JOURNAL OF FUNCTIONAL ANALYSIS 13, 1–12 (1973)

An Application of Tomita's Theory of Modular Hilbert Algebras: Duality for Free Bose Fields

JEAN-PIERRE ECKMANN*

Université de Genève, Switzerland

AND

KONRAD OSTERWALDER**

Harvard University, Cambridge, Massachusetts 02138

Communicated by J. Dixmier

Received January 10, 1972

Tomita's theory of modular Hilbert algebras is applied to prove the duality theorem for local von Neumann algebras in a free field theory.

The algebraic relation between a von Neumann algebra R and its commutant R' has found its deepest analysis in Tomita's theory of modular Hilbert algebras, which has been elaborated and applied to many interesting problems by Takesaki [11]. In [4] Dixmier has given a short and very clear summary of the theory.

In this paper we give a further application of Tomita's theory: we use it to prove the duality theorem for local algebras in a quantum theory of free Bose fields. In axiomatic field theory one associates with every region O in spacetime a von Neumann algebra $R(O)$. Let O' be the causal complement of O. Then duality means that the commutant $R(O)'$ of $R(O)$ is equal to $R(O')$.

By 1964 Araki [1] had already shown that duality for the free Bose field holds. Subsequently other proofs have been given in references [3] and [10].

The purpose of this paper is to give more insight into the mechanism of Tomita's theory and to show that for the particular example we are

* Supported by the Fonds National Suisse.

** Supported by National Science Foundation GP 31239X.

looking at, all relevant quantities of this theory can be computed explicitly.

We first give a short review of the Fock space formalism of Cook [2]. Let $\mathscr{K}$ be a real Hilbert space and set

$$\mathscr{F}_r^{(0)} = \mathbb{R}$$

and

$$\mathscr{F}_r^{(n)} = \left(\bigotimes_n \mathscr{K}\right)_s,$$

the symmetrized n-fold tensor product of $\mathscr{K}$. Then we define

$$\mathscr{F}_r = \bigoplus_{n=0}^{\infty} \mathscr{F}_r^{(n)}$$

and the Fock space $\mathscr{F}$ is the complexification $\mathscr{F} = \mathscr{F}_r + i\mathscr{F}_r$ of $\mathscr{F}_r$. Also we write $\mathscr{F}^{(n)}$ for $\mathscr{F}_r^{(n)} + i\mathscr{F}_r^{(n)}$. Given $f \in \mathscr{K}$ we define the creation operator $a^*(f)$ on $\mathscr{F}$ with $a^*(f)\colon \mathscr{F}^{(n)} \to \mathscr{F}^{(n+1)}$ by

$$a^*(f)\,\Psi_n = (n+1)^{1/2}\,\mathrm{sym}(f \otimes \Psi_n) \tag{1}$$

for $\Psi_n \in \mathscr{F}^{(n)}$, and sym means symmetrization. The adjoint of $a^*(f)$ is denoted by $a(f)$. On $\mathscr{D}_0 = \bigcup_{N=0}^{\infty} \bigoplus_{n=0}^{N} \mathscr{F}^{(n)}$ the operators $a(f)$ and $a^*(g)$ obey the canonical commutation relations. Define for g, $f \in \mathscr{K}$ the operators

$$\begin{aligned} \varphi_0(f) &= \frac{1}{\sqrt{2}}\,(a^*(f) + a(f)), \\ \pi_0(g) &= \frac{-i}{\sqrt{2}}\,(a^*(g) - a(g)). \end{aligned} \tag{2}$$

Then $\varphi_0(f)$ and $\pi_0(g)$ have all of $\mathscr{D}_0$ as analytic vectors, and are essentially selfadjoint on this domain. Their closures, denoted by $\varphi(f)$ and $\pi(g)$ respectively, satisfy on $\mathscr{D}_0$

$$[\varphi(f), \pi(g)] = -i(f, g),$$
$$[\varphi(f), \varphi(f')] = [\pi(g), \pi(g')] = 0.$$

By a theorem of Nelson [9], we are allowed to write for any core $\mathscr{E}$ of analytic vectors

$$e^{i\varphi(f)}\Phi = \sum_{n=0}^{\infty} \frac{(i\varphi(f))^n}{n!}\,\Phi, \qquad \text{for} \quad \Phi \in \mathscr{E},$$

and similarly for $\varphi(f)$ replaced by $\pi(g)$. We denote the element 1 in $\mathscr{F}^{(0)}$ by Ω (the Fock vacuum).

Note that by (1) and (2) we may identify

$$\varphi(f)\Omega = (1/\sqrt{2})f \in \mathscr{F}_r^{(1)}, \qquad i\pi(g)\Omega = (1/\sqrt{2})g \in \mathscr{F}_r^{(1)}, \tag{3}$$

for $f, g \in \mathscr{K}$.

We now define the von Neumann algebras $R(\mathscr{K}_1, \mathscr{K}_2)$. Let $\mathscr{K}_1$ and $\mathscr{K}_2$ be closed subspaces of $\mathscr{K}$ and denote by $\mathscr{K}_1{}^\perp$ and $\mathscr{K}_2{}^\perp$ their orthogonal complement in $\mathscr{K}$, respectively. We shall always assume that any two of the spaces $\mathscr{K}_1$, $\mathscr{K}_2$, $\mathscr{K}_1{}^\perp$, and $\mathscr{K}_2{}^\perp$ have zero intersection. Then we define $R(\mathscr{K}_1, \mathscr{K}_2)$ to be the von Neumann algebra generated by

$$\{e^{i\varphi(f)}e^{i\pi(g)}, f \in \mathscr{K}_1, g \in \mathscr{K}_2\}.$$

The following properties of $R(\mathscr{K}_1, \mathscr{K}_2)$ are well known (see e.g. [1]).

LEMMA 1. *Under the above assumptions the set of vectors* $R(\mathscr{K}_1, \mathscr{K}_2)\Omega$ *is dense in* $\mathscr{F}$ *(Reeh–Schlieder theorem). Furthermore*

$$R(\mathscr{K}_2^\perp, \mathscr{K}_1^\perp) \subset R(\mathscr{K}_1, \mathscr{K}_2)' \qquad (locality).$$

The vector Ω *is thus cyclic and separating for* $R(\mathscr{K}_1, \mathscr{K}_2)$.

The theorem we want to prove is the following.

THEOREM 2. *Under the above assumptions*

$$R(\mathscr{K}_2^\perp, \mathscr{K}_1^\perp) = R(\mathscr{K}_1, \mathscr{K}_2)' \qquad (duality).$$

Remark. In [1], Araki considers the more general situation where the zero intersection property for $\mathscr{K}_1$, $\mathscr{K}_2$, $\mathscr{K}_1{}^\perp$, and $\mathscr{K}_2{}^\perp$ does not necessarily hold. He shows that by a simple analysis one can reduce the problem to the special case considered in this paper. It is easy to see that the local algebras $R(O)$ usually considered by physicists are of the form $R(\mathscr{K}_1, \mathscr{K}_2)$, and the zero intersection property holds (see [1]).

As we want to use Tomita's theory to prove Theorem 2, we recapitulate those parts of it which enter the proof explicitly. However, we assume the reader is familiar with Takesaki's lecture notes [11].

THEOREM 3 (part of Theorem 12.1 in [11]). *Let* R *be a von Neumann algebra acting on a Hilbert space* $\mathscr{F}$, *admitting a cyclic and separating vector* Ω *in* $\mathscr{F}$. *Then there exists an isometric antilinear involution* J *in* $\mathscr{F}$ *such that*

$$JRJ = R' \qquad and \qquad JR'J = R.$$

In order to construct the operator J one first shows that $R\Omega = \{A\Omega, A \in R\}$ is an achieved generalized Hilbert algebra. For $A \in R$ put $\Phi(A) = A\Omega$. Define a product and an involution in $R\Omega = \mathfrak{A}$ by

$$\Phi(A)\Phi(B) = \Phi(AB), \qquad A, B \in R$$
$$\Phi(A)^{\#} = \Phi(A^*), \qquad A \in R.$$

$\mathfrak{A}$ is a generalized Hilbert algebra by Theorem 12.1 in [11]. Using Takesaki's notation we have $\pi(\mathfrak{A}'') \subset \mathscr{L}(\mathfrak{A}) = R = \pi(\mathfrak{A}) \subset \pi(\mathfrak{A}'')$. Thus $\pi(\mathfrak{A}'') = \pi(\mathfrak{A})$ or $\mathfrak{A}'' = \mathfrak{A}$, which proves that $\mathfrak{A} = R\Omega$ is achieved (definition 5.1 in [11]). According to Tomita's fundamental theorem (Theorem 10.1 in [11]) there is an isometric involution J such that $J\mathscr{L}(\mathfrak{A})J = \mathscr{L}(\mathfrak{A})'$ or $JRJ = R'$. Its explicit construction is given in [11, Section 7]: Denote by S the closure of the involution $\Phi(A) \to \Phi(A)^{\#}$, $A \in R$. The domain of S, which is a dense subset of $\mathscr{F}$, is called $\mathscr{D}^{\#}$. With the inner product

$$(\Phi, \Psi)_{\#} = (\Phi, \Psi) + (S\Psi, S\Phi), \qquad \Phi, \Psi \in \mathscr{D}^{\#},$$

$\mathscr{D}^{\#}$ becomes a Hilbert space which we will denote by $\mathscr{D}_h^{\#}$. Let I be the natural injection of $\mathscr{D}_h^{\#}$ into $\mathscr{F}$. Then [11, p. 31] there exists a bounded positive operator H on $\mathscr{D}_h^{\#}$ such that

$$(H\Phi, \Psi)_{\#} = (I\Phi, I\Psi), \qquad \text{for} \quad \Phi, \Psi \in \mathscr{D}_h^{\#}, \tag{4}$$

and $H = I^*I$. Let $I = VH^{1/2} = K^{1/2}V$ be the polar decomposition of I, where $K = II^*$ is a bounded positive operator in $\mathscr{F}$ and V is an isometry from $\mathscr{D}_h^{\#}$ onto $\mathscr{F}$. Then [11, p. 33]

$$\begin{aligned} J &= VI^{-1}SIV^* \\ &= (K^{-1/2}I)I^{-1}SI(I^{-1}K^{1/2}) \end{aligned} \tag{5}$$

is the isometric involution of Theorem 3. (Note that in [11] the operator I is sometimes omitted).

To prove duality (Theorem 2) we shall subsequently compute the operators S, K, V, and J associated with the von Neumann algebra $R(\mathscr{K}_1, \mathscr{K}_2)$ and the vector Ω, which is cyclic and separating for $R(\mathscr{K}_1, \mathscr{K}_2)$. Then we use Theorem 3, which says that

$$R(\mathscr{K}_1, \mathscr{K}_2)' = JR(\mathscr{K}_1, \mathscr{K}_2)J.$$

It will turn out that there is an isometric involution $\hat{J}$ on $\mathscr{K}$ ($\hat{J}$ is the restriction of J to the real one particle subspace $\mathscr{F}_r^{(1)} = \mathscr{K}$ of $\mathscr{F}$)

such that $\hat{J}\mathscr{K}_1 = \mathscr{K}_2{}^{\perp}$, $\hat{J}\mathscr{K}_2 = \mathscr{K}_1{}^{\perp}$ and for any $f \in \mathscr{K}_1$, $g \in \mathscr{K}_2$ we find

$$\begin{aligned} Je^{i\varphi(f)}J &= e^{i\varphi(\hat{J}f)}, \\ Je^{i\pi(g)}J &= e^{i\pi(\hat{J}g)}. \end{aligned} \tag{6}$$

This obviously suffices to prove that $R(\mathscr{K}_1, \mathscr{K}_2)' = R(\mathscr{K}_2{}^{\perp}, \mathscr{K}_1{}^{\perp})$, which is Theorem 2.

We start our computations by deriving a suitable formula for $K = IHI^{-1}$. Define the "linear version" S_0 of the antilinear operator S as the linear extension of $S \upharpoonright \mathscr{D}^{\#} \cap \mathscr{F}_r$ (the restriction of S to the real part of its domain) to $\mathscr{D}^{\#}$. Obviously S_0 is closed and

$$(S\Psi, S\Phi) = (S_0\Phi, S_0\Psi) \qquad \text{for} \quad \Phi, \Psi \in \mathscr{D}^{\#}.$$

Using S_0 instead of S we can rewrite the defining Eq. (4) for H as

$$\begin{aligned} (H\Psi, \Phi)_{\#} &= (IH\Psi, I\Phi) + (SI\Phi, SIH\Psi) \\ &= (IH\Psi, I\Phi) + (S_0 IH\Psi, S_0 I\Phi) \\ &= (I\Psi, I\Phi). \end{aligned}$$

Comparing the last two lines we conclude that

$$IHI^{-1} = K = (1 + S_0{}^*S_0)^{-1}, \tag{7}$$

(see e.g. [7, p. 1245]). This is the desired expression for K. Note that K is a bounded linear operator; thus it is sufficient to compute it on vectors in $\mathscr{F}_r$.

We continue by exhibiting a structure which is typical for the Fock representation.

LEMMA 4. *Let $E^{(k)}$ be the projection onto $\mathscr{F}^{(k)}$, $k = 0, 1,\ldots$ Then*

$$SE^{(k)} \supset E^{(k)}S. \tag{8}$$

The restriction $S^{(k)}$ of S to $E^{(k)}\mathscr{D}^{\#}$ is a densely defined closed operator in $\mathscr{F}^{(k)}$ with domain $E^{(k)}\mathscr{D}^{\#} = \mathscr{D}^{\#} \cap \mathscr{F}^{(k)}$. It leaves $\mathscr{D}^{\#} \cap \mathscr{F}_r^{(k)}$ invariant. The bounded operators K, V and J commute with $E^{(k)}$ and leave $\mathscr{F}_r^{(k)}$ invariant.

Proof. As we need Lemma 4 only for $k = 0, 1$ we restrict our proof to these two cases. The proof for general k is similar. By definition of S, $S\Omega = \Omega$; thus for $k = 0$, Eq. (8) is trivial.

Let R_0 be the *-algebra generated by

$$\{e^{i(\varphi(f)+\pi(g))}, f \in \mathscr{K}_1, g \in \mathscr{K}_2\}.$$

Then $R_0'' = R(\mathscr{K}_1, \mathscr{K}_2)$ and $R_0\Omega$ is dense in $R(\mathscr{K}_1, \mathscr{K}_2)\Omega$. Namely for $A \in R(\mathscr{K}_1, \mathscr{K}_2)$ there is a sequence A_n, $n = 1, 2,...$, of operators in R_0 such that $A = w - \lim_{n\to\infty} A_n$ and $A^* = w - \lim_{n\to\infty} A_n{}^*$. Thus $\Phi(A) = A\Omega = \lim_{n\to\infty} \Phi(A_n)$ and $\Phi(A)^\# = \lim_{n\to\infty} \Phi(A_n)^\#$, which is equivalent to $\Phi(A) = \# - \lim_{n\to\infty} \Phi(A_n)$. This implies that $R_0\Omega$ is also dense as a subset of $\mathscr{D}_h{}^\#$. Let

$$\Phi = e^{i(\varphi(f)+\pi(g))}\Omega, \tag{9}$$

for some $f \in \mathscr{K}_1$, $g \in \mathscr{K}_2$. Then

$$\begin{aligned} E^{(1)}\Phi &= ic(\varphi(f) + \pi(g))\Omega \\ &= c \lim_{\lambda\to 0} \lambda^{-1}(e^{\lambda i(\varphi(f)+\pi(g))} - 1)\Omega, \end{aligned} \tag{10}$$

where $c = \exp(-\frac{1}{4}\|f - ig\|^2)$, but also

$$\begin{aligned} SE^{(1)}\Phi &= c \lim_{\lambda\to 0} S\lambda^{-1}(e^{\lambda i(\varphi(f)+\pi(g))} - 1)\Omega \\ &= c \lim_{\lambda\to 0} \lambda^{-1}(e^{-\lambda i(\varphi(f)+\pi(g))} - 1)\Omega \\ &= -ic(\varphi(f) + \pi(g))\Omega \\ &= E^{(1)}S\Phi. \end{aligned} \tag{11}$$

Thus $E^{(1)}\Phi$ is in $\mathscr{D}^\#$, the domain of S, and as all vectors in $R_0\Omega$ are finite linear combinations of vectors of the form (9), we conclude that

$$E^{(1)}R_0\Omega \subset \mathscr{D}^\#.$$

Equation (11) shows that

$$E^{(1)}SA_0\Omega = SE^{(1)}A_0\Omega, \qquad \text{for all} \quad A \in R_0. \tag{12}$$

Now let Φ be an arbitrary vector in $\mathscr{D}^\#$. Then there exists a sequence $A_n \in R_0$, $n = 1, 2...$, such that

$$\Phi = \# - \lim_{n\to\infty} A_n\Omega, \qquad \text{i.e.,}$$

$$\Phi = \lim_{n\to\infty} A_n\Omega \qquad \text{and} \qquad S\Phi = \lim_{n\to\infty} SA_n\Omega.$$

Hence

$$\lim_{n\to\infty} E^{(1)}A_n\Omega = E^{(1)}\Phi,$$

and

$$\begin{aligned} E^{(1)}S\Phi &= \lim_{n\to\infty} E^{(1)}SA_n\Omega \\ &= \lim_{n\to\infty} SE^{(1)}A_n\Omega, \qquad \text{using Eq. (12),} \\ &= SE^{(1)}\Phi, \qquad \text{as} \quad S \text{ is closed.} \end{aligned}$$

Thus $SE^{(1)} \supset E^{(1)}S$, which is (8) for $k = 1$. The above computations also show that

$$E^{(1)}\mathscr{D}^{\#} = \mathscr{D}^{\#} \cap \mathscr{F}^{(1)}.$$

The operator $S^{(1)}$ is certainly closed. To show that its domain $E^{(1)}\mathscr{D}^{\#}$ is dense in $\mathscr{F}^{(1)}$ it suffices to prove that $E^{(1)}R_0\Omega(\subset E^{(1)}\mathscr{D}^{\#})$ is dense. We note that

$$\begin{aligned} E^{(1)}R_0\Omega &= \{(\varphi(f_1) + i\varphi(f_2) + \pi(g_1) + i\pi(g_2))\Omega, f_i \in \mathscr{K}_1, g_i \in \mathscr{K}_2\} \\ &= \left\{\frac{1}{\sqrt{2}}(f_1 + if_2 - ig_1 + g_2), f_i \in \mathscr{K}_1, g_i \in \mathscr{K}_2\right\} \\ &= \text{complexification of } (\mathscr{K}_1 + \mathscr{K}_2). \end{aligned} \tag{13}$$

The first line of Eq. (13) holds because any vector in $R_0\Omega$ is a finite linear combination (with complex coefficients) of vectors of the form (9) and because of Eq. (10). The second line follows from Eq. (3). By assumption $\mathscr{K}_1{}^\perp \cap \mathscr{K}_2{}^\perp = \{0\}$; thus $\mathscr{K}_1 + \mathscr{K}_2$ is dense in $\mathscr{K}$ and $E^{(1)}R_0\Omega$ is dense in the complexification of $\mathscr{K}$, which is $\mathscr{F}^{(1)}$.

Now take $\Phi \in E^{(1)}R_0\Omega$,

$$\begin{aligned} \Phi &= \frac{1}{\sqrt{2}}(f_1 + if_2 - ig_1 + g_2), \qquad f_i \in \mathscr{K}_1, \quad g_i \in \mathscr{K}_2 \\ &= (\varphi(f_1) + i\varphi(f_2) + \pi(g_1) + i\pi(g_2))\Omega. \end{aligned} \tag{14}$$

Then

$$\begin{aligned} S^{(1)}\Phi &= S \lim_{\lambda\to 0} \lambda^{-1}[-i(e^{\lambda i(\varphi(f_1)+\pi(g_1))} - 1) + (e^{\lambda i(\varphi(f_2)+\pi(g_2))} - 1)]\Omega \\ &= \lim_{\lambda\to 0} \lambda^{-1}[i(e^{-\lambda i(\varphi(f_1)+\pi(g_1))} - 1) + (e^{-\lambda i(\varphi(f_2)+\pi(g_2))} - 1)]\Omega \\ &= (\varphi(f_1) - i\varphi(f_2) + \pi(g_1) - i\pi(g_2))\Omega \\ &= \frac{1}{\sqrt{2}}(f_1 - if_2 - ig_1 - g_2). \end{aligned} \tag{15}$$

Thus $S^{(1)}$ leaves $\mathscr{D}^{\#} \cap \mathscr{F}_r^{(1)} = \mathscr{K}_1 + \mathscr{K}_2$ invariant. The last part of

Lemma 4 follows easily from the simple relations involving S and $\#$-scalar products which define K, V, and J. This concludes the proof of the lemma.

Remark. The proof of Lemma 4 shows that $E^{(1)}R_0\Omega = \mathscr{K}_1 + \mathscr{K}_2$ is a core for $S^{(1)}$. In fact one can show that $\mathscr{K}_1 + \mathscr{K}_2$ is equal to $\mathscr{D}(S^{(1)})$.

In view of Lemma 4 it is possible and as we will see later also sufficient to construct the restrictions $\hat{K}$, $\hat{V}$, and $\hat{J}$ of K, V, and J to the real one particle space $\mathscr{F}_r^{(1)}$ only. The restriction $\hat{S}$ of S to $\mathscr{F}_r^{(1)}$ is identical with the restriction of S_0 to $\mathscr{F}_r^{(1)}$. We use the following lemma.

LEMMA 5. *Let $\mathscr{K}_1$ and $\mathscr{K}_2$ be closed subspaces of a Hilbert space $\mathscr{K}$ such that any two of $\mathscr{K}_1$, $\mathscr{K}_2$, $\mathscr{K}_1^\perp$, and $\mathscr{K}_2^\perp$ have only the zero vector in common (i.e., $\mathscr{K}_1$ and $\mathscr{K}_2$ are in a "generic position"). Then there exists a Hilbert space $\mathscr{K}_*$ and there exists a positive contraction T on $\mathscr{K}_*$ with* $\ker T = \ker(1 - T) = 0$, *such that the pair $\{\mathscr{K}_1, \mathscr{K}_2\}$ is unitarily equivalent to the pair $\{$graph T, graph $(-T)\}$.*

Remark. There is a large literature on pairs of subspaces of a Hilbert space, but the basic reference is J. Dixmier's thesis [5] and a paper related to it [6]. Our Lemma 5 is quoted from a paper by Halmos [8], where a proof is given.

Lemma 5 makes the computation of $\hat{K}$, $\hat{V}$, and $\hat{J}$ simple. We may identify the space $\mathscr{K}$ with a direct sum $\mathscr{K}_* \oplus \mathscr{K}_*$. Using vector notation, i.e., $\binom{h}{0} \in \mathscr{K}_* \oplus 0$, $\binom{0}{h} \in 0 \oplus \mathscr{K}_*$ for $h \in \mathscr{K}_*$ we conclude from Lemma 5 that

$$\begin{aligned} \mathscr{K}_1 &= \left\{\begin{pmatrix} h \\ Th \end{pmatrix}, h \in \mathscr{K}_*\right\}, & \mathscr{K}_1^\perp &= \left\{\begin{pmatrix} -Th \\ h \end{pmatrix}, h \in \mathscr{K}_*\right\}, \\ \mathscr{K}_2 &= \left\{\begin{pmatrix} h \\ -Th \end{pmatrix}, h \in \mathscr{K}_*\right\}, & \mathscr{K}_2^\perp &= \left\{\begin{pmatrix} Th \\ h \end{pmatrix}, h \in \mathscr{K}_*\right\}. \end{aligned} \tag{16}$$

Using Eqs. (14) and (15) we get

$$\hat{S}\begin{pmatrix} h \\ Th \end{pmatrix} = \begin{pmatrix} h \\ Th \end{pmatrix}; \qquad \hat{S}\begin{pmatrix} h \\ -Th \end{pmatrix} = -\begin{pmatrix} h \\ -Th \end{pmatrix},$$

or equivalently

$$\hat{S}\begin{pmatrix} h \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ Th \end{pmatrix}, \qquad \hat{S}\begin{pmatrix} 0 \\ Th \end{pmatrix} = \begin{pmatrix} h \\ 0 \end{pmatrix},$$

which allows us to write $\hat{S}$ as

$$\hat{S} = \begin{pmatrix} 0 & T^{-1} \\ T & 0 \end{pmatrix}, \tag{17}$$

and its domain $\mathscr{D}(\hat{S})$ as $\{\binom{h_1}{Th_2}, h_1, h_2 \in \mathscr{K}_*\}$. Now we get

$$\hat{S}^* = \begin{pmatrix} 0 & T \\ T^{-1} & 0 \end{pmatrix}, \qquad \text{and} \qquad \hat{K}^{-1} = 1 + \hat{S}^*\hat{S} = \begin{pmatrix} 1 + T^2 & 0 \\ 0 & 1 + T^{-2} \end{pmatrix}$$

on

$$\mathscr{D}(\hat{K}^{-1}) = \left\{ \begin{pmatrix} h_1 \\ T^2 h_2 \end{pmatrix}, h_1, h_2 \in \mathscr{K}_* \right\},$$

which is dense in $\mathscr{K}$. Now we find immediately

$$\begin{aligned} \hat{K}^{1/2} &= \begin{pmatrix} (1 + T^2)^{-1/2} & 0 \\ 0 & (1 + T^{-2})^{-1/2} \end{pmatrix}, \\ \hat{K}^{-1/2} &= \begin{pmatrix} (1 + T^2)^{1/2} & 0 \\ 0 & (1 + T^{-2})^{1/2} \end{pmatrix}. \end{aligned} \tag{18}$$

$\hat{K}^{1/2}$ is a bounded operator on $\mathscr{K}$ and $I^{-1}\hat{K}^{1/2} = \hat{V}^*$ is an isometry from $\mathscr{K}$ onto the real one particle subspace $\mathscr{D}_{hr}^{\#(1)}$ of $\mathscr{D}_h{}^{\#}$. On the other hand $\hat{K}^{-1/2}I = \hat{V}$ is an isometry from $\mathscr{D}_{hr}^{\#(1)}$ onto $\mathscr{K}$ and $I^{-1}\hat{S}I$ is a bounded operator on $\mathscr{D}_{hr}^{\#(1)}$. Hence we can write, according to Eq. (5),

$$\begin{aligned} \hat{J} &= (\hat{K}^{-1/2}I)I^{-1}\hat{S}I(I^{-1}\hat{K}^{1/2}) \\ &= \begin{pmatrix} (1 + T^2)^{1/2} & 0 \\ 0 & (1 + T^{-2})^{1/2} \end{pmatrix} \begin{pmatrix} 0 & T^{-1} \\ T & 0 \end{pmatrix} \begin{pmatrix} (1 + T^2)^{-1/2} & 0 \\ 0 & (1 + T^{-2})^{-1/2} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{19}$$

Remember that we have $\hat{J}$ defined to be the restriction of the isometric antilinear involution J to the real one particle subspace $\mathscr{F}_r^{(1)} = \mathscr{K}$. To find out what it does to elements in $\mathscr{K}_1$ we take $f \in \mathscr{K}_1$. Then according to (16) there is an $h \in \mathscr{K}_*$ such that

$$f = \begin{pmatrix} h \\ Th \end{pmatrix} \qquad \text{and} \qquad \hat{J}f = \begin{pmatrix} Th \\ h \end{pmatrix} \in \mathscr{K}_2^{\perp},$$

by Eq. (19). Similarly for $g \in \mathscr{K}_2$ we find an $h \in \mathscr{K}_*$, such that

$g = \binom{h}{-Th}$ and $\hat{J}g = \binom{-Th}{h} \in \mathscr{K}_1^{\perp}$. Using $J\Omega = \Omega$ we get for $f \in \mathscr{K}_1$, $g \in \mathscr{K}_2$,

$$J\varphi(f)\, J\Omega = \frac{1}{\sqrt{2}}\, \hat{J}f = \varphi(\hat{J}f)\Omega,$$

and

$$J\pi(g)\, J\Omega = \frac{-i}{\sqrt{2}}\, \hat{J}g = \pi(\hat{J}g)\Omega. \tag{20}$$

LEMMA 6. *Let* $f \in \mathscr{K}_1$, $g \in \mathscr{K}_2$. *Then*

$$J\varphi(f)J = \varphi(\hat{J}f), \qquad J\pi(g)J = \pi(\hat{J}g), \tag{21}$$

where $\hat{J}f \in \mathscr{K}_2^{\perp}$ *and* $\hat{J}g \in \mathscr{K}_1^{\perp}$.

Proof. Let A be an arbitrary operator in $R(\mathscr{K}_1, \mathscr{K}_2)$. Then using $J^2 = 1$ and $J\Omega = \Omega$ we get

$$AJ\varphi(f)\, J\Omega = J(JAJ)\, \varphi(f)\Omega,$$

but $JAJ \in R(\mathscr{K}_1, \mathscr{K}_2)'$ by Tomita's theory, thus

$$JAJ\varphi(f) \subset \varphi(f)\, JAJ,$$

($\varphi(f)$, $f \in \mathscr{K}_1$, is affiliated with $R(\mathscr{K}_1, \mathscr{K}_2)$). Thus

$$\begin{aligned} J\varphi(f)\, JA\Omega &= J\varphi(f)(JAJ)\Omega \\ &= AJ\varphi(f)\, J\Omega \\ &= A\varphi(\hat{J}f)\Omega, \end{aligned}$$

by Eq. (20). We note that $\varphi(\hat{J}f)$ is affiliated with $R(\mathscr{K}_2^{\perp}, \mathscr{K}_1^{\perp})$ for $\hat{J}f \in \mathscr{K}_2^{\perp}$, and as $R(\mathscr{K}_2^{\perp}, \mathscr{K}_1^{\perp})' \supset R(\mathscr{K}_1, \mathscr{K}_2)$ by locality, we have $A\varphi(\hat{J}f) \subset \varphi(\hat{J}f)A$. Thus we get $J\varphi(f)\, JA\Omega = \varphi(\hat{J}f)A\Omega$, and

$$J\varphi(f)\, J \upharpoonright R(\mathscr{K}_1, \mathscr{K}_2)\Omega = \varphi(\hat{J}f) \upharpoonright R(\mathscr{K}_1, \mathscr{K}_2)\Omega. \tag{22}$$

Take the closure of both sides of (22). The expression on the right-hand side gives simply $\varphi(\hat{J}f)$, as $R(\mathscr{K}_1, \mathscr{K}_2)\Omega$ is a core for this operator, (show by an explicit estimate that the domain of the closure of $\varphi(\hat{J}f) \upharpoonright R(\mathscr{K}_1, \mathscr{K}_2)\Omega$ contains $\mathscr{D}_0$, which is a core for $\varphi(\hat{J}f)$). One can show that for any selfadjoint operator A, JAJ is again selfadjoint. Thus the closure of $J\varphi(f)J \upharpoonright R(\mathscr{K}_1, \mathscr{K}_2)\Omega$ is a selfadjoint operator, namely $\varphi(\hat{J}f)$, and has the selfadjoint operator $J\varphi(f)J$ as an extension. Therefore $J\varphi(f)J = \varphi(\hat{J}f)$, and by the same arguments $J\pi(g)J = \pi(\hat{J}g)$, which proves the lemma.

Now let $\Phi = A\Omega$, $A \in R(\mathscr{K}_1, \mathscr{K}_2)$, and let f be in $\mathscr{K}_1$. Then

$$Je^{i\varphi(f)}J\Phi = AJe^{i\varphi(f)}J\Omega,$$

since $\qquad Je^{i\varphi(f)}J \in R(\mathscr{K}_1, \mathscr{K}_2)' \qquad$ by Tomita's theory.

$$\begin{aligned} Je^{i\varphi(f)}J\Phi &= AJ\sum_{n=0}^{\infty}\frac{(i\varphi(f))^n}{n!}J\Omega \\ &= A\sum_{n=0}^{\infty}\frac{(J(i\varphi(f))J)^n}{n!}\Omega \\ &= Ae^{i\varphi(\mathcal{J}f)}\Omega \\ &= e^{i\varphi(\mathcal{J}f)}\Phi \end{aligned}$$

as $e^{i\varphi(\mathcal{J}f)} \in R(\mathscr{K}_1, \mathscr{K}_2)'$, by locality. Thus on the dense set $R(\mathscr{K}_1, \mathscr{K}_2)\Omega$ we have

$$Je^{i\varphi(f)}J = e^{i\varphi(\mathcal{J}f)}, \qquad \mathcal{J}f \in \mathscr{K}_2^{\perp}, \tag{28}$$

and by continuity, Eq. (28) holds on all of $\mathscr{F}$. The same argument shows that for $g \in \mathscr{K}_2$, $Je^{i\pi(g)}J = e^{i\pi(\mathcal{J}g)}$, $\mathcal{J}g \in \mathscr{K}_1^{\perp}$. This proves duality (Theorem 3).

Let us note for completeness that the operators H, J, and V can be calculated with little additional work on all of $\mathscr{F}$; also Δ can be given explicitly.

Acknowledgments

This work has been done during our pleasant stay at the I.H.E.S. in Bures-sur-Yvette. We thank Professor N. Kuiper and Professor D. Ruelle for the invitation which made our collaboration possible. We also thank Professor J. Dixmier for helpful criticism of an earlier version of this paper; especially for bringing to our attention Lemma 5.

References

1. H. Araki, A lattice of von Neumann algebras associated with the quantum theory of a free bose field, *J. Math. Phys.* **4** (1963), 1343; Von Neumann algebras of local observables for the free scalar field, *J. Math. Phys.* **5** (1964), 1.
2. J. Cook, The mathematics of second quantization, *Trans. AMS* **74** (1953), 222.
3. G. F. Dell'Antonio, Structure of the algebra of some free systems, *Comm. Math. Phys.* **9** (1968), 81.

4. J. DIXMIER, Les algèbres hilbertiennes modulaires de Tomita (d'après Takesaki), Séminaire Bourbaki 1969/70. Lecture Notes in Mathematics Vol. 180, Springer–Verlag, Berlin, 1971.
5. J. DIXMIER, Étude sur les variétés et les opérateurs de Julia, avec quelques applications, *Bull. Soc. Math. France* **77** (1949), 11.
6. J. DIXMIER, Positions relative de deux variétés linéaires fermées dans un espace de Hilbert, *Rev. Sci.* **86** (1948), 387.
7. N. DUNFORD AND J. SCHWARTZ, "Linear Operators," Vol. II, Interscience, New York, 1958.
8. P. R. HALMOS, Two Subspaces, *Trans. Amer. Math. Soc.* **144** (1969), 381.
9. E. NELSON, Analytic vectors, *Ann. Math.* **67** (1959), 572.
10. K. OSTERWALDER, Duality for free bose fields, *Comm. Math. Phys.*, **29** (1973), 1.
11. M. TAKESAKI, Tomita's theory of modular Hilbert algebras and its applications, "Lecture Notes in Mathematics," Vol. 2, Springer–Verlag, 1970, Berlin.