)}{t - t' + \lambda}$$

and that

$$\partial_{\omega'}\big(t - \frac{t'\omega'}{t' + \omega'} + \lambda\big) \;=\; -\frac{t'}{t' + \omega'} + \frac{t'\omega'}{(t' + \omega')^2} \;=\; -\frac{t'^2}{(t' + \omega')^2} \;. \tag{5.13}$$

Therefore,

$$\delta\big(t - \frac{t'\omega'}{t' + \omega'} + \lambda\big) \;=\; \delta\big(\omega' + \frac{t'(t+\lambda)}{t - t' + \lambda}\big) \cdot \left|\frac{t' + \omega'}{t'}\right|^2 \;. \tag{5.14}$$

Using now

$$\frac{t'}{t' + \omega'} \;=\; \frac{t'}{t' - \frac{t'(t+\lambda)}{t - t' + \lambda}} \;=\; -\frac{t - t' + \lambda}{t'} \;, \tag{5.15}$$

we see finally that

$$U_{j+1}(t) \;=\; \int dt'\, F(\frac{t'(t+\lambda)}{t'-t-\lambda})\; U_j(t') \left|\frac{t'}{t'-t-\lambda}\right|^2 \;. \qquad (5.16)$$

We define the operator T (which depends on F) by

$$(Tu)(t) \;=\; \int dt'\, F(\frac{t'(t+\lambda)}{t'-t-\lambda})(\frac{t'}{t'-t-\lambda})^2 u(t') \;. \qquad (5.17)$$

Note that T maps non-negative functions to non-negative functions, in fact, it is "positivity improving", i.e., $(Tu)(t) > 0$ for all t, if $u(t) \geq 0$ for all t (cf. the remark after (5.26)).

We want to find the fixed points of Eq.(5.16) which is equivalent to finding the fixed points of (5.17). This will allow us to find the density of states of $\mathbf{\Omega}$ by applying ideas which are originally due to Schmidt [11], and have been more recently elaborated by Simon and Taylor [12]. Let $\{e_\omega^N(j)\}_{j=1,\dots,N}$ be the eigenvalues of the system (5.1). If we define

$$L(\lambda) \;=\; \lim_{N\to\infty} \frac{1}{N}\big\{\text{number of } j \,|\, e_\omega^N(j) \;<\; \lambda\big\} \;,$$

then $L(\lambda)$ exists and is almost surely independent of ω, cf. Benderskii, Pastur [1], Kirsch, Martinelli [5].

If we rewrite (5.1) as (5.2), and are given the distribution $G_1(z_1)$, then (5.3) determines $G_n(z_n)$. Using now the condition F3. on F, i.e., that the support of F lies on one side of zero, we see that $\mathbf{\Omega}$ is a definite matrix and by Corollary 2.2 of [12], we have

$$1 - L(-\lambda) \;=\; \lim_{M\to\infty} \frac{1}{M} \sum_{n=1}^{M} \int_0^\infty dz\, G_n(z) \;.$$

(In [12], the operator is considered with opposite signs in the off-diagonal elements, hence we have here $1 - L(-\lambda)$ rather than $L(\lambda)$.) By Eq.(5.7) and the definition of U_n, we have

$$G_n(z) \;=\; \int_{-\infty}^{\infty} d\omega F(\omega)|\omega| U_n(\omega(z-1)) \;.$$

On the other hand, given G_1 we can determine U_1 and then $U_{n+1} = T^n U_1$. Thus,

$$G_{n+1}(z) \;=\; \int_{-\infty}^{\infty} d\omega F(\omega)|\omega| T^n U_1(\omega(z-1)) \;.$$

In particular, we have the

Proposition 5.1. *If* $\lim_{M\to\infty} \frac{1}{M}\sum_{n=0}^{M-1} T^n U_1 = U^*$, *then the integrated density of states for* $\mathbf{\Omega}$ *is given by*

$$1 - L(-\lambda) = \int_0^\infty dz \int_{-\infty}^\infty d\omega\, U^*(\omega(z-1))|\omega| F(\omega)\ .$$

(Recall that T*, and hence* U^* *depend on* λ*.)*

We now prove that the above limit exists. We will use the notation $\|f\|_p = \left(\int |f(x)|^p dx\right)^{1/p}$, for $1 \le p < \infty$, and $\|f\|_\infty = \sup_x |f(x)|$. Our first result is

Theorem 5.2. *Assume that* $\sup_\xi F(\xi)\xi^2 < \infty$ *and that* $\int dx\, F(x) = 1$. *If* $\lambda \neq 0$, *then for every function* U_1 *in* L^1, *with* $\|U_1\|_1 = 1$, *the limit* $\lim_{M\to\infty} \frac{1}{M}\sum_{n=0}^{M-1} T^n U_1$ *exists and is equal to a function* U^* *in* L^1. *In fact,* U^* *is the unique eigenvector of* T *with eigenvalue* 1.

Proof. This proof is similar to the one given by Delyon, Kunz, and Souillard [2]. It is based on the following two lemmas:

Lemma 5.3. *The operator* T *is a contraction on* L^1.

and

Lemma 5.4. *For* $\lambda \neq 0$, *the operator* T^2 *is bounded on* L^∞ *by* const.λ^{-2}, *where the constant depends only on* F.

Proof of Lemma 5.3. We have

$$\|Tu\|_1 \le \int dt \int dt'\, F\Big(\frac{t'(t+\lambda)}{t'-t-\lambda}\Big) \left|\frac{t'}{t'-t-\lambda}\right|^2 |u(t')|\ . \tag{5.18}$$

Changing variables to t' and z with

$$z = \frac{t'(t+\lambda)}{t'-t-\lambda}\ , \tag{5.19}$$

we have

$$dz \;=\; (\frac{t'}{t'-t-\lambda})^2\,dt\ , \tag{5.20}$$

and hence

$$\|Tu\|_1 \;\le\; \int dz\,dt'\,F(z)|u(t')| \;\le\; \|u\|_1\ ,$$

since F is a probability measure. The proof is complete.

Remark. The Eq.(5.20) also shows that if $u \ge 0$, then $\|Tu\|_1 = \|u\|_1$.

Proof of Lemma 5.4. We assume, without loss of generality, that $u \ge 0$. Then

$$\begin{aligned}(Tu)(t) \;&=\; \int dt'\,F(\frac{t'(t+\lambda)}{t'-t-\lambda})\left|\frac{t'}{t'-t-\lambda}\right|^2 u(t') \\ &\le\; \frac{\|u\|_1}{|t+\lambda|^2}\sup_\xi F(\xi)\xi^2 \\ &\equiv\; \frac{B_0\|u\|_1}{|t+\lambda|^2}\ .\end{aligned} \tag{5.21}$$

If $t+\lambda$ is bounded away from 0, (5.21) and Lemma 5.3 lead to a bound for T^2u:

$$|T^2u(t)| \;\le\; \frac{B_0\|u\|_1}{|t+\lambda|^2}\ . \tag{5.22}$$

We next consider the integral

$$(Tu)(t) \;=\; \int dt'\,F(\frac{t'(t+\lambda)}{t'-t-\lambda})\left|\frac{t'}{t'-t-\lambda}\right|^2 u(t')\ , \tag{5.23}$$

which we separate into the regions $|t'| \ge 2|t+\lambda|$ and $|t'| \le 2|t+\lambda|$. It is convenient to introduce the notation $q = t+\lambda$. In the region $|t'| \ge 2|q|$, we find

$$\left|\frac{t'}{t'-t-\lambda}\right| \;\le 2\ ,$$

and hence

$$\int_{|t'|\ge 2|q|} dt'\,F(\frac{t'(t+\lambda)}{t'-t-\lambda})\left|\frac{t'}{t'-t-\lambda}\right|^2 (Tu)(t') \;\le\; 4\|F\|_\infty\|Tu\|_1\ . \tag{5.24}$$

To discuss the region $|t'| \le 2|q|$ we change variables to

$$z = \frac{t'q}{t'-q}\ , \tag{5.25}$$

so that $t' = zq/(z-q)$. We get the bound, using (5.21),

$$\int_{|t'|\le 2|q|} dt'\, F(\frac{t'(t+\lambda)}{t'-t-\lambda}) \left|\frac{t'}{t'-t-\lambda}\right|^2 (Tu)(t')$$

$$= \int_{|z/(z-q)|\le 2} dz\, F(z) \left|\frac{z}{z-q}\right|^2 (Tu)(\frac{zq}{z-q})$$

$$\le \int_{|z/(z-q)|\le 2} dz\, F(z) \left|\frac{z}{z-q}\right|^2 \frac{B_0\|u\|_1}{\left|\frac{zq}{z-q}+\lambda\right|^2}$$

$$= B_0\|u\|_1 \int_{|z/(z-q)|\le 2} dz\, F(z) \left|\frac{z}{zq+z\lambda-q\lambda}\right|^2 .$$

If $|q| > \lambda/4$, the assertion of the lemma follows already from (5.22). So we may assume $|q| < \lambda/4$. This, together with $|z/(z-q)| \le 2$ implies

$$\left|\frac{z}{zq+z\lambda-q\lambda}\right|^2 < \frac{B_1}{\lambda^2} .$$

The proof of Lemma 5.4 is complete.

We now proceed to the **Proof of Theorem 5.2.** We have already seen that

$$\|T\|_1 \le 1 , \tag{a}$$

and that

$$(T^n u) \text{ is uniformly bounded in } L^\infty \text{ for } n = 2, 3, \ldots . \tag{b}$$

By Theorem IV.8.9 in [3], a set $C \subset L^1(\mathbf{R}, \mu(ds))$ is weakly sequentially compact if it is bounded and if for each decreasing sequence of sets $\{E_n\}$ with empty intersection, the limit

$$\lim_{n\to\infty} \int_{E_n} f(s)\mu(ds) = 0 ,$$

is uniform for f in C. If the functions f in C are all uniformly bounded in L^∞, this condition is satisfied. Thus, the sequence $\{T^n U\}_{n\ge 2}$ is weakly sequentially compact for any U in the unit ball of L^1. By (a), $\frac{1}{M}\sum_{n=0}^{M-1} T^n U$ is bounded for U in the unit ball, and $T^n U/n \to 0$ as $n \to \infty$. Since the unit ball is a fundamental set of L^1, these facts, plus Corollary VII.5.3 in [3] imply that

$$\lim_{M\to\infty} \frac{1}{M} \sum_{n=0}^{M-1} T^n U = U^* .$$

Since $\|TU\|_1 = \|U\|_1 = 1$ and T maps positive functions to positive functions, we see that for positive U the limit U^* is not zero. Thus 1 is an eigenvalue of T. Suppose now that the eigenvalue 1 is not simple. Then there exist u and v in L^1, both real, and normalized to $\|u\|_1 = \|v\|_1 = 1$ with $u \neq \pm v$ such that $Tu = u$ and $Tv = v$. Then

$$\|u - v\|_1 \;=\; \|Tu - Tv\|_1 \;=\; \int dt\, \big|(Tu)(t) \;-\; (Tv)(t)\big|$$

$$= \int dt \left| \int dt'\, F\Big(\frac{t'(t+\lambda)}{t'-t-\lambda}\Big)\Big(\frac{t'}{t'-t-\lambda}\Big)^2 \big(u(t') - v(t')\big) \;\right| \;.$$

Since $u \neq v$, and $\|u\|_1 = \|v\|_1$, $u(x) - v(x)$ is not everywhere positive, (nor everywhere negative). Thus, assume $u(x) - v(x) \geq 0$ for $x \in E_1$ and $u(x) - v(x) < 0$ for $x \in E_2$, with E_1 and E_2 of positive measure. By the triangle inequality,

$$\|u - v\|_1 \;\leq\; \int dt \int dt'\, F\Big(\frac{t'(t+\lambda)}{t'-t-\lambda}\Big)\Big(\frac{t'}{t'-t-\lambda}\Big)^2 |u(t') - v(t')| \;,$$

and this inequality is *strict* if

$$\int dt \int_{E_2} dt'\, F\Big(\frac{t'(t+\lambda)}{t'-t-\lambda}\Big)\Big(\frac{t'}{t'-t-\lambda}\Big)^2 |u(t') - v(t')| \;\neq\; 0 \;. \tag{5.26}$$

Note now that (5.26) holds if, for each $t' \in E_2$, we can find a t such that

$$t'(t+\lambda)/(t'-t-\lambda) \in \mathrm{supp}\; F \;.$$

Clearly, such a t always exists, and hence (5.26) holds. Set now $z = t'(t+\lambda)/(t'-(t+\lambda))$. Then

$$dz \;=\; \frac{t'^2\, dt}{(t'-t-\lambda)^2} \;.$$

Thus

$$\|u - v\|_1 \;<\; \int dt' \int dz\, F(z) |u(t') - v(t')| \;=\; \|u - v\|_1 \;.$$

This contradiction implies that $u = v$. The proof of Theorem 5.2 is complete.

Remark. The following argument shows that the operator T^4 is in fact compact: By Definition VI.4.1 in [3] an operator is weakly compact iff it maps bounded sets into weakly sequentially compact sets. By our previous results T^2 is therefore weakly compact. We may now apply Corollary VI.8.13 of [3] which implies that the product of weakly compact operators in L^1 is compact. Thus, the assertion is proven.

6. Numerical Experiments for Symplectic Matrices

In this section, we first outline how numerical experiments could be done to test our theory to its full extent. We then present results of a much less sophisticated test, involving especially the equidistribution hypothesis R1. Our results are encouraging, but some serious open questions remain unsolved.

6.1. A Complete Algorithm

In the preceding sections, we have in fact already outlined the algorithmic aspects of every step of the argument, with the exception of the problem of solving the integral equation leading from the probability density F for the ω_i to the density of the eigenvalues, L of the corresponding matrix Ω. We suggest now a method for doing this, but we have not tested this method.

We want to solve the integral equation $Tu = u$ with T as defined in Eq.(5.17). The kernel of this integral equation is quite singular, and thus the numerical calculation will become inefficient. It is however possible, to devise a numerically acceptable algorithm by considering a system of equations of the form

$$(T^* v_n - \lambda v_n, u) = 0, \quad \text{for } n = 1, \ldots, n_{\max} ,$$

for a suitable basis of vectors v_n. Here, T^* is the adjoint of T and $(\cdot, \cdot)$ is the usual scalar product. The integral $T^* v_n(t)$ can then be computed to high accuracy for any given t by using a conventional integrator, and then we can choose a finite set $t_1, \ldots, t_{n_{\max}}$ of points for which we get a system of linear equations

$$\left(\sum_{i=1}^{n_{\max}} T^* v_n(t_i) - \lambda v_n(t_i) \right) u(t_i) = 0 ,$$

for $i = 1, \ldots, n_{\max}$. This gives a numerical sampling of $u = u_\lambda$. One then integrates, according to A2, and recovers a function $L(\lambda)$.

In practice, there is a more efficient "Monte-Carlo" method, in which one determines numerically the density of states of a matrix of the form of Eq.(1.2), by generating a random matrix of this form on the computer and determining numerically its eigenvalues. One fixes a large N (we have taken up to $N = 1000$, which is possible due to the tri-diagonal form of the problem) and one determines the spectrum of $\mathbf{\Omega}$. The eigenvalues λ_i of $|\mathbf{S}| = (\mathbf{S}^T\mathbf{S})^{1/2}$ are then easily obtained through the relation Eq.(2.1). (One can then even solve the identity Eq.(2.3), by replacing it by the corresponding numerical problem

$$\sum_{j=1}^{2N} \frac{\lambda_j^2}{H(\mu)e^{2\mu} + (1-H(\mu))\lambda_j^2} = 2N \ . \tag{6.1}$$

Note that because of the symplectic nature of $\mathbf{S}$, H satisfies the relation

$$H(-\mu) = 1 - H(\mu) \ ,$$

which we prove in the Appendix.)

6.2. A Simplified Test

We have tested the predictions of the theory for the simplified problem of a product of random matrices which approximate matrices of the form

$$\mathbf{S} = \exp\left(\tau \begin{pmatrix} \mathbf{0} & -\mathbf{\Omega} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}\right) \ . \tag{6.2}$$

The approximation we use, is valid for small τ. It is

$$\mathbf{M}(\mathbf{\Omega}) = \begin{pmatrix} \mathbf{1} & -\tau\mathbf{\Omega} \\ \tau\mathbf{1} & \mathbf{1} \end{pmatrix} \ .$$

Note that the matrices $\mathbf{M}(\mathbf{\Omega})$ are only approximately symplectic. We have performed extensive runs for uniform distributions of the random variables ω_i taking values in intervals $[a,b]$ with $(a+b)/2 \in \{0.25n, n = 1,\ldots,10\}$ and $\tau \in \{0.025k, k = 1,2,3\}$. In these cases, $(a-b)/2$ was chosen to be $0.25m$ with $m = 1,\ldots,n$. The matrix $\mathbf{\Omega}$ was taken 100×100. We computed the largest Liapunov exponent in all these cases and compared it to the prediction given by A4. We simultaneously tested the statistical independence of the components of the image vectors as follows. We start with an initial vector, whose components are normally distributed, and apply 10000 matrices $\mathbf{M}(\mathbf{\Omega})$ to it, keeping the resulting vector normalized at all times. For the final vector, we consider the components w_i and compute $v_i = w_i - \langle w \rangle$, where $\langle \cdot \rangle$ denotes the average over the components. We then compute $\Delta = \langle w^4 \rangle - 3\langle w^2 \rangle^2$. We also repeated this calculation for 20000 up to 40000 matrices. For those values of the parameters n, m, and k for which Δ is about as small as it can be for a normal distribution, we find that the value of the largest Liapunov exponent predicted by our theory is off from the numerically computed value by between a factor 1.01 and 2 with no apparent correlation with Δ. We have no explanation for this discrepancy with our theory. On the other hand, if Δ really reflects a non-normal distribution, then the result is off by a factor of around 10, which is more satisfactory. This seems to happen especially when the random variables have support near 0.

In the paper [10], Paladin and Vulpiani do not consider matrices of the form (1.1) but rather matrices of the form

$$\mathbf{S}'(\mathbf{\Omega}) = \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{\Omega} & \mathbf{1}+\mathbf{\Omega} \end{pmatrix} \tag{6.3}$$

which are easily seen to be symplectic. One should think of $\mathbf{S}'$ as the *time one* integral of the matrices of (1.1). Since, under our assumptions E1–E3, the matrices change rapidly in time, these matrices are not a good approximation for the situation we consider. They are, of course, interesting in their own right, but we do not expect our theory to apply in this case. It thus remains mysterious why the experimentally observed Liapunov spectrum systems is so close to that of the systems considered above.

The question of the *shape* of the function H representing the density of Liapunov exponents seems to be a less delicate question than the estimate of their absolute size. In fact, using the identity (6.1), one observes that for a very large class of distributions of the λ_i one obtains mildly S-shaped, but essentially straight lines and this agrees well with the known general behaviour in all examples, in other words, our theory confirms and explains the *essentially uniform density* of Liapunov exponents.

Appendix

In this appendix, we prove that the integrated density of Liapunov exponents H satisfies $H(-\mu) = 1 - H(\mu)$ if the eigenvalues of the random matrices come in pairs λ and $1/\lambda$.

Let K be the integrated density of eigenvalues. We consider a density $K' : \mathbf{R}^+ \to \mathbf{R}^+$ satisfying

$$\int_0^\infty da\, K'(a) \;=\; 1 \;. \tag{A.1}$$

The symmetry $\lambda \leftrightarrow 1/\lambda$ implies

$$K'(a) \;=\; K'(1/a)\, a^{-2} \;. \tag{A.2}$$

Note that by (A.2),

$$\int_0^1 da\, K'(a) f(a) \;=\; \int_1^\infty \frac{dx}{x^2} K'(1/x) f(x) \;=\; \int_1^\infty dx\, K'(1/x) f(1/x) \;.$$

Define

$$B \;=\; \int_1^\infty (a + a^{-1}) da\, K'(a)$$

Assume $b \in [B^{-1}, B]$ and assume that $x = x(b)$ is a non–zero solution of the equation

$$\int_1^\infty \Big(\frac{a}{bx + (1-x)a} + \frac{a^{-1}}{bx + (1-x)a^{-1}}\Big) da\, K'(a) \;=\; 1 \;. \tag{A.3}$$

Lemma A.1. *We have the relation* $x(b^{-1}) \;=\; 1 - x(b)$.

Proof. Assuming that b and $x = x(b)$ are given we deduce from Eq.(A.3) that

$$1 \;=\; \int_1^\infty da\, K'(a) \frac{\alpha b x \;+\; 2(1-x)}{b^2x^2 \;+\; (1-x)^2 + \alpha b x(1-x)} \;, \tag{A.4}$$

where $\alpha = a + a^{-1}$. Subtracting 1 on both sides we find

$$\begin{aligned} 0 \;&=\; \int_1^\infty da\, K'(a)\; \frac{\alpha b x + 2(1-x) - 2b^2x^2 - 2(1-x)^2 - 2\alpha b x(1-x)}{b^2x^2 \;+\; (1-x)^2 + \alpha b x(1-x)} \\ &=\; x \cdot \int_1^\infty da\, K'(a)\; \frac{2(1-x) - 2b^2 x - \alpha b + 2\alpha b x}{b^2x^2 \;+\; (1-x)^2 + \alpha b x(1-x)} && (A.5) \\ &=\; \int_1^\infty da\, K'(a)\; \frac{2b^{-2}(1-x) - 2x - \alpha b^{-1} + 2\alpha b^{-1} x}{x^2 + b^{-2}(1-x)^2 + \alpha b^{-1} x(1-x)} \;. && (A.6) \end{aligned}$$

Upon setting $x = 1 - y$ and $b^{-1} = c$ we see that the expression in Eq.(A.6) equals

$$
\begin{aligned}
0 &= \int_1^\infty da\, K'(a)\, \frac{2c^2y - 2(1-y) - \alpha c + 2\alpha c(1-y)}{c^2y^2 \;+\; (1-y)^2 + \alpha cy(1-y)} \\
&= -\int_1^\infty da\, K'(a)\, \frac{-2c^2y + 2(1-y) - \alpha c + 2\alpha cy}{c^2y^2 \;+\; (1-y)^2 + \alpha cy(1-y)}\ .
\end{aligned}
\tag{A.7}
$$

Working our way back to Eq.(A.4), the assertion follows.

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References

[1] M. Benderskii and L. Pastur: On the spectrum of the one dimensional Schrödinger equation with a random potential. Mat. Sb. **82**, 245–256 (1970).

[2] F. Delyon, H. Kunz, and B. Souillard: One-dimensional wave equations in disordered media. J. Phys. **A16**, 25–42 (1983).

[3] N. Dunford and J.T. Schwartz: "*Linear Operators*", New York, Interscience Publishers (1958).

[4] J.-P. Eckmann and D. Ruelle: Ergodic theory of chaos and strange attractors. Rev. Mod. Phys. **57**, 617–656 (1985).

[5] W. Kirsch and F. Martinelli: On the density of states of Schrödinger operators with a random potential. J. Phys. **A15**, 2139–2156 (1982).

[6] R. Livi, M. Pettini, S. Ruffo, M. Sparpaglione, and A. Vulpiani: Equipartition threshold in non linear large Hamiltonian systems: The Fermi-Pasta-Ulam model. Phys. Rev. **A31**, 1039–1045 (1985).

[7] R. Livi, A. Politi, and S. Ruffo: Distribution of characteristic exponents in the thermodynamic limit. J. Phys. **A19**, 2033–2040 (1986).

[8] C.M. Newman: The distribution of Liapunov exponents. Commun. Math. Phys. **103**, 121–126 (1986).

[9] C.M. Newman: Liapunov exponents for some products of random matrices: Exact expressions and asymptotic distributions. In "*Random Matrices and Their Applications*", (J.E. Cohen, H. Kesten, and C.M. Newman, Eds.). AMS Providence, Contemporary Mathematics **50**, 121 (1986).

[10] G. Paladin and A. Vulpiani: Scaling law and asymptotic distribution of Lyapunov exponents in conservative dynamical systems with many degrees of freedom. J. Phys. **A19**, 1881–1888 (1986).

[11] H. Schmidt: Disordered one-dimensional crystals. Phys. Rev. **105**, 425–441 (1957).

[12] B. Simon and M. Taylor: Harmonic analysis on $SL(2,\mathbf{R})$ and smoothness of the density of states in the one-dimensional Anderson model. Commun. Math. Phys. **101**, 1–19 (1985).

[13] K.W. Wachter: The strong limits of random matrix spectra for sample matrices of independent elements. Ann. of Prob. **6**, 1–18 (1978).