A Rigorous Upper Bound on the Propagation Speed for the Swift-Hohenberg and Related Equations

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For D.Ruelle and Y.Sinai, with best wishes

Abstract. We prove that if the initial condition of the Swift-Hohenberg equation

$$\partial_t u(x,t) = \left(\varepsilon^2 - (1+\partial_x^2)^2\right) u(x,t) - u^3(x,t)$$

is bounded in modulus by $Ce^{-\beta x}$ as $x \to +\infty$, the solution cannot propagate to the right with a speed greater than

$$\sup_{0<\gamma\le\beta} \gamma^{-1}(\varepsilon^2 + 4\gamma^2 + 8\gamma^4) .$$

This settles a long-standing conjecture about the possible asymptotic propagation speed of the Swift-Hohenberg equation. The proof does not use the maximum principle and is simple enough to generalize easily to other equations. We illustrate this with an example of a modified Ginzburg-Landau equation, where the critical speed is not determined by the linearization alone.

1. Introduction

The marginal stability conjecture deals with the possible propagation speed of solutions of dissipative partial differential equations. It was formulated in the late 1970's by several authors. Its clearest form is obtained for the Ginzburg-Landau equation

$$\partial_t u(x,t) \;=\; \partial_x^2 u(x,t) + u(x,t) - u^3(x,t) \;, \tag{1.1}$$

where $u : \mathbf{R} \times \mathbf{R}^+ \to \mathbf{R}$. When the initial data have compact support, then the solution cannot propagate with a speed faster than some critical speed c, which happens to be 2 for this example. The number 2 can be understood as follows. One writes $u(x,t) = v(x-ct)$, and looks for a solution of (1.1) expressed for v:

$$0 \;=\; \partial_\xi^2 v + c\partial_\xi v + v - v^3 \;. \tag{1.2}$$

If one makes the assumption that $v(\xi) = C_1 e^{-\beta\xi}$ as $\xi \to +\infty$, one finds that β and c should be related through the equation

$$0 \;=\; \beta^2 - \beta c + 1 \;, \tag{1.3}$$

since the non-linear term is irrelevant at $\xi = \infty$ in this case. For fixed β we clearly find $c = (\beta^2 + 1)/\beta$, and since functions which are (in absolute value) bounded by $C \exp(-\beta x)$ are also bounded by $C' \exp(-\beta' x)$ for $0 < \beta' < \beta$ one finds in this case an upper bound

$$c_\beta^{\mathrm{GL}} = \inf_{0<\gamma<\beta} \frac{\gamma^2 + 1}{\gamma} , \tag{1.4}$$

and this is equal to 2 for $\beta \geq 1$. Using the maximum principle for parabolic PDE's, Aronson and Weinberger were able to show [AW] that no positive solution starting from initial conditions with compact support can move faster than the speed $c_2^{\mathrm{GL}} = 2$. Using essentially the same argument, it was also shown in [CE] that if the initial condition decays like $e^{-\beta x}$ with $\beta < 1$, then the solution cannot move faster than c_β^{GL}. However, in cases where the maximum principle does not apply, such as in (1.5), the maximum possible speed was only conjectured, and tested numerically, but no rigorous result was obtained, see *e.g.*, [LMK, DL, BBDKL]. Further, more modern, references are [EvS] and [PT].

In a somewhat different direction, there is the important, and difficult, issue on whether there is actually a solution moving with the maximal allowed velocity. In general, its realization depends on the details of the nonlinearity, and this question has been extensively discussed in the literature [AW, B, DL, BBDKL, vS]. It will not be treated here.

The main result of our paper is an *upper bound* on the speed of propagation of solutions to the Swift-Hohenberg equation

$$\partial_t u = \left(\varepsilon^2 - (1 + \partial_x^2)^2\right) u - u^3 . \tag{1.5}$$

The polynomial equation analogous to (1.3) turns out to be

$$0 = \varepsilon^2 + 4\beta^2 + 8\beta^4 - c\beta , \tag{1.6}$$

and we define in this case

$$c_\beta = c_\beta^{\mathrm{SH}} = \inf_{0<\gamma\leq\beta} \frac{\varepsilon^2 + 4\gamma^2 + 8\gamma^4}{\gamma} . \tag{1.7}$$

The polynomial is an absolute maximum of the real part of the polynomial $P(x) = \varepsilon^2 - (1+x^2)^2$, as we explain at the end of the introduction and in the Appendix. This will be the minimal speed.* Our result can be expressed informally as follows: *If the initial data for the problem are bounded in absolute value by $Ce^{-\beta x}$ as $x \to +\infty$ then the solution cannot advance faster to the right than c_β in the sense that*

$$\lim_{t\to\infty} u(x + ct, t) = 0 ,$$

* While it looks different from the standard discussion in [BBDKL], we explain in the Appendix that the two definitions coincide. The current formulation has the advantage of being expressed in terms of real variables, although the traveling wave in this case is actually modulated [CE].

for all $c > c_\beta$. In particular, if the initial condition has compact support, the above hypotheses are satisfied for any $\beta > 0$ and we find an upper bound on the speed which is $c_* = \inf_\beta c_\beta$: This is the absolute minimum of $(\varepsilon^2 + 4\beta^2 + 8\beta^4)/\beta$.

Remark. The precise formulation is given in Theorem 4.1.

Before explaining the main steps of the proof we note a well-known result [CE2], namely that if the initial condition u_0 is bounded in $\mathcal{C}^3$, *i.e.*,

$$\max_{j=0,\dots,3} \sup_{x\in\mathbf{R}} |\partial_x^j u_0(x)| \le K , \tag{1.8}$$

then there is a constant $L = L(K)$ such that for all $t > 0$ one has

$$\max_{j=0,\dots,3} \sup_{x\in\mathbf{R}} |\partial_x^j u(x,t)| \le L(K) . \tag{1.9}$$

The proof of the main result is really quite easy and consists of 3 steps:

i) An a priori bound on the Green's function of the semigroup generated by the linear part $\varepsilon^2 - (1 + \partial_x^2)^2$ of the Swift-Hohenberg equation.

ii) The observation that if the initial condition satisfies $\lim_{x\to\infty} e^{\beta x}\partial_x^j u_0(x) = 0$, for $j = 0, \dots, 3$, then the same holds for $u(x,t)$. This is needed later on to ensure that integration by parts does not produce boundary terms at infinity.

iii) An energy-like estimate which shows that

$$\lim_{t\to\infty} \int_{ct}^{\infty} \mathrm{d}x\, |u(x,t)|^2 e^{2\beta(x-ct)} = 0 ,$$

when $c > c_\beta$ (if it is finite at $t = 0$, see below for details). Thus, the solution is outrun by a frame moving with speed $c > c_\beta$. In the case of second order problems, this is a well-known consequence of the maximum principle, see *e.g.* [CE]. In our context, where the maximum principle cannot be applied, we show that this phenomenon has a different origin of dynamical nature.

In Section 5, we consider the case of the Ginzburg-Landau equation when the nonlinearity $u - u^3$ is replaced by a general function $f(u)$ with the properties $f(0) = 0$, $0 < f'(0) < \infty$ and $\limsup_{z\to\infty} f(z)/z < 0$. In such a case, the bound (1.4) is replaced by

$$c_\beta^{\mathrm{GL}'} = \inf_{0<\gamma<\beta} \frac{\gamma^2 + \sup_u \frac{f(u)}{u}}{\gamma} .$$

In the case of the Swift-Hohenberg equation the bound generalizes as follows: Assume the equation is

$$\partial_t u = \big(\varepsilon^2 - (1 + \partial_x^2)^2\big)u + f(u) .$$

Then we get for the maximal possible speed:

$$c_\beta^{\mathrm{SH}'} = \inf_{0<\gamma\le\beta} \frac{\varepsilon^2 + 4\gamma^2 + 8\gamma^4 + \sup_u \frac{f(u)}{u}}{\gamma} .$$

In an appendix, we show that the expression (1.7) is nothing but

$$\sup_{k^*_\beta} \mathrm{Re}\, P(z)|_{z=-\beta+ik^*_\beta}\ ,$$

where the sup is over the solutions k^*_β of

$$\frac{\mathrm{dRe}\, P(-\beta+ik)}{\mathrm{d}k}\ =\ 0\ .$$

We also show that these conditions are the same as those found in [BBDKL].

Finally, it should be noted that the method is not restricted to 1-dimensional problems, and can also be applied to questions of growth of "bubbles" in the 2-dimensional Swift-Hohenberg equation.

2. A pointwise bound on the Green's function

Here we bound the Green's function of the operator $\varepsilon^2-(1+\partial_x^2)^2$ by a method which generalizes immediately to other problems of similar type. Let P be a polynomial in k which is of the form

$$P(ik)\ =\ -a_n k^n + \sum_{m=0}^{n-1} a_m k^m\ \equiv\ -a_n k^n + R(k)\ ,$$

and assume n even and $a_n>0$. (For the Swift-Hohenberg equation, $P(z)=\varepsilon^2-(1+z^2)^2$.) Then the Green's function

$$G_t(x)\ =\ \int \mathrm{d}k\, e^{ikx} e^{P(ik)t}\ ,$$

satisfies:

Lemma 2.1. *Given $0<\beta<\infty$, there is a constant $C(\beta)$ such that for all $t\in(0,1]$ one has the bound*

$$t^{1/n}|G_t(x)|e^{(\beta'+2t^{-1/n})|x|}\ \le\ C(\beta)\ , \tag{2.1}$$

for all $\beta'\in[0,\beta]$.

Remark. This clearly also implies, for all $t\in(0,1]$ and all $\beta'\in[0,\beta]$:

$$\int \mathrm{d}x\, |G_t(x)|e^{\beta'|x|}\ \le\ C(\beta)\ , \tag{2.2}$$

since $\int \mathrm{d}x\, |G_t(x)|e^{\beta'|x|} \le C(\beta) \int \mathrm{d}x\, t^{-1/n} e^{-2|x|t^{-1/n}} \le C(\beta)$.

Proof. We will show the bound in the form

$$t^{1/n}|G_t(zt^{1/n})|e^{\gamma t^{1/n} z}\ \le\ C(\beta)\ , \tag{2.3}$$

with $\gamma = \beta + 2t^{-1/n}$, and it clearly suffices to consider $z > 0$. Proving (2.3) is a straightforward calculation which is probably well-known. Indeed, the l.h.s. of (2.3) equals (without the absolute values)

$$\begin{aligned} &\int \mathrm{d}k\, t^{1/n} \exp\bigl(\gamma t^{1/n} z + ikt^{1/n} z - a_n k^n t + R(k)t\bigr) \\ &= \int \mathrm{d}\ell\, \exp\bigl(\gamma t^{1/n} z + i\ell z - a_n \ell^n + R(\ell t^{-1/n})t\bigr)\ . \end{aligned}$$

Since the integrand is an entire function in ℓ we can shift the contour from ℓ to $\ell' = \ell - i\gamma t^{1/n}$ and the last expression is seen to be equal to

$$\int \mathrm{d}\ell'\ \exp\bigl(i\ell' z - a_n(\ell' + i\gamma t^{1/n})^n + R(\ell' t^{-1/n} + i\gamma)t\bigr)\ .$$

Note now that

$$\begin{aligned} &\bigl|\exp\bigl(i\ell' z - a_n(\ell' + i\gamma t^{1/n})^n + R(\ell' t^{-1/n} + i\gamma)t\bigr)\bigr| \\ &= \bigl|\exp\bigl(-a_n(\ell' + i\gamma t^{1/n})^n + R(\ell' t^{-1/n} + i\gamma)t\bigr)\bigr|\ , \end{aligned} \tag{2.4}$$

and for bounded β and $t \in (0, 1]$ we find that $\gamma t^{1/n} = (\beta + 2t^{-1/n})t^{1/n} \le \beta + 2$, and hence (2.4) is uniformly integrable in ℓ', since $a_n > 0$. The proof of Lemma 2.1 is complete. $\square$

3. Exponential decay of solutions

In this section, we prove a bound in the laboratory frame, showing that if the initial condition goes exponentially to 0 then the solution at time t goes to zero as well, *with the same rate*.

Theorem 3.1. *Assume that u_0 is bounded in $\mathcal{C}^3$ and that*

$$\lim_{x\to\infty} e^{\beta x} \partial_x^j u_0(x) \;=\; 0\ , \tag{3.1}$$

for $j = 0, \ldots, 3$ and some $\beta > 0$. Then the solution $u(x, t)$ of (1.5) with initial data u_0 satisfies for all $t > 0$:

$$\lim_{x\to\infty} e^{\beta x} \partial_x^j u(x,t) \;=\; 0\ , \tag{3.2}$$

for $j = 0, \ldots, 3$.

Proof. The proof is in steps of some (fixed) time τ_*. We define first

$$g_\xi(x) \;=\; 1 + e^{\beta(x-\xi)}\ .$$

The assumption means that u_0 satisfies (1.8) for some K. From (3.1), and because $L(K) \ge K$, we conclude that there is a $\xi > 0$ for which

$$\sup_{x\in\mathbf{R}} g_\xi(x)|\partial_x^j u_0(x)| \;\le\; 2K \;\le\; 2L(K)\ , \tag{3.3}$$

for $j = 0, \dots, 3$. Note that we do not have any control on the size of ξ, but such a control is not needed.

From (1.8) we also conclude (see (1.9)) that

$$\sup_{t \ge 0} \sup_{x \in \mathbf{R}} |\partial_x^j u(x,t)| \le L(K) , \tag{3.4}$$

for $j = 0, \dots, 3$.

The crucial step in the proof of Theorem 3.1 is

Lemma 3.2. *There are a $\tau_* > 0$ and a ρ, independent of ξ, such that for $t \in [0, \tau_*]$ one has*

$$\sup_{j=0,\dots,3} \sup_{x \in \mathbf{R}} g_\xi(x) |\partial_x^j u(x,t)| \le \rho . \tag{3.5}$$

Proof. We use the estimates on the convolution kernel G_t associated with the semigroup $t \mapsto \exp\bigl(t(\varepsilon^2 - (1+\partial_x^2)^2\bigr)$ which were proven in Section 2. One has

$$u_t = G_t \star u_0 - \int_0^t \mathrm{d}s \, G_{t-s} \star u_s^3 ,$$

where $u_s(x) = u(x,s)$. We define $\mathcal{B}_\xi$ as the space of uniformly continuous functions f for which

$$\|f\|_\xi = \sup_{x \in \mathbf{R}} g_\xi(x) |f(x)| < \infty .$$

Using this quantity as a norm makes $\mathcal{B}_\xi$ a Banach space. Consider next the space $\mathcal{K} = \mathcal{K}_{\xi,\tau_*} = \mathcal{C}^0([0,\tau_*], \mathcal{B}_\xi)$ of functions $h : (x,t) \mapsto h(x,t)$, with $h(x,0) = 0$, and with the norm

$$\|h\|_{\xi,\tau_*} = \sup_{t \in [0,\tau_*]} \|h(\cdot,t)\|_\xi .$$

This is again a Banach space. For $v \in \mathcal{K}$ we define the map $v \mapsto \mathcal{Q}v$ by

$$\bigl(\mathcal{Q}v\bigr)(x,t) = \bigl(G_t \star u_0\bigr)(x) - u_0(x) - \int_0^t \mathrm{d}s \, \bigl(G_{t-s} \star (v_s + u_0)^3\bigr)(x) . \tag{3.6}$$

Note that if $\mathcal{Q}v = v$, then $v(x,t) + u_0(x)$ is a solution to (1.5) with initial condition u_0. To find v, we will show that for sufficiently small $\tau_* > 0$ the operator $\mathcal{Q}$ contracts a ball of $\mathcal{K}_{\xi,\tau_*}$ to itself. The center of this ball is the function $(x,t) \mapsto 0$.

First we bound $G_t \star u_0$. Note that from the definition of g_ξ we find

$$\frac{g_\xi(x)}{g_\xi(y)} \le e^{\beta|x-y|} ,$$

since for $x < y$ the quotient is bounded by 1 and for $x > y$ we have the (very rough) bound $e^{\beta(x-y)}$. From Lemma 2.1, we have for all $t \in (0,1]$ and all $x \in \mathbf{R}$:

$$|G_t(x)|e^{2\beta|x|} \le C(\beta)t^{-1/4}e^{-2|x|t^{-1/4}} , \tag{3.7}$$

and, clearly, $C(\beta)$ can be chosen the same value for all smaller β. Using this, we find

$$\begin{aligned}
|(G_t \star u_0)(x)\, g_\xi(x)| &\le \int \mathrm{d}y\, |G_t(x-y)u_0(y)|\, g_\xi(y)\frac{g_\xi(x)}{g_\xi(y)} \\
&\le \int \mathrm{d}y\, |G_t(x-y)u_0(y)|\, g_\xi(y)e^{\beta|x-y|} \\
&\le \int \mathrm{d}z\, |G_t(z)e^{\beta|z|}| \sup_{z' \in \mathbf{R}} |u_0(z')|\, g_\xi(z') \\
&\le C(\beta) \sup_{z' \in \mathbf{R}} |u_0(z')|\, g_\xi(z') .
\end{aligned} \tag{3.8}$$

Combining these bounds with (3.3) we get

$$|(G_t \star u_0)(x)\, g_\xi(x)| \le C_2 L(K) .$$

In fact, we can do a little better in (3.8) by extracting a factor of $e^{-\beta|x-y|}$. The last two lines in (3.8) are replaced by

$$\begin{aligned}
&|(G_t \star u_0)(x)\, g_\xi(x)| \\
&\quad\le \int \mathrm{d}z\, |G_t(z)e^{2\beta|z|}| \sup_{y \in \mathbf{R}} |u_0(y)|\, g_\xi(y)e^{-\beta|x-y|} \\
&\quad\le C(2\beta) \sup_{y \in \mathbf{R}} |u_0(y)|\, g_\xi(y)e^{-\beta|x-y|} .
\end{aligned} \tag{3.9}$$

Since $|u_0(y)|\, g_\xi(y)$ is bounded and converges to 0 as $y \to +\infty$, we conclude that the quantity in (3.9) tends to 0 as $x \to +\infty$. Thus, we also have

$$\lim_{x \to \infty} |(G_t \star u_0)(x)\, g_\xi(x)| = 0 . \tag{3.10}$$

We next bound the non-linear term. Let $\hat{u}_0(x,t) = u_0(x)$. Assume $v \in \mathcal{K}_{\xi,\tau_*}$ and $\|v + \hat{u}_0\|_{\xi,\tau_*} < \rho$. Then for any power ($\ge 1$) of $v + \hat{u}_0$ one has a bound of the form

$$\|(v + \hat{u}_0)^3\|_{\xi,\tau_*} \le C_3\rho^3 .$$

Therefore, the method leading to (3.8) now yields

$$\left| \int_0^t \mathrm{d}s\, \bigl(G_{t-s} \star (v_s + u_0)^3\bigr)(x) g_\xi(x) \right| \le C_4\rho^3 t ,$$

and if also $\|w+\hat{u}_0\|_{\xi,\tau_*}<\rho$, then a variant of that method gives:

$$\begin{aligned}\Big|\int_0^t \mathrm{d}s\,\big(G_{t-s}\star(v_s+u_0)^3\big)(x)g_\xi(x) - \int_0^t \mathrm{d}s\,\big(G_{t-s}\star(w_s+u_0)^3\big)(x)g_\xi(x)\Big|\\ \le\; C_5\rho^2 t \sup_{s\in[0,t]}\sup_{x\in\mathbf{R}}\big|v_s(x)-w_s(x)\big|\,g_\xi(x)\;.\end{aligned}$$

Taking the center of the ball at $(x,t)\mapsto 0$ and the radius $\rho = 2C_2K$ and then $\tau_* < \min\{(4C_4\rho^3)^{-1},(4C_5\rho^2)^{-1}\}$, we have a contraction and hence a unique fixed point v for $\mathcal{Q}$. For $j=1,2,3$, we use the same methods since we can push all derivatives from the operator G_t to the function v, because $G_t\star$ is a convolution. The details are left to the reader. The existence of this fixed point clearly shows Lemma 3.2. □

We come back to the proof of Theorem 3.1. We define

$$\Gamma(t) \;=\; \limsup_{x\to\infty}|u(x,t)g_\xi(x)|\;.$$

By assumption, we have $\Gamma(0)=0$ and by Lemma 3.2 we have

$$|u(x,t)|\;\le\;\rho/g_\xi(x)\;,$$

so that $\Gamma(t)\le\rho$ for $t\le\tau_*$. We now show it is actually 0 for those t. Consider $\mathcal{Q}$ as in (3.6). Note that

$$\begin{aligned}\Gamma(t) &= \limsup_{x\to\infty}|u(x,t)g_\xi(x)|\\ &= \limsup_{x\to\infty} g_\xi(x)|\big(G_t\star u_0\big)(x)| + \limsup_{x\to\infty} g_\xi(x)|\int_0^t \mathrm{d}s\,\big(G_{t-s}\star u_s^3\big)(x)|\;.\end{aligned}$$

The first term vanishes by (3.9). Thus, Γ only depends on the nonlinear part. Using (3.7), that part can be bounded as

$$\begin{aligned}g_\xi(x)\Big|\int_0^t \mathrm{d}s\,G_{t-s}\star u_s^3(x)\Big| &\le \int_0^t \mathrm{d}s\int \mathrm{d}y\,\frac{g_\xi(x)}{g_\xi^3(y)}\,\big|G_{t-s}(x-y)\big|\,\big|g_\xi(y)u_s(y)\big|^3\\ &\le \int_0^t \mathrm{d}s\int \mathrm{d}y\,\big|G_{t-s}(x-y)\big|\,e^{\beta|x-y|}\,\big|g_\xi(y)u_s(y)\big|^3\\ &\le C(\beta)\int_0^t \mathrm{d}s\int \mathrm{d}z\,(t-s)^{-1/4}e^{-2|z|(t-s)^{-1/4}}\\ &\qquad\cdot\big|g_\xi(x-z)u_s(x-z)\big|^3\;.\end{aligned}\tag{3.11}$$

We need an upper bound for the $\limsup_{x\to\infty}$ of this expression. Fix an $\varepsilon>0$. For $s\in[0,t]$, we can find an $\eta(s,\varepsilon)>0$ such that

$$\sup_{y\ge\eta(s,\varepsilon)}|g_\xi(y)u_s(y)|\le\Gamma(s)+\varepsilon\;.$$

There is also a number $\zeta(\varepsilon) > 0$ such that for any $s \in [0, t]$:

$$\int_{|z|>\zeta(\varepsilon)} \mathrm{d}z\,(t-s)^{-1/4} e^{-2|z|(t-s)^{-1/4}} \le \varepsilon\ .$$

If $x > \zeta(\varepsilon) + \eta(s, \varepsilon)$, we have

$$\int \mathrm{d}z\,(t-s)^{-1/4} e^{-2|z|(t-s)^{-1/4}} |g_\xi(x-z) u_s(x-z)|^3 \le (\Gamma(s)+\varepsilon)^3 + \rho^3 \varepsilon\ ,$$

by Lemma 3.2. We cannot conclude directly by integration over s because η depends on s. However, $\eta(s, \varepsilon)$ is finite for almost every s (in reality for every s). Therefore, we can find a finite number $\Theta(\varepsilon)$ such that the set

$$E(\varepsilon) \;=\; \bigl\{ s \in [0, t] \mid \eta(s, \varepsilon) \;>\; \Theta(\varepsilon) \bigr\}$$

has Lebesgue measure at most ε (note that $E(\varepsilon)$ is measurable). Therefore, if $x > \Theta(\varepsilon) + \zeta(\varepsilon)$ we have

$$\begin{aligned}
&\int_0^t \mathrm{d}s \int \mathrm{d}z\,(t-s)^{-1/4} e^{-2|z|(t-s)^{-1/4}} |g_\xi(x-z) u_s(x-z)|^3 \\
&\quad = \int_{([0,t]\setminus E(\varepsilon))\cup E(\varepsilon)} \mathrm{d}s \int \mathrm{d}z\,(t-s)^{-1/4} e^{-2|z|(t-s)^{-1/4}} |g_\xi(x-z) u_s(x-z)|^3 \\
&\quad \le C_6 \int_0^t \mathrm{d}s\, \bigl((\Gamma(s)+\varepsilon)^3 + \rho^3 \varepsilon\bigr) + C_7 \rho^3 \int_{E(\varepsilon)} \mathrm{d}s\ .
\end{aligned}$$

The last integral is of order ε. Since $\varepsilon > 0$ is arbitrary, we get

$$\Gamma(t) \;\le\; C_8 \int_0^t \mathrm{d}s\, \Gamma(s)^3\ .$$

Since Γ is bounded by what we said above and $\Gamma(0) = 0$, it follows from Gronwall's lemma that $\Gamma(t) = 0$ for $t \le \tau_*$. One then repeats the argument for all consecutive intervals of length τ_*. The proof of the corresponding bounds on the derivatives is similar and is left to the reader. □

4. Bound on the speed

We define J_ξ by

$$J_\xi(t) \;=\; \int_\xi^\infty \mathrm{d}x\, |u(x,t)|^2 e^{2\beta(x-\xi)} \;, \tag{4.1}$$

where $u(x,t)$ is the solution of the Swift-Hohenberg equation. The main result of this paper is

Theorem 4.1. *Let $u(x,t)$ be a solution of the Swift-Hohenberg equation (1.5) for an initial condition $u_0(x) = u(x,0)$ which is in $\mathcal{B}$, which satisfies $J_0(0) < \infty$ for some $\beta > 0$ and which satisfies the assumptions of Theorem 3.1. Then one has*

$$\lim_{t\to\infty} \int_{ct}^\infty \mathrm{d}x\, |u(x,t)|^2 e^{2\beta(x-ct)} \;=\; 0 \,, \tag{4.2}$$

for all $c > \left(\varepsilon^2 + 4\beta^2 + 8\beta^4\right)/\beta$.

Remark. If one is willing to pay a price of slightly more complicated formulations and proofs, one can omit the condition on $J_0(0)$ in Theorem 4.1. One would then assume the pointwise bounds of Theorem 3.1 fore some $\beta > 0$ and work throughout the proof with a $J_\xi(t)$ defined with some $\beta' < \beta$, but arbitrarily close to it, since the condition on c is open.

Proof. We define $v_\xi(x,t) = u(x,t)e^{\beta(x-\xi)}$, so that $J_\xi(t) = \int_\xi^\infty \mathrm{d}x\, |v_\xi(x,t)|^2$, and v_ξ solves the equation

$$\partial_t v_\xi(x,t) \;=\; \varepsilon^2 v_\xi(x,t) - \left(1 + (\partial_x - \beta)^2\right)^2 v_\xi(x,t) - v_\xi^3(x,t)e^{-2\beta(x-\xi)} \;. \tag{4.3}$$

Since u is real, the absolute values in the definition of $J_\xi(t)$ can be omitted. Differentiating (4.1) with respect to time, we get

$$\tfrac{1}{2}\partial_t J_\xi(t) \;=\; \int_\xi^\infty \mathrm{d}x\, v_\xi(x,t)\partial_t v_\xi(x,t) \;.$$

Since ξ is fixed throughout the calculation, we omit the index of v_ξ. We also omit the arguments (x,t). Note that by Theorem 3.1, $\lim_{x\to\infty} \partial_x^j v_\xi(x,t) \;=\; 0$, for $j \;=\; 0,\ldots,3$, so that we can freely integrate by parts in the following calculation. We find, using $\partial_x v = v'$:

$$\begin{aligned}
\tfrac{1}{2}\partial_t J_\xi(t) \;&=\; \int_\xi^\infty \mathrm{d}x\, v\Big(\varepsilon^2 v - \big(1+(\partial_x-\beta)^2\big)^2 v - v^3 e^{-2\beta(x-\xi)}\Big) \\
&=\; \int_\xi^\infty \mathrm{d}x\, v\Big(\varepsilon^2 v - \big(1+\partial_x^2-2\beta\partial_x+\beta^2\big)^2 v - v^3 e^{-2\beta(x-\xi)}\Big) \\
&=\; \int_\xi^\infty \mathrm{d}x\, v\Big(\varepsilon^2 v - \partial_x^4 v + 4\beta\partial_x^3 v - 2(1+3\beta^2)\partial_x^2 v + 4\beta(1+\beta^2)\partial_x v - (1+\beta^2)^2 v \\
&\qquad\qquad - v^3 e^{-2\beta(x-\xi)}\Big) \\
&=\; \int_\xi^\infty \mathrm{d}x\,\Big(\big(\varepsilon^2-(1+\beta^2)^2\big)v^2 - e^{-2\beta(x-\xi)}v^4 - 2(1+3\beta^2)vv'' \\
&\qquad\qquad +\; v'v''' - 4\beta v'v''\Big) \\
&\qquad\qquad + \big(vv''' - 4\beta vv'' - 2\beta(1+\beta^2)v^2\big)\big|_{x=\xi,t}\;.
\end{aligned}$$

We integrate by parts some more and get

$$\begin{aligned}
\tfrac{1}{2}\partial_t J_\xi(t) \;=\; \int_\xi^\infty \mathrm{d}x\,\Big(\big(\varepsilon^2-(1+\beta^2)^2\big)v^2 - e^{-2\beta(x-\xi)}v^4 - 2(1+3\beta^2)v''v - (v'')^2\Big) \\
+ \big(vv''' - 4\beta vv'' - 2\beta(1+\beta^2)v^2 - v'v'' + 2\beta(v')^2\big)\big|_{x=\xi,t}\;.
\end{aligned} \tag{4.4}$$

We write $B_\xi(t)$ for the boundary term obtained above:

$$B_\xi(t) \;=\; \big(vv''' - 4\beta vv'' - 2\beta(1+\beta^2)v^2 - v'v'' + 2\beta(v')^2\big)\big|_{x=\xi,t}\;.$$

Finally, we rewrite (4.4) by completing a square:

$$\begin{aligned}
\tfrac{1}{2}\partial_t J_\xi(t) \;=\; \int_\xi^\infty \mathrm{d}x\,\Big(\big(\varepsilon^2-(1+\beta^2)^2+(1+3\beta^2)^2\big)v^2 - e^{-2\beta(x-\xi)}v^4 \\
- \big(v''+(1+3\beta^2)v\big)^2\Big) + B_\xi(t)\;.
\end{aligned} \tag{4.5}$$

Note that (4.5) leads immediately to a differential inequality:

$$\tfrac{1}{2}\partial_t J_\xi(t) \;\le\; G(\beta)J_\xi(t) + B_\xi(t)\;, \tag{4.6}$$

with

$$G(\beta) \;=\; \varepsilon^2 - (1+\beta^2)^2 + (1+3\beta^2)^2 \;=\; \varepsilon^2 + 4\beta^2 + 8\beta^4\;. \tag{4.7}$$

This is the origin of the polynomial in (1.7). We bound first the boundary term.

Lemma 4.2. *There is a* C_9 *such that for all* $u_0 \in \mathcal{B}$, *all* ξ, *and all* $t > 0$ *one has*

$$B_\xi(t) \le C_9 . \tag{4.8}$$

Proof. Recall that $v_\xi(x,t) = e^{\beta(x-\xi)} u(x,t)$. Using elementary calculus, we find

$$\partial_x^j v_\xi(x,t) = \sum_{k=0}^{j} \binom{j}{k} \beta^j e^{\beta(x-\xi)} \partial_x^{j-k} u(x,t) .$$

Therefore,

$$\partial_x^j v_\xi(\xi,t) = \sum_{k=0}^{j} \binom{j}{k} \beta^j \partial_\xi^{j-k} u(x,t)|_{x=\xi} ,$$

and the assertion follows because $u \in \mathcal{B}$.

Using Lemma 4.2, we conclude from (4.6) that

$$\partial_t J_\xi(t) \le 2G(\beta) J_\xi(t) + 2C_9 .$$

Solving the differential inequality from t to t', we obtain for $t' > t$,

$$J_\xi(t') \le e^{2G(\beta)(t'-t)} J_\xi(t) + 2\frac{e^{2G(\beta)(t'-t)} - 1}{2G(\beta)} C_9 . \tag{4.9}$$

We need this inequality in a slightly different form. Note that for $\xi' > \xi$, one has

$$\begin{aligned} J_{\xi'}(t) = \int_{\xi'}^{\infty} \mathrm{d}x\, u^2(x,t) e^{2\beta(x-\xi')} &= e^{-2\beta(\xi'-\xi)} \int_{\xi'}^{\infty} \mathrm{d}x\, e^{2\beta(x-\xi)} u^2(x,t) \\ &\le e^{-2\beta(\xi'-\xi)} \int_{\xi}^{\infty} \mathrm{d}x\, e^{2\beta(x-\xi)} u^2(x,t) \\ &= e^{-2\beta(\xi'-\xi)} J_\xi(t) . \end{aligned} \tag{4.10}$$

Combining this with (4.9) we get for $\xi' > \xi$ and $t' > t$:

$$J_{\xi'}(t') \le e^{-2\beta(\xi'-\xi)} \Big(e^{2G(\beta)(t'-t)} J_\xi(t) + \frac{e^{2G(\beta)(t'-t)} - 1}{G(\beta)} C_9 \Big) . \tag{4.11}$$

To complete the proof of Theorem 4.1, it suffices to set $\xi' = c\tau$, $t' = \tau$, $\xi = 0$ and $t = 0$ in (4.11). Then we get

$$J_{c\tau}(\tau) \le e^{2(G(\beta)-\beta c)\tau} \Big(J_0(0) + \frac{C_9}{G(\beta)} \Big) . \tag{4.12}$$

Clearly, if $c > G(\beta)/\beta$, then $J_{c\tau}(\tau) \to 0$ as $\tau \to \infty$. Thus, if $J_0(0) < \infty$, and the assertion of Theorem 4.1 follows. $\square$

Remark. One can do a little better than (4.12). Namely, consider the case where $c = G(\beta)/\beta$, that is, the case of a critical speed. Then one finds from (4.11) that

$$J_{c\tau+\lambda}(\tau) \le e^{-2\beta\lambda}\big(J_0(0) + \frac{C_9}{G(\beta)}\big) ,$$

and in particular $\lim_{\lambda\to\infty} J_{c\tau+\lambda}(\tau) = 0$, if $J_0(0)$ is finite. This means that in the frame moving with exactly the critical speed, no amplitude "leaks" far ahead in that frame in $\mathrm{L}^2(e^{2\beta x}\mathrm{d}x)$. One can compare this with the results of Bramson [B] who showed (for positive solutions of the Ginzburg-Landau equation) that such a leakage is only possible if the initial data decay like $e^{-x}x^{\alpha}$ with $\alpha > 1$. In that case, he gets positive amplitudes at $ct + (\alpha - 1)\log t$. Note that the condition $J_0(0) < \infty$ can only hold for $\alpha < -\frac{1}{2}$, and then the correction term will push the amplitude behind the position of ct. Thus, in the case of the Ginzburg-Landau equations the two results are consistent.

5. An example of a non-linear velocity bound

Consider the semi-linear parabolic equation

$$\partial_t u = P(\partial_x)u + f(u) , \tag{5.1}$$

where P is a real polynomial, $\mathrm{Re}\, P(ik)$ diverges to $-\infty$ as $|k| \to \infty$ and $\mathrm{Im}\,(ik)$ is a polynomial of lower order.* We also assume that f is a $\mathcal{C}^2$ function for which $f(0) = 0$, and $f'(0) = 0$. This implies that $u = 0$ is an unstable fixed point of (5.1) . We also assume that

$$\limsup_{|u|\to\infty} \frac{f(u)}{u} < 0 .$$

This assumption ensures global existence and regularity of the semiflow (see [CE]). (If $\vec{u}$ is vector valued we impose $\limsup_{\|\vec{u}\|\to\infty} \vec{u}\cdot\vec{f}(u)/\|\vec{u}\|^2 < 0$.)

Define

$$\sigma = \sup_u \frac{f(u)}{u} .$$

This is a finite positive quantity from the above assumptions (if $\vec{u}$ is vector valued we define it as the sup of $\vec{u}\cdot\vec{f}(u)/\|\vec{u}\|^2$.) Note that one can have $\sigma > f'(0)$, and if this happens Aronson and Weinberger [AW] showed that the minimal speed is bounded above by $\sqrt{4\sigma}$, when $P(ik) = -k^2$. In this section we show that the same result can be recovered for this, and many

* The complex Ginzburg-Landau equation is somewhat more complicated because in that case P is a 2×2 matrix polynomial. But it is covered by our methods.

other equations using the methods of Section 4, again without any recourse to the maximum principle.

In this case, Eq.(4.7) becomes

$$G(\beta) \;=\; Q(\beta) + \sigma \;,$$

where Q is given by

$$Q(\beta) \;=\; \sup_{k_\beta^*} \operatorname{Re} P(-\beta + ik_\beta^*)$$

where the k_β^* are the solutions of

$$\left.\frac{\mathrm{dRe}\, P(-\beta + ik)}{\mathrm{d}k}\right|_{k=k_\beta^*} \;=\; 0 \;.$$

The remainder of the proof is the same, except that in (4.5) the term $-\exp(-2\beta(x-\xi))v^4$ is replaced by

$$e^{\beta(x-\xi)} v f\left(e^{-\beta(x-\xi)} v\right) \leq \sigma v^2$$

After this modification the proof proceeds as before.

Appendix: The determination of the critical speed

Let P be a real polynomial for which $\operatorname{Re} P(ik)$ diverges to $-\infty$ as $|k| \to \infty$ and $\operatorname{Im} P(ik)$ is of lower order. In the case of SH, we have $P(z) = \varepsilon^2 - (1 + z^2)^2$. For $\beta > 0$ we consider $P(-\beta + ik)$, take the real part and look for an extremum in k. In other words, we solve

$$\frac{\mathrm{dRe}\, P(-\beta + ik)}{\mathrm{d}k} \;=\; 0 \;,$$

in the unknown k. Since P is analytic, one can write this as

$$0 \;=\; \operatorname{Im}\left(\left.\frac{\mathrm{d}P(z)}{\mathrm{d}z}\right|_{z=-\beta+ik}\right) \;. \tag{A.1}$$

For each β we find solutions k_β^*. The velocity c_β^* is related to the critical value of P in (A.1) by

$$c_\beta^* \;=\; \sup_{k_\beta^*} \operatorname{Re} P(-\beta + ik_\beta^*)/\beta \;. \tag{A.2}$$

Then, the minimal speed is

$$c_* \;=\; \inf_{\beta\in(0,\infty]} c_\beta^* \;,$$

which is determined by (A.3). To simplify the discussion, we will assume from now on that for all k_β^* one obtains the same critical value. This is the case for the Ginzburg-Landau and Swift-Hohenberg equations.

Note that there is at least one β_* solving

$$\partial_\beta \big(\mathrm{Re}\, P(-\beta + ik^*_\beta)/\beta\big)\big|_{\beta=\beta_*} = 0 \,, \tag{A.3}$$

for which $c_* = c^*_{\beta_*}$.

In the approach of [BBDKL] the authors consider $\omega_0(k) = -P(ik)$. They determine $\bar{k}(c) \in \mathbf{C}$ by

$$(\mathrm{d}\omega_0/\mathrm{d}\bar{k})|_{k=\bar{k}(c)} \;=\; ic \,, \tag{A.4}$$

and then $c_* \in \mathbf{R}$ by the condition

$$\mathrm{Re}\,\big(\omega(\bar{k}(c_*)) - i\bar{k}(c_*)c_*\big) \;=\; 0 \,. \tag{A.5}$$

To compare the two approaches, note that $P(-\beta + ik) = -\omega_0(k + i\beta)$. Clearly the equations (A.2) and (A.5) are equivalent. To see that (A.1) and (A.4) say the same thing, note that since c is real one has

$$\mathrm{Im}\,(P'(z)) \;=\; \mathrm{Re}\,(-\omega_0'(-iz)) \;=\; \mathrm{Re}\,(-\omega_0'(-iz) + ic) \,. \tag{A.6}$$

In particular, if ω_0 is an even function, the relation $\mathrm{Re}\,(\omega_0'(\bar{k}) - ic) = 0$ is equivalent to requiring $\omega_0'(\bar{k}) = ic$, which is (A.4). Using (A.6), we conclude that the solution $\bar{k}$ of $\omega_0'(\bar{k}) = ic$ of [BBDKL] is the same as $-i$ times the solution z of $\mathrm{Im}\,(P'(z)) = 0$, which is (A.1). Therefore $\bar{k} = k^*_{\beta_*} + i\beta_*$. Finally, to find $c^*_{\beta_*}$ one can solve

$$\begin{aligned} 0 \;=\; \mathrm{Re}\,(P(-\beta_* + ik^*_{\beta_*}) - \beta_* c^*_{\beta_*}) \;&=\; \mathrm{Re}\,(-\omega_0(k^*_{\beta_*} + i\beta_*) - ic^*_{\beta_*}(k^*_{\beta_*} + i\beta_*)) \\ &=\; \mathrm{Re}\,(-\omega_0(\bar{k}) - ic^*_{\beta_*}\bar{k}) \,. \end{aligned}$$

Remark. The same kind of calculation can be done for multi-component problems (such as reaction diffusion), where P would be a matrix.

The example of the SH equation. In this case

$$P(z) \;=\; \varepsilon^2 - (1 + z^2)^2 \,,$$

and so

$$\omega_0(k) \;=\; -P(ik) \;=\; -\varepsilon^2 + (1 - k^2)^2 \,.$$

In [BBDKL], it is found that

$$\begin{aligned} \bar{k} \;&=\; \bar{k}_1 + i\bar{k}_2 \,, \\ \bar{k}_2 \;&=\; \frac{\sqrt{\sqrt{1 + 6\varepsilon^2} - 1}}{12} \;=\; \frac{\varepsilon^2}{4} + \ldots \,, \\ \bar{k}_1 \;&=\; 1 + 3\bar{k}_2^2 \,, \\ c_* \;&=\; 8\bar{k}_2(1 + 4\bar{k}_2^2) \;=\; 4\varepsilon + \ldots \,. \end{aligned} \tag{A.7}$$

In our formulation, we find

$$P(-\beta+ik) \;=\; \varepsilon^2 - \bigl(1+(ik-\beta)^2\bigr)^2 \;.$$

The real part of the derivative w.r.t. k yields

$$\frac{\mathrm{dRe}\,P(-\beta+ik)}{\mathrm{d}k} \;=\; 4k-4k^3+12k\beta^2 \;.$$

The solutions of $\mathrm{dRe}\,P(-\beta+ik)/\mathrm{d}k \;=0$ are $k_\beta^* \;=\pm\sqrt{1+3\beta^2}$ (and $k_\beta^* \;=0$ which leads to less stringent bounds). Substituting back into $\mathrm{Re}\,P$, we get

$$\mathrm{Re}\,P(-\beta+ik_\beta^*) \;=\; \varepsilon^2+4\beta^2+8\beta^4 \;,$$

which is what we announced in (1.6) and got as a result of integration by parts in Eqs.(4.5)–(4.7). Solving now

$$\mathrm{Re}\,P(-\beta+ik_\beta^*)-c\beta \;=\; 0 \;,$$

for $c=c_\beta^*$ leads to $c_\beta^* \;=(\varepsilon^2+4\beta^2+8\beta^4)/\beta$. To find the absolutely minimal speed, we find that β for which c_β^* is extremal, that is $\partial_\beta c_\beta^* =0$. The only positive solution is

$$\beta_* \;=\; \frac{\sqrt{3\sqrt{1+6\varepsilon^2}-3}}{6} \;,$$

and hence,

$$c_{\beta_*}^* \;=\; \frac{4\bigl(\sqrt{1+6\varepsilon^2}-1+6\varepsilon^2\bigr)}{3\sqrt{3\sqrt{1+6\varepsilon^2}-3}} \;.$$

This quantity is the same as $\inf_{\beta\in\mathbf{R}} c_\beta$ where c_β is given by (1.7).

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