Oscillations of Observables in 1-Dimensional Lattice Systems

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Abstract: Using, and extending, striking inequalities by V.V. Ivanov on the down-crossings of monotone functions and ergodic sums, we give universal bounds on the probability of finding oscillations of observables in 1-dimensional lattice gases in infinite volume. In particular, we study the finite volume average of the occupation number as one runs through an increasing sequence of boxes of size $2n$ centered at the origin. We show that the probability to see k oscillations of this average between two values β and $0 < \alpha < \beta$ is bounded by CR^k, with $R < 1$, where the constants C and R do *not* depend on any detail of the model, nor on the state one observes, but only on the ratio α/β.

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1. Introduction

In two recent papers, V.V. Ivanov [I1, I2] derived a novel theorem on down-crossings of monotone functions. Theorems of this kind are useful as key elements of "constructive" proofs of the Birkhoff Ergodic Theorem [B1, B2]. For example, let h be a *non-negative* measurable function on Ω, and let T be a measurable map $T:\Omega\to\Omega$ which preserves a probability measure μ. We denote by $s_n(\omega)$ the sum

$$s_n(\omega)\;=\;\sum_{j=0}^{n-1}h(T^j\omega)\;.$$

Let $\beta>\alpha>0$ be given. A down-crossing is defined as a pair of integers $n<m$ such that

$$s_n(\omega)/n\;\ge\;\beta\;,\quad\text{and}\quad s_m(\omega)/m\;\le\;\alpha\;.$$

Let Ω_k denote the set of ω for which $\{s_n(\omega)/n\}_{n=1,2,\dots}$ makes at least k successive down-crossings, *i.e.,* there is a sequence $n_1<m_1<n_2<m_2<\ldots<n_k<m_k$, such that each pair n_i, m_i defines a down-crossing. The surprising result of Ivanov is the

Theorem 1.1. *One has the bound*

$$\mu(\Omega_k)\;\le\;(\alpha/\beta)^k\;. \tag{1.1}$$

Note that there is no constant in front of $(\alpha/\beta)^k$, and that the result is independent of Ω, μ, T and $h\ge 0$. Several (relatively straightforward) generalizations and consequences have been pointed out in [I1, I2] and in the review paper [K]. We list two of them for the convenience of the reader, and will mention related work at the end of the introduction.

1) If $h\in L^\infty$—there is no assumption on $h\ge 0$ here—then, for all β and $\alpha=\beta-\varepsilon$ one has the bound $\mu(\Omega_k)\;\le\;Ae^{-Bk}$, where A and B depend only on $q=\varepsilon/\|h\|_\infty$. One has $B=\mathcal{O}(q^2)$.
2) The above results can easily be used to actually prove the ergodic theorem, even for $h\in L_1$,

In this paper, we give a partially new proof of Ivanov's theorem, and we extend it in such a way that it applies to 1-dimensional models of statistical mechanics. Indeed, it suffices to consider any translation invariant state of a spin system [R]. To be specific, we might consider an Ising-like model with spin 0, 1 (in a particle interpretation) and long-range interaction. Then $\Omega=\{0,1\}^{\mathbf{Z}}$, T is lattice translation and μ is the Gibbs state, not necessarily pure. For $\omega=\{\omega_n\}_{n\in\mathbf{Z}}\in\Omega$, we let $h(\omega)=\omega_0$ be the value of the spin at the site 0 and then $s_n(\omega)/n$ has the meaning of the average "occupation number" on the interval $[0,n-1]$. Ivanov's theorem has then the interpretation:

Proposition 1.2. *The probability that the mean occupation number (as a function of the volume n) makes more than k oscillations between β and α, $0<\alpha<\beta$ is bounded by $(\alpha/\beta)^k$.*

Note that this statement is independent of the spin system under consideration, of the temperature considered, of boundary conditions or any other parameter of the system. In particular, it also holds if the system is not in a pure state. Thus, it is a kind of geometrical constraint on ergodic sums, or on the fluctuations of physical observables. If these observables can take negative values, the results will be modified as in 1) above, but the bound will still be exponential in k.

In the statement above, we considered "boxes" which are given by the intervals $[0, n-1]$. However, the statement can be extended to *symmetric intervals* by the following new result:

Assume that T, as defined above, is invertible. Define for $n \in \mathbf{Z}$,

$$S_n(\omega) = \sum_{j=-n+1}^{n} h(T^j\omega) . \tag{1.2}$$

We now let Θ_k denote the set of those ω for which the sequence $\{S_n(\omega)/(2n+1)\}_{n=0,1,2,\dots}$ makes at least k down-crossings from β to α, $0 < \alpha < \beta$. We will show:

Theorem 1.3. *There are two constants $C = C(\alpha/\beta)$ and $R = R(\alpha/\beta) < 1$ such that one has the bound*

$$\mu(\Theta_k) \le CR^k .$$

The constants C and R are independent of μ, Ω, T, and $h \ge 0$.

Remark. We will describe R in Section 5, but note that $R < 1$, $R(x) \to 0$ as $x \to 0$ and $R(x) \approx \exp\bigl(-\mathcal{O}(\varepsilon/4^{1/\varepsilon})\bigr)$ when $x = 1 - \varepsilon$. (This is certainly not the best possible bound.)

The Theorem 1.3 can be extended to sequences of volumes which tend to infinity in a more general way as $n \to \infty$: Let $p_1 \ge 0$, $p_2 \ge 0$, $r_1 \ge 0$, $r_2 \ge 0$ be given integers with $p_1 + p_2 > 0$ and define now

$$S_n(\omega) = \sum_{j=-np_1-r_1}^{np_2+r_2-1} h(T^j\omega) . \tag{1.3}$$

Let Θ_k be the set of ω for which the sequence $\{S_n(\omega)/(n(p_1+p_2)+r_1+r_2)\}_{n\in\mathbf{N}}$ makes at least k down-crossings from β to α.

Theorem 1.4. *There are two constants*

$$C = C(p_1, p_2, r_1, r_2, \alpha/\beta) , \quad R = R(p_1, p_2, r_1, r_2, \alpha/\beta) < 1 ,$$

such that one has the bound

$$\mu(\Theta_k) \le CR^k .$$

The constants C and R are independent of μ, Ω, T, and $h \ge 0$.

Our paper is organized as follows. In Section 2, we show the basic inequality, "Ivanov's theorem" which is used in proving Theorem 1.1 for $k = 1$. In Section 3, we extend these results to arbitrary k. In Section 4, we use the results of Section 3 to prove Theorem 1.1 for all k. To make the paper self-contained, we give complete proofs, even when they are essentially just rewordings of Ivanov's work. In Section 5, we give the proof of Theorem 1.3 and Theorem 1.4.

1.1. Some other work on oscillations

The results described and obtained here are among a rather large list of related literature, obtained earlier by other authors. While our intention is not to review here the literature, we just note some highlights which seem relevant in the context of our work.

1) Kalikow and Weiss [KW] have obtained bounds of the form Ae^{-Bk} (by a clever extension of a Vitali type covering argument) not only for intervals, as in the present paper, but also for "rectangles" when one considers $\mathbf{Z}^d$-actions. Thus, in particular, the statements made above about statistical mechanics extend to models in *any dimension* (with a rate of decay depending only on α/β and on the dimension d). However, their methods seem not to give the astonishing "best possible" result of Ivanov in 1 dimension. Since the methods of Ivanov, used here, are typically 1-dimensional in nature and use that the real line is ordered, it is hard to imagine how to extend them to 2 or more dimensions. The best rate of decay in more than 1 dimensions remains thus an open question, and will depend on the nature of the geometric embeddings of the rectangles. The approach used here to go from one sided to symmetric intervals is quite different from the approach used in [KW].

2) In his review paper [K], Kachurovskii extends the fluctuations theorems to variations of the mean of "size" ε, but not necessarily across a fixed gap $\beta = \alpha + \varepsilon$, α. He gets bounds again of the form Ae^{-Bk}, based on Ivanov's work, when $h \in L_\infty$ and square root decay (up to logarithms) when $h \in L_1$. These results hold for semi-intervals $[0, n]$ and can probably be extended by our methods to symmetric intervals. At this point, it seems ([W]) that the methods of Kachurovskii can be combined with those of [KW] to obtain similar results (but only with $\mathcal{O}(k^{-1/3})$ decay) in the case of Z^d actions.

3) There is an extensive literature on L_p bounds, with $p > 1$ on which we shall not dwell here.

2. A proof of Ivanov's theorem

We consider non-decreasing (not necessarily continuous) functions f on $\mathbf{R}$. Let $E = \cup_\ell E_\ell$ be a closed bounded subset of $\mathbf{R}$ which is a finite disjoint union of closed intervals E_ℓ. Furthermore, we assume that β and α are given constants satisfying $\beta > \alpha > 0$.

Definition. Let E' be a subset of E. A point $x \in \mathbf{R}$ is said to be in the *shadow* of E' (relative to E) if it is in E and if there are two numbers y, z in E' satisfying:

i) $x < y < z$,

ii) the interval (y, z) is contained in E',

iii) $f(z^-) - f(x) \leq \alpha(z - x)$, and $f(y^+) - f(x) \geq \beta(y - x)$.

Remark. This definition is slightly different from the one by Ivanov.

Let $S(E') = S(E', E)$ denote the set of x which are in the shadow of E' (relative to E). We assume throughout that E is a fixed set and omit mostly the second argument of S. If A is a set in $\mathbf{R}$ we let $|A|$ denote its Lebesgue measure. The proof of Theorem 1.1 is based on the following basic bound by Ivanov [I1,I2]:

Ivanov's Theorem. *Under the above hypotheses, one has the inequality*

$$|S(E,E)| \le \frac{\alpha}{\beta}|E| \ . \tag{2.1}$$

Proof. Our proof relies heavily on Ivanov's ideas, but presents some simplifications. We will first prove the following

Theorem 2.2. *Assume f is a non-decreasing, piecewise affine, continuous function. Then one has the inequality*

$$|S(E,E)| \le \frac{\alpha}{\beta}|E| \ . \tag{2.2}$$

Postponing the proof of this theorem, we now show how Theorem 2.2 implies Ivanov's Theorem. We first assume that the boundary of E does not contain points of discontinuity of f. To make things clearer, we indicate the function, and the limits of the shadow, *i.e.,* we write $S_{f,\alpha,\beta}(E)$. Let f be an arbitrary non-decreasing function, and let f_n be a sequence of continuous, piecewise affine, functions approximating f (pointwise). We consider the sequences $S_{n,m}(E) = S_{f_n,\alpha(1+1/m),\beta(1-1/m)}(E)$, for $n = 2, 3, \ldots$, and large m. Let $U_{p,m} = \cap_{n>p} S_{n,m}(E)$. Clearly, $U_{p,m} \subset U_{p+1,m}$. Furthermore, every $x \in S(E)$ is in $\cap_{n>n_0(x,m)} S_{n,m}(E)$ for some $n_0(x,m) < \infty$, as one can see from the definition of shadows. Thus, we find

$$S(E) \subset \cup_p U_{p,m} = \lim_{p\to\infty} U_{p,m} \ ,$$

and therefore

$$|S(E)| \le \left|\cup_p U_{p,m}\right| = \lim_{p\to\infty} |U_{p,m}| \le \lim_{p\to\infty} \sup_{n>p} |S_{n,m}(E)| \le \frac{1+1/m}{1-1/m} \cdot \frac{\alpha}{\beta}|E| \ ,$$

by Theorem 2.2. Taking $m \to \infty$, the proof of Ivanov's Theorem is complete, when the discontinuities of f do not coincide with the boundary of E.

If the boundary of E contains discontinuity points of f we can find for each ℓ a decreasing sequence of closed intervals E_ℓ^p such that $E \subset E_\ell^p$, E_ℓ^p converges to E_ℓ and the boundary of each E_ℓ^p is made up of points of continuity of f. Let $E^p = \cup_\ell E_\ell^p$, then obviously $E \subset E^p$, hence $S(E) \subset S(E^p)$, and therefore

$$|S(E)| \le \liminf_{p\to\infty} |S(E^p)| \le \frac{\alpha}{\beta} \liminf_{p\to\infty} |E^p| = \frac{\alpha}{\beta}|E| \ .$$

This completes the proof of Ivanov's Theorem in all cases.

Proof of Theorem 2.2. As we have said before, we can at this point work with piecewise affine, non-decreasing continuous functions defined on $\mathbf{R}$, with a *finite* number of straight pieces.

We start by defining *regular* and *maximal regular* intervals. If A is a subset of E we denote by $F(A)$ the graph of f above A, *i.e.*, $F(A) = \{(x, f(x)) \mid x \in A\}$.

Definition. An interval $[a, b]$ in $\mathbf{R}$ is called *regular* if it is contained in E and if for all $x \in [a, b]$ one has

$$f(a) - \beta(a - x) \geq f(x) \ , \text{ and } f(x) \geq f(b) - \alpha(b - x) \ . \tag{2.3}$$

This means that the graph $F([a, b])$ lies entirely in the cone spanned by the two straight lines of (2.3), see Fig. 1.

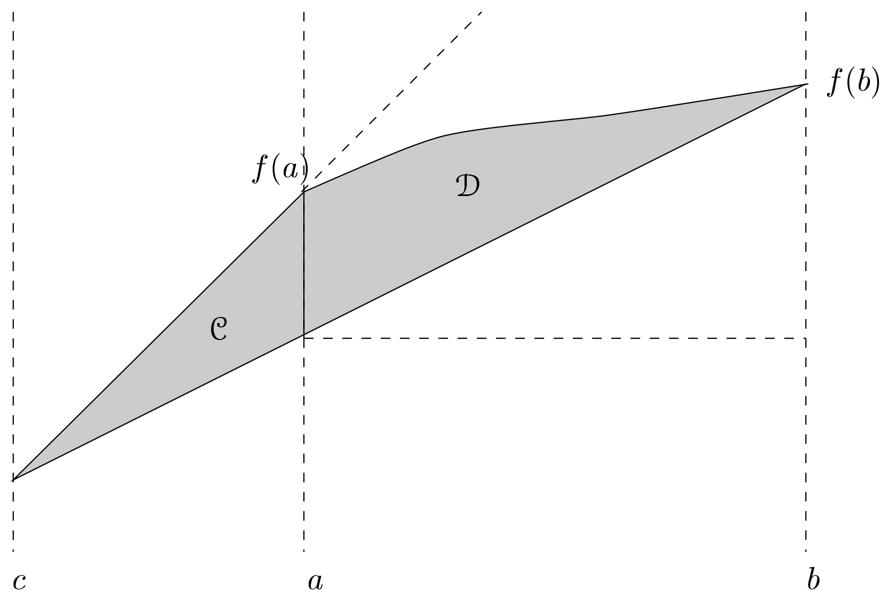

Fig. 1: The shadow cast by a (maximal) regular interval $[a, b]$, the cone $\mathcal{C}$, and the region $\mathcal{D}$.

It will be useful to talk about the sets $\mathcal{C}([a, b])$ and $\mathcal{D}([a, b])$ spanned in this figure: Define first $c = c(a, b)$ by

$$c(a, b) = \frac{f(b) - f(a) + \beta a - \alpha b}{\beta - \alpha} \ , \tag{2.4}$$

this is the x-coordinate of the tip of the cone. Then we define

$$\begin{aligned} \mathcal{C}([a, b]) &= \{(x, y) \mid x \in [c, a] \ , \quad f(a) + \beta(x - a) \geq y \geq f(b) + \alpha(x - b)\} \ , \\ \mathcal{D}([a, b]) &= \{(x, y) \mid x \in [a, b] \ , \quad f(x) \geq y \geq f(b) + \alpha(x - b)\} \ . \end{aligned}$$

Definition. An interval $[a, b]$ in $\mathbf{R}$ is called *maximal regular* if it is regular and is contained in no larger regular interval. It should be noted that this definition depends on the function f *and* on the set E.

Lemma 2.3. *Different maximal regular intervals are disjoint.*

Proof. Since parallel lines do not intersect, one verifies easily that the union of two regular intervals with non-empty intersection is regular. The assertion follows.

We denote by $E_{\mathrm{M}} \subset E$ the disjoint union of the maximal regular intervals:

$$E_{\mathrm{M}} = \cup_j \Delta_j \ . \tag{2.5}$$

The next lemma shows that it suffices to consider only shadows which are cast by maximal regular intervals:

Lemma 2.4. *One has the identity $S(E_{\mathrm{M}}) = S(E)$, more precisely $S(E_{\mathrm{M}}, E) = S(E, E)$.*

Proof. If $x \in S(E, E)$, then there is at least one interval $I \subset E$ for which $x \in S(I, E)$. By the continuity of f, there is a minimal such interval in I, which we call J. This interval is regular. The assertion follows, because every regular interval is contained in a maximal regular interval, as follows from the proof of Lemma 2.3.

Lemma 2.5. *The set E_{M} is a finite union of maximal regular intervals.*

Proof. It is here that we use the restricted class of piecewise affine, continuous functions. A minutes' reflection shows that the endpoints of the Δ_j are either points of discontinuity in the slope of f or boundary points of E. The assertion follows because there are a finite number of such points.

We define an auxiliary function g.* For $\Delta_j = [a_j, b_j]$, let $c_j = c(a_j, b_j)$ as above and define intervals $G_j(x)$ by

$$G_j(x) = \begin{cases} \emptyset, & \text{when } x \le c_j, \\ [f(b_j) + \alpha(x - b_j), f(a_j) + \beta(x - a_j)], & \text{when } x \in (c_j, a_j], \\ [f(b_j) + \alpha(x - b_j), f(x)], & \text{when } x \in (a_j, b_j], \\ \emptyset, & \text{when } x > b_j. \end{cases}$$

Note that $G_j(x)$ is simply the intersection of a vertical line at x with the cone $\mathcal{C}([a_j, b_j])$ or the set $\mathcal{D}([a_j, b_j])$, and $|G_j(x)|$ is continuous. We define

$$g(x) = \left|\cup_j G_j(x)\right| ,$$

and note that this is finite, since each $|G_j(x)|$ is bounded by $\alpha(b_j - a_j)$, so that $g(x)/\alpha$ is bounded by the diameter of E. By construction, g measures the length of the vertical cuts across the system of cones $\mathcal{C}$ and sets $\mathcal{D}$ generated by the Δ_j, *not* including multiplicities if the cones overlap.

* This definition is similar to, but different from, the one given for the function H in [I2]. Our definition makes the proofs somewhat easier.

Our next operation consists in partitioning the shadow into those pieces Δ_j' generated by a Δ_j *under itself*, and those cast by a cone associated with a Δ_i to the right of Δ_j. In formulas:

$$\Delta_j' = S(\Delta_j) \cap \Delta_j = S(\Delta_j, E) \cap \Delta_j ,$$

and

$$\Delta_j'' = \big(S(E) \cap \Delta_j\big) \backslash \Delta_j' .$$

See Fig. 3 below for a typical arrangement. We first argue that Δ_j' can be characterized by *looking only at slopes* β.

Lemma 2.6. *One has*

$$\Delta_j' = \big\{ x \in \Delta_j \mid \exists\, y \in \Delta_j, y > x, \text{ for which } f(y) - f(x) \geq \beta(y - x) \big\} .$$

Proof. It suffices to show that the second set is included in Δ_j'. Consider the ray $\{(z, f(x) + \alpha(z - x) \mid z > x\}$. If it intersects $F(\Delta_j \cap [y, b_j])$ then $x \in S(\Delta_j)$. If not, then $x \notin \Delta_j$, since Δ_j is regular. Hence $x \notin \Delta_j'$ either and the proof is complete.

We now can use the Riesz lemma to give a *bound* on the size of Δ_j':

Lemma 2.7. *One has the inequality*

$$|\Delta_j'| \leq \frac{f(b_j) - f(a_j)}{\beta} . \tag{2.6}$$

Proof. Define $s(x) = f(x) - \beta x$. Then, by Lemma 2.6, we see that

$$\Delta_j' = \big\{ x \in \Delta_j \mid \exists\, y \in \Delta_j, y > x, \text{ for which } s(y) \geq s(x) \big\} .$$

We apply here a variant of the Riesz lemma [RN, Chapter 1.3].* It tells us that Δ_j' is a finite disjoint union

$$\Delta_j' = \cup_k [a_{j,k}, b_{j,k}] ,$$

and that furthermore, for every of these intervals one has the inequality

$$s(x) \leq s(b_{j,k}) ,$$

when $x \in [a_{j,k}, b_{j,k}]$. Taking $x = a_{j,k}$, we get

$$f(a_{j,k}) - \beta a_{j,k} \leq f(b_{j,k}) - \beta b_{j,k} ,$$

and thus

$$|\Delta_j'| = \sum_k (b_{j,k} - a_{j,k}) \leq \beta^{-1} \sum_k \big(f(b_{j,k}) - f(a_{j,k})\big) \leq \beta^{-1} \big(f(b_j) - f(a_j)\big) .$$

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* The Riesz lemma is formulated in [RN] for arbitrary functions, with *open* intervals. Because we have piecewise affine functions, we can go over the proof and obtain the result for closed intervals.

The last inequality is a consequence of the monotonicity of f. The proof of Lemma 2.7 is complete.

We next study Δ''_j.

Lemma 2.8. *One has the following inequality:*

$$\beta|\Delta''_j| \leq g(b_j) - g(a_j) - f(b_j) + f(a_j) + \alpha(b_j - a_j) \ .$$

Proof. First observe that if $x \in \Delta''_j$, then by Lemma 2.6 the infinite ray

$$\{(x+s, f(x)+\beta s) \mid 0 < s\} \tag{2.7}$$

does *not* meet the graph $F(\Delta_j)$. Consider next any vertical line. To be specific, we take the line whose abscissa is b_j, and, since each of the previous rays emanates from a unique point of $F(\Delta_j)$, this provides a bijection between Δ''_j and its projection D''_j along the slope β onto the vertical line of abscissa b_j. See Fig. 2.

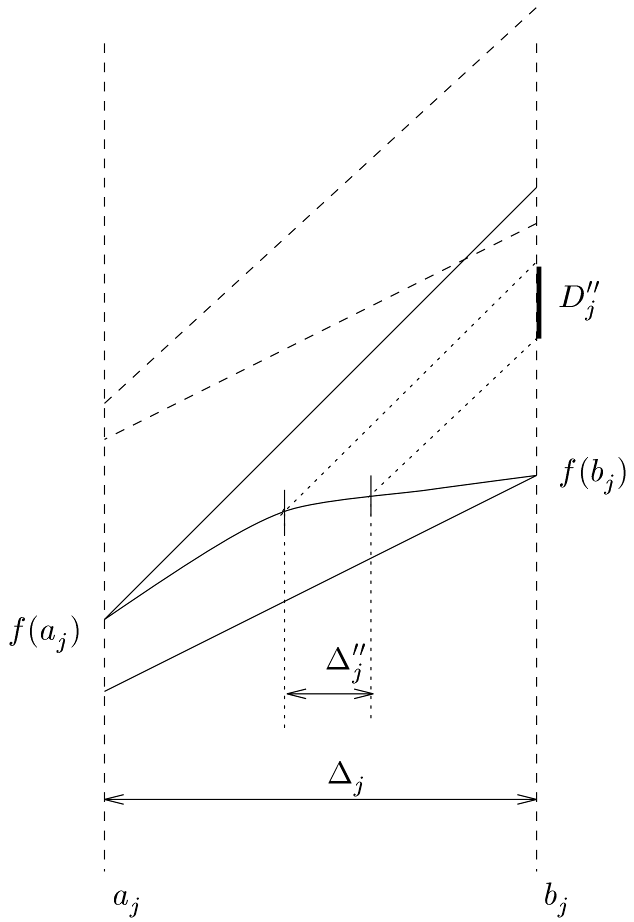

Fig. 2: The bijection between Δ''_j and D''_j. The size of Δ''_j is taken here symbolically. See Fig. 3 for a realistic arrangement.

Note that D''_j is a union of disjoint intervals and satisfies $|D''_j| = \beta|\Delta''_j|$. To understand the following construction, it is useful to consider Fig. 3.

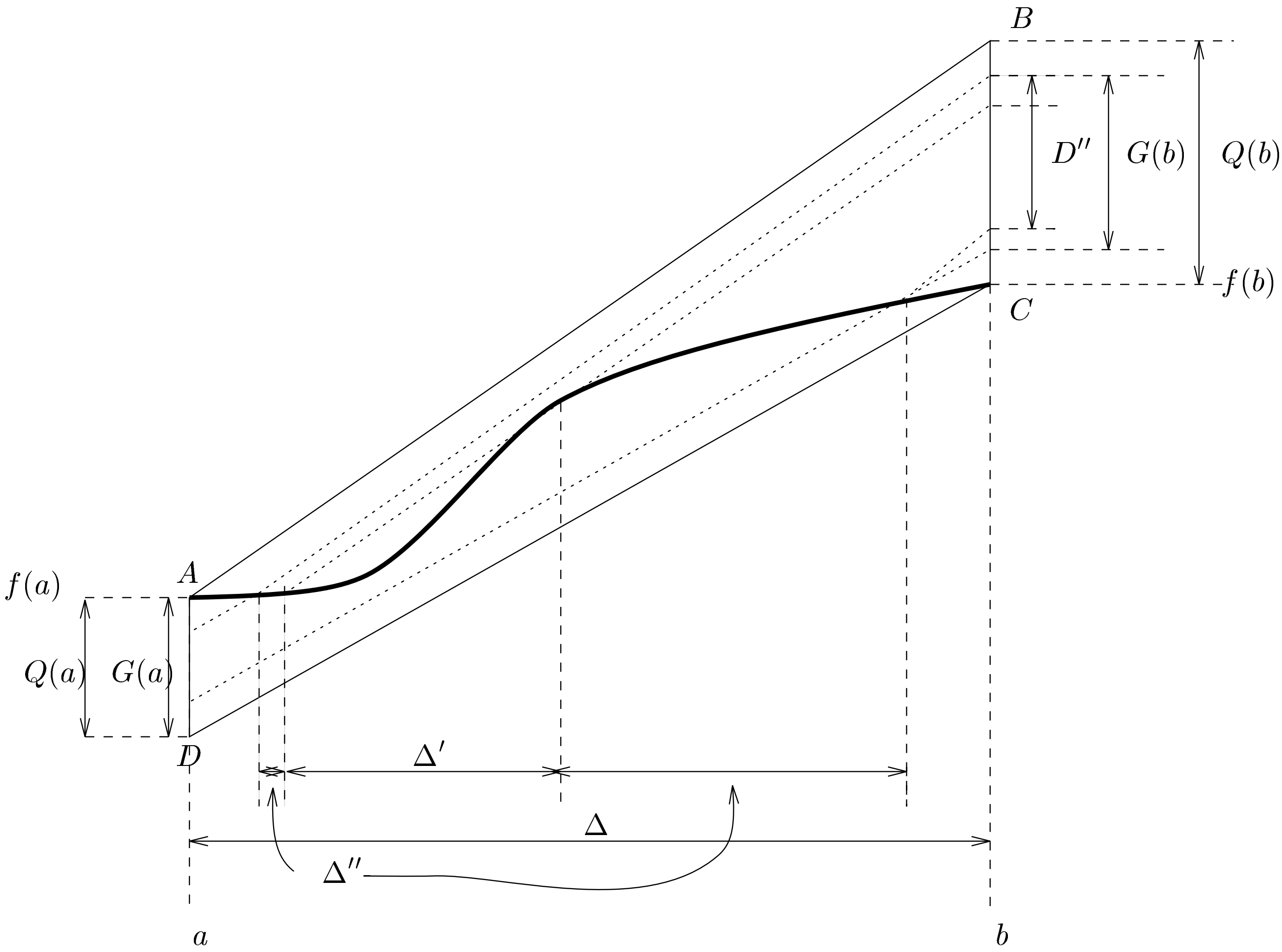

Fig. 3: The region $ABCD$, through which a cone passes. The intersection of the cone with the vertical line at b is $G(b)$. Inside this cone there is the bijection between Δ'' (which has 2 pieces) and D'', and there is a piece of shadow, Δ', which is generated from the (maximal) regular interval $[a, b]$ itself. Note that $G(a)$ and $G(b)$ will in general contain pieces from other cones as well.

Consider a fixed Δ_j, we will omit the index j in this argument. We define two intervals:

$$Q(a) = [f(b) + \alpha(a - b), f(a)] ,$$
$$Q(b) = [f(b), f(a) + \beta(b - a)] ,$$

and we let $q(a) = |Q(a)|$. We have the following chain of inequalities:

1) $g(a) - q(a) \leq |G(a) \setminus Q(a)|$,
2) $|G(a) \setminus Q(a)| \leq |G(b) \setminus Q(b)|$,
3) $|G(b) \setminus Q(b)| \leq |G(b) \setminus D''|$,
4) $|G(b) \setminus D''| \leq g(b) - |D''|$.

Inequality 1) follows from $Q(a) \subset G(a)$, 3) follows from $D'' \subset Q(b)$ and 4) from $D'' \subset G(b)$ which holds by the definition of Δ'' and the bijection constructed above. The inequality 2) describes the intersections of the cones outside of the interesting sets $Q(a)$ resp. $Q(b)$. If the cones do not intersect $ABCD$ in Fig. 2, the statement is trivial. If they intersect this region partially, the statement follows by examining the (rather obvious) cases which can occur.

Combining 1)–4), we see that

$$\beta|\Delta_j''| = |D_j''| \leq g(b_j) - g(a_j) + q(a_j) \ . \tag{2.8}$$

Since $q(a_j) = f(a_j) - f(b_j) + \alpha(b_j - a_j)$, the claim Lemma 2.8 follows.

Combining Lemma 2.7 and Lemma 2.8, and using again the definition of $q(a_j)$, we get immediately

Corollary 2.9. *One has the bound*

$$|S(E) \cap \Delta_j| \leq \frac{g(b_j) - g(a_j)}{\beta} + \frac{\alpha}{\beta}|\Delta_j| \ .$$

We next consider a maximal interval $E' = [a', b']$ of $E \backslash E_{\mathrm{M}}$.

Lemma 2.10. *One has the inequality*

$$|S(E) \cap E'| \leq \frac{g(b') - g(a')}{\beta} + \frac{\alpha}{\beta}|E'| \ . \tag{2.9}$$

Proof. We distinguish two cases. Assume first that at least one cone "traverses" E' completely, *i.e.,* its tip "c" is to the left of the interior of E' and its point "a" is to the right. Then

$$|E'| = b' - a' \leq \frac{g(b') - g(a')}{\beta - \alpha} \ ,$$

or equivalently

$$\beta|E'| \leq \alpha|E'| + g(b') - g(a') \ .$$

Since $S(E) \cap E' \subset E'$ the assertion follows. If no cone traverses E' completely, but some penetrate into it, we consider instead of the interval $S(E) \cap E'$ the shortest subinterval $[c, b']$ containing the projection of all the cones onto the x-axis. Since $S(E) \cap E' \subset [c, b']$, the assertion follows as before.

It is now straightforward to complete the proof of Theorem 2.2: First observe that if $X = [x_1, x_2]$ is an interval of $\mathbf{R} \setminus E$, then $0 = |S(E, E) \cap X| \leq g(x_2) - g(x_1)$, since the widths of the cones is increasing in the gaps of E. Combining this with Corollary 2.9 and Lemma 2.10, and observing that the intervals E', Δ_j', and X have contiguous boundaries, we get a telescopic sum in which the $g(\cdot)$ all cancel, except the first and the last. The first is subtracted, and the last is zero. The other terms add up to $(\alpha/\beta)|E|$, and the proof is complete.

3. The iterated theorem

We now give a bound, analogous to Ivanov's Theorem for the case of k oscillations.

Theorem 3.1. *Let E_k the set of $x \in E$ for which the function f has k successive down-crossings—as defined in Section 1—from β to $\alpha < \beta$ to the right of x. Then*

$$|E_k| \leq (\alpha/\beta)^k |E| \ .$$

Proof. The case $k = 1$ is an immediate consequence of Ivanov's Theorem, because if x is in $S(E)$ it is in the shadow of some regular interval J, and this means there is (at least) one down-crossing from β to α. The proof proceeds by induction. Assume we have shown the claim for all $k < k^*$. If $x \in E_{k^*}$, we let $[y_j, z_j]$, $j = 1, \dots, k^*$ denote the intervals of successive crossings. Each of the cones $\mathcal{C}([y_j, z_j])$ contains a smaller cone which has its apex at the point $(x, f(x))$. Therefore x is in the shadow of all the other cones. But this means that if $x \in E_{k^*}$ then $x \in S(E_{k^*-1})$. The assertion follows.

4. Proof of Theorem 1.1

We first need to define the notion of down-crossing of sequences more precisely.

Definition. For every $\beta > \alpha > 0$ and every $k \in \mathbf{N}$ we define $C_{k,\alpha,\beta}$ as the set of monotone sequences $\mathbf{c} = \{c_n\}_{n=0,1,\dots}$ for which $\{c_n/n\}_{n\in\mathbf{N}}$ makes k down-crossings from β to α:

$$\begin{aligned} C_{k,\alpha,\beta} = \Big\{ & \{c_n\}_{n\geq 0} \mid c_j \geq c_{j-1} \text{ for } j = 1, 2, \dots \ , \\ & \text{there are numbers } \ 0 < n_1 < m_1 < n_2 < m_2 < \cdots < m_k \ \text{ for which} \\ & c_{n_i}/n_i \geq \beta, \quad c_{m_i}/m_i \leq \alpha, \ \text{for } i = 1, \dots, k \Big\} \ . \end{aligned} \tag{4.1}$$

We shall say that $\mathbf{c} \in C_{k,\alpha,\beta}$ has k oscillations of amplitude β/α.*

Given a sequence $\mathbf{c}$, and $\ell \geq 0$, we define a new sequence $\mathbf{d}^{(\ell,L)}$ by $d_n^{(\ell,L)} = c_{n+\ell} - c_\ell$, $n = 0, \dots, L-\ell$. We denote by $I(\mathbf{c}, k, \alpha, \beta, L)$ the set of those indices ℓ, for which $\mathbf{d}^{(\ell,L)} \in C_{k,\alpha,\beta}$. Thus, $I(\mathbf{c}, k, \alpha, \beta, L)$ counts how many "shifted" subsequences of $\{c_0, \dots, c_L\}$ make at least k oscillations. In other words, for $\ell \in I(\mathbf{c}, k, \alpha, \beta, L)$, the sequence

$$\left\{ \frac{c_{n+\ell} - c_\ell}{n} \right\}_{n=0,\dots,L-\ell} ,$$

makes at least k down-crossings between β and α.

* This terminology is adequate since all bounds will be functions of the amplitude β/α alone, *i.e.*, they only depend on the relative size of α and β.

Proposition 4.1. *One has the inequality:*

$$|I(\mathbf{c},k,\alpha,\beta,L)| \;\leq\; (\alpha/\beta)^k(L+1)\ .$$

Remark. See Ivanov [I1] for the manipulations—essentially a "periodic" extension of the sequence $\{c_0,\ldots c_L\}$—which lead to the bound $(\alpha/\beta)^k L$.

Proof. We apply Theorem 3.1 to the following setting. We let $E=[0,L+1)$, and we let $f(x)=c_j$ for $x\in[j,j+1)$. It is easy to verify that if an index j is such that the sequence $\mathbf{c}$ has k down-crossings from β to α to the right of j, then the same is true for the function f on the interval $[j,j+1)$. In other words,

$$|I(\mathbf{c},k,\alpha,\beta,L)| \;\leq\; |E_k|\ ,$$

and the result follows from Theorem 3.1.

Proof of Theorem 1.1. At this point, we use the invariance of the measure μ under T. For every $\omega\in\Omega$, we consider sequences $\mathbf{s}(\omega)=\{s_n(\omega)\}$, where $s_n(\omega)=\sum_{\ell=0}^{n-1}f(T^\ell\omega)$. We let $\Omega_{k,\alpha,\beta}$ denote the set of those ω for which the sequence $\mathbf{s}(\omega)$ makes k oscillations of amplitude β/α, and we let $\Omega_{k,\alpha,\beta,m}$ be the subset of those ω where this happens for the subsequence $\{s_1(\omega),\ldots,s_m(\omega)\}$. We then have, since $\mu(A)=\mu(T^{-1}A)$,

$$\begin{aligned}
X \;\equiv\; \mu(\Omega_{k,\alpha,\beta}) \;&=\; \lim_{m\to\infty}\mu(\Omega_{k,\alpha,\beta,m})\\
&=\; \lim_{m\to\infty} L^{-1}\sum_{j=0}^{L-1}\mu(T^{-j}\Omega_{k,\alpha,\beta,m})\\
&=\; \lim_{m\to\infty} L^{-1}\int \mathrm{d}\mu(\omega)\sum_{j=0}^{L-1}\chi_{T^{-j}\Omega_{k,\alpha,\beta,m}}(\omega)\\
&=\; \lim_{m\to\infty} L^{-1}\int \mathrm{d}\mu(\omega)\sum_{j=0}^{L-1}\chi_{\Omega_{k,\alpha,\beta,m}}(T^j\omega) \;\equiv\; \lim_{m\to\infty}X_{m,L}\ .
\end{aligned} \tag{4.2}$$

Note now that $\chi_{\Omega_{k,\alpha,\beta,m}}(\omega')=1$, if the sequence $\{s_n(\omega')\}_{n=1,\ldots,m}$ makes k oscillations of amplitude β/α, and 0 otherwise.

The crucial observation by Ivanov is now that if

$$\chi_{\Omega_{k,\alpha,\beta,m}}(T^j\omega)=1,\ \text{ then }\ j\in I(\mathbf{s}(\omega),k,\alpha,\beta,L+m-1)\ , \tag{4.3}$$

as one can see just from the definitions. Therefore, by Proposition 4.1, we find

$$\sum_{j=0}^{L-1}\chi_{\Omega_{k,\alpha,\beta,m}}(T^j\omega) \;\leq\; |I(\mathbf{c}(\omega),k,\alpha,\beta,L+m-1)| \;\leq\; (\alpha/\beta)^k(L+m)\ .$$

Coming back to $X_{m,L}$, we see that

$$X_{m,L} \leq L^{-1} \int \mathrm{d}\mu(\omega)(\alpha/\beta)^k \cdot (L+m) ,$$

for all L, and therefore

$$X_m \equiv \limsup_{L\to\infty} X_{m,L} \leq \limsup_{L\to\infty} (\alpha/\beta)^k \frac{L+m}{L} = (\alpha/\beta)^k . \tag{4.4}$$

Since $X \leq \lim_{m\to\infty} X_m$, the assertion of Theorem 1.1 follows.

5. Symmetric intervals

In this section, we prove Theorem 1.3, and Theorem 1.4. The proofs leading to Theorem 1.1 are not quite applicable, because the device used in Eq.(4.3) does not work in the case of symmetric intervals, since a subsequence will cut a “hole” in the original sequence. However, we shall work with the decomposition of the sequence $S_n(\omega) = \sum_{j=-n}^{n-1} h(T^j\omega)$ as the sum of two sequences $\mathbf{a}$ and $\mathbf{b}$ to be defined below. We first show that if s_n oscillates, then at least one of the sequences $\mathbf{a}$ or $\mathbf{b}$ must oscillate as well, but a little less. We study this as a general problem:

We assume $\mathbf{c} = \{c_n\}_{n\geq 0} \in C_{k,2\alpha,2\beta}$ and further that $c_n = a_n + b_n$, where $\mathbf{a} = \{a_n\}$ and $\mathbf{b} = \{b_n\}$ are monotone sequences of non-negative numbers. We are going to show that *either* $\mathbf{a}$ *or* $\mathbf{b}$ *must have oscillations*, and we will give bounds on the number and size of these oscillations. (Our bounds are not optimal, and we do not know the optimal bounds, but we will give a reasonable set of bounds for the cases when α/β is close to 0 or 1.)

To describe the nature of the oscillations, we set

$$\tau = \frac{1+(\beta/\alpha)}{2} ,$$

so that $1 < \tau < \beta/\alpha$. Then we define for $j = 1, 2, \ldots$,

$$\begin{aligned} \alpha_j &= \alpha + 2(j-1)(\alpha - \beta/\tau) , \\ \beta_j &= \tau\alpha_j , \\ \gamma_j &= 2\beta/\tau - \alpha - 2(j-1)(\alpha - \beta/\tau) . \end{aligned} \tag{5.1}$$

We also define $k_0 = k$ and $k_n = 1 + [k_{n-1}/2^n]$, where [ ] denotes the integer part. We can now formulate our result:

Proposition 5.1. *If* $\mathbf{c} \in C_{k,2\alpha,2\beta}$ *and* $\mathbf{c} = \mathbf{a} + \mathbf{b}$ *as above, then at least one of the sequences* $\mathbf{a}$ *or* $\mathbf{b}$ *is in*

$$C'_{k,\alpha,\beta} \equiv \left(\bigcup_{p^*\geq n\geq 1} C_{k_{2n+1},\gamma_n,\tau\gamma_n} \right) \bigcup \left(\bigcup_{p^*\geq n\geq 1} C_{k_{2n},\alpha_n,\tau\alpha_n} \right) ,$$

where p^* is the smallest integer satisfying

$$p^* \geq \frac{\alpha+\beta}{2(\beta-\alpha)} + 1 \ .$$

Remark. The meaning of this inclusion is that either $\mathbf{a}$ or $\mathbf{b}$ make at least k_{2p^*+1} oscillations of "amplitude" τ. Thus, the theorem says that if $\mathbf{c}$ has k oscillations of amplitude β/α, then, for large k, $\mathbf{a}$ *or* $\mathbf{b}$ *have at least* $\mathcal{O}(k/4^{p^*})$ *oscillations of amplitude* τ. Note that if β/α diverges then τ diverges as well, while for $\beta/\alpha = 1+\varepsilon$ we have $\tau = 1 + \varepsilon/2$.

Proof. Before we start with the proof, we note that the definitions of α_j, β_j have been chosen such that for $j \geq 1$, one has

$$\tau\alpha_j = \beta_j \ , \quad \tau\gamma_j = 2\beta - \beta_j \ , \quad \alpha_{j+1} = 2\alpha - \gamma_j \ . \tag{5.2}$$

We will construct recursively the possible sets of indices for which oscillations occur. Assume $\mathbf{c} \in C_{k,2\alpha,2\beta}$, with the oscillating indices m_j, n_j as in Eq.(4.1). Define $I_0 = J_0 = \{1,\ldots,k\}$, and

$$J_0^a = \{i \in J_0 \mid a_{m_i} \leq \alpha_1 m_i\} \ ,$$
$$J_0^b = \{i \in J_0 \mid b_{m_i} \leq \alpha_1 m_i\} \ .$$

Since $a_{m_i} + b_{m_i} = c_{m_i} \leq 2\alpha m_i = 2\alpha_1 m_i$, we see that each $i \in J_0$ must be in at least one of the sets J_0^a, J_0^b. Therefore the cardinalities satisfy $|J_0^a| + |J_0^b| \geq |J_0| = k = k_0$, and we conclude that $\max(|J_0^a|,|J_0^b|) \geq k_1$. We assume for definiteness that $|J_0^a| \geq k_1$; in the other case, the proof is obtained by exchanging the rôles of $\mathbf{a}$ and $\mathbf{b}$. We define next

$$I_1^a = \{i \in J_0^a \mid a_{n_i} \geq \beta_1 n_i\} \ .$$

Assume first $|I_1^a| \geq k_2$. By the definition of J_0^a and I_1^a, this means—cf. Eq.(5.2)—that $\mathbf{a} \in C_{k_2,\alpha_1,\beta_1} = C_{k_2,\alpha_1,\alpha_1\tau}$, which is part of the set $C'_{k,\alpha,\beta}$, and we stop the induction. In the other case, we define $I_1^b = J_0^a \setminus I_1^a$. Clearly, $|I_1^b| \geq k_2$, but furthermore we have for all $i \in I_1^b$ the inequalities

$$a_{n_i} < \beta_1 n_i \ ,$$
$$a_{n_i} + b_{n_i} \geq 2\beta n_i \ ,$$

and therefore

$$b_{n_i} \geq (2\beta - \beta_1) n_i \ . \tag{5.3}$$

We now define

$$J_1^b = \{i \in I_1^b \mid b_{m_i} \leq \gamma_1 m_i\} \ .$$

If $|J_1^b| \geq k_3$, then we have, using Eqs.(5.3) and (5.2),

$$\mathbf{b} \in C_{k_3,\gamma_1,2\beta-\beta_1} = C_{k_3,\gamma_1,\tau\gamma_1} \ ,$$

and we stop the induction. In the other case, we let $J_1^a = I_1^b \setminus J_1^b$, and then for all $i \in J_1^a$ we have

$$\begin{aligned} b_{m_i} \;&>\; \gamma_1 m_i \;, \\ a_{m_i} + b_{m_i} \;&\leq\; 2\alpha m_i \;, \end{aligned}$$

and therefore

$$a_{m_i} \;\leq\; (2\alpha - \gamma_1) m_i \;=\; \alpha_2 m_i \;. \tag{5.4}$$

If $2\alpha - \gamma_1 < 0$, the inequality (5.4) contradicts the positivity of the a_j and hence $|J_1^b| < k_3$ will never occur and the induction stops.

Otherwise, we continue, defining for $\ell \geq 2$,

$$\begin{aligned} I_\ell^a \;&=\; \{i \in J_{\ell-1}^a \mid a_{n_i} \geq \beta_\ell n_i\} \;, \\ I_\ell^b \;&=\; J_{\ell-1}^a \setminus I_\ell^a \;, \\ J_\ell^b \;&=\; \{i \in I_\ell^b \mid b_{m_i} \leq \gamma_\ell m_i\} \;, \\ J_\ell^a \;&=\; I_\ell^b \setminus J_\ell^b \;. \end{aligned}$$

There are now four cases.

1) If $|I_\ell^a| \geq k_{2\ell}$, then $I_\ell^a \subset J_{\ell-1}^a$ implies $a_{n_i} \geq \beta_\ell n_i$ and $a_{m_i} \leq \alpha_\ell m_i$ for $i \in I_\ell^a$, and hence $\mathbf{a} \in C_{k_{2\ell},\alpha_\ell,\beta_\ell} = C_{k_{2\ell},\alpha_\ell,\tau\alpha_\ell}$, and the induction stops.
2) If $|I_\ell^a| < k_{2\ell}$, then we have for $i \in I_\ell^b$ the inequality $b_{n_i} \geq (2\beta - \beta_\ell) n_i$, since $a_{n_i} < \beta_\ell n_i$ and $a_{n_i} + b_{n_i} \geq 2\beta n_i$, and we continue the induction.
3) If $|J_\ell^b| \geq k_{2\ell+1}$, then $J_\ell^b \subset I_\ell^b$ implies $b_{m_i} \leq \gamma_\ell m_i$ and $b_{n_i} \geq (2\beta - \beta_\ell) n_i$ for $i \in J_\ell^b$, and hence $\mathbf{b} \in C_{k_{2\ell+1},\gamma_\ell,2\beta-\beta_\ell} = C_{k_{2\ell+1},\gamma_\ell,\tau\gamma_\ell}$, and the induction stops.
4) In the last case, $|J_\ell^b| < k_{2\ell+1}$, and then we have for $i \in J_\ell^a$ the inequality $a_{m_i} \leq (2\alpha - \gamma_\ell) m_i$, since $b_{m_i} > \gamma_\ell m_i$ and $a_{m_i} + b_{m_i} \leq 2\alpha m_i$. If $(2\alpha - \gamma_\ell) \geq 0$, we continue the induction, while in the opposite case, we see that $|J_\ell^b| < k_{2\ell+1}$ cannot occur, and the induction stops.

Since $2\alpha - \gamma_{p^*} < 0$, as one checks easily from the definitions, the induction must stop for some $\ell \leq p^*$. The proof of Proposition 5.1 is complete.

We can now complete the proof of Theorem 1.3 by applying Proposition 5.1. We write the sum S_n of Eq.(1.2) as

$$S_n(\omega) \;=\; a_n(\omega) + b_n(\omega) \;,$$

where

$$a_n(\omega) \;=\; \sum_{j=0}^{n-1} h(T^j \omega) \;, \quad b_n(\omega) \;=\; \sum_{j=1}^{n} h(T^{-j} \omega) \;.$$

By Proposition 5.1, if $\mathbf{S}(\omega) \in C_{k,2\alpha,2\beta}$ then at least one of the sequences $\mathbf{a}(\omega)$, $\mathbf{b}(\omega)$ is in $C'_{k,\alpha,\beta}$. Therefore

$$\mu(\{\omega \mid \mathbf{S}(\omega) \in C_{k,\alpha,\beta}\}) \;\leq\; \mu(\{\omega \mid \mathbf{a}(\omega) \in C'_{k,\alpha,\beta}\}) \;+ \mu(\{\omega \mid \mathbf{b}(\omega) \in C'_{k,\alpha,\beta}\}) \;.$$

Since μ is invariant under T and T^{-1}, we can apply Theorem 1.1 to *both* sequences and we get a bound:

$$\mu(\{\omega \mid \mathbf{S}(\omega) \in C_{k,\alpha,\beta}\}) \leq 2 \sum_{n=1}^{2p^*+1} (1/\tau)^{k_n} \leq 4(p^*+1)(1/\tau)^{k/4^{p^*+1}} .$$

Since both τ and p^* are functions of α/β and $\tau > 1$, the Theorem 1.3 follows.

Proof of Theorem 1.4. This proof will be straightforward combination of the 2 following lemmas.

Lemma 5.2. *Let $p \geq 0$. There are a $k' = k'(p, \alpha/\beta)$ and a $\beta' = \alpha B(\alpha/\beta)$, with $B > 1$ when $\alpha < \beta$, such that if $\{q_n\} \in C_{k,\alpha,\beta}$, then the sequence with elements $t_n = q_n \frac{n}{n+p}$ is in $C_{k-k',\alpha,\beta'}$.*

Remark. It will be obvious from the proof that similar statements hold in the following cases:

$$\begin{aligned} \{t_n\} &= \{q_n \max(0, (n-p))/n\} \in C_{k-k',\alpha,\beta'} , \\ \{t_n\} &= \{q_n n/ \max(1, (n-p))\} \in C_{k-k',\alpha',\beta} , \\ \{t_n\} &= \{q_n (n+p)/n\} \in C_{k-k',\alpha',\beta} , \end{aligned} \tag{5.5}$$

where $\alpha' = \beta A(\alpha/\beta)$ with $A < 1$ if $\alpha < \beta$.

Proof. We will actually construct k' and β'. Let n_i and m_i be defined as the crossing points of the sequence s_n, cf. Eq.(4.1). Since $n_1 \geq 1$, and the s_n form an increasing sequence, we have

$$\alpha m_i \geq s_{m_i} \geq s_{n_i} \geq \beta n_i ,$$

so that $m_i \geq (\beta/\alpha) n_i > (\beta/\alpha) m_{i-1}$ and thus

$$m_i \geq (\beta/\alpha)^i . \tag{5.6}$$

Therefore,

$$t_{n_i} = s_{n_i} \frac{n_i}{n_i + p} \geq \beta n_i \frac{n_i}{n_i + p} = \beta n_i (1 + \frac{p}{n_i})^{-1} \geq n_i \frac{\beta}{1 + p(\alpha/\beta)^{i-1}} .$$

We choose

$$B(\alpha/\beta) = \frac{1 + (\beta/\alpha)}{2} ,$$

so that $\beta' = \alpha B(\alpha/\beta) > \alpha$, and there is clearly a $k' = k'(p, \alpha/\beta)$ for which $\beta/\big(1 + p(\alpha/\beta)^{k'-1}\big) > \beta'$. Then we have for $i > k'$,

$$t_{n_i} \geq n_i \beta' .$$

On the other hand,

$$t_{m_i} = q_{m_i} \frac{m_i}{m_i + p} \le q_{m_i} \le \alpha m_i \ ,$$

so that the assertion follows.

We next study sequences with increments of more than 1. Fix $r \in \mathbf{N}$ and define

$$t_n(\omega) = \sum_{j=0}^{rn-1} h(T^j \omega) \ .$$

We are interested in the oscillations of $t_n/(nr)$. This question is reduced to the one described in Proposition 1.2: Let

$$h_r(\omega) = \frac{1}{r} \sum_{j=0}^{r-1} h(T^j \omega) \ , \quad T_r = T^r \ ,$$

and

$$s_n(\omega) = \sum_{j=0}^{n-1} h_r\bigl((T_r)^j \omega\bigr) \ .$$

By construction, $s_n(\omega) = t_n(\omega)$. Since $h_r \ge 0$ and T_r preserves the measure μ if T preserves it, we conclude

Lemma 5.3. *The probability that the sequence $\{t_n/(nr)\}$ (defined with h and T) makes at least k oscillations is the same as the probability that $\{s_n/n\}$ (defined with h_r and T_r) makes at least k oscillations, and this quantity is bounded by $(\alpha/\beta)^k$.*

Remark. The Lemma 5.3 is a little too strong for our purpose, since it would have sufficed to observe that the sequence $\{s_n/n\}$ makes more oscillations than $\{t_n/(nr)\}$.

We can now complete the proof of Theorem 1.4 by a painful but somehow obvious combination of the results above. Recall the definition of S_n in Eq.(1.3):

$$S_n(\omega) = \sum_{j=-np_1-r_1}^{np_2+r_2-1} h(T^j \omega) \ .$$

We want to bound the probability that the sequence $S_n/\bigl(n(p_1+p_2)+r_1+r_2\bigr)$ makes k down-crossings from β to α. So assume the sequence with elements $q_n \equiv n \cdot S_n/\bigl(n(p_1 + p_2) + r_1 + r_2\bigr)$ is in $C_{k,\alpha,\beta}$. We let

$$t_n = q_n \frac{n+p}{n} = \frac{S_n}{p_1+p_2} \ , \quad \text{where} \quad p = \frac{r_1+r_2}{p_1+p_2} \ .$$

Applying Lemma 5.2, (actually Eq.(5.5)), we see that the sequence with elements $S_n/(p_1+p_2)$ is in $C_{k-k',\alpha',\beta}$, and thus the sequence with elements S_n is in $C_{k'',\alpha'',\beta''}$, where $k'' = k - k'$, $\alpha'' = \alpha'/(p_1 + p_2)$, $\beta'' = \beta/(p_1 + p_2)$. We next use the "splitting" mechanism and write $S_n = a_n + b_n$, where

$$a_n = \sum_{j=1}^{np_1+r_1} h(T^{-j}\omega) \ , \quad \text{and} \quad b_n = \sum_{j=0}^{np_2+r_2-1} h(T^{j}\omega) \ .$$

By Proposition 5.1, we conclude that one of the two sequences $\mathbf{a} = \{a_n\}$ or $\mathbf{b} = \{b_n\}$ must oscillate; we discuss here the case where it is $\mathbf{a}$ and leave the other case to the reader. Then we conclude that there are a $k^{(3)}$, $\alpha^{(3)}$ and $\beta^{(3)}$ for which $\mathbf{a} \in C_{k^{(3)},\alpha^{(3)},\beta^{(3)}}$ and these constants depend only on α/β, and furthermore $k^{(3)} = \mathcal{O}(k)$ as $k \to \infty$. Finally, $\alpha^{(3)}/\beta^{(3)} < 1$ when $\alpha < \beta$. (We will construct further such constants and they will possess the same properties. Of course, with some more work one can see that the quotient $\alpha^{(3)}/\beta^{(3)}$ goes to 0 when $\alpha/\beta \to 0$.) If $\mathbf{a} \in C_{k^{(3)},\alpha^{(3)},\beta^{(3)}}$, then the sequence with elements a_n/p_1 is in $C_{k^{(3)},\alpha^{(3)}/p_1,\beta^{(3)}/p_1}$, and, applying again Eq.(5.5), we see that the sequence with elements $(a_n/p_1) \cdot n/\bigl(n + (r_1/p_1)\bigr)$ is in $C_{k^{(4)},\alpha^{(4)},\beta^{(4)}}$. This means that the sequence with elements

$$\frac{s_m(\omega)}{m} = \frac{1}{np_1 + r_1} \sum_{j=1}^{np_1+r_1} h(T^{-j}\omega) \ ,$$

where $m = np_1 + r_1$, makes at least $k^{(4)}$ down-crossings from $\beta^{(4)}$ to $\alpha^{(4)}$. The probability that this happens for $m = r_1, r_1 + n, r_1 + 2n, \ldots$ is certainly less than the probability that this happens for the sequence $s_m(\omega)/m$ when $m = 1, 2, \ldots$. But this probability is bounded, using Theorem 1.1, by $\bigl(\alpha^{(4)}/\beta^{(4)}\bigr)^{k^{(4)}}$. Since $s_m(\omega)$ has been derived from the original sequence $S_n(\omega)$ by successive modifications, the proof of Theorem 1.4 is complete.

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