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A Global Attracting Set for the Kuramoto-Sivashinsky Equation

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Summary. New bounds are given for the L^2-norm of the solution of the Kuramoto-Sivashinsky equation

$$\partial_t U(x,t) \;=\; -(\partial_x^2 + \partial_x^4) U(x,t) - U(x,t)\partial_x U(x,t) \;,$$

for initial data which are periodic with period L. There is no requirement on the antisymmetry of the initial data. The result is

$$\limsup_{t\to\infty} \|U(\cdot,t)\|_2 \;\le\; \text{const. } L^{8/5} \;.$$

Introduction

In this paper, we prove new bounds on the Kuramoto-Sivashinsky equation (KS) by extending the ingenious method of Nicolaenko, Scheurer, and Temam [NST]. We study the KS-equation in its "derivative form:"

$$\partial_t U(x,t) \;=\; -(\partial_x^2 + \partial_x^4) U(x,t) - U(x,t)\partial_x U(x,t) \;. \tag{1.1}$$

The "original equation" is for the integral, $H(x,t) = \int_0^x d\xi U(\xi,t)$. Before we start with the bounds, we give some background material. The interest in the KS-equation is based on its relation as a phase equation for hydrodynamic problems, see Manneville [M] for a derivation. We consider the equation on the interval $[-L/2, L/2]$, with periodic boundary conditions. Since U should be thought of as the derivative of a periodic function, we always require $\int_{-L/2}^{L/2} U = 0$. In the paper [NST] it is shown that if the initial data are in L^2, and are *antisymmetric with respect to the origin,* then the evolution leaves them in L^2, forever, and there is a global attracting set whose diameter in L^2 is bounded. This bound depends on the size L of the system, and the bound given by [NST] is

$$\limsup_{t\to\infty} \|U(\cdot,t)\|_2 \;\leq\; \text{const. } L^{5/2} \;. \tag{1.2}$$

We shall *drop the requirement of antisymmetry* and prove that

$$\limsup_{t\to\infty} \|U(\cdot,t)\|_2 \;\leq\; \text{const. } L^{8/5} \;, \tag{1.3}$$

see the Main Theorem (4.2) in Section 4. A similar result, proved with different methods, and less stringent bounds, can be found in Ilyashenko [Il].

The mathematical and physical interest for this equation has to do with the mechanism which stabilizes this problem. To understand the problematics, we consider the equation for the Fourier transform of U (with a factor i):

$$U(x,t) \;=\; i\sum_{n\in\mathbf{Z}} u_n(t)e^{inqx} \;.$$

Note that

$$\|U(\cdot,t)\|_2^2 \;=\; L\sum_{n\in\mathbf{Z}} |u_n(t)|^2 \;\equiv\; L\|u(t)\|_2^2 \;, \tag{1.4}$$

where $u(t) = \{u_n\}_{n\in\mathbf{Z}}$. The Eq.(1.1) takes now the form

$$\partial_t u_n(t) \;=\; \mathcal{L}_n u_n(t) + \tfrac{1}{2}qn \sum_{n'+n''=n} u_{n'}u_{n''} \;, \quad n\in\mathbf{Z}\;, \tag{1.5}$$

where $q = 2\pi/L$ and $\mathcal{L}_n = (nq)^2 - (nq)^4$. Note that the spectrum of the linear operator $\mathcal{L}$ is *unstable* for $|n| \leq 1/q = L/(2\pi)$. If $|q| > 1$, all initial data converge to zero in L^2 since

$$\tfrac{1}{2}\partial_t \int U^2 \;=\; L\sum_{n\in\mathbf{Z}} \mathcal{L}_n|u_n|^2 \;\leq\; (q^2-q^4)\int U^2 \;,$$

since $u_0 = 0$. Henceforth, we can thus assume $|q| \leq 1$. In that case, the nonlinearity will stabilize the potentially growing modes. But there is an important difference to other non-linear equations such as the Ginzburg-Landau equation:

$$\partial_t u_n(t) \;=\; (1-q^2n^2)u_n(t) - \sum_{n'+n''+n'''=n} u_{n'}u_{n''}u_{n'''} \;, \quad n\in\mathbf{Z}\;. \tag{1.6}$$

In Eq.(1.6), the stabilization is through the amplitude of each *individual* mode u_n (through the diagonal term $u_n \cdot \sum_{n'+n''=0} u_{n'}u_{n''}$). In contrast, in the KS-equation, it is only the coupling of many modes together which *collectively* stabilize the equation. Note also that, in contrast to the Ginzburg-Landau equation, the sign of the nonlinear term is not related to the sign of the linear term. The mechanism of collective stabilization is the source of all complications in proving bounds for the KS-equation, and is also the basic reason for the absence of bounds in the infinite volume. The "nonlocal stabilization mechanism" is beautifully illustrated in the construction of stationary (i.e., time-independent) solutions by Frisch, She, and Thual [FST]. In our normalizations, they construct periodic, time-independent solutions as follows: Let n_0 be such that $0 < 1-(qn_0)^2 = \epsilon^2$, with ϵ sufficiently small. Then one can find a stationary solution of the form

$$\begin{aligned} u_{\pm n_0} &= \pm\sqrt{12}\epsilon + \mathcal{O}(\epsilon^2)\;, \\ u_{\pm 2n_0} &= \mp\epsilon^2 + \mathcal{O}(\epsilon^3)\;, \\ u_{\pm pn_0} &= \mathcal{O}(\epsilon^p)\;, \quad p\geq 2\;. \end{aligned}$$

Here, the first mode, u_{n_0} is linearly unstable, but it is coupled in such a way to the second mode u_{2n_0} to make the whole situation stable (and, in fact, stationary).

The Antisymmetric Case

We consider the class of real, L-periodic functions, with vanishing integral:

$$\mathcal{P}_L^0 \;=\; \left\{ U \;:\; U(x+L) = U(x)\;,\; \int_{-L/2}^{L/2} dx\, U(x) = 0 \right\} .$$

We define the operator $\mathcal{L}$ by $\mathcal{L}f = -\partial_x^2 f - \partial_x^4 f$, so that the KS-equation is

$$\partial_t U \;=\; \mathcal{L}U - UU' \;. \tag{2.1}$$

We shall omit the arguments of U whenever no confusion is possible. Rewrite U as $U(x,t) = V(x,t) + \Phi(x)$, where Φ, $V \in \mathcal{P}_L^0$. Then, V satisfies the equation

$$\partial_t V \;=\; \mathcal{L}V + \mathcal{L}\Phi - VV' - V\Phi' - \Phi V' - \Phi\Phi' \;. \tag{2.2}$$

Remark. Suppose that the function $\Phi(x)$ equals x. Then, we get the equation

$$\partial_t V \;=\; (\mathcal{L}-1)V - VV' - \Phi V' - \Phi\Phi' \;. \tag{2.3}$$

The operator $\mathcal{L}-1$ is negative definite, and this is the source of the convergence proof of [NST]. The "error terms" $-VV' - \Phi V' - \Phi\Phi'$ will have to be bounded carefully. Note that x is not a periodic function, but we shall choose a periodic function Φ for which $\mathcal{L}-\Phi'$ is a negative definite operator.

Denoting always by $\int$ the integral over a period we get from Eq.(2.2), using integration by parts:

$$\tfrac{1}{2}\partial_t \int V^2 \;=\; \int V\mathcal{L}V + \int V\mathcal{L}\Phi - \int V^2V' - \tfrac{1}{2}\int V^2\Phi' - \int V\Phi\Phi' \;. \tag{2.4}$$

The term $\int V^2V'$ vanishes. We will use the Eq.(2.4) and variants thereof to bound V, as a function of time. If V_1, V_2 are sufficiently smooth, we can define the bilinear form

$$(V_1,V_2)_{\gamma\Phi} \;\equiv\; \int_{-L/2}^{L/2} V_1''V_2'' - \int_{-L/2}^{L/2} V_1'V_2' + \gamma\int_{-L/2}^{L/2} V_1V_2\Phi' \;. \tag{2.5}$$

Note that this definition is formally equivalent to

$$(V_1,V_2)_{\gamma\Phi} \;=\; -\int V_1(\mathcal{L}-\gamma\Phi')V_2 \;, \tag{2.6}$$

but we shall always refer to the form of Eq.(2.5) in manipulations below. We will show that this form is positive definite. Note now that the Eq.(2.4) takes the form:

$$\tfrac{1}{2}\partial_t \int V^2 \;=\; -(V,V)_{\Phi/2} - (V,\Phi)_{\Phi} \;. \tag{2.7}$$

We define the space $\mathcal{A}_L$ of antisymmetric functions of period L:

$$\mathcal{A}_L = \{U : U(x) = -U(-x) , \ U(x+L) = U(x)\} . \tag{2.8}$$

The KS-equation leaves this space invariant, but it does not leave the space of symmetric functions invariant. Finally, we define the two quadratic forms

$$\begin{aligned} R_{\gamma\Phi}(U) &= (U,U)_{\gamma\Phi} , \\ Q(U) &= \tfrac{1}{4}\int_{-L/2}^{L/2} dx \left(U''\right)^2(x) + \tfrac{1}{4}\int_{-L/2}^{L/2} dx\, U^2(x) . \end{aligned} \tag{2.9}$$

In these definitions, L is fixed once and for all. Our first main result is

Proposition 2.1. *There is a constant K such that the following holds for every $L > 0$: There is a function $\Phi \in \mathcal{A}_L$ such that for all $\gamma \in [\frac{1}{4}, 1]$ and all $V \in \mathcal{A}_L$ one has the inequality*

$$R_{\gamma\Phi}(V) \geq Q(V) . \tag{2.10}$$

Furthermore,

$$R_{\gamma\Phi}(\Phi) \leq KL^{16/5} . \tag{2.11}$$

Remark. The quantity $R_{\gamma\Phi}(\Phi)$ is in fact independent of γ, since the γ-dependent term is $\gamma \int \Phi^2\Phi' = 0$, by the periodicity of Φ. Thus, $R_{\gamma\Phi}(\Phi) = R_0(\Phi)$.

Remark. The preceding result is inspired by the proof of [NST], but with a better bound. The proof will be given in the next section.

We can use the preceding proposition for a quick proof of the following result which is an improvement of the bound Eq.(1.2):

Theorem 2.2. *If the initial data $U_0(x) = U(x,0)$ of the KS-equation are in $\mathcal{A}_L$, then the solution is attracted to a ball of radius* const. $L^{8/5}$ *in* L^2. *More precisely, there is a constant K_1 (independent of L and U_0) such that*

$$\limsup_{t\to\infty} \|U(\cdot,t)\|_2 \leq K_1 L^{8/5} .$$

Remark. We can apply the bound of Theorem 2.2 to improve a series of known bounds for the KS-equation. For example, in [T], it is shown that the Hausdorff dimension d_H of the universal attractor is bounded above by $\mathcal{O}(L^{3/2})$. In fact, it is shown that this dimension is related to the bound $\mathcal{O}(L^\beta)$ of the L^2 norm by $L^{(35+10\beta)/40}$. Thus, we obtain a bound $d_H \leq \mathcal{O}(L^{51/40})$ from our results. The conjectured "best bound" is about $\mathcal{O}(L)$.

Proof of Theorem 2.2. We can write the Eq.(2.7) as follows, choosing suitable constants ϵ, ϵ', e.g., $\epsilon = 2/3$, $\epsilon' = 2/3$:

$$\begin{aligned} \tfrac{1}{2}\partial_t \int V^2 &= -(V,V)_{\Phi/2} - (V,\Phi)_\Phi \\ &\leq -(V,V)_{\Phi/2} + \tfrac{\epsilon}{2}(V,V)_\Phi + \tfrac{1}{2\epsilon}(\Phi,\Phi)_\Phi \\ &= \int V\left((1-\tfrac{\epsilon}{2})\mathcal{L} - \Phi'(\tfrac{1}{2}-\tfrac{\epsilon}{2})\right) V + \tfrac{1}{2\epsilon} R_\Phi(\Phi) \end{aligned} \tag{2.12}$$

$$
\begin{aligned}
&= -\epsilon' R_{(\frac{1}{2}-\frac{\epsilon}{2})/(1-\frac{\epsilon}{2})\Phi}(V) + \tfrac{1}{2\epsilon} R_0(\Phi) \\
&\leq -\epsilon' Q(V) + \tfrac{1}{2\epsilon} R_0(\Phi) \ .
\end{aligned}
$$

Here, we have used the inequality (2.10), and the choice of $\gamma \geq \frac{1}{4}$. The assertion follows from the definition of Q and from Eq.(2.11).

Proof of Proposition 2.1 and construction of Φ

The reader who is only interested in the general case, can skip this section at a first reading except for the construction of Φ which will also be used in the general case. We fix throughout $L > 0$ and let $q = 2\pi/L$. If $V \in \mathcal{A}_L$, then we can write V as $V(x) = i \sum_{n\in\mathbf{Z}} v_n e^{inqx}$ with

$$v_n \ = \ -\bar{v}_{-n} \ = \ -v_{-n} \ ,$$

so that $v_n = -v_{-n} \in \mathbf{R}$, and $v_0 = 0$. Similarly, if $\Phi \in \mathcal{A}_L$, then

$$\Phi'(x) \ = \ -\sum_{n\in\mathbf{Z}} \psi_n e^{inqx} \ , \tag{3.1}$$

and, since $\Phi' \in \mathbf{R}$ we have $\psi_n = \psi_{-n} \in \mathbf{R}$. We also require $\psi_0 = 0$.

We now exploit, as in [NST, Appendix], these symmetries to simplify the expression for $R_{\gamma\Phi}(V) = (V,V)_{\gamma\Phi}$. We have

$$
\begin{aligned}
\frac{1}{L}\int dx\, V^2(x)\Phi'(x) \ &= \ \frac{1}{L}\sum_{k,\ell,m}\int dx\, e^{iq(k+\ell+m)x} v_k v_\ell \psi_m \\
&= \sum_{k+\ell+m=0} v_k v_\ell \psi_m \ = \ \sum_{k,\ell} v_k v_\ell \psi_{-k-\ell} \ = \ \sum_{k,\ell} v_k v_\ell \psi_{|k+\ell|} \ .
\end{aligned}
$$

Using now $v_k = -v_{-k}$ and $\psi_k = \psi_{-k}$, $\psi_0 = 0$, we get, with $E_n = -(nq)^2 + (nq)^4$,

$$
\begin{aligned}
\frac{1}{L}(V,V)_{\gamma\Phi} \ &= \ 2\sum_{n>0} E_n v_n^2 + \gamma \sum_{k,m>0} v_k v_m \big(\psi_{|k+m|} - \psi_{|k-m|} + \psi_{|-k-m|} - \psi_{|-k+m|}\big) \\
&= \ 2\sum_{n>0} E_n v_n^2 + 2\gamma \sum_{k,m>0} v_k v_m \big(\psi_{|k+m|} - \psi_{|k-m|}\big) \\
&= \ 2\sum_{n>0} (E_n + \gamma\psi_{2n}) v_n^2 + 2\gamma \sum_{\substack{k,m>0\\ k\neq m}} v_k v_m \big(\psi_{|k+m|} - \psi_{|k-m|}\big) \\
&= \ 2\left[\sum_{n>0} (E_n + \gamma\psi_{2n}) v_n^2 + 2\gamma \sum_{k>m>0} v_k v_m \big(\psi_{|k+m|} - \psi_{|k-m|}\big)\right] \ .
\end{aligned}
$$

We shall bound the bracket $J = [\ ]$ from below.

Here, our method varies with respect to that of [NST], without being radically different. We assume henceforth that $\psi_{2n} = 4$ for $n \leq 2/q$ and $\psi_{2n} \geq 0$ for $n > 2/q$. Then,

$$E_n + \gamma\psi_{2n} \ \geq \ \tfrac{1}{2}\big((qn)^4 + 1\big) \ ,$$

as follows easily from $\gamma \geq \frac{1}{4}$ and from the definition of ψ_n. We now define

$$\tau_n \;=\; \sqrt{\tfrac{1}{2}\big((qn)^4+1\big)}\;,$$

so that $E_n + \gamma\psi_{2n} \geq \tau_n^2$. The main idea is now to set $v_n\tau_n = w_n$, so that J is bounded below by

$$\sum_{n>0} w_n^2 + 2\gamma \sum_{k>m>0} w_k \frac{\psi_{|k+m|}-\psi_{|k-m|}}{\tau_k\tau_m} w_m \;\equiv\; (w, (\mathrm{Id}+2\gamma\mathit{\Gamma})w)\;.$$

To show Eq.(2.10) of Proposition 2.1, we will show that $J \geq \frac{1}{2}(w,w)$ for $\gamma \in [1/4, 1]$. (The assertion follows then at once from the definition of Q.) For this, it suffices to check that the Hilbert-Schmidt norm of $2\gamma\mathit{\Gamma}$ is less than $\frac{1}{2}$. But this means that we only have to verify that

$$\|\mathit{\Gamma}\|_{\mathrm{HS}}^2 \;\equiv\; \sum_{k>m>0} \left|\frac{\psi_{|k+m|}-\psi_{|k-m|}}{\tau_k\tau_m}\right|^2 \;<\; \frac{1}{16}\;. \tag{3.2}$$

We want to choose ψ such that the inequality (3.2) holds, while, on the other hand, we want to minimize $(\mathit{\Phi},\mathit{\Phi})_{\gamma\mathit{\Phi}}$ as a function of L. In view of the first requirement the choice $\psi_n = \mathrm{const.}$ will be the best, however, in order to make the norms of $\mathit{\Phi}$ finite, the Fourier coefficients of ψ have to vanish sufficiently fast as n tends to infinity. Therefore we choose $\psi(n) = \psi_n$ to be a non-increasing $\mathcal{C}^1$ function having a small derivative. We do this in the following way: For a natural number M to be chosen later, we define $\psi_{2n+1} = 0$ and, for even n,

$$\psi_n \;=\; \begin{cases} 4, & \text{when } 1 \leq |n| \leq 2M \\ 4f(|n|/2M-1), & \text{when } 2M \leq |n| \end{cases}\;,$$

where f is a non-increasing $\mathcal{C}^1$ function satisfying $f(0) = 1$, $f'(0) = 0$ and

$$f \;\geq\; 0\;, \quad \sup|f'| \;<\; 1\;, \quad \int_0^\infty dk\,(1+k^2)|f(k)|^2 \;<\; \infty\;. \tag{3.3}$$

Then we have, for all $k > m > 0$,

$$|\psi_{k-m} - \psi_{k+m}| \;=\; 0\;, \quad \text{if } k+m \;\leq\; 2M\;,$$

and

$$|\psi_{k-m} - \psi_{k+m}| \;\leq\; 4m/M\;, \quad \text{for all } k \;>\; m\;.$$

Therefore, we have the following estimate of the Hilbert-Schmidt norm of $\mathit{\Gamma}$

$$\begin{aligned}
\|\mathit{\Gamma}\|_{\mathrm{HS}}^2 \;&\leq\; \frac{16}{M^2}\sum_{m=1}^{M} m^2\,\tau_m^{-2} \sum_{k=2M-m+1}^{\infty} \tau_k^{-2} \;+\; \frac{16}{M^2}\sum_{m=M+1}^{\infty} m^2\,\tau_m^{-2} \sum_{k=m+1}^{\infty} \tau_k^{-2} \\
&\leq\; \frac{16}{M^2}\sum_{m=1}^{M} m^2\,\tau_m^{-2} \int_{2M-m}^{\infty} dk\,\tau_k^{-2} \;+\; \frac{16}{M^2}\sum_{m=M+1}^{\infty} m^2\,\tau_m^{-2} \int_{m}^{\infty} dk\,\tau_k^{-2} \\
&\leq\; \frac{32}{3}q^{-4}M^{-5}\sum_{m=1}^{M} m^2\,\tau_m^{-2} \;+\; \frac{32}{3}q^{-4}M^{-2}\sum_{m=M+1}^{\infty} m^{-1}\,\tau_m^{-2}\;.
\end{aligned}$$

Taking again integrals as upper bounds for the sums we obtain

$$
\begin{aligned}
&\|\Gamma\|_{\mathrm{HS}}^2 \\
&\leq \frac{64}{3} q^{-6} M^{-5} \int_0^\infty dm\,(1+q^4m^4)^{-1/2} \;+\; \frac{64}{3} q^{-4} M^{-2} \int_M^\infty dm\, m^{-1}(1+q^4m^4)^{-1} \\
&\leq \frac{128}{3} q^{-7} M^{-5} \;+\; \frac{16}{3} q^{-8} M^{-6}\,. \qquad (3.4)
\end{aligned}
$$

Definition. We choose M as the smallest integer bigger than $4q^{-7/5} = 4(2\pi/L)^{-7/5}$.

Then Eq.(3.4) implies the desired bound on the Hilbert-Schmidt norm of Γ. Substituting into Eq.(3.4), we find, since we only have to consider the case $|q| \leq 1$,

$$\|\Gamma\|_{\mathrm{HS}}^2 \;\leq\; 11/256 \;<\; 1/16\,.$$

So far, we have only used the first condition on f in Eq.(3.3). Using the second condition, we estimate the scalar product of Φ. There is a constant K such that

$$
\begin{aligned}
(\Phi,\Phi)_{\gamma\Phi} \;=\; R_0(\Phi) \;=\; &\frac{4\pi}{q} \sum_{n=1}^{\infty} E_n (qn)^{-2} \psi_n^2 \\
&\leq\; K\, L^{16/5} \left(1 \;+\; \int_0^\infty dk\,(k+1)^2 f^2(k)\right)\,.
\end{aligned}
$$

The proof of Proposition 2.1 is complete.

The general case

We now come to the extension of the method to the case of functions $U \in \mathcal{P}^0_{L/2}$. The reduction to *half* the interval will be exploited below. The idea will be to consider a generalization of the quantity V, namely $V = U - \Phi_b$, where $\Phi_b(x) = \Phi(x+b)$. The translation b will be carefully chosen below. We consider now $Z = Z(V) = \int V^2$. Note that the integral extends still over a length L, i.e., *twice* the period of U. All integrals below are over $[-L/2, L/2]$. Note that, in the antisymmetric case, we have really studied Z with $b \equiv 0$.

Since U does not have symmetry properties in the general case, we can choose in principle any comparison function Φ_b; these comparison functions form a closed curve in $\mathcal{P}^0_L \cap \mathrm{L}^2$. The idea of the proof is to show that the distance of U to this curve will diminish until it reaches some saturation value. The point b will be chosen *as a function of time,* in such a way that the gradient of the distance function is essentially parallel to a line connecting U to the closest point on the curve. More precisely, we define $b(0) = 0$ and we define $b(t)$ as the solution of the equation

$$\partial_t b \;=\; \frac{3}{2L} \int_{-L/2}^{L/2} dx\, U(x,t) \Phi'_{b(t)}(x)\,. \qquad (4.1)$$

Note that

$$-\int U \Phi'_b \;=\; \tfrac{1}{2}\partial_b \int (U - \Phi_b)^2\,,$$

and that

$$\frac{1}{L}\int_{-L/2}^{L/2} dx\, U(x,t)\Phi'_{b(t)}(x) \;=\; \frac{1}{L}\int_{-L/2}^{L/2} dx\, V(x,t)\Phi'_{b(t)}(x)\;.$$

Also, if $U \in \mathcal{P}^0_{L/2}$ then so is V, since Φ has only even Fourier coefficients. The existence and uniqueness of solutions to the Eq.(4.1) and to the KS-equation are shown for completeness in Appendix A. Our main bound is:

Theorem 4.1. *There are constants $\alpha > 0$, and β such that for all $L > 0$ and all $V_0 = V(\cdot, t = 0) \in \mathcal{P}^0_{L/2}$ one has*

$$\partial_t Z \;\le\; -\alpha Z + \beta L^{16/5}\;. \tag{4.2}$$

Clearly, this implies $\limsup_{t\to\infty} Z(t) \le (\beta/\alpha)L^{16/5}$. Since $Z = \int(U - \Phi_b)^2$ and since $\frac{1}{4}\|\Phi_b\|_2^2 \le R_{\gamma\Phi_b}(\Phi_b) \le KL^{16/5}$, by Eq.(2.11), this implies our main result:

Main Theorem 4.2. *Let the initial data $U_0 = U(\cdot, 0)$ of the KS-equation be in $\mathcal{P}^0_L$, i.e., L-periodic, and of integral 0. There is a K_2 (independent of L and U_0) such that*

$$\limsup_{t\to\infty} \|U(\cdot,t)\|_2 \;\le\; K_2 L^{8/5}\;.$$

Proof of Theorem 4.1. The proof is very similar to the one for the case of antisymmetric functions. Instead of Eq.(2.7), we now have,

$$\tfrac{1}{2}\partial_t \int V^2 \;=\; -(V,V)_{\Phi_b/2} - (V,\Phi_b)_{\Phi_b} - (\partial_t b)\int V\Phi'_b\;, \tag{4.3}$$

with a new term coming from the derivative of b. There is an analogue of the inequality (2.10): We define, similarly to Eq.(2.5), the bilinear form

$$B_{\gamma\Phi_b}(V_1,V_2) \;=\; -\int_{-L/2}^{L/2} V_1\mathcal{L}V_2 + \gamma\int_{-L/2}^{L/2} V_1\Phi'_b V_2 + \frac{4\gamma^2}{L}\int_{-L/2}^{L/2} V_1\Phi'_b\cdot\int_{-L/2}^{L/2} V_2\Phi'_b\;.$$

Then, one has

Proposition 4.3. *For all $V \in \mathcal{P}^0_{L/2}$, all $b \in \mathbf{R}$, all $\gamma \in [\frac{1}{4}, 1]$, and all $L > 0$, one has*

$$B_{\gamma\Phi_b}(V,V) \;\ge\; Q(V)\;. \tag{4.4}$$

The proof will be given in Section 5, and we continue with the proof of Theorem 4.1. Note first the identities $(V,\Phi_b)_{\Phi_b} = B_{\Phi_b}(V,\Phi_b)$ and $B_{\Phi_b}(\Phi_b,\Phi_b) = R_0(\Phi)$. Furthermore, the inequality (4.4) has just one more term than Eq.(2.10), and thus we find, as in Eq.(2.12):

$$\tfrac{1}{2}\partial_t Z \;\le\; -\epsilon' Q(V) + \tfrac{1}{2\epsilon}R_0(\Phi) + \tfrac{3}{2L}\left(\int V\Phi'_b\right)^2 - (\partial_t b)\int V\Phi'_b\;. \tag{4.5}$$

By the construction of b in Eq.(4.1), the last two terms in (4.5) cancel and we get

$$\tfrac{1}{2}\partial_t Z \;\le\; -\epsilon' Q(V) + \tfrac{1}{2\epsilon} R_0(\Phi)\ .$$

Since $Q(V) \geq \frac{1}{4}\int V^2$, the assertion of Theorem 4.1 follows.

Proof of Proposition 4.3

It suffices to prove the proposition for $b = 0$ since the inequality (4.4) is invariant under translation. When $b = 0$, it is useful to define the spaces $\mathcal{A}_L$ and $\mathcal{S}_L$ of antisymmetric and symmetric functions of period L:

$$\begin{aligned} \mathcal{A}_L \;&=\; \bigl\{V \;:\; V(x) = -V(-x)\ ,\;\; V(x+L) = V(x)\bigr\}\ ,\\ \mathcal{S}_L \;&=\; \bigl\{V \;:\; V(x) = V(-x)\ ,\;\; V(x+L) = V(x)\bigr\}\ . \end{aligned}$$

(The space $\mathcal{A}_L$ was used before.) We also define

$$\mathcal{S}_L^0 \;=\; \bigl\{V \in \mathcal{S}_L \;:\; V(0) = 0\bigr\}\ .$$

If $V \in \mathcal{P}_{L/2}^0$ then we can decompose V as follows:

$$V(x) \;=\; V(0) + V_s(x) + V_a(x)\ ,$$

with $V_s \in \mathcal{S}_{L/2}^0$ and $V_a \in \mathcal{A}_{L/2}$. We define now the operation $\mathcal{T} : \mathcal{S}_{L/2}^0 \to \mathcal{A}_L$ by

$$(\mathcal{T}F)(x) \;=\; \begin{cases} F(x) & \text{if } x \in [0, L/2] \\ -F(x) & \text{if } x \in [-L/2, 0] \end{cases}\ ,$$

see Fig 1.

Fig. 1. A function $F \in \mathcal{S}_{L/2}^0$ and its antisymmetrized part $\mathcal{T}F \in \mathcal{A}_L$

Note now that from the definition of $R_{\gamma\Phi}$ and Q it follows that

$$R_{\gamma\Phi}(\mathcal{T}V_s) = R_{\gamma\Phi}(V_s)\,, \quad Q(\mathcal{T}V_s) = Q(V_s)\,.$$

This is checked easily by using the definition Eq.(2.9) of the quadratic forms. Therefore, we can apply Proposition 2.1 to $\mathcal{T}V_s$ and we see that for $V_s \in \mathcal{S}^0_{L/2}$, one has

Lemma 5.1. *If $V_s \in \mathcal{S}^0_{L/2}$ and $R_{\gamma\Phi}$, Q are defined over the original interval $[-L/2, L/2]$ then one has*

$$R_{\gamma\Phi}(V_s) \ge Q(V_s)\,, \tag{5.1}$$

when $\gamma \in [\frac{1}{4}, 1]$.

We now write $\Phi_{b=0} = \Phi$, with Φ defined in Section 3. Note that

$$\begin{aligned}
-\int V^2\Phi' &= -\int V_s^2\Phi' - \int V_a^2\Phi' - 2\int V_s V(0)\Phi' - \int V(0)^2\Phi' \\
&= -\int V_s^2\Phi' - \int V_a^2\Phi' - 2\int V(0)V\Phi'\,,
\end{aligned}$$

since the term $\int V(0)^2\Phi'$ vanishes. Using Lemma 5.1, we get

$$\begin{aligned}
\int V\mathcal{L}V - \gamma\int V^2\Phi' &= \int V_a\mathcal{L}V_a - \gamma\int V_a^2\Phi' \\
&\quad + \int V_s\mathcal{L}V_s - \gamma\int V_s^2\Phi' - 2\gamma V(0)\int V\Phi' \\
&= -R_{\gamma\Phi}(V_a) - R_{\gamma\Phi}(V_s) - 2\gamma V(0)\int V\Phi' \\
&\le -(Q(V_a) + Q(V_s)) - 2\gamma V(0)\int V\Phi'\,.
\end{aligned}$$

Note now that

$$\begin{aligned}
Q(V_a) + Q(V_s) &= \tfrac{1}{4}\int (V_a'')^2 + (V_s'')^2 + V_s^2 + V_a^2 \\
&= \tfrac{1}{4}\int \big(V_s + V_a + V(0)\big)''^2 + \tfrac{1}{4}\int \big(V_s + V_a + V(0)\big)^2 + \tfrac{L}{4}V(0)^2 \\
&= Q(V) + \tfrac{L}{4}V(0)^2\,,
\end{aligned}$$

where we have used the fact that $V \in \mathcal{P}^0_{L/2}$ and therefore $0 = \int V = LV(0) + \int V_s$. Thus,

$$-R_{\gamma\Phi}(V) = \int V\mathcal{L}V - \gamma V^2\Phi' \le -Q(V) - \frac{L}{4}V(0)^2 - 2\gamma V(0)\int V\Phi'\,. \tag{5.2}$$

Completing the square, we observe that

$$-\frac{L}{4}V(0)^2 - 2\gamma V(0)\int V\Phi' \le \frac{4\gamma^2}{L}\left(\int V\Phi'\right)^2\,. \tag{5.3}$$

Combining the inequalities (5.2) and (5.3) the assertion of Proposition 4.3 follows.

Appendix A. A Priori Bounds

It is well-known, see e.g. [T], that the KS-equation has a unique solution U_t for all initial data U_0 in $H = \mathrm{L}^2 \cap \mathcal{P}_L^0$. In particular, we have the estimate

$$\tfrac{1}{2}\partial_t \int U^2 = \int U\mathcal{L}U \le \tfrac{1}{4}\int U^2 ,$$

which implies

$$\int U_t^2 \le e^{t/2} \int U_0^2 . \tag{A.1}$$

We next prove the existence of a unique solution of Eq.(4.1). This equation is of the form

$$\partial_t b = F(t,b) . \tag{A.2}$$

It follows from the existence of strong solutions of the KS-equation that F is continuous in t. Furthermore, F is differentiable in b and

$$\partial_b F = \frac{3}{2L} \int U \Phi_b'' ,$$

which is finite because of the estimate (A.1) and because $\Phi_b'' \in \mathrm{L}^2$, and $\|\Phi_b''\|_2$ is independent of b. Therefore a unique solution satisfying $b(0) = 0$ exists for all $T > 0$.

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