Extensive Properties of the

Complex Ginzburg-Landau Equation

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Abstract: We study the set of solutions of the complex Ginzburg-Landau equation in $\mathbf{R}^d$, $d < 3$. We consider the global attracting set (*i.e.*, the forward map of the set of bounded initial data), and restrict it to a cube Q_L of side L. We cover this set by a (minimal) number $N_{Q_L}(\varepsilon)$ of balls of radius ε in $\mathrm{L}^\infty(Q_L)$. We show that the Kolmogorov ε-entropy per unit length, $H_\varepsilon = \lim_{L\to\infty} L^{-d} \log N_{Q_L}(\varepsilon)$ exists. In particular, we bound H_ε by $\mathcal{O}\big(\log(1/\varepsilon)\big)$, which shows that the attracting set is *smaller* than the set of bounded analytic functions in a strip. We finally give a positive lower bound: $H_\varepsilon > \mathcal{O}\big(\log(1/\varepsilon)\big)$.

1. Introduction

In the last few years, considerable effort has been made towards a better understanding of partial differential equations of parabolic type in *infinite space*. A typical equation is for example the complex Ginzburg-Landau equation (CGL) on $\mathbf{R}^d$:

$$\partial_t A \,=\, (1+i\alpha)\Delta A + A - (1+i\beta)A|A|^2 \,. \tag{1.1}$$

Such equations show, at least numerically, in certain parameter ranges, interesting "chaotic" behavior, and our aim here is to discuss notions of *chaoticity per unit length* for such systems. Our discussion will be restricted to the CGL, but it will become clear from the methods of the proofs that the results can be extended without much additional work to other problems in which high frequencies are strongly damped.

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A first idea which comes to mind in the context of measuring chaoticity is the notion of "dimension per unit length." As we shall see, this quantity is a well-defined and useful concept in dynamical systems with finite-dimensional phase space. While the "standard" definition leads to infinite dimensions for finite segments of infinite systems, we shall see that an adequate definition, first introduced by Kolmogorov and Tikhomirov [KT], leads to finite bounds which measure the "complexity" of the set under study.

2. Attracting Sets

In the study of PDE's, there are several definitions of "attractors." In this work, we concentrate our attention onto attracting sets (which may be larger than attractors).

Definition. A set $\mathcal{G}$ is called an *attracting set* with fundamental neighborhood $\mathcal{U}$ for the flow Φ_t if

i) The set $\mathcal{G}$ is compact.

ii) For every open set $\mathcal{V} \supset \mathcal{G}$ we have $\Phi_t \mathcal{U} \subset \mathcal{V}$ when t is large enough.

iii) The set $\mathcal{G}$ is invariant.

The open set $\cup_{t>0}(\Phi_t)^{-1}(\mathcal{U})$ is called the basin of attraction of $\mathcal{G}$. If the basin of attraction is the full space, then $\mathcal{G}$ is called a *global attracting set*.

Remark. One finds a large number of definitions of "attractors" in the literature [T], [MS]. Our terminology is inspired from the theory of dynamical systems. In particular, an attracting set is *not* an attractor in the sense of dynamical systems, it is usually larger than the attractor. See also [ER] for a discussion of these issues.

We will consider the Eq.(1.1) in a (large) box Q_L of side L in $\mathbf{R}^d$, with periodic boundary conditions. Let $\mathcal{G}_{Q_L}$ denote the global attracting set for this problem. It has been shown [GH] that $\mathcal{G}_{Q_L}$ is a compact set in $\mathrm{L}^\infty_{\mathrm{per},Q_L}$ (since the set is made up of functions analytic in a strip around the real axis).

For the CGL on the *infinite space* the situation is somewhat more complicated. A non-trivial invariant set $\mathcal{G}$ can be defined in the topology of uniformly continuous functions as follows: First, if B is a large enough ball of uniformly continuous functions in L^∞, there is a finite time $T_0(B)$ such that for any $T > T_0(B)$ one has

$$\Theta^T(B) \subset B ,$$

where $t \mapsto \Theta^t$ is the flow defined by the CGL. The set $\mathcal{G}(B,T)$ is then defined by

$$\mathcal{G}(B,T) = \bigcap_{n\geq 0} \Theta^{nT}(B) . \tag{2.1}$$

It can be shown (see [MS]) that this set is invariant and that it does not depend on the initial ball B (if it is large enough) nor on the (large enough) time $T > T_0(B)$. Thus, we *define* $\mathcal{G} = \mathcal{G}(B,T)$. It is made up of functions which extend to bounded analytic functions in a strip. Its width and the bound on the functions only depend on the parameters of the problem. These facts can be found scattered in the literature, but are "well-known," see, *e.g.*, [C]. The set $\mathcal{G}$ probably lacks properties i) and ii) above in the topology of uniformly continuous functions. We will nevertheless call it a globally attracting set since in [MS] it was proven that in local and/or weaker topologies conditions of the type of i) and ii) are satisfied. The set $\mathcal{G}$ defined by Eq.(2.1) will be our main object of study.

3. Dimension in Finite Volume

We define $M_{Q_L}(\varepsilon)$ to be the minimum number of balls of radius ε in $\mathrm{L}^\infty_{\mathrm{per},Q_L}$ needed to cover $\mathcal{G}_{Q_L}$. One can then define

$$\mathcal{C}_{Q_L} = \limsup_{\varepsilon\to 0} \frac{\log M_{Q_L}(\varepsilon)}{\log(1/\varepsilon)} .$$

The technical term [M, 5.3] for this is the "upper Minkowski dimension." This dimension is an upper bound for the Hausdorff dimension. It is also equal to the (upper) box-counting dimension (in which the positions of the boxes are centered on a dyadic grid).

It has been shown by Ghidaglia and Héron [GH] that $\mathcal{C}_{Q_L}$ satisfies an "extensive bound:"

Proposition 3.1. *For CGL one has in dimensions $d = 1, 2$, the bound*

$$\limsup_{L\to\infty} \frac{\mathcal{C}_{Q_L}}{L^d} < \infty . \tag{3.1}$$

To our knowledge, it is an open problem to show the existence of the limit in (3.1). The difficulty in obtaining a proof is that the familiar methods of statistical mechanics of matching together pieces of configurations to obtain a subadditivity bound of the form (written for simplicity for the case of dimension $d = 1$ and with L instead of Q_L)

$$\mathcal{C}_{L_1+L_2} \le \mathcal{C}_{L_1} + \mathcal{C}_{L_2} + \mathcal{O}(1) ,$$

do not seem to work.

One can try to define a sort of "local" dimension by restricting the global problem to a local window. But this idea does not work either as we show now: For example consider the global attracting set $\mathcal{G}$ for CGL on the *infinite line*. As we have said before, this set is compact in a local topology which is not too fine. Take again a cube Q_L of side L in $\mathbf{R}^d$ and then denote by $N_{Q_L}(\varepsilon)$ the minimum number of balls of radius ε in $\mathrm{L}^\infty(Q_L)$ needed to cover $\mathcal{G}|_{Q_L}$. Again, this number is finite. But we have the following

Lemma 3.2. *For every $L > 0$ we have*

$$\liminf_{\varepsilon\to 0} \frac{\log N_{Q_L}(\varepsilon)}{\log(1/\varepsilon)} = \infty . \tag{3.2}$$

Remark. In other words, this lemma shows that the lower Minkowski dimension for the restriction of $\mathcal{G}$ to Q_L is infinite. Thus, there are many more functions in $\mathcal{G}|_{Q_L}$ than in $\mathcal{G}_{Q_L}$. In fact, our proof will show a little more, namely

Corollary 3.3. *The Hausdorff dimension of $\mathcal{G}|_{Q_L}$ is infinite for every $L > 0$. See also [BV].*

Proof. The proof will be given in Section 5.

The example of Lemma 3.2 and Corollary 3.3 teaches us that the restriction to nice functions on the infinite line produces "too many" functions on a *finite* interval, as the observation (the ε) becomes infinitely accurate. This fact calls for a new kind of definition. Such a possibility is offered by the considerations of Kolmogorov and Tikhomirov [KT].

4. The ε-entropy per Unit Length

The basic idea is to take the limit of infinite L *before* considering the behavior as ε goes to zero. Thus, with the definitions of the preceding section, we now define

$$H_\varepsilon = \lim_{L\to\infty} \frac{\log N_{Q_L}(\varepsilon)}{L^d} .$$

In the paper [KT], this quantity was studied for different sets of functions. The authors considered in particular three classes of functions on the real line:

i) The class $\mathcal{E}_\sigma(C)$ of entire functions f which are bounded by $|f(z)| \le Ce^{\sigma|\mathrm{Im}\, z|}$.
ii) The class $\mathcal{F}_{p,\sigma}(C)$ of entire functions f with growth of order $p > 1$, which are bounded by $|f(z)| \le Ce^{\sigma|\mathrm{Im}\, z|^p}$.
iii) The class $\mathcal{S}_h(C)$ of bounded analytic functions in the strip $|\mathrm{Im}\, z| < h$ with a bound $|f(z)| < C$.

For these classes the following result holds

Theorem 4.1. *[KT]. One has the bounds:*

$$H_\varepsilon \sim \begin{cases} (2\sigma/\pi)\cdot \log(1/\varepsilon) & \textit{for the class } \mathcal{E}_\sigma(C), \\ \frac{2\sigma^{1/p}p^2}{\pi(2p-1)(p-1)^{1-1/p}} \cdot \big(\log(1/\varepsilon)\big)^{2-1/p} & \textit{for the class } \mathcal{F}_{p,\sigma}(C), \\ \frac{1}{\pi h}\cdot \big(\log(1/\varepsilon)\big)^2 & \textit{for the class } \mathcal{S}_h(C), \end{cases}$$

as $\varepsilon \to 0$ in the sense that the l.h.s divided by the r.h.s has limit equal to 1.

Notation. It will sometimes be convenient to write the dependence on the space such as $H_\varepsilon\big(\mathcal{E}_\sigma(C)\big)$.

Our main result is the following

Theorem 4.2. *The global attracting set $\mathcal{G}$ of CGL satisfies a bound*

$$H_\varepsilon(\mathcal{G}) \le \text{const.} \log(1/\varepsilon) , \tag{4.1}$$

where the constant depends only on the parameters of the equation.

Remark. The reader should note that this result contains new information about the set $\mathcal{G}$ of limiting states. It is for example well known that the solutions of CGL are analytic and bounded in a strip, that is, they are in the class $\mathcal{S}_h(C)$ for some $h > 0$ and some $C < \infty$. This alone, however would only give a bound

$$\frac{1}{\pi h}\big(\log(1/\varepsilon)\big)^2 ,$$

as we have seen in Theorem 4.1. Therefore, Theorem 4.2 shows that the long-time solutions are not only analytic in a strip, but form a proper subset of $\mathcal{S}_h(C)$ with smaller ε-entropy per unit length. On the other hand, the set $\mathcal{G}$ is in general *not contained in* the class $\mathcal{E}_\sigma(C)$, because some stationary solutions are not entire. For example for the real Ginzburg-Landau equation, the function $\tanh(x/\sqrt{2})$ is a stationary solution with a singularity in the complex plane. For the CGL, Hocking and Stewartson [HS, Eq.(5.2)] describe time-periodic solutions which exist in certain parameter ranges of α and β, and which are again not entire in x and are of the form

$$\text{const.}e^{ia_1 t}\text{sech}(a_2 x)^{1+ia_3} ,$$

where $a_i = a_i(\alpha, \beta)$ can be found in [HS].

5. Proof of Lemma 3.2 and Corollary 3.3

We fix $L > 0$, and we want to show that

$$\liminf_{\varepsilon\to 0} \frac{\log N_{Q_L}(\varepsilon)}{\log(1/\varepsilon)} = \infty , \tag{5.1}$$

where $N_{Q_L}(\varepsilon)$ is the minimum number of balls needed to cover $\mathcal{G}|_{Q_L}$.

The idea of the proof is to observe that $\mathcal{G}|_{Q_L}$ contains subsets of arbitrarily high Hausdorff dimension. These subsets are essentially parts of the unstable manifold of the 0 solution.

We begin, as in [GH], by considering periodic solutions of period Λ for various Λ. In that space, for Λ large enough, the origin is an unstable fixed point and the spectrum of the generator for the linearized evolution is

$$\left\{ 1 - (1+i\alpha)\frac{4\pi^2}{\Lambda^2}(n_1^2 + \cdots + n_d^2) \;\middle|\; n_i \in \mathbf{Z} \right\} .$$

Thus, the origin is a hyperbolic fixed point if $2\pi/\Lambda$ is irrational. In that case the local unstable manifold W of the origin has dimension $D_\Lambda \equiv \mathcal{O}(1)\Lambda^d$. In other words, we have a $\mathcal{C}^1$ map Ψ_Λ from a neighborhood U of 0 in $\mathbf{R}^{D_\Lambda}$ to W which is injective (and in fact has differentiable inverse). This construction can be justified in a Sobolev space with sufficiently high index [GH, Remark 3.2, p. 289], [G].

This unstable manifold is of course contained in the global attracting set $\mathcal{G}_\Lambda$. But it is also in $\mathcal{G}$. We can consider W as a subset $\Phi(U)$ in $\mathcal{G}$ and look at it in $\mathrm{L}^\infty(Q_L)$ (with $L \ll \Lambda$). We would like to prove that there also it has a dimension equal to D_Λ. Note that there is a $\mathcal{C}^1$ map Φ which maps U to W. We claim Φ is injective. Indeed, assume not, then there are two different points u_1 and u_2 in U such that on Q_L the functions $\Phi(u_1)$ and $\Phi(u_2)$ coincide. But since these functions are analytic in a strip they coincide everywhere and hence $u_1 = u_2$: we have a contradiction.

This implies that in $\mathrm{L}^\infty(Q_L)$, the local unstable manifold W has also dimension D_Λ. Therefore, for ε small enough, we need at least

$$\left(\frac{1}{\varepsilon}\right)^{D_\Lambda - 1}$$

balls of radius ε to cover it. The assertion (5.1) follows by letting Λ tend to infinity. The proof of Lemma 3.2 is complete.

□

Since we have constructed a lower bound for every L, the Corollary 3.3 follows at once.

6. Upper Bound on the ε-entropy per Unit Length

We study in this section the quantity $H_\varepsilon(\mathcal{G})$ for the global attracting set on $\mathcal{G}$ for the CGL on the whole space. We begin by

Theorem 6.1. *For fixed $\varepsilon > 0$, the sequence $\log N_{Q_L}(\varepsilon)/L^d$ has a limit when L goes to infinity, and there exists a constant C such that*

$$\lim_{L\to\infty} \frac{\log N_{Q_L}(\varepsilon)}{L^d} \leq C \log(1/\varepsilon) . \tag{6.1}$$

The constant C only depends on the parameters of the CGL.

We first prove the existence of the limit.

Lemma 6.2. *For any fixed $\varepsilon > 0$, the sequence $\log N_{Q_L}(\varepsilon)/L^d$ has a limit when L goes to infinity.*

Proof. Let B and B' denote two disjoint bounded sets of $\mathbf{R}^d$. We denote by $N_B(\varepsilon)$ the minimum number of balls in $\mathrm{L}^\infty(B)$ of radius ε which is needed to cover $\mathcal{G}|_B$. Since we are using the sup norm, it is easy to verify that

$$N_{B\cup B'}(\varepsilon) \leq N_B(\varepsilon) N_{B'}(\varepsilon) , \tag{6.2}$$

because one can choose the functions in B and B' independently. The lemma follows by the standard sub-additivity argument, see [R], since the Q_L form a van Hove sequence.

□

We now begin working towards a bound relating $N_{Q_L}(\varepsilon)$ and $N_{Q_L}(\varepsilon/2)$. The bound will be inefficient for small L but becomes asymptotically better. We let the CGL semi-flow act on balls in $\mathrm{L}^\infty(Q_L)$, and we will analyze the deformation of these balls by looking at the difference between the trajectory of the center and the trajectory of the other points.

We begin by considering functions f and g, both in $\mathcal{G}$. We set

$$w_0 \,=\, g - f \,.$$

It is left to the reader to verify that there are bounded functions R and S of space and time such that

$$\partial_t w \,=\, (1+i\alpha)\Delta w + Rw + S\overline{w} \,, \tag{6.3}$$

more precisely, we set $w(t=0) = w_0$, and

$$R \,=\, 1-(1+i\beta)(f_t+g_t)\bar{f}_t \,, \quad S \,=\, -(1+i\beta)g_t^2 \,,$$

where $f_t = \Theta^t(f)$, and $g_t = \Theta^t(g)$. Note that since $\mathcal{G}$ is bounded in a suitable space of analytic functions, there is a constant $K > 1$ which depends only on α and β such that

$$\sup_t \|w(t,\,\cdot\,)\|_\infty + \sup_t \|\nabla w(t,\,\cdot\,)\|_\infty \,\le\, K \,. \tag{6.4}$$

We want to show that if w_0 is small in Q_L then the same is true for the solution of (6.3) up to time 1. This might seem not to be true because a large perturbation may reach Q_L from the outside. However, using localization techniques, we now show that this effect can only take place near the boundary.

We will therefore introduce a layer of width ℓ near the boundary of the cube Q_L, and we assume $\ell < L$. We assume Q_L to be centered at the origin and consider the cube $Q_{L-\ell}$ also centered at the origin.

We use as in [CE] the family of space cut-off functions

$$\varphi_a(x) \,=\, Z\frac{1}{(1+|x-a|^4)^d} \,\equiv\, \varphi(x-a) \,,$$

where

$$Z^{-1} \,=\, \int dx\,\frac{1}{(1+|x|^4)^d} \,.$$

Lemma 6.3. *Let f and g be in $\mathcal{G}$, and let $w_0 = f - g$. In dimension $d \le 3$, if $\ell > 1/\varepsilon$ and w is a solution of Eq.(6.3) with initial data w_0 satisfying*

$$\|w_0\|_\infty \,\le\, 2K \,, \qquad \text{and} \qquad \sup_{x\in Q_L} |w_0(x)| \,\le\, \varepsilon \,,$$

then

$$\sup_{0\le t\le 1}\ \sup_{a\in Q_{L-\ell/2}} \int dx\,\varphi_a(x)\,|w(t,x)|^2 \,\le\, \mathcal{O}(\varepsilon^2) \,, \tag{6.5}$$

$$\sup_{0\le t\le 1}\ \sup_{a\in Q_{L-\ell}} |w(t,a)| \le \mathcal{O}(\varepsilon)\ , \tag{6.6}$$

and

$$\sup_{x\in Q_{L-\ell}} |\nabla w(t=1,x)| \le \mathcal{O}(\varepsilon)\ . \tag{6.7}$$

These bounds depend on K but are independent of $\ell > 1/\varepsilon$.

Remark. The constant $K = K(\alpha,\beta)$ in this lemma is the one found in Eq.(6.4). Below, the notation $\mathcal{O}_{\alpha,\beta}(1)$ will stand for a bound which depends only on α, β and this $K(\alpha,\beta)$, but not on L, ℓ or ε.

Proof. We begin by bounding $X \equiv \partial_t \int dx\, \varphi_a(x)|w(t,x)|^2$. Using Eq.(6.3) we have:

$$X = \int dx\, \bar{w}\varphi_a\big((1+i\alpha)\Delta w + Rw + S\overline{w}\big) + cc\ ,$$

where cc denotes the complex conjugate. Integrating by parts we get

$$\begin{aligned} X &= -(1+i\alpha)\int dx\, \varphi_a|\nabla w|^2 - (1+i\alpha)\int dx\, \bar{w}(\nabla\varphi_a\cdot\nabla w) \\ &\quad + \int dx\, \varphi_a\bar{w}\big(Rw + S\bar{w}\big) + cc\ . \end{aligned} \tag{6.8}$$

By the choice of φ_a we have $|\nabla\varphi_a(x)| \le \text{const.}\varphi_a(x)$, uniformly in x and a. Therefore X can be bounded above by

$$X \le -2\int dx\, \varphi_a|\nabla w|^2 + \mathcal{O}_{\alpha,\beta}(1)\int dx\, \varphi_a|w||\nabla w| + \mathcal{O}_{\alpha,\beta}(1)\int dx\, \varphi_a|w|^2\ .$$

By polarization, and using that $\varphi_a > 0$, we get a bound

$$\partial_t\int dx\, \varphi_a|w|^2 \le -\int dx\, \varphi_a|\nabla w|^2 + \mathcal{O}_{\alpha,\beta}(1)\int dx\, \varphi_a|w|^2\ . \tag{6.9}$$

Therefore we see that there is a constant C which depends only on α and β, for which we have the differential inequality

$$\partial_t\int dx\, \varphi_a(x)|w(t,x)|^2 \le C\int dx\, \varphi_a(x)\,|w(t,x)|^2\ . \tag{6.10}$$

Since $w(0,x)$ is bounded on $\mathbf{R}^d$ and small on Q_L, we have for $\ell > 1/\varepsilon$,

$$\sup_{a\in Q_{L-\ell}}\int dx\, \varphi_a(x)|w(0,x)|^2 \le \mathcal{O}(1+K^2)\varepsilon^2\ .$$

To see this, split the integration region into Q_L and $\mathbf{R}^d \setminus Q_L$. In the first region, w is small and in the second region the integral of φ_a is small and $|w| \le K$. Using Eq.(6.10), we find

$$\sup_{0\le t\le 1}\ \sup_{a\in Q_{L-\ell}} \int dx\, \varphi_a(x)\, |w(t,x)|^2 \;\le\; e^C \mathcal{O}_{\alpha,\beta}(1)\varepsilon^2 \;=\; \mathcal{O}_{\alpha,\beta}(1)\varepsilon^2 \,.$$

Thus we have shown Eq.(6.5).

We next bound the solutions in L^∞. Let G_t denote the convolution kernel of the semigroup generated by the operator $(1+i\alpha)\Delta$. We have

$$w(t,\,\cdot\,) \;=\; G_t \star w_0 + \int_0^t ds\, G_{t-s} \star \big(R(s,\,\cdot\,)w(s,\,\cdot\,) + S(s,\,\cdot\,)\bar w(s,\,\cdot\,)\big) \,. \tag{6.11}$$

We first bound the term

$$Y_{t,s} \;\equiv\; G_{t-s} \star \big(R(s,\,\cdot\,)w(s,\,\cdot\,)\big) \,.$$

We rewrite it as

$$Y_{t,s}(x) \;=\; \int dy\, \frac{G_{t-s}(x-y)}{\sqrt{\varphi(x-y)}}\, \sqrt{\varphi(x-y)}\, R(s,y)w(s,y) \,.$$

By the Schwarz inequality, we get a bound

$$Y_{t,s}^2 \;\le\; \int dy\, \frac{|G_{t-s}|^2(x-y)}{\varphi(x-y)} \cdot \mathcal{O}_{\alpha,\beta}(1) \int dz\, \varphi_x(z)|w(s,z)|^2 \,. \tag{6.12}$$

Using Eq.(6.5), the second factor in (6.12) is bounded by $\mathcal{O}(\varepsilon^2)$. The complex heat kernel G can be bounded as follows:

Lemma 6.4. *For every $n > 0$ there is a constant C_n such that*

$$|G_t(z)| \;\le\; \frac{C_n}{(1+z^2/t)^{n/2}}\, \frac{1}{t^{d/2}} \,, \tag{6.13}$$

and

$$|\nabla G_t(z)| \;\le\; \frac{1}{t^{d/2}}\, \frac{C_n}{(1+z^2/t)^n}\, \frac{|z|}{t} \,. \tag{6.14}$$

Proof. Use the stationary phase method [H].

□

Using this lemma, the first factor in (6.12) is bounded for $t-s<1$ and for n large enough, by

$$\int dy\, \frac{|G_{t-s}|^2(x-y)}{\varphi(x-y)} \;\le\; \int dy \frac{C_n}{\big(1+\frac{(x-y)^2}{t-s}\big)^n}\, \frac{1}{(t-s)^d}(1+|x-y|^4)^d \;\le\; \mathcal{O}\big((t-s)^{-d/2}\big) \,.$$

Inserting in (6.12), and integrating over s, we get the bound

$$\int_0^t ds\, Y_{t,s} \;\le\; \mathcal{O}(\varepsilon)\;,$$

provided $d < 4$. The term involving S is bounded in the same manner. The inhomogeneous term in (6.11) is bounded by splitting the convolution integral into the regions $y \in Q_L$ and $y \in \mathbf{R}^d \setminus Q_L$. The first term gives a small contribution because w_0 is $\mathcal{O}(\varepsilon)$ on Q_L and the second contribution is small because the kernel G_t is small for $x \in Q_{L-\ell}$ and $y \in \mathbf{R}^d \setminus Q_L$. This proves Eq.(6.6).

It remains to show Eq.(6.7). We have

$$\nabla w(t,\,\cdot\,) \;=\; \nabla G_t \star w_0 + \int_0^t ds\, \nabla G_{t-s} \star \big(R(s,\,\cdot\,)w(s,\,\cdot\,) + S(s,\,\cdot\,)\bar{w}(s,\,\cdot\,)\big)\;.$$

We deal first with the inhomogeneous term. Using the same splitting as before, and Lemma 6.4, we get

$$\sup_{x\in Q_{L-\ell}} \big|\big((\nabla G_{t=1}) \star w_0\big)(x)\big| \;\le\; \mathcal{O}(\varepsilon)\;.$$

The homogeneous term I involving R is:

$$I \;=\; \int_0^t ds\, \nabla G_{t-s} \star \big(wR\big)\;.$$

We want to bound I for $t = 1$ and rewrite it as

$$I \;=\; \int_0^{1/2} ds\, \nabla G_{1-s} \star \big(wR\big) + \int_{1/2}^1 ds\, G_{1-s} \star \big(w\nabla R\big) + \int_{1/2}^1 ds\, G_{1-s} \star \big(R\nabla w\big) \;=\; I_1 + I_2 + I_3\;.$$

The term I_2 is bounded in the same way as the integral of $Y_{t,s}$. To bound the term I_1 we observe that there is no singularity in the kernel (6.14), since $s < \frac{1}{2}$, and furthermore,

$$|\nabla G_{1-s}(z)| \;\le\; \text{const.}\varphi(z)\;.$$

Then the Schwarz inequality and the results on w yield

$$I_1 \;\le\; \mathcal{O}(\varepsilon)\;. \tag{6.15}$$

Finally, consider I_3. Integrating Eq.(6.9) over s from 0 to $\frac{1}{2}$, we have

$$\begin{aligned}&\int_0^{1/2} ds \int dx\, \varphi_a(x)\, |\nabla w(s,x)|^2 \\ &\qquad \le\; \mathcal{O}(1) \int_0^{1/2} ds \int dx\, \varphi_a(x)\, |w(s,x)|^2 + \int dx\, \varphi_a(x)\, |w(0,x)|^2\;.\end{aligned}$$

Our previous bounds show that the r.h.s. is bounded by $\mathcal{O}(\varepsilon^2)$. Therefore there is a value of $s^* \in (0, \frac{1}{2})$ for which

$$\int dx\, \varphi_a(x)\, |\nabla w(s^*, x)|^2 \;\leq\; \mathcal{O}(\varepsilon^2)\;. \tag{6.16}$$

Furthermore, we have

Lemma 6.5. *We have the bounds*

$$\partial_t \int dx\, \varphi_a(x)\, |\nabla w(t,x)|^2 \;\leq\; \mathcal{O}(1) \int dx\, \varphi_a(x)\, |\nabla w(t,x)|^2 + \mathcal{O}(1) \int dx\, \varphi_a(x)\, |w(t,x)|^2\;. \tag{6.17}$$

Proof. We start with

$$\begin{aligned}
\partial_t \int dx\, \varphi_a\, |\nabla w|^2 \;&=\; \int dx\, \varphi_a\, \nabla \bar{w} \cdot \partial_t \nabla w + cc \\
&=\; \int dx\, \varphi_a\, \nabla \bar{w} \cdot \nabla \Big((1+i\alpha)\Delta w + Rw + S\bar{w} \Big) + cc \\
&=\; -\int dx\, \varphi_a\, \Delta \bar{w} \Big((1+i\alpha)\Delta w + Rw + S\bar{w} \Big) \\
&\quad -\int dx\, (\nabla \varphi_a \cdot \nabla \bar{w}) \Big((1+i\alpha)\Delta w + Rw + S\bar{w} \Big) + cc\;.
\end{aligned}$$

Using again the explicit form of φ_a, completing the square and polarization, as in the proof of Eq.(6.10), the assertion follows.

$\square$

We continue with the proof of Lemma 6.3. Let $s \in (\frac{1}{2}, 1]$ and $T_s = \int dx\, \varphi_a(x) |\nabla w(s,x)|^2$. Then we integrate the differential inequality (6.17) which reads $\partial_t T_t \leq \mathcal{O}(1) T_t + \mathcal{O}(\varepsilon^2)$ from s^* to s. This yields, using (6.16),

$$\begin{aligned}
\int dx\, \varphi_a(x) |\nabla w(s,x)|^2 \;&\leq\; \exp\big(\mathcal{O}(1)(s-s^*)\big) \int dx\, \varphi_a(x) |\nabla w(s^*,x)|^2 + \mathcal{O}(\varepsilon^2) \\
&\leq\; \mathcal{O}(\varepsilon^2)\;.
\end{aligned} \tag{6.18}$$

Using this bound, we rewrite

$$I_3 \;=\; \int_{1/2}^1 ds \int dy\, \frac{G_{1-s}(x-y)}{\sqrt{\varphi(x-y)}}\, \sqrt{\varphi(x-y)} R(s,y) \nabla w(s,y)\;.$$

Using the Schwarz inequality as in Eq.(6.12), we get a bound

$$I_3 \;\leq\; \mathcal{O}(\varepsilon)\;.$$

Combining the bounds on I_1, I_2 and I_3 completes the proof of Eq.(6.7). The proof of Lemma 6.3 is complete.

□

Lemma 6.3 gives us control over the evolution of differences in $\mathcal{G}$, when they are small in $\mathcal{G}|_{Q_L}$. We shall now use this information to study the deformation of balls covering $\mathcal{G}|_{Q_L}$. To formulate the next result we need the following notation: Consider the universal attracting set $\mathcal{G}$. The quantity $N_B^{(t)}(\varepsilon)$ denotes the number of balls of radius ε needed to cover the set $\Theta^t(\mathcal{G})|_B$, in $\mathrm{L}^\infty(B)$, where Θ^t is the semi-flow defined by the CGL equation.

Proposition 6.6. *There are constants $c < \infty$ and D, $D_1 < \infty$ such that for all sufficiently small $\varepsilon > 0$ and all $L > 3/\varepsilon$ one has the bound*

$$N_{Q_L}^{(t+1)}(\varepsilon/2) \leq \left(\frac{c}{\varepsilon}\right)^{D_1 L^{d-1} \varepsilon^{-(1+d)}} D^{L^d} N_{Q_L}^{(t)}(\varepsilon) . \tag{6.19}$$

Before we prove this proposition, we need a geometric lemma:

Lemma 6.7. *Let Q be a set of diameter r in $\mathbf{R}^d$ and assume that $\mathcal{F}$ is a family of complex functions f on Q which satisfy the bounds*

$$|f| \leq a , \quad |\nabla f| \leq b ,$$

with $br \leq c/2$. Then one can cover $\mathcal{F}$ with not more than

$$(4a/c)^2$$

balls of radius c in $\mathrm{L}^\infty(Q)$.

Proof. On a disk in $\mathbf{R}^d$ of diameter r, the function f varies no more than br which is bounded by $c/2$. On the other hand, one can find a set $\mathcal{S}$ of $(4a/c)^2$ complex numbers of modulus less than a such that every complex number of modulus less than a is within $c/2$ of $\mathcal{S}$. Since f varies less than $c/2$ one can find a constant function f^* with value in $\mathcal{S}$ such that $\sup_Q |f - f^*| < c$.

□

Proof of Proposition 6.6. By definition we can find, for every $t \geq 0$, $N_{Q_L}^{(t)}(\varepsilon)$ balls of radius ε in $\mathrm{L}^\infty(Q_L)$ which cover $\Theta^t(\mathcal{G})|_{Q_L}$. Therefore we can find a collection $\mathcal{B}$ of $N_{Q_L}^{(t)}(\varepsilon)$ balls of radius 2ε in $L^\infty(Q_L)$ *with center in* $\Theta^t(\mathcal{G})|_{Q_L}$, which cover $\Theta^t(\mathcal{G})|_{Q_L}$. Let B be a ball (*i.e.*, an element of $\mathcal{B}$). We denote by $B \cap \Theta^t(\mathcal{G})$ those functions in $\Theta^t(\mathcal{G})$ whose restriction to Q_L is in B. We have obviously $\cup_{B\in\mathcal{B}} \big(B \cap \Theta^t(\mathcal{G})\big) \supset \Theta^t(\mathcal{G})$, and therefore

$$\Theta^{t+1}(\mathcal{G})|_{Q_L} \subset \bigcup_{B\in\mathcal{B}} \Theta^1\big(B \cap \Theta^t(\mathcal{G})\big)|_{Q_L} .$$

Thus, we can move the time forward by one unit without changing the set we cover. This will be the crux of our argument, which will use the smoothing properties of Θ^1 described in Lemma 6.3.

We are going to cover every set $\Theta^1\big(B \cap \Theta^t(\mathcal{G})\big)|_{Q_L}$ by balls of radius $\varepsilon/2$ in $\mathrm{L}^\infty(Q_L)$. Counting all these balls will give the result. So we fix a $B \in \mathcal{B}$ and consider $\Theta^1\big(B\cap\Theta^t(\mathcal{G})\big)|_{Q_L}$. Since $B \in \mathcal{B}$, its center f is in $\Theta^t(\mathcal{G})|_{Q_L}$, and, since $\Theta^t(\mathcal{G}) \subset \mathcal{G}$, we also have $f \in \mathcal{G}|_{Q_L}$. (In fact f is the restriction of a function in $\mathcal{G}$ to Q_L.) Let g be an arbitrary point in $\big(B\cap\Theta^t(\mathcal{G})\big)|_{Q_L}$. Our construction makes sure that both f and g satisfy the assumptions of Lemma 6.3 (with 2ε instead of ε). From Lemma 6.3, there are constants c_1 and c_2 (which do not depend on ε, f, or g) such that in $Q_{L-\ell}$ the following holds: If $w_0 = g - f$ and $w = \Theta^1(g) - \Theta^1(f)$, then

$$|w| \le c_1\varepsilon\ , \qquad |\nabla w| \le c_2\varepsilon\ .$$

Let

$$r_1 = \min\big(1, 1/(4c_2)\big)\ .$$

We partition $Q_{L-\ell}$ into disjoint cubes Q of side r_1 (except at the boundary where we take possibly a strip of smaller cubes if necessary). In each of these cubes we can apply Lemma 6.7 with $c = \varepsilon/2$ since

$$c_2\varepsilon r_1 \le \varepsilon/4\ .$$

Therefore we can cover the restriction of $\Theta^1\big(B \cap \Theta^t(\mathcal{G})\big)$ to each cube Q by

$$(4c_1\varepsilon/(\varepsilon/2))^2 = 64c_1^2$$

balls of radius $\varepsilon/2$ in $\mathrm{L}^\infty(Q)$. We shall now use the same method in the corridor $Q_L\backslash Q_{L-\ell}$ but with balls at a different scale. In $Q_L\backslash Q_{L-\ell}$ we have only inequality (6.4) and not a bound $\mathcal{O}(\varepsilon)$ as in $Q_{L-\ell}$. Therefore we define

$$r_2 = \varepsilon/(4K)\ ,$$

and again $c = \varepsilon/2$. This leads to

$$Kr_2 = c/2\ .$$

We now cover the corridor $Q_L\backslash Q_{L-\ell}$ by cubes Q' of side r_2 (again a smaller strip at the boundary may be needed). In each of these cubes Q' the Lemma 6.7 applies and we can cover $\Theta^1\big(B \cap \Theta^t(\mathcal{G})\big)$ restricted to these cubes by

$$64K^2\varepsilon^{-2}$$

balls of radius $\varepsilon/2$ in $\mathrm{L}^\infty(Q')$. We now have a covering of Q_L by disjoint cubes. If we have a ball of radius $\varepsilon/2$ in L^∞ in each cube, this defines a ball in $\mathrm{L}^\infty(Q_L)$ since in L^∞ the product of two independent covers is a cover of the union of the sets, see Eq.(6.2).

To get a covering of $\Theta^1\big(B \cap \Theta^t(\mathcal{G})\big)$ in $\mathrm{L}^\infty(Q_L)$ we have to consider all these possible balls and in particular count them. It is easy to verify that the number of such balls is bounded by

$$(64c_1^2)^{(1+(L-\ell)/\min(1,1/4c_2))^d}(64K^2\varepsilon^{-2})^{2(1+4KL/\varepsilon)^{d-1}(1+4K\ell/\varepsilon)}\ , \tag{6.20}$$

and the inequality (6.19) follows. The proof of Proposition 6.6 is complete.

□

Proof of Theorem 6.1. Finally, we can prove Theorem 6.1, and hence also Theorem 4.2. We use Proposition 6.6 recursively by starting at time $t = 1$ with $\varepsilon = 1$. For this case, we can apply Lemma 6.7 with $a = K$, $b = K$, $r = 1/(4K)$ to get

$$N_{Q_L}^{(t=1)}(\varepsilon = 1) \le e^{\mathcal{O}_{\alpha,\beta}\left((2L+1)^d\right)} ,$$

and using inequality (6.19) inductively, we get

$$N_{Q_L}^{(n+1)}(2^{-n}) \le e^{\mathcal{O}_{\alpha,\beta}\left((2L+1)^d\right)} D^{nL^d} \prod_{j=0}^{n-1} (2^j c)^{D_1 L^{d-1} 2^{j(d+1)}} .$$

Taking logarithms and dividing by $(2L)^d$ we get

$$\frac{\log N_{Q_L}^{(n+1)}(2^{-n})}{(2L)^d} \le n \log D + L^{-d}\mathcal{O}_{\alpha,\beta}(L^d) + L^{-d}\mathcal{O}_{\alpha,\beta}(nL^{d-1}2^{n(d+1)}) .$$

Clearly, Theorem 6.1 follows by taking n as the integer part of $\log(1 + 1/\varepsilon)$.

□

Remark. As asserted, D only depends on the parameters of the equation, as can be seen from Eq.(6.20):

$$D = \mathcal{O}\left(64 c_1^{1/\min(1,1/(4c_2))}\right) ,$$

where c_1 and c_2 can be found in the proof of Lemma 6.3. Note also that there is a crossover point (for our bound) between the behavior described in Theorem 4.2, and the divergence described in (3.2), at about $\varepsilon = L^{-1/(1+d)}$.

7. Lower Bound on the ε-entropy per Unit Length

In this section, we construct a lower bound on $H_\varepsilon(\mathcal{G})$. The idea is to construct a subset of the "local unstable manifold" of the origin with large enough ε-entropy per unit length. Working in space dimension 1 is enough, because such solutions are also solutions (in L^∞) in higher dimensions which do not depend on the other variables (of course the lower bounds are not very accurate). The main result of this section is then

Theorem 7.1. *There is a constant $A > 0$ such that for sufficiently small $\varepsilon > 0$, the ε-entropy per unit length of the unstable manifold of 0 (and hence of the global attracting set $\mathcal{G}$ of CGL) satisfies the bound*

$$H_\varepsilon(\mathcal{G}) \ge A \log(1/\varepsilon) . \tag{7.1}$$

7.1. The idea of the proof

To obtain a lower bound on the ε-entropy (always per unit length), we exhibit a large enough set of functions for which we prove that they are in the global attracting set. This set is built by observing that the 0 solution $u = 0$ has an unstable linear subspace which is made up of functions with momenta k in $[-1, 1]$. For these functions to be in the strongly unstable region, we restrict our attention to the class $\mathcal{E}_b(\eta)$ with $b = 1/3$ of entire functions in $z = x + iy$ which are bounded by $|f(z)| \le \eta e^{b|\mathrm{Im}\, z|}$. The Fourier transform $\widehat{f}$ of a function f in this class is a distribution with support in $[-b, b]$ (see [S]) and is therefore strongly unstable. Furthermore, by Theorem 4.1 we have the bound

$$\lim_{\varepsilon \to 0} \frac{H_\varepsilon\big(\mathcal{E}_b(\eta)\big)}{\frac{2b}{\pi}\log(1/\varepsilon)} = 1 , \tag{7.2}$$

so there are "many" such functions. (See [KT], Theorem XXII and beginning of §3).

We want to use the set $\mathcal{E}_b(\eta)$ as the starting point for the construction of a set in $\mathcal{G}$ with positive ε-entropy. Thus, we want to evolve $\mathcal{E}_b(\eta)$ *forward* in time to reach $\mathcal{G}$, using the evolution operator Θ^t defined above. However, this would move us far away from the solution 0 and we would lose control of the non-linearity. To overcome this difficulty, we first evolve the set $\mathcal{E}_b(\eta)$ *backward* in time by a linearized evolution. Thus, we use the method known from the usual construction of unstable manifolds, adapted to the case of continuous spectrum.

We begin by defining the linear evolution. Given $T > 0$ we let $\widehat{\Theta}_0^T(k) = e^{(1-(1+i\alpha)k^2)T}$ and then

$$(\widehat{\Theta_0^T f})(k) = \widehat{\Theta}_0^T(k)\widehat{f}(k) = e^{(1-(1+i\alpha)k^2)T}\widehat{f}(k) .$$

Note that the map (in x-space) $\Theta_0^T : f \mapsto \Theta_0^T f$ is the evolution generated by the linearized CGL. Inspired by scattering theory, we will then consider the quantity

$$S(f) = \lim_{T\to\infty} \Theta^T \Theta_0^{-T}(f) .$$

Since we consider the unstable manifold of 0 and stay in a vicinity of $f = 0$, the nonlinearities should be negligible and thus the following result seems very natural:

Theorem 7.2. *Let* $b = 1/3$. *There is an* $\eta_* > 0$ *such that for* $\eta \le \eta_*$ *the following limit exists in* $\mathrm{L}^\infty(\mathbf{R})$ *for* $f \in \mathcal{E}_b(\eta)$*:*

$$S(f) = \lim_{T\to\infty} \Theta^T \Theta_0^{-T} f .$$

Moreover,

$$S(f) = f + Z(f) ,$$

where Z *is Lipshitz continuous in* f*, with a Lipshitz constant of order* $\mathcal{O}(\eta)$.

In other words, S is close to the identity. Using this kind of information, we shall see that if two functions are separated by ε the functions $S(f) - S(f')$ are separated almost as much. Therefore, knowing that the set $\mathcal{E}_b(\eta)$ of f has positive ε-entropy implies that the set $S(\mathcal{E}_b(\eta))$—which is in the global attracting set—also has positive ε-entropy, as we shall show later.

7.2. The regularized linear evolution

In this subsection, we construct a somewhat more regular representation of Θ_0^T, which is needed because we consider negative T.

We consider the class $\mathcal{E}_b(\eta)$, with $b = 1/3$. It is clear from the Paley-Wiener-Schwartz [S] theorem that the functions $f \in \mathcal{E}_b(\eta)$ have a Fourier transform $\widehat{f}(k) = \int dx\, e^{ikx} f(x)$ which is a distribution with support in $[-b, b]$. If $\widehat{f}$ were a function, we could freely go back and forth between k-space and x-space. To deal with this problem, we use a regularizing device. Let $c > b$ and let $\widehat{\psi}$ be a positive $\mathcal{C}^\infty$ function with support in $[-c, c]$ and equal to 1 on $[-b, b]$. We shall take $b = 1/3$, $c = 1/\sqrt{3}$. Clearly $\widehat{\psi}(k)\widehat{f}(k) = \widehat{f}(k)$ (as a distribution) and therefore $\psi \star f = f$ (in x-space), where $\star$ denotes the convolution product. We define a regularized linear evolution kernel

$$g_T(x) = \int dk\, e^{ikx} \widehat{\psi}(k) e^{T(1-(1+i\alpha)k^2)} ,$$

and then we define

$$\bigl(\Theta_{0,\psi}^T f\bigr)(x) \equiv \bigl(g_T \star f\bigr)(x) .$$

This is our regularized representation of the linear evolution. By construction, it has the property: If $f \in \mathcal{E}_b(\eta)$, then

$$\Theta_{0,\psi}^T f = \Theta_0^T f , \tag{7.3}$$

as a distribution. But, as we shall see below, the l.h.s. is a well defined function and thus we can use either of the definitions, whichever is more convenient. Henceforth, we use the notation f_t for $\Theta_0^t f = \Theta_{0,\psi}^T f$.

7.3. Proof of the first part of Theorem 7.2

This theorem is relatively conventional, but tedious, to prove. We will therefore only sketch the standard estimates and describe in detail only the general sequence of estimates which are needed.

We begin the proof of the first part of Theorem 7.2 with a study of Θ^t. First we would like to prove that $\Theta^t(f_{-T}) - f_{t-T}$ remains small for $0 \le t \le T$.

Lemma 7.3. *For η small enough, there is a $\rho > 0$ such that for any $T > 0$ and any $t \in [0, T]$ we have for all $f \in \mathcal{E}_b(\eta)$, the bound*

$$\|\Theta^t(f_{-T}) - f_{t-T}\|_\infty \le \eta^2 e^{-\rho(T-t)} .$$

Proof. First observe that by assumption $\|f\|_\infty \le \eta$. By definition, we have

$$\bigl(\Theta_0^{-T} f\bigr)(x) = \int dy\, dk\, e^{ik(x-y)} e^{-T(1-k^2(1+i\alpha))} \widehat{\psi}(k) f(y) .$$

Since $\widehat{\psi}$ is smooth and supported in $|k| \le c$, we get from this the easy but useful bound

$$\|f_{-T}\|_\infty \le \mathcal{O}(\eta) e^{-(1-c^2)T} . \tag{7.4}$$

Using Eq.(7.3), we see that $\Theta_{0,\psi}^{t} f_{-T} = f_{t-T}$ satisfies

$$\partial_t f_{t-T} \;=\; (1+i\alpha)\partial_x^2 f_{t-T} + f_{t-T} \;.$$

We let $v = \Theta^t(f_{-T}) - f_{t-T}$, and then we find

$$\partial_t v \;=\; (1+i\alpha)\partial_x^2 v + v - (1+i\beta)(v+f_{t-T})|(v+f_{t-T})|^2 \;.$$

We write this as an integral equation using $v(0,x)=0$. We get

$$v(t,\cdot) \;=\; -(1+i\beta)\int_0^t ds\, \Theta_{0,\psi}^{t-s}\big((v(s,\cdot)+f_{s-T})\cdot |v(s,\cdot)+f_{s-T}|^2\big) \;. \tag{7.5}$$

In particular there is an inhomogeneous term

$$-(1+i\beta)\int_0^t ds\, \Theta_{0,\psi}^{t-s}\big(f_{s-T}|f_{s-T}|^2\big) \;. \tag{7.6}$$

This term can be bounded by using Eq.(7.4) and the bound $\|\Theta_0^\tau g\|_\infty \le e^\tau \|g\|_\infty$ (which follows from Lemma 6.4). We get

$$\eta^3 \mathcal{O}(1)\int_0^t ds\, e^{T-s} e^{-3(1-c^2)(T-s)} \;\le\; \mathcal{O}(\eta^3)\; e^{-\rho(T-t)} \;, \tag{7.7}$$

and here the restriction on the choice of c implies

$$\rho \;=\; 3(1-c^2) - 1 > 0 \;.$$

Thus we have bounded the inhomogeneous term (7.6).

We next consider the set of functions satisfying

$$\sup_{0\le t\le T} e^{\rho(T-t)} \sup_x |v(t,x)| \;\le\; \eta^2 \;,$$

with the associated metric. A standard argument using the bound (7.7) shows that in Eq.(7.5) we have a contraction (for η small enough, independent of t, T) in this space and therefore a unique solution v for the Eq.(7.5). Furthermore, the asserted bounds of Lemma 7.3 follow at once. We leave the (trivial) details to the reader. The proof of Lemma 7.3 is complete.

□

We now come to the proof of convergence of $\Theta^T f_{-T}$, as $T \to \infty$. We shall show that the derivative of this quantity is integrable in T. We recall that if we have a vector field X with flow φ_t then

$$\frac{d}{dt}\varphi_t(x) \;=\; \mathrm{D}\varphi_t[x]X(x) \;.$$

We use throughout the notation $\mathrm{D}F[x]$ for the derivative of F evaluated at x; this is usually an operator. In our case, we get

$$\begin{aligned}\frac{d}{dT}\Theta^T(f_{-T}) &= \mathrm{D}\Theta^T[f_{-T}] \\ &\quad\cdot\Big((1+i\alpha)\partial_x^2 f_{-T} + f_{-T} - (1+i\beta)f_{-T}|f_{-T}|^2 - (1+i\alpha)\partial_x^2 f_{-T} - f_{-T}\Big) \\ &= -(1+i\beta)\mathrm{D}\Theta^T[f_{-T}]\big(f_{-T}|f_{-T}|^2\big)\ .\end{aligned} \tag{7.8}$$

We want to prove that this quantity is integrable over T. For this purpose we have to control the linear operator $\mathrm{D}\Theta^T[f_{-T}]$.

Lemma 7.4. *We have the inequality*

$$\|\mathrm{D}\Theta^T[f_{-T}]w\|_\infty \;\leq\; \mathcal{O}(1)e^{T(1+\mathcal{O}(\eta))}\|w\|_\infty\ . \tag{7.9}$$

Proof. It is easy to verify that $\mathrm{D}\Theta^T[f_{-T}]w_0$ is given as the value at time T of the solution of the linear equation

$$\partial_t w \;=\; (1+i\alpha)\partial_x^2 w + w + R_\beta\, w + S_\beta\, \overline{w}\ , \tag{7.10}$$

with initial condition $w(t=0,\cdot) = w_0(\cdot)$. The coefficients R_β and S_β are given by

$$R_\beta(t,x) \;=\; -2(1+i\beta)|\Theta^t(f_{-T})(x)|^2\ ,$$

and

$$S_\beta(t,x) \;=\; -(1+i\beta)\left(\Theta^t(f_{-T})(x)\right)^2\ .$$

The assertion of Lemma 7.4 follows now, using a contraction argument, as in the study of Eq.(7.5), from Lemma 7.3 and the previous formula. The details are again left to the reader.

□

As a consequence, combining the inequalities (7.4) and (7.9), the right hand side of Eq.(7.8) is exponentially small in T and therefore integrable and we have a limit. So our map S is well-defined by

$$S(f) \;=\; \lim_{T\to\infty}\Theta^T(f_{-T}) \;=\; f + Z(f)\ ,$$

where

$$Z(f) \;=\; \lim_{T\to\infty}\int_0^T dt\,\frac{d}{dt}\Theta^t(f_{-t})\ ,$$

and in fact we have proven that this last term is of order η^2 (in reality η^3). This completes the proof of the first part of Theorem 7.2. It remains to prove that it is Lipshitz and to estimate its Lipshitz constant in L^∞. This will be done in the next subsection, together with some even more detailed information on Z which we need later.

7.4. Proof of the second part of Theorem 7.2

In this subsection, we prove the second part of Theorem 7.2, in fact even more. We first need some notation:

Remark. It will be more convenient to work with the intervals $[-L, L]$ instead of $[-L/2, L/2]$ as in the earlier sections. We shall use the following notations:

$$\begin{aligned} \mathbf{B} &= [-L, L] , \\ \mathbf{S} &= [-L + \ell, L - \ell] , \\ \mathbf{S}' &= [-L + \ell/2, L - \ell/2] , \\ \mathbf{S}'' &= [-L + \ell/4, L - \ell/4] , \\ \mathbf{B} \setminus \mathbf{S} &= [-L, -L + \ell) \cup (L - \ell, L] . \end{aligned}$$

These letters stand for "big" and "small." Our result is

Proposition 7.5. *The function Z is Lipshitz continuous in f in a neighborhood of 0 in $\mathcal{E}_b(\eta)$, $b = 1/3$, with a Lipshitz constant $\mathcal{O}(\eta)$:*

$$\|Z(f) - Z(f')\|_{\mathrm{L}^\infty(\mathbf{R})} \le \mathcal{O}(\eta)\|f - f'\|_{\mathrm{L}^\infty(\mathbf{R})} . \tag{7.11}$$

Moreover, for $\ell \ge 1/\varepsilon$ and $L > \ell$, one has the inequality

$$\|Z(f) - Z(f')\|_{\mathrm{L}^\infty(\mathbf{S})} \le \mathcal{O}(\eta)\|f - f'\|_{\mathrm{L}^\infty(\mathbf{B})} + \mathcal{O}(\varepsilon^2)\|f - f'\|_{\mathrm{L}^\infty(\mathbf{R})} . \tag{7.12}$$

Clearly, this result states more than what is asserted in Theorem 7.2, and thus, proving Proposition 7.5 will at the same time complete the proof of Theorem 7.2.

Proof. Using Eq.(7.8), we have the expression

$$Z(f) = \lim_{T\to\infty} Z_T(f) ,$$

where

$$Z_T(f) = -(1 + i\beta) \int_0^T dt\, \mathrm{D}\Theta^t[f_{-t}] \left(f_{-t}|f_{-t}|^2\right) .$$

To prove the first part of Proposition 7.5, we would like to obtain a bound uniform in T on the differential of $Z_T(f)$ with respect to f. Due to the presence of the absolute value, this function is not differentiable in f. One should therefore consider the expression obtained by taking the real and imaginary parts (note that we are only dealing with the values on the real axis and analyticity is not used in the following argument). To make the exposition simpler we will only explain the proof for the real Ginzburg-Landau equation (the field is real and $\alpha = \beta = 0$), and for a space dimension equal to one, but the general case only presents notational complications.

We have then, since we assume $\beta = 0$,

$$Z_T(f) = -\int_0^T dt\, \mathrm{D}\Theta^t[f_{-t}] \left(f_{-t}^3\right) .$$

From this formula we have

$$\begin{aligned} \mathrm{D}Z_T[f]w \;=\; & -\int_0^T dt\, \mathrm{D}^2\Theta^t[f_{-t}](f_{-t}^3, w) \\ & -\, 3\int_0^T dt\, \mathrm{D}\Theta^t[f_{-t}]\left(f_{-t}^2(\mathrm{D}f_{-t})w\right) \;\equiv\; X_1 + X_2\;. \end{aligned} \tag{7.13}$$

The second term X_2 is easier to handle and we first prove both Eq.(7.11) and (7.12) for the contributions coming from this term. Since f_{-t} is linear in f we have

$$(\mathrm{D}f_{-t})w \;=\; \Theta_0^{-t}w \;=\; w_{-t}\;.$$

Using Lemma 7.4, and Eq.(7.4), the integrand is bounded by

$$\|\mathrm{D}\Theta^t[f_{-t}]\left(f_{-t}^2 w_{-t}\right)\|_\infty \;\le\; \mathcal{O}(1)e^{t(1+\mathcal{O}(\eta))}\mathcal{O}(\eta)e^{-2(1-c^2)t}\mathcal{O}(1)e^{-(1-c^2)t}\|w\|_\infty\;, \tag{7.14}$$

and therefore we get a bound for the integral which is of the form

$$\|3\int_0^T dt\, \mathrm{D}\Theta^t[f_{-t}]\left(f_{-t}^2 w_{-t}\right)\|_\infty \;\le\; \mathcal{O}(\eta)\|w\|_\infty\;,$$

which shows that the contribution from X_2 to Eq.(7.11) is of the desired form, by linearity.

We now come to the localized bound Eq.(7.12) for the contribution coming from the term X_2. It is enough to assume T large enough and for example $T > t_0\log(1/\varepsilon)$. Using the exponential estimates of Eq.(7.14), we have for a large enough constant t_0 (independent of ε small enough),

$$\|3\int_{t_0\log\varepsilon^{-1}}^T dt\, \mathrm{D}\Theta^t[f_{-t}]\left(f_{-t}^2 w_{-t}\right)\|_\infty \;\le\; \mathcal{O}(\varepsilon^2)\|w\|_\infty\;.$$

For the other part of the integral, from 0 to $t_0\log\varepsilon^{-1}$, we proceed as in the proof of Lemma 7.4. We want to bound

$$X_{2,+} \;=\; \int_0^{t_0\log\varepsilon^{-1}} dt\, \mathrm{D}\Theta^t[f_{-t}]\left(f_{-t}^2 w_{-t}\right)\;.$$

In particular we will control the solution of the equation

$$\partial_t v \;=\; \partial_x^2 v + v + Rv\;,$$

where $R = \mathcal{O}(\eta^2)$. Note that this is very similar to the estimate in Lemma 6.3, but the proof is more delicate.

We can write an integral equation, namely if K_t is the heat kernel (associated with the Laplacian), we have

$$v_t \;=\; e^t K_t \star v_0 + \int_0^t ds\, e^{t-s}K_{t-s}\star(R_s v_s)\;. \tag{7.15}$$

It is now convenient to define, as in the proof of Lemma 6.3,

$$y_t \;=\; e^{-t(1+\eta)}v_t \;, \tag{7.16}$$

and to prove uniform bounds in t for y_t. This leads to the integral equation

$$y_t = K_t \star v_0 + \int_0^t ds\, e^{-\eta(t-s)} K_{t-s} \star (R_s y_s) \;. \tag{7.17}$$

In particular, if we consider this equation in the space of functions bounded in space and time, the last term gives an operator of norm $\mathcal{O}(\eta)$ because $R = \mathcal{O}(\eta^2)$. Therefore we can solve this equation for η small by iteration (*i.e.*, the Neumann series converges). This is really the proof of Lemma 6.3. We are going to use this idea in a slightly more subtle way, taking advantage of the decay properties of the heat kernel. We first choose a number $c_1 > 0$ large enough, basically $c_1^2/t_0 \gg 1$, where t_0 was defined above. We then choose an integer n such that

$$\frac{(\log \varepsilon)^2}{\log \eta^{-1}} \;\ll\; n \;, \quad \textit{and} \quad n c_1 \log(1/\varepsilon) \;\le\; \ell/4 \;.$$

Clearly, for our choice of $\ell \ge 1/\varepsilon$, and since η is a fixed (but small) constant, we can choose n for example of order $\bigl(\log(1/\varepsilon)\bigr)^3$, if $\varepsilon > 0$ is small enough.

We next define a sequence of domains for $0 \le j \le n$ by

$$\mathbf{S}'_j \;=\; [-L + \ell/2 - j c_1 \log(1/\varepsilon), L - \ell/2 + j c_1 \log(1/\varepsilon)] \;.$$

Note that the distance between $\mathbf{S}'_j$ and the complement of $[-L, L]$ is at least $\ell/4$ (for ε small enough), that $\mathbf{S}'_j \subset \mathbf{S}'_{j+1}$, that $\mathbf{S}_0 = \mathbf{S}'$ and that $\mathbf{S}_n \subset [-L + \ell/4, L - \ell/4] = \mathbf{S}''$. Using the integral equation Eq(7.17) and $t \le t_0 \log \varepsilon^{-1}$ we find, upon splitting the convolution integrals in the space variable, and writing $t^* = t_0 \log \varepsilon^{-1}$:

$$\begin{aligned} \sup_{t\in[0,t^*]} \|y_t\|_{\mathrm{L}^\infty(\mathbf{S}'_j)} \;&\le \|v_0\|_{\mathrm{L}^\infty(\mathbf{B})} + \mathcal{O}(\varepsilon^2)\|v_0\|_{\mathrm{L}^\infty(\mathbf{R})} \\ +\mathcal{O}(\eta) \sup_{t\in[0,t^*]} \|y_t\|_{\mathrm{L}^\infty(\mathbf{S}'_{j+1})} &+ \mathcal{O}(\varepsilon^2) \sup_{t\in[0,t^*]} \|y_t\|_{\mathrm{L}^\infty(\mathbf{R})} \;. \end{aligned} \tag{7.18}$$

For example, the term $K_t \star v_0$ is bounded as follows: Writing $t^* = t_0 \log \varepsilon^{-1}$we have

$$\sup_{t\in[0,t^*]} \sup_{x\in\mathbf{S}'_j} \bigl|\bigl(K_t \star v_0\bigr)(x)\bigr| \;\le\; \sup_{t\in[0,t^*]} \sup_{x\in\mathbf{S}'_j} \int_{z\in\mathbf{B}\cup(\mathbf{R}\backslash\mathbf{B})} dz\, |K_t(x-z)v_0(z)| \;\equiv\; X_{\mathbf{B}} + X_{\mathbf{R}\backslash\mathbf{B}} \;.$$

The term $X_{\mathbf{B}}$ leads to the bound $\|v_0\|_{\mathrm{L}^\infty(\mathbf{B})}$, since the integral of $|K_t| = K_t$ equals 1. Using $K_t(z) \le 2^{1/2} e^{(z^2/(2t))} K_{2t}(z)$, the term $X_{\mathbf{R}\backslash\mathbf{B}}$ is bounded by the supremum of v_0 times

$$\begin{aligned} &\sup_{t\in[0,t^*]} \int_{|z|>\ell/2 - j c_1 \log(1/\varepsilon)} dz\, K_t(z) \\ \le\; &\sup_{t\in[0,t^*]} \; \sup_{|x|>\ell/2 - j c_1 \log(1/\varepsilon)} \mathcal{O}(1) \exp\bigl(-\mathrm{const.}x^2/(2t)\bigr) \cdot \int_{\mathbf{R}} dz\, (K_{2t}(z)) \;. \end{aligned}$$

Our choice of n and c_1 implies that $x^2/t \geq \log(1/\varepsilon^2)$ (in fact a much better bound holds here, but later, when we iterate the argument, we shall use a bound which essentially saturates this inequality) and thus the bound of the first term in Eq.(7.17) follows. The bound on the second term follows using the same techniques and the contraction mapping principle as in our treatment of Eq.(7.5), and using that $R_s = \mathcal{O}(\eta^2)$ to compensate for a factor of η^{-1} which comes from the bound on the s-integral.

Using the estimate on the whole line (Lemma 6.3), we conclude that the last term in (7.18) is of the same size as the second term and we get

$$\sup_{t\in[0,t^*]} \|y_t\|_{\mathrm{L}^\infty(\mathbf{S}'_j)} \leq \|v_0\|_{\mathrm{L}^\infty(\mathbf{B})} + \mathcal{O}(\varepsilon^2)\|v_0\|_{\mathrm{L}^\infty(\mathbf{R})} + \mathcal{O}(\eta) \sup_{t\in[0,t^*]} \|y_t\|_{\mathrm{L}^\infty(\mathbf{S}'_{j+1})} \,.$$

We now iterate n times this inequality (and here we only get a bound $x^2/t > \log(1/\varepsilon^2)$ which comes from the lower bound on the separation of $\mathbf{R} \setminus \mathbf{S}'_{j+1}$ from $\mathbf{S}'_j$) to obtain an estimate on $\mathbf{S}_0 = \mathbf{S}'$. Since we have chosen the constant n such that $\eta^n = o(\varepsilon^2)$, we find

$$\sup_{t\in[0,t^*]} \|y_t\|_{\mathrm{L}^\infty(\mathbf{S}')} \leq \mathcal{O}(1)\|v_0\|_{\mathrm{L}^\infty(\mathbf{B})} + \mathcal{O}(\varepsilon^2)\|v_0\|_{\mathrm{L}^\infty(\mathbf{R})} \,.$$

We can now undo the effect of the exponential of Eq.(7.16). If we furthermore replace v_0 by the initial data $f_{-t}^2 w_{-t}$, and use the information we have on $f_{-t}^2 w_{-t}$, we get the bound for this part of the integral:

$$\|X_{2,+}\|_{\mathrm{L}^\infty(\mathbf{S}')} \;\leq\; \mathcal{O}(\eta)\|w\|_{\mathrm{L}^\infty(\mathbf{B})} + \mathcal{O}(\varepsilon^2)\|w\|_{\mathrm{L}^\infty(\mathbf{R})} \,.$$

Since $\mathbf{S}' \supset \mathbf{S}$, this is the desired bound, and we have completed the bound on X_2.

We finally consider the term X_1 of Eq.(7.13). Here, we estimate

$$\mathrm{D}^2\Theta^t[f](w_1, w_2) \,.$$

Again, this is a function z which is a solution of

$$\partial_t z = \partial_x^2 z + z - 3\big(\Theta^t f\big)^2 z - 6\Theta^t f \cdot \mathrm{D}\Theta^t[f]w_1 \cdot \mathrm{D}\Theta^t[f]w_2 \,, \tag{7.19}$$

with initial data $z = 0$, which is the analog of the Eq.(7.10) which we found for the first derivative. Its estimate is analogous to the previous one. To deal with the localization problem for the non-homogeneous term in (7.19), we now exploit that the bound on X_2 was done on a region $\mathbf{S}'$ which is larger (by $\ell/2$) than the region $\mathbf{S}$ on which we really need the bounds.

Details are left to the reader.

Interpretation. The inequality Eq.(7.12) serves to localize the bounds of the previous subsection. If ε is small enough (depending only on the bounds Theorem 7.2 on the derivative of Z which are global), we have for any two functions f, f' in $\mathcal{E}_b(\eta)$ the inequality

$$\|Z(f) - Z(f')\|_\infty \;\leq\; \mathcal{O}(\eta)\|f - f'\|_\infty \,.$$

Therefore,

$$\|S(f) - S(f')\|_\infty \;\geq\; (1 - \mathcal{O}(\eta))\|f - f'\|_\infty \,.$$

Basically, we want to use Eq.(7.12) to show that if f and f' differ by at least ε somewhere on $\mathbf{B}$ this implies that $S(f)$ and $S(f')$ differ by at least $\varepsilon/2$ somewhere on $\mathbf{S}$. While this is not true in general, we will see in the next section that it must be true for enough functions among those which form the centers of the balls which cover $\mathcal{G}$. This will be exploited in the next subsection.

7.5. Proof of Theorem 7.1

The idea of the proof is to show that because $S(f) \approx f$ and because the ε-entropy of the set $\mathcal{E}_b(\eta_*)$ of f is $\mathcal{O}\big(\log(1/\varepsilon)\big)$, the same will hold for the set $S\big(\mathcal{E}_b(\eta_*)\big)$. Here, and in the sequel we fix η_* to the value found as a bound in Theorem 7.2. Basically, we are going to show that if $\|f - f'\|_\infty \geq \varepsilon$, then not only

$$\|S(f) - S(f')\|_\infty \geq \varepsilon/2 , \tag{7.20}$$

but also that we can find enough functions for which $\sup_{x\in\mathbf{B}} |f(x) - f'(x)| > \varepsilon$ and

$$\sup_{x\in\mathbf{S}} |S(f)(x) - S(f')(x)| \geq \varepsilon/4 . \tag{7.21}$$

Here, we shall choose $\ell \geq 1/\varepsilon$.

Note that we cannot prove (7.21) for *individual* pairs of functions, but only for a (large enough) subset of them. The mechanism responsible for that is a "crowding lemma" in the following setting: Let $\mathcal{S}$ be a set of $N \gg 1$ functions which are pairwise at a distance at least α from each other, when considered on a set I_{big} which is a finite union of intervals. Let I_{small} be another finite union of intervals contained in I_{big}.

Lemma 7.6. *Under the above assumptions at least one of the following alternatives holds:*
- *At least $N^{1/2}/2$ functions in $\mathcal{S}$ differ pairwise by α on $I_{\text{big}} \setminus I_{\text{small}}$.*
- *At least $N^{1/2}$ functions in $\mathcal{S}$ differ pairwise by $\alpha/3$ on I_{small}.*

Remark. We can symmetrize the statement. We formulate this as a corollary for further use:

Corollary 7.7. *Under the above assumptions at least one of the following alternatives holds:*
- *At least $N^{1/2}/2$ functions in $\mathcal{S}$ differ pairwise by $\alpha/3$ on $I_{\text{big}} \setminus I_{\text{small}}$.*
- *At least $N^{1/2}/2$ functions in $\mathcal{S}$ differ pairwise by $\alpha/3$ on I_{small}.*

Proof. We first need the following auxiliary

Lemma 7.8. *Let $\mathcal{E}$ be a set of $M^2 > 4$ points in a metric space. Assume that for a given $\rho > 0$ we can find in $\mathcal{E}$ no more than M points which are pairwise at a distance at least ρ. Then there is a point x_* in $\mathcal{E}$ such that at least $M/2$ points of $\mathcal{E}$ are within a distance ρ of x_*.*

Proof. Let $\mathcal{E}_0$ be a maximal set of points in $\mathcal{E}$ with pairwise distance at least ρ. By assumption, the cardinality of $\mathcal{E}_0$ satisfies $|\mathcal{E}_0| \leq M$. Adding any point $x_0 \in \mathcal{E} \setminus \mathcal{E}_0$ to $\mathcal{E}_0$, we can find a point $x_0' \in \mathcal{E}_0$ such that $d(x_0, x_0') < \rho$, where d is the distance. We continue in this fashion with every point x_j of $\mathcal{E} \setminus \mathcal{E}_0$, finding a partner x_j' in $\mathcal{E}_0$ with $d(x_j, x_j') < \rho$. There are thus $|\mathcal{E} \setminus \mathcal{E}_0| = M^2 - M$ choices of x_j'. But since there are at most M points in $\mathcal{E}_0$, there must be at least one point in $\mathcal{E}_0$ which has at least $(M^2 - M)/M$ partners. Clearly, this point can be chosen as x_*. Since $(M^2 - M)/M > M/2$, the proof of Lemma 7.8 is complete.

□

Proof of Lemma 7.6. We assume that the second alternative does not hold and show that then the first must hold. If the second alternative does not hold, then we can apply Lemma 7.8 with $(M+2)^2 > N \geq (M+1)^2$ and $\rho = \alpha/3$ on the set $\mathcal{S}$ of functions with the sup norm on I_{small} and conclude that there is a function, f^*, such that on I_{small} we can find $M/2$ others within distance at most $\alpha/3$ from f^*. Call those functions f_i ($i = 1, \dots, K$ with $K \geq M$). Therefore,

$$\sup_{x \in I_{\text{small}}} |f_j(x) - f_{j'}(x)| < 2\alpha/3 ,$$

for all pairs $j, j' \in \{1, \dots, K\}$. This implies that these $M/2$ functions f_i must differ pairwise by at least α on $I_{\text{big}} \setminus I_{\text{small}}$ since they have to differ pairwise by α on the whole interval I_{big}. The proof of Lemma 7.6 is complete.

□

With these tools in place, we can now start the proof of Theorem 7.1 proper. We first make precise the limiting process in the definition of $H_\varepsilon\big(\mathcal{E}_b(\eta_*)\big)$. Using the definition of H_ε we have the following information about the set $\mathcal{E}_b(\eta_*)$: Let $N_{[-L,L]}(\varepsilon)$ denote again the minimum number of balls of radius ε in $\mathrm{L}^\infty([-L,L])$ needed to cover $\mathcal{E}_b(\eta_*)$ (restricted to $[-L,L]$). Then we know that

$$\lim_{\varepsilon \to 0} \frac{1}{\log(1/\varepsilon)} \lim_{L \to \infty} \frac{\log N_{[-L,L]}(\varepsilon)}{2L} = \frac{2b}{\pi} .$$

This leads to upper and lower bounds of the following form: *For every $\delta > 0$ there is an $\varepsilon(\delta) > 0$ and for every ε satisfying $0 < \varepsilon < \varepsilon(\delta)$ there is an $L(\delta, \varepsilon)$ such that for all $L > L(\delta, \varepsilon)$ one has*

$$\left(\frac{1}{\varepsilon}\right)^{\sigma_* L(1-\delta)} \leq N_{[-L,L]}(\varepsilon) \leq \left(\frac{1}{\varepsilon}\right)^{\sigma_* L(1+\delta)} , \tag{7.22}$$

where $\sigma_* = 4b/\pi$. Given $\delta > 0$, we pick ε and L as above and can find therefore in $\mathcal{E}_b(\eta_*)$ a set $\mathcal{S}_1$ of

$$N_1(\varepsilon, L) \geq \left(\frac{1}{\varepsilon}\right)^{\sigma_* L(1-\delta)} , \tag{7.23}$$

functions which are pairwise at distance at least ε in $\mathrm{L}^\infty(\mathbf{B})$.

Lemma 7.9. *When $\ell \gg 1/\varepsilon$ and $L \gg \ell$ on can find in $\mathcal{S}_1$ a set $\mathcal{S}_2$ of at least $N_2 = \frac{1}{2} N_1^{1/2}$ functions which differ pairwise by $\varepsilon/3$ on $\mathrm{L}^\infty(\mathbf{S})$.*

Proof. We apply Corollary 7.7 with $I_{\text{big}} = [-L, L]$ and $I_{\text{small}} = [-L+\ell, L-\ell]$ and with $N = [(1/\varepsilon)^{\sigma_* L(1-\delta)/2}]$ and $\alpha = \varepsilon$. If the conclusion of Lemma 7.9 does not hold, then by Lemma 7.6 we can find N_2 functions which are pairwise at a distance at least ε on $[-L, -L+\ell] \cup [L-\ell, L]$. Applying Corollary 7.7 with $I_{\text{big}} = [-L, -L+\ell] \cup [L-\ell, L]$ and $I_{\text{small}} = [-L, -L+\ell]$ we conclude that in at least one of the intervals $[-L, -L+\ell]$ and $[L-\ell, L]$ we can find at least $N_3 = \frac{1}{2} N_2^{1/2}$ functions which are pairwise at a distance $\varepsilon/3$ when considered on that interval. Since we are considering a subset of $\mathcal{E}_b(\eta_*)$, we see by Eq.(7.22), there can be no more than $N_4 \equiv (1/\varepsilon)^{\sigma_*(1+\delta)\ell}$ such functions. Since $\delta > 0$ is arbitrarily small and we have seen that there are at least N_3 such functions, we find for $L/5 > \ell(1+\delta)/(1-\delta)+1$ the inequality $N_3 > N_4$. This is a contradiction and the proof of Lemma 7.9 is complete.

□

Continuing the proof of Theorem 7.1, we take the set $\mathcal{S}_2$ of N_2 functions among the initial ones which differ pairwise at least by $\varepsilon/3$ on $\mathbf{S}$. Note that this is different from looking at functions which differ by $\varepsilon/3$ only on that interval because in $\mathcal{S}_2$ we have some information outside, namely that the functions differ by at least ε when considered on $\mathbf{B}$. We consider the different $S(f)$ for these functions. Assume first that at least

$$N_5 \equiv \frac{1}{2} N_2^{1/2} = \mathcal{O}\big(\varepsilon^{-L\sigma_*(1-\delta)/4}\big)$$

of these $S(f)$ differ pairwise by at least $\varepsilon/12$ on $\mathbf{S}$. This means that N_5 balls of radius $\varepsilon/25$ in $\mathrm{L}^\infty(\mathbf{S})$ do *not* cover the set $S(\mathcal{S}_1)$. In the terminology of [KT, pp 86–87], this means that the minimal number of points in an $\varepsilon/25$-net is at least N_5. Thus the ε-entropy per unit length of $S(\mathcal{S}_1)$ is bounded below by $\mathcal{O}\big(\log(1/\varepsilon)\big)$, we have a lower bound and we are done, *i.e.*, Theorem 7.1 is proved in this case.

For the opposite case, we are going to derive a contradiction, and this will complete the proof of Theorem 7.1 for all cases. By Lemma 7.8, with $\rho = \varepsilon/12$, if we cannot find at least N_5 of the $S(f_i)$ which differ pairwise by at least $\varepsilon/12$ on $\mathbf{S}$, there is an f^{**} such that in a neighborhood of radius $\varepsilon/36$ around $S(f^{**})$ we can find at least N_5 of the other $S(f)$. This implies that we have a sub-collection $\{f_i\}$ of N_5 functions for which

$$\sup_{x\in\mathbf{S}} |S(f_j)(x) - S(f_{j'})(x)| < \varepsilon/36\ ,$$

for all choices of j and j'. Therefore, by the definition of S and Z we have

$$\sup_{x\in\mathbf{S}} |f_j(x) - f_{j'}(x)| < \varepsilon/36 + \sup_{x\in\mathbf{S}} |Z(f_j)(x) - Z(f_{j'})(x)|\ .$$

We now apply Eq.(7.12) to bound this quantity by

$$\sup_{x\in\mathbf{S}} |f_j(x) - f_{j'}(x)| < \varepsilon/36 + \mathcal{O}(\varepsilon^2) + \mathcal{O}(\eta_*) \sup_{x\in\mathbf{B}} |f_j(x) - f_{j'}(x)|\ .$$

Now if

$$\sup_{x\in\mathbf{B}\backslash\mathbf{S}} |f_j(x) - f_{j'}(x)| \le \sup_{x\in\mathbf{S}} |f_j(x) - f_{j'}(x)|\ ,$$

the previous inequality implies

$$\sup_{x\in\mathbf{S}} |f_j(x) - f_{j'}(x)| < (1 + \mathcal{O}(\eta_*))^{-1}(\varepsilon/6 + \mathcal{O}(\varepsilon^2))\ .$$

Combining the last two inequalities we have

$$\sup_{x\in\mathbf{B}} |f_j(x) - f_{j'}(x)| < (1 + \mathcal{O}(\eta_*))^{-1}(\varepsilon/36 + \mathcal{O}(\varepsilon^2)) ,$$

and we have a contradiction since the distance should be at least ε. (It is here that we use the additional information we have on the set $\mathcal{S}_2$ of N_2 functions constructed in Lemma 7.9.) Therefore we conclude that

$$\sup_{x\in\mathbf{B}\setminus\mathbf{S}} |f_j(x) - f_{j'}(x)| > \sup_{x\in\mathbf{S}} |f_j(x) - f_{j'}(x)| ,$$

but since the sup over the whole interval must be ε we conclude that the sup on the l.h.s. is at least ε. Applying again Corollary 7.7 we can find among the $\{f_i\}$ at least $\frac{1}{2}N_5^{1/2}$ functions such that on one of the intervals $[-L, -L + \ell]$ or $[L - \ell, L]$ of $\mathbf{B} \setminus \mathbf{S}$ they are pairwise at a distance at least $\varepsilon/36$. As before this leads to a contradiction if $L \gg \ell$ because there should be at most $\varepsilon^{-\ell\sigma_*(1+\delta)}$ such functions. The proof of Theorem 7.1 is complete.

$\square$

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