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Controllability for chains of dynamical scatterers

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Abstract
In this paper, we consider a class of mechanical models which consists of a linear chain of identical chaotic cells, each of which has two small lateral holes and contains a freely rotating disc at its centre. Particles are injected at characteristic temperatures and rates from stochastic heat baths located at both ends of the chain. Once in the system, the particles move freely within the cells and will experience elastic collisions with the outer boundary of the cells as well as with the discs. They do not interact with each other but can transfer energy from one to another through collisions with the discs. The state of the system is defined by the positions and velocities of the particles and by the angular positions and angular velocities of the discs. We show that each model in this class is *controllable* with respect to the baths, i.e. we prove that the action of the baths can drive the system from any state to any other state in a finite time. As a consequence, one obtains the existence of, at most, one *regular* invariant measure characterizing its states (out of equilibrium).

Mathematics Subject Classification: 70Q05, 37D50, 82C70

1. Introduction

The study of heat conduction in (one-dimensional) solids remains a fascinating topic in theoretical physics. Various models have been developed to describe this phenomenon [1, 2]. In particular, the Lorentz gas model has been investigated and has been shown rigorously to satisfy Fourier's law [3]. However, since this model does not satisfy local thermal equilibrium (LTE) one cannot give a precise meaning to the temperature parameter involved in Fourier's law. To resolve this problem, a modified Lorentz gas was proposed, where the scatterers (represented by discs) are still fixed in place but are now free to rotate [4]. In this manner, the (non-interacting) particles can exchange energy from one to another through collisions with the scatterers. One clearly sees from numerical simulations that LTE is indeed satisfied and that heat conduction is accurately described by Fourier's law. To investigate such systems further, a class of models consisting of a chain of chaotic billiards, each containing a freely rotating scatterer, was introduced in [5]. The authors developed a theory that allows one, under physically reasonable assumptions (such as LTE), to derive rigorously Fourier's law as well as profiles for macroscopic quantities related to heat transport. They applied this theory to concrete examples that are either stochastic or deterministic. In particular, they established a detailed analysis of a mechanical modified Lorentz gas (MMLG) in which they assumed the existence and unity of an invariant measure describing its non-equilibrium steady state (section 4 in [5]). To obtain a complete description of the MMLG model it thus remains to prove the existence and unity of an invariant measure. While the question of existence is still a very challenging open problem (see the discussion in section 6), we shall show that there can be at most one *regular* invariant measure.

In this paper, we consider a class of mechanical models, extending the MMLG, and show that every model in this class is *controllable* with respect to the baths, i.e. we prove that the action of the baths can drive the system from any state to any other state in a finite time. The result is formulated as theorem 5.5, where we show that, starting from any initial state (comprising n particles), the system can be emptied of any particle, with all discs stopped at zero angular position. The system being *time-reversible*, this implies that one can fill it again with any number of particles and thus that one can drive the system between any two states. (A set of states of zero Liouville measure has to be excluded; this set consists of all states for which some particles stay forever in the system without hitting the disc or in the course of time, will have simultaneous or tangential collisions with the discs or will realize corner collisions with the outer boundary of the cells.) As a consequence, one obtains for each model in the considered class, assuming the existence and enough regularity of an invariant measure characterizing its states (out of equilibrium), the *uniqueness* of that invariant measure (see remark 5.7). The organization of this paper is as follows. In sections 2 and 3 we present our assumptions on the baths and introduce the class of mechanical models considered. Sections 4 and 5 are devoted to the controllability of the one-cell and N-cell systems, respectively. In the conclusion we make some comments on possible generalizations.

2. Heat baths

Although our discussion is mainly about the mechanical aspects of the models, the notion of controllability is of course relative to properties of the heat baths. Here, the exact details of the measure describing the (stochastic) heat baths are not of importance. What counts are only the sets of velocities and injection points into the system. More precisely, we assume throughout the paper that, at any time, any open set of injection points and velocities (including the direction) has *positive measure*. In particular, we shall exploit in a crucial way that any (open) set of realizations of the injection process with very high velocity indeed has positive measure. We shall use this positivity to inject 'driver' particles to help in emptying the system and thus obtain controllability as explained in the introduction.

3. Mechanical models

The class of mechanical models considered in this paper consists of a linear chain of identical chaotic cells, each of which has two small lateral holes and contains a freely rotating disc at its centre (see figure 1). Particles are injected at characteristic temperatures T_L, T_R and rates ϱ_L, ϱ_R from stochastic heat baths located at both ends of the chain (see section 2). Once in the

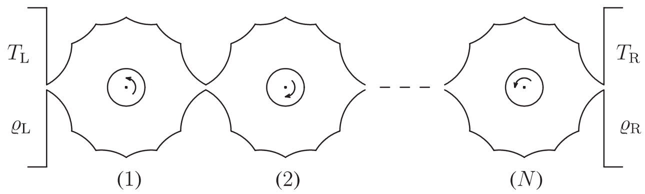

Figure 1. The system composed of N cells.

system, the particles move freely within the cells and will experience elastic collisions with the outer boundary of the cells as well as with the discs. They do not interact with each other but can exchange energy through collisions with the discs. The state of the system is defined by the positions and velocities of the particles and by the angular positions and angular velocities of the discs. We give a more precise definition of phase space in section 3.2.

We next specify the dynamics of the system (composed of N cells) in more detail: when there are n particles in the system, we number them as $i = 1, \ldots, n$ and denote by $q_1, \ldots, q_n$ and $v_1, \ldots, v_n$ their positions and velocities, respectively. Their trajectories are made of straight line segments joined at the outer boundary of the cells or at the boundary of the discs. If a particle reaches one of the two openings $\partial\Gamma_L^{(1)}$ or $\partial\Gamma_R^{(N)}$, it leaves the system (and the remaining particles are arbitrarily renumbered). Particles are injected into the system (from the baths) through these boundary pieces as well. We write $\omega_1, \ldots, \omega_N$ for the angular velocities of the discs and φ_j for the angle a marked point on the rim of disc j makes with the horizontal line passing through the centre of disc j ($j = 1, \ldots, N$).

To describe the rules of the dynamics, let us focus on one of the N cells, say the jth cell $\Gamma = \Gamma^{(j)}$, and assume that $q_i \in \partial\Gamma$ for some $1 \leqslant i \leqslant n$. We denote by D the disc at the centre of Γ, by $\partial\Gamma_{\text{box}}$ the outer boundary of Γ and by $\partial\Gamma_L$ and $\partial\Gamma_R$ its openings; they are exits either to the adjacent cells or to the heat baths. For a piecewise regular boundary $\partial\Gamma = \partial\Gamma_{\text{box}} \cup \partial D$, there are unit vectors e_n and e_t, respectively, normal outwards and tangent to $\partial\Gamma$ at q_i, and one can write $v_i = v_i^n e_n + v_i^t e_t$. We assume that the particles collide specularly from the boundary $\partial\Gamma_{\text{box}} \backslash (\partial\Gamma_L \cup \partial\Gamma_R)$ and that the collisions between the particles and the disc are elastic, so that for appropriate values of the parameters (i.e. the mass of the particles, the mass and the radius of the disc), one obtains the following dynamical rules, where primes denote the values after the collision:

1. If $q_i \in \partial\Gamma_L \cup \partial\Gamma_R$, then the ith particle keeps moving in a straight line to the adjacent cell or leaves the system.
2. If $q_i \in \partial\Gamma_{\text{box}} \backslash (\partial\Gamma_L \cup \partial\Gamma_R)$, then
$$(v_i^n)' = -v_i^n, \qquad (v_i^t)' = v_i^t. \tag{1}$$
3. If $q_i \in \partial D$, then
$$(v_i^n)' = -v_i^n, \qquad (v_i^t)' = \omega, \qquad \omega' = v_i^t. \tag{2}$$

The position of the ith particle and the angular position of the disc after the collision are left unchanged.

3.1. Geometry of the cell

In this subsection we describe the class of cells for which we can prove controllability. Our definition is a compromise between generality and tractability. In particular, this definition

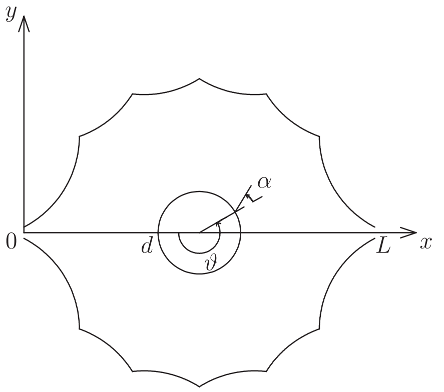

Figure 2. A typical cell.

will allow for a relatively simple controllability strategy. The reader who wants to proceed to the controllability can just look at figure 2 and use that example as a typical cell.

Let Γ_{box} be a bounded connected closed domain in $\mathbb{R}^2$ and let L denote its width, that is $(x, y) \in \Gamma_{\text{box}}$ implies $x \in [0, L]$. We assume

1. The boundary $\partial\Gamma_{\text{box}}$ of Γ_{box} is made of two straight segments (the 'openings') and a finite number of arcs of circle, i.e.
$$\partial\Gamma_{\text{box}} = \partial\Gamma_{\text{L}} \bigcup \partial\Gamma_{\text{R}} \bigcup \left(\bigcup_{k=1}^{b} \partial\Gamma_k\right), \tag{3}$$
where $\partial\Gamma_{\text{L}} = \{(0, y) \mid y \in [-a, a]\}$, $\partial\Gamma_{\text{R}} = \{(L, y) \mid y \in [-a, a]\}$ ($2a$ corresponds to the size of the openings) and each $\partial\Gamma_k$ is an arc of circle. The arcs of circle are oriented so that $\partial\Gamma_{\text{box}}$ is everywhere dispersing (see figure 2).
2. In the interior of Γ_{box} lies a disc D of centre $c = (L/2, 0)$ and radius r. The disc does not intersect the boundary of Γ_{box}, i.e. $\partial D \cap \partial\Gamma_{\text{box}} = \emptyset$.
3. Every ray from the centre of the disc intersects the boundary $\partial\Gamma_{\text{box}}$ only once: for every $z \in \partial\Gamma_{\text{box}}$ the segment $[c, z]$ intersects $\partial\Gamma_{\text{box}}$ only at z, i.e. $[c, z] \cap \partial\Gamma_{\text{box}} = z$.

Definition 3.1. *The closed domain* $\Gamma = \Gamma_{\text{box}} \backslash D$ *(with boundary* $\partial\Gamma = \partial\Gamma_{\text{box}} \cup \partial D$*) is called a cell.*

Our construction of $\partial\Gamma_{\text{box}}$ is motivated by the study of the *return map* R from the disc to the disc under the dynamics of the particle (see figure 3). We parametrize the points on ∂D by the angle $\vartheta \in [0, 2\pi)$ and denote by $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ the angle a line makes with the outward normal to the circle at ϑ (see figure 2). The return map R is defined for (ϑ, α) satisfying the following property: when a particle leaves the disc from ϑ in the direction α, it returns to the disc after *one* collision with the boundary $\partial\Gamma_{\text{box}}$ (and lands at ϑ'). In that case, we define $R(\vartheta, \alpha) = \vartheta'$. For other values of (ϑ, α), we say that R is undefined. The domain of R obviously depends on the boundary $\partial\Gamma_{\text{box}}$.

We next narrow the construction of acceptable domains by introducing the notion of *illumination*. For each $k \in \{1, \ldots, b\}$, we denote by I_k the set of ϑ for which $R(\vartheta, \alpha_k(\vartheta)) = \vartheta$ for some value $\alpha_k(\vartheta)$ of α and so that the reflection occurs on $\partial\Gamma_k$. Since the collisions with the corner points of $\partial\Gamma_k$ are undefined and the line connecting the boundary points

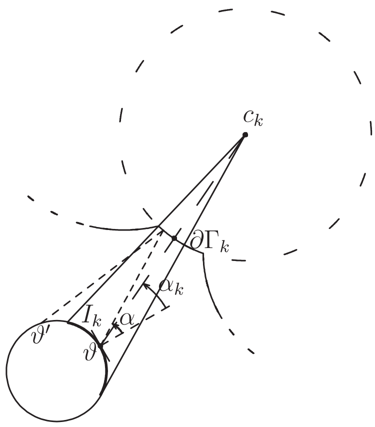

Figure 3. The illumination construction: the illuminated segment I_k is the part of the disc (thick line) delimited by the two rays arising from c_k and going through the kth arc $\partial\Gamma_k$. The return path corresponding to $R : (\vartheta, \alpha) \mapsto \vartheta'$ is shown as a dashed line.

of I_k to the centre c_k (see figure 3) may be tangent to the disc, we actually neglect the boundary points of I_k, i.e. we define I_k as the largest *open* (connected) set satisfying the above criteria.

This set can be more easily understood as follows: Let C_k be the circle on which $\partial\Gamma_k$ lies and let $c_k \in \mathbb{R}^2$ be its centre. If we 'shine' light from that centre to the disc, with only the kth arc $\partial\Gamma_k$ letting the light go through, then I_k is in fact that portion of the boundary of the disc on which light shines from c_k (and $\alpha_k(\vartheta)$ is the direction pointing from ϑ to the centre c_k). Thus, I_k is illuminated from c_k. See figures 3 and 4.

Remark 3.2. Note that if a particle leaves the disc at ϑ in the direction α and hits the kth arc $\partial\Gamma_k$, then $R(\vartheta, \alpha) > \vartheta$ if $\alpha > \alpha_k(\vartheta)$ and $R(\vartheta, \alpha) < \vartheta$ if $\alpha < \alpha_k(\vartheta)$; see figure 3. In other words, the return map R maps *away* from the line pointing to the centre c_k.

Remark 3.3. Note that the illuminated segments $I_1, \ldots, I_b$ will in general overlap.

Definition 3.4. *A cell is called 1-controllable if the illuminated segments cover the entire boundary of the disc, i.e.*

$$\bigcup_{k=1}^{b} I_k = \partial D .$$

Remark 3.5. We chose the term 1-controllable because our controllability proof will involve exactly one collision with $\partial\Gamma_{\text{box}}$ between any two consecutive collisions with the disc. One can imagine controllability proofs for domains with returns to the disc after several collisions with $\partial\Gamma_{\text{box}}$, and this would allow for more general domains. However, the gain of generality is perhaps not worth the effort.

Remark 3.6. Note that, since the illuminated regions $I_1, \ldots, I_b$ are *open* sets, one needs at least three generating circles to make a 1-controllable cell. There are domains which are *not* 1-controllable. See figure 4.

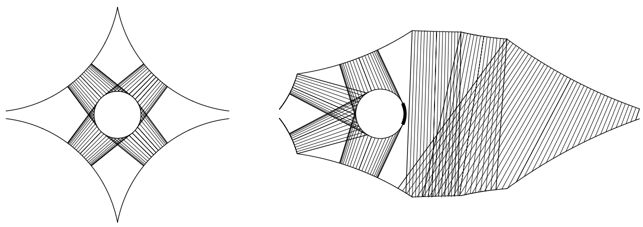

Figure 4. The illuminated segments are the parts of the disc delimited by the outermost pairs of rays emanating perpendicularly from the arcs. (Left) A 1-controllable cell. (Right) This cell is not 1-controllable since the illuminations do not cover the part of the disc shown in thick line. The illuminations on the right are shown for the arcs on the top only. Basically, domains with long 'tails' will not be 1-controllable.

3.2. Phase space

We next turn to the characterization of the phase space of the system consisting of one cell and an arbitrary number of particles. We denote by

$$\Omega_n = (\Gamma^n \times [0, 2\pi) \times \mathbb{R}^{2n+1})/\sim \tag{4}$$

the state space with n particles, where $\boldsymbol{q} = (q_1, \ldots, q_n) \in \Gamma^n$ denotes the positions of the n particles, $\varphi \in [0, 2\pi)$ denotes the angular position of a (marked) point on the boundary of the turning disc, $\boldsymbol{v} = (v_1, \ldots, v_n) \in \mathbb{R}^{2n}$ denotes the velocities of the n particles, $\omega \in \mathbb{R}$ denotes the angular velocity of the turning disc (measured in the clockwise direction), and $\sim$ is the relation that identifies pairs of points in the collision manifold $M_n = \{(\boldsymbol{q}, \varphi, \boldsymbol{v}, \omega) \mid q_i \in \partial\Gamma \text{ for some } i\}$.

The phase space of the system (for one cell) is

$$\Omega = \bigcup_{n=0}^{\infty} \Omega_n \qquad \text{(disjoint union)},$$

where now n is the current number of particles in the cell. When a particle is injected into the cell, the state of the system changes from $\xi \in \Omega_n$ to a state in Ω_{n+1} obtained by adding to ξ a particle with position $q_{n+1} \in \partial\Gamma_L \cup \partial\Gamma_R$ and velocity $v_{n+1} \in \mathbb{R}^2$ pointing into the cell. Similarly, when a particle leaves the cell, the corresponding two coordinates q_i and v_i are dropped. We refer to [5] for a detailed discussion of the numbering of the particles.

We denote by Φ_n^t the flow on Ω_n. As long as no collisions are involved, we have

$$\Phi_n^t(\boldsymbol{q}, \varphi, \boldsymbol{v}, \omega) = (\boldsymbol{q} + \boldsymbol{v}t, \varphi + \omega t \ (\text{mod } 2\pi), \boldsymbol{v}, \omega). \tag{5}$$

Clearly, if one specifies a realization $\mathcal{I}$ of the injection process in the time interval $[0, T]$ then, by applying (5) as well as the rules (1) and (2) at collisions, one obtains a flow $\Phi^t(\cdot, \mathcal{I})$ on the full state space Ω. Thus, if the system is in the state $\xi_0 \in \Omega$ at time $t = 0$, then its state at any later time $t \in (0, T]$ is given by

$$\xi(t) \equiv \Phi^t(\xi_0, \mathcal{I}) = (\boldsymbol{q}(t), \varphi(t), \boldsymbol{v}(t), \omega(t)) \in \Omega. \tag{6}$$

The scheme described above leaves collisions with the corners $\partial\Gamma^*$ of the cell Γ undetermined. When we discuss controllability, such orbits will not be considered. Similarly, we shall only consider dynamics so that at most one particle collides with the disc at any given time. The state space associated with the N-cell system will be introduced in section 5.

3.3. The strategy

Here, we outline the strategy adopted to show the controllability of our class of systems. Note first that the mechanical nature of the class of systems considered in this paper makes them *time-reversible*. Thus, one obtains controllability of any system in our class by establishing a way to drive (in a finite time) the system from any state to the ground state, i.e. the state in which there is no particle and all discs have zero angular positions and zero angular velocities. We shall start with the one-cell system and easily obtain its controllability from the following three crucial properties:

1. Given an initial state $\xi_0 \in \Omega$, there is a way to set the angular velocity and the angle of the disc to any prescribed value in an arbitrary short time (in particular before any particle collides with the disc). This operation can be achieved by particles which fly into the cell from outside, hit the disc, and exit again (all this before the next collision of another particle with the disc). The particles used for this process exist because of our assumptions on the nature of the heat baths: they will be called *drivers*.
2. Any *admissible path* in the cell (to be defined) can be realized by a particle in the system, which we shall call a *tracer*, by controlling its trajectory by acting adequately with driver particles on the disc.
3. If the cell is 1-controllable, then there in fact exists an admissible path between any point ϑ on the disc and one of the openings $\partial\Gamma_L$ or $\partial\Gamma_R$ (one can choose which one).

In the N-cell situation, we will obtain controllability by generalizing the strategy described above.

4. One-cell analysis

4.1. Paths of a particle

In this subsection, we consider one particle in one cell and characterize the set of possible paths it can follow (with the help of other particles) under the collision rules (1) and (2) at $\partial\Gamma$. We will extend that later in a straightforward way to an arbitrary number of particles.

Definition 4.1. *A curve* $\gamma : s \mapsto \gamma(s) \in \Gamma$, $s \in [0, 1]$, *is called an admissible path if it is continuous on* $[0, 1]$, *piecewise differentiable on* $(0, 1)$ *and satisfies the following properties:*

1. *It consists of a finite sequence of straight segments meeting at the boundary* $\partial\Gamma = \partial\Gamma_{box} \cup \partial D$ *of the cell.*
2. *The incoming and outgoing angles of two consecutive segments of* γ *meeting on the outer boundary* $\partial\Gamma_{box}$ *of the cell are equal.*
3. *Only its end points* $\gamma(0)$ *and* $\gamma(1)$ *can be in the openings* $\partial\Gamma_L$ *and* $\partial\Gamma_R$.
4. *It does not meet any corners of the cell, i.e.* $\gamma(s) \notin \partial\Gamma^*$ *for all* $s \in [0, 1]$.
5. *It is nowhere tangent to the boundary of the disc* ∂D.

An example of an admissible path is shown in figure 5. In the subsequent development, we shall denote by $|\gamma|$ the length of an admissible path γ, i.e. $|\gamma| = \sum_{i=0}^{m-1} \int_{s_i}^{s_{i+1}} |\gamma'(s)|\mathrm{d}s$ if γ is made up of m straight segments ($0 = s_0 < s_1 < \ldots < s_m = 1$).

Remark 4.2. Note that an admissible path does not need to satisfy any particular 'law of reflection' on the boundary ∂D of the disc (see figure 5).

We will show that, by shooting in 'driver' particles from the opening $\partial\Gamma_L$ (or $\partial\Gamma_R$) in a well chosen way, any admissible path can be realized as the orbit of a 'tracer' particle moving

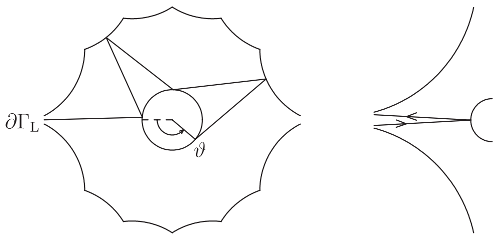

Figure 5. (Left) An admissible path. (Right) One possible orbit of the driver particle.

according to the laws (1) and (2) we gave earlier and that this is possible for any initial speed of the tracer particle (provided it is strictly positive) and any initial angular velocity of the disc.

We start with the following crucial lemma which shows that very fast particles coming from the baths can set the disc to any prescribed angular velocity ω and leave the system in a very short time δ. In what follows, these fast particles will be called *drivers*.

Lemma 4.3. *Assume that at time* 0 *the disc rotates with angular velocity* $\hat{\omega}$ *and that none of the particles which are inside the cell will collide with the disc before time* $\tau > 0$. *Then, given any* $\omega \in \mathbb{R}$ *and* $0 < \delta < \tau$, *there exists a way to inject a particle into the cell from the left entrance* $\partial\Gamma_{\mathrm{L}}$ *at time* 0 *such that at time* δ *the disc has angular velocity* ω *and the particle has left the system (through* $\partial\Gamma_{\mathrm{L}}$*). The same holds for* $\partial\Gamma_{\mathrm{R}}$.

Remark 4.4. The choice of the initial time equal to 0 is for convenience, and we will use the lemma for other initial times as well.

Remark 4.5. Assume we want to describe a strategy which should achieve some goal within a lapse of time δ. Then, by lemma 4.3, we can use a fraction of this time, say $\delta/2$, to stop the disc, and the other half of the time to do the actual task. So, without loss of generality, we may assume that the disc is at rest when the actual task begins.

Remark 4.6. Note that lemma 4.3 actually permits one to set both the angular velocity ω *and* the angular position φ of the disc at time δ. Assume for illustration that the disc is initially in the state ($\hat{\varphi} = 0, \hat{\omega} = 0$) and proceed as follows: send a driver to set the velocity of the disc to ω_1 at time $\delta_1 < \delta$ and send a second driver to set its velocity to ω at time δ such that $\omega_1(\hat{\delta} - \hat{\delta}_1)/2 + \omega(\delta - \hat{\delta}/2) = \varphi$, where $\hat{\delta}_1/2$ and $\hat{\delta}/2$ denote (as in the proof of lemma 4.3) the collision times of the first and, respectively, second driver with the disc.

Proof of lemma 4.3. To simplify the discussion, we assume $\hat{\omega} \geqslant 0$. Consider the general setup of figure 2. The axes are chosen such that the injection takes place in the segment $\partial\Gamma_{\mathrm{L}}$ (of length $2a$ and at x-coordinate 0), the centre of the disc has y-coordinate 0 and has its leftmost point at $(d, 0)$. The process we shall realize is sketched in figure 5 (the arrows correspond to the case $\omega \geqslant 0$). Choose $\hat{\delta}$ such that

$$0 < \hat{\delta} < \delta \qquad \text{and} \qquad \frac{2}{\hat{\delta}} > \max\left\{\frac{|\omega|}{a}, \frac{\hat{\omega}}{a}\right\}. \tag{7}$$

Define v_x and ε by

$$v_x = \frac{2d}{\hat{\delta}} \qquad \text{and} \qquad \varepsilon = \frac{\omega d}{v_x}\,. \tag{8}$$

Clearly, $|\varepsilon| < a$. We inject a particle into the cell at time 0 at the point $(0, -\varepsilon)$, with velocity (v_x, ω). No other particles are injected in the time interval $[0, \delta]$. Before the collision with the disc the particle follows the path:

$$\{x(t) = v_x t,\ \ y(t) = \omega t - \varepsilon \text{ for } t \in [0, \hat{\tau}]\}\ ,$$

where $\hat{\tau}$ denotes the collision time. By construction, the particle hits the disc at the point $(d, 0)$ at time $\hat{\tau} = \hat{\delta}/2$. At the collision, the tangent velocity of the particle is exactly ω and the disc rotates at angular velocity $\hat{\omega}$. After the collision, the particle has velocity $(-v_x, \hat{\omega})$ and follows the path:

$$\{x(t) = d - v_x(t - \hat{\tau}),\ \ y(t) = \hat{\omega}(t - \hat{\tau}) \text{ for } t \in [\hat{\tau}, 2\hat{\tau}]\}.$$

At time $2\hat{\tau} = \hat{\delta}$, the particle is at $(0, \tilde{y} = \hat{\omega}\hat{\delta}/2)$. Since $0 \leqslant \tilde{y} < a$ by (7) the particle will have reached $\partial\Gamma_{\mathrm{L}}$ at time $\hat{\delta}$ and will exit the cell. Note that if $v_x = \hat{\omega}d/a$ then $\tilde{y} = a$, so that the second condition in (7) demands that the (x-component of the) incoming velocity is sufficiently large so that the particle will not miss the exit. □

Proposition 4.7. *Let γ be an admissible path and assume that a particle starts at time 0 from $\gamma(0)$ with velocity $v_0 \neq 0$ in the positive direction along γ. Then one can find a sequence of drivers such that the particle will follow γ to its end in a finite time. In particular, if the end of γ is in $\partial\Gamma_{\mathrm{L}}$ or $\partial\Gamma_{\mathrm{R}}$ the particle will leave the cell.*

Proof. Consider first the case where γ does not intersect the boundary ∂D of the disc. In this situation the admissible path γ is automatically followed by the particle, since by (1) the reflections on the outer boundary of the cell are specular. Moreover, the entire path γ is realized in a finite time $T = |\gamma|/|v_0|$ since the norm of the particle's velocity $|v_0|$ is conserved at all times and initially non-zero. It thus suffices to discuss the intersections of the admissible path γ with the disc. Here, we will use drivers to direct the particle along γ. It will become clear that if one can do this for one collision with the disc one can do it for any finite number of them.

Assume that γ hits ∂D for the first time at $s_1 \in (0, 1)$ and decompose γ into two parts: the path before the intersection $\gamma_0 := \{\gamma(s) \mid s \in [0, s_1]\}$ and the path after the intersection $\gamma_1 := \{\gamma(s) \mid s \in [s_1, 1]\}$. Since there are only specular reflections up to time $t_1 = |\gamma_0|/|v_0|$, the particle will follow the path γ_0 without driver intervention and will arrive at the impact point $\gamma(s_1) \in \partial D$ at time t_1 with some velocity v_{in} satisfying $|v_{\mathrm{in}}| = |v_0|$. Let $\mathrm{e_n}$ and $\mathrm{e_t}$ be unit vectors, respectively, normal (outwards) and tangent to ∂D at $\gamma(s_1)$, and let us write $v_{\mathrm{in}} = v_{\mathrm{n}}\mathrm{e_n} + v_{\mathrm{t}}\mathrm{e_t}$. Note that $v_{\mathrm{n}} > 0$. If the disc has angular velocity $\hat{\omega}$ at the impact time t_1, then, by the collision rule (2) the particle will leave the disc with velocity $v_{\mathrm{out}} = -v_n\mathrm{e_n} + \hat{\omega}\mathrm{e_t}$. Let $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ be the angle between v_{out} and $-\mathrm{e_n}$ (figure 2). Clearly, one has

$$\alpha = \arctan(\hat{\omega}/v_{\mathrm{n}}). \tag{9}$$

Hence, in order to force the particle to emerge from the impact point in any prescribed direction α (which is not tangent to the impact point), in particular in the direction of γ_1, it suffices to let a driver arrive at the disc at time τ_1 before t_1 to give the disc the appropriate angular velocity $\hat{\omega}$.

To follow the full path γ we proceed by induction over the intersections with the disc and this concludes the proof. Note that the norm of the particle's velocity is not conserved along the orbit, so that the total time T the particle takes to complete the entire path γ is not $|v_0|/|\gamma|$. Note however that because γ is nowhere tangent to the disc the normal component v_{n} is non-zero at each collision so that the total time T is anyhow finite. □

Remark 4.8. The precise details used in proposition 4.7 to constrain the tracer particle along the path γ are not unique. Note first that given an admissible path γ and an initial velocity v_0, the speed of the tracer in each straight segment of γ is determined by the rules (1) and (2) of collision. Therefore, there is a sequence of times $t_1 < \cdots < t_m$ at which the tracer will hit the disc. The times $\{\tau_1, \ldots, \tau_m\}$ at which the drivers set the angular velocity of the disc to the appropriate value only have to satisfy

$$\tau_1 < t_1 \qquad \text{and} \qquad t_{i-1} < \tau_i < t_i. \tag{10}$$

Indeed, any sequence $\{\tau_1, \ldots, \tau_m\}$ satisfying these conditions is acceptable in the context of proposition 4.7 and for every $j \in \{1, \ldots, m\}$ there exist infinitely many $\delta_j \in (0, t_j - \tau_j)$ that can be considered in lemma 4.3.

4.2. Repatriation of particles

In this subsection, we use the specific properties of the cell (section 3.1) to control the trajectories of the particles after they have encountered the disc. In particular, the results established here will be necessary in the N-cell analysis to bring back the drivers from a given cell to one of the baths.

Lemma 4.9. *Let $\vartheta \in \partial D$ and assume that the cell is 1-controllable. Then there exists an admissible path between ϑ and $\partial\Gamma_{\mathrm{L}}$ (or $\partial\Gamma_{\mathrm{R}}$).*

Remark 4.10. Note that ergodicity is not a sufficient condition to obtain the above result. Indeed, consider the following system: a particle in a cell with closed entrances ($a = 0$) and with a circular inner boundary. Assume that all collisions of the particle in the cell are specular. Note that our model can be reduced to this system by using the drivers of lemma 4.3 (before each collision with the disc, use a driver to set $\omega = v_{\mathrm{t}}$, where $v = v_{\mathrm{n}}e_{\mathrm{n}} + v_{\mathrm{t}}e_{\mathrm{t}}$ is the velocity of the particle at the collision time; this will mimic a specular reflection). Then, even though it is well known that such a system is ergodic [6, 7], one still cannot conclude that there exists a trajectory between ϑ and $\partial\Gamma_{\mathrm{L}}$ that does not intersect $\partial\Gamma_{\mathrm{R}}$ in between. For this one needs to control the trajectory (see the proof below).

Proof. We shall exploit the properties of the illuminated segments $I_1, \ldots, I_b$ (section 3.1). Consider a point ϑ in I_k, for some $k \in \{1, \ldots, b\}$. A particle leaving this point in the direction of the centre c_k will return to ϑ after one collision with $\partial\Gamma_k$. Clearly, if one changes the direction sufficiently little, the particle will return to a point ϑ' which is still in I_k. Consider the union of the open intervals (ϑ, ϑ') (respectively (ϑ', ϑ) if $\vartheta' < \vartheta$) obtained in this fashion. Since every illuminated segment is an open connected set, one obtains, by varying the index k over $\{1, \ldots, b\}$, an open cover $\mathcal{O}$ of the illuminated region $I = \cup_{k=1}^{b} I_k$.

By assumption of 1-controllability, one has $I = \partial D$ and it follows, by the Heine–Borel theorem, that there exists a finite subset of $\mathcal{O}$ which covers the entire boundary of the disc. Therefore one finds, for *any* two points $\vartheta_{\mathrm{initial}}$ and $\vartheta_{\mathrm{final}}$ on the boundary of the disc, a sequence $(\vartheta_1, \ldots, \vartheta_m)$ of angles, with $\vartheta_1 = \vartheta_{\mathrm{initial}}$ and $\vartheta_m = \vartheta_{\mathrm{final}}$, such that an admissible path from $\vartheta_{\mathrm{initial}}$ to $\vartheta_{\mathrm{final}}$ can be realized by 'jumping' from ϑ_i to ϑ_{i+1}, for $i = 1, \ldots, m-1$ (each time via some $\partial\Gamma_k$ with a specular reflection).

Finally, if the orbit has reached an angle from which there is a direct line joining the left exit (without intersecting the boundary $\partial\Gamma_{\mathrm{box}} \backslash (\partial\Gamma_{\mathrm{L}} \cup \partial\Gamma_{\mathrm{R}})$), we choose that line and we are done (see figure 5). □

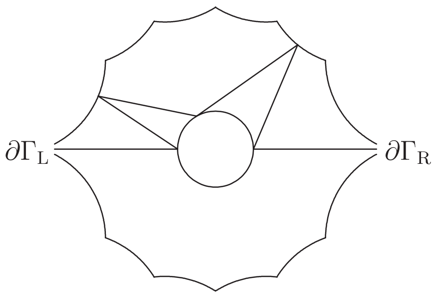

Figure 6. An admissible path linking the two openings.

Remark 4.11. Note that the set of intermediate points $(\vartheta_1, \ldots, \vartheta_m)$ between $\vartheta_{\text{initial}}$ and ϑ_{final} is open in $\mathbb{R}^m$. It follows that there actually exists an *open* set of admissible paths between a given point ϑ on the disc and the left exit $\partial\Gamma_{\text{L}}$, each of which has different intermediate intersection points with the disc.

Remark 4.12. While the proof of lemma 4.9 uses the Heine–Borel theorem, which in its standard form is non-constructive, it is in principle easy for any given region to actually invent a constructive proof. For example, one can proceed as follows: fix any pair of points $\vartheta_{\text{initial}}$ and ϑ_{final} in a given illuminated region I_k and determine a *uniform* lower bound $\Delta\vartheta > 0$ for the displacement of a particle within $[\vartheta_{\text{initial}}, \vartheta_{\text{final}}]$ through specular reflections from $\partial\Gamma_k$. Such a uniform bound can be obtained by considering the worst possible situation in $[\vartheta_{\text{initial}}, \vartheta_{\text{final}}]$. This shows that there exists an admissible path between any two points in a given illuminated region. One then concludes, as in the above proof, by using the assumption of 1-controllability. Since the arithmetic is somewhat involved, we omit this construction.

Corollary 4.13. *If the cell is* 1*-controllable, then there exists an admissible path between* $\partial\Gamma_{\text{L}}$ *and* $\partial\Gamma_{\text{R}}$ *so that its end points are located at the centre of the straight boundary pieces and its first and last straight segments are orthogonal to them (see figure 6). Furthermore, such a path exists also for which the first and last straight segments make a 'small' angle with the horizontal.*

Proof. The statements are obvious, by considering the proof of lemma 4.9 with the angles $\vartheta_{\text{initial}}$ and ϑ_{final} corresponding to the points where the first and, respectively, last straight segment intersect the disc.

4.3. Orbits of the system

We define the *ground state* $\xi_{\text{g}} \in \Omega$ of the system as the state in which the system is empty ($\xi_{\text{g}} \in \Omega_0$) and the disc is at rest ($\omega = 0$) at zero angular position ($\vartheta = 0$). In this subsection, we show that a suitable realization of the injection process can drive the system from any (admissible) initial state $\xi_0 \in \Omega$ to the ground state.

Definition 4.14. *A state* $\xi_0 = (q_{0,1}, \ldots, q_{0,n}, \varphi_0, v_{0,1}, \ldots, v_{0,n}, \omega_0) \in \Omega_n$ *is called an admissible initial state (at time* 0*) if it satisfies the following properties* ($i, j = 1, \ldots, n$)*:*

1. *The particles are initially inside the cell with non-zero velocities:* $q_{0,i} \in \Gamma\backslash\partial\Gamma$ *and* $v_{0,i} \neq 0$.

2. *The particles will either hit the disc or exit: for each i there is a finite time $t_i > 0$ such that $q_i(t_i) \in \partial D \cup \partial\Gamma_L \cup \partial\Gamma_R$ and $q_i(t) \notin \partial D \cup \partial\Gamma_L \cup \partial\Gamma_R$ for $0 < t < t_i$.*
3. *No tangent collisions with the disc: if $q_i(t_i) \in \partial D$, then the normal component $v_i^n(t_i)$ of $v_i(t_i)$ to ∂D at $q_i(t_i)$ is non-zero.*
4. *No simultaneous collisions with the disc: if $q_i(t_i) \in \partial D$ and $q_j(t_j) \in \partial D$ with $i \neq j$, then $t_i \neq t_j$.*
5. *No collisions with the corner points of the cell: $q_i(t) \notin \partial\Gamma^*$ for $0 < t \leqslant t_i$.*

Remark 4.15. The second condition in property 1 as well as properties 2 and 3 are necessary to prevent particles from staying forever in the system. (Note that a tangential collision with the disc at rest would stop the particle forever.) The other properties are necessary to get rid of all undefined events. Using the well known fact that the cell without the disc constitutes an ergodic system [6, 7], one easily sees that the set of states in Ω_n which do not satisfy these properties is negligible with respect to Liouville measure.

Definition 4.16. *An admissible movie is a set of n admissible paths $\gamma_1, \ldots, \gamma_n$ each of which being equipped with a tracer initially located at $\gamma_i(0)$ with velocity $\overline{v}_i(0)$ directed positively along γ_i such that*

1. *Each γ_i ends at the exits: $\gamma_i(1) \in \partial\Gamma_L \cup \partial\Gamma_R$.*
2. *Each tracer follows its corresponding admissible path up to the end in a finite time.*
3. *The scattering events on the disc are not simultaneous.*

Theorem 4.17. *Let $\xi_0 \in \Omega_n$ be an admissible initial state and assume the cell to be 1-controllable. Then there exists an admissible movie with $\gamma_i(0) = q_{0,i}$ and $\overline{v}_i(0) = v_{0,i}$ for $i = 1, \ldots, n$.*

Proof. Let us put a tracer at each position $q_{0,i}$ with velocity $\overline{v}_i(0) = v_{0,i}$ for $i = 1, \ldots, n$. Then, by definition 4.14, there exist finite times $t_i > 0$ $(1 \leqslant i \leqslant n)$ at which each tracer either leaves the cell (without making any collision with the disc) or hits the disc:

(a) If the ith tracer is in the first alternative, we consider its path $\gamma_i = \{q_i(t) \mid t \in [0, t_i]\}$ which is clearly admissible.

(b) In the second alternative, we denote by γ_i^- the path realized by the ith tracer between time 0 and the collision time t_i (along which there is no collision with the disc). By lemma 4.9, there exists an admissible path γ_i^+ between the collision point on the disc and the left exit. We then consider the following admissible path: $\gamma_i = \gamma_i^- \cup \gamma_i^+$.

We denote by $\mathcal{C} \subset \{1, \ldots, b\}$ the set of subscripts corresponding to the particles which are in case (b). Then, by proposition 4.7 combined with remarks 4.8 and 4.11, one can choose the admissible paths γ_j^+ $(j \in \mathcal{C})$ and inject the drivers that are used to constrain the jth tracer along γ_j^+ in such a way that all drivers and tracers involved in the movie do not make any simultaneous collisions with the disc. More precisely, there exist admissible paths and a set of drivers so that the tracers will hit the disc at distinct times $\tau_1 < \ldots < \tau_m$ and the drivers will be in the system only in the time intervals $[\tau_i, \tau_{i+1})$, for $i = 1, \ldots, m-1$, during each of which they control the disc in such a way that the tracer leaving the disc at time τ_{i+1} has the appropriate direction. This ends the proof. □

Taking into account remarks 4.6 and 4.15 as well as remarks 4.8 and 4.11, one obtains the following result as a consequence of the preceding theorem:

Corollary 4.18. *Assume the cell to be 1-controllable. Then, for almost every initial state $\xi_0 \in \Omega$ (with respect to Liouville) there exist a finite time $T > 0$ and an open set $\mathcal{B}([0, T])$ of realizations of the injection process in the time interval $[0, T]$ such that $\Phi^T(\xi_0, \mathcal{I}) = \xi_g$ for all $\mathcal{I} \in \mathcal{B}([0, T])$.*

5. N-cell analysis

We now extend the preceding results to the N-cell system. For this we need to introduce the corresponding notations and terminologies.

A system composed of N identical 1-controllable cells is said to be 1-controllable. A continuous path in the system which is composed of finitely many admissible paths is also called an admissible path. The particles that will be used to control the angular velocity of a given disc in the system will still be called drivers and those which will follow admissible paths will again be called tracers.

We write $\Gamma^N = \Gamma^{(1)} \times \cdots \times \Gamma^{(N)}$ for the domain accessible to the particles in the system composed of N identical cells, where each $\Gamma^{(\ell)} = \Gamma^{(\ell)}_{\text{box}} \backslash D^{(\ell)}$ can be identified with Γ, and denote by $\Omega^N = \prod_{\ell=1}^{N} \bigcup_{n=0}^{\infty} \Omega_n^{(\ell)}$ the corresponding state space, where each $\Omega_n^{(\ell)}$ is defined as in (4). We also define $\Omega^N_{n_1,\ldots,n_N} = \Omega^{(1)}_{n_1} \times \cdots \times \Omega^{(N)}_{n_N}$ so that $\Omega^N = \cup_{n_1,\ldots,n_N=0}^{\infty} \Omega^N_{n_1,\ldots,n_N}$. A state $\xi \in \Omega^N_{n_1,\ldots,n_N}$ is written as follows:

$$\xi = (q_1, \ldots, q_n, \varphi_1, \ldots, \varphi_N, v_1, \ldots, v_n, \omega_1, \ldots, \omega_N) \,, \tag{11}$$

where the total number of particles within the system is $n = n_1 + \ldots + n_N$. As in (6) we denote by $\Phi^t(\cdot, \mathcal{I})$ the flow on Ω^N. Note that the openings corresponding to the baths are now $\partial\Gamma_{\text{L}}^{(1)}$ and $\partial\Gamma_{\text{R}}^{(N)}$. Clearly, the notions of ground state $\xi_{\text{g}} \in \Omega^N$ and that of admissible movie can be generalized in a straightforward way to the N-cell system. Finally, the notion of admissible initial state, given in definition 4.14, is generalized as follows:

Definition 5.1. *A state $\xi_0 \in \Omega^N_{n_1,\ldots,n_N}$, written as in (11), is called an admissible initial state if it satisfies the following properties ($\ell, \ell' = 1, \ldots, N$ and $i, j = 1, \ldots, n$):*

1. *The particles are initially inside the system with non-zero velocities: $q_{0,i} \in \Gamma \backslash \partial\Gamma$ and $v_{0,i} \neq 0$.*
2. *The particles will either hit a disc or exit the system: for each i there is a finite time $t_i > 0$ and an index ℓ such that $q_i(t_i) \in \partial D^{(\ell)} \cup \partial\Gamma_{\text{L}}^{(1)} \cup \partial\Gamma_{\text{R}}^{(N)}$ and $q_i(t) \notin \partial D^{(1)} \cup \ldots \cup \partial D^{(N)} \cup \partial\Gamma_{\text{L}}^{(1)} \cup \partial\Gamma_{\text{R}}^{(N)}$ for $0 < t < t_i$.*
3. *No tangent collisions with the discs: if $q_i(t_i) \in \partial D^{(\ell)}$, then the normal component $v_i^{\text{n}}(t_i)$ of $v_i(t_i)$ to $\partial D^{(\ell)}$ at $q_i(t_i)$ is non-zero.*
4. *No simultaneous collisions with the discs: if $q_i(t_i) \in \partial D^{(\ell)}$ and $q_j(t_j) \in \partial D^{(\ell')}$ with $i \neq j$, then $t_i \neq t_j$.*
5. *No collisions with the corner points of the system: $q_i(t) \notin \partial\Gamma^{N,*}$ for $0 < t \leqslant t_i$.*

Remark 5.2. Note that property 4 excludes simultaneous collisions with any given disc ($\ell = \ell'$), which is necessary since such events are undefined, but it also excludes simultaneous collisions of particles with different discs ($\ell \neq \ell'$). This requirement is actually not necessary but, since such events are negligible (with respect to Liouville), we decided for a matter of convenience to exclude them.

From lemma 4.9 and corollary 4.13 one immediately obtains the following generalized result:

Lemma 5.3. *Let $\vartheta_j \in \partial D^{(j)}$ for some $1 \leqslant j \leqslant N$ and assume that the system is 1-controllable. Then there exists an admissible path between ϑ_j and $\partial\Gamma_{\text{L}}^{(1)}$ (or $\partial\Gamma_{\text{R}}^{(N)}$).*

Let us now generalize the second crucial result, namely lemma 4.3. We want to achieve the controlling of disc j in a very short time. Basically, one should think that one wants to control disc j *before* some time when a particle hits it, but this controlling should happen *after* any collision of any other particle with one of the discs $1, \ldots, j-1$.

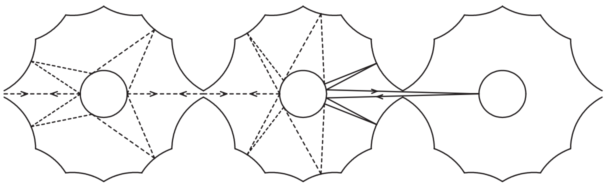

Figure 7. The admissible incoming and outgoing paths in the case $j = 3$ ($\hat{\omega}_j \leqslant 0, \omega_j \leqslant 0$): $\gamma_{\rm in}$ is the upper path and $\gamma_{\rm out}$ the lower one.

Proposition 5.4. *Assume that the system is* 1*-controllable and that at time* 0 *the discs rotate with angular velocities* $\hat{\omega}_1, \ldots, \hat{\omega}_N$ *and that none of the particles which are inside the system will collide with any disc before time* $\tau > 0$*. Then, given* $j \in \{1, \ldots, N\}$*,* $\omega_j \in \mathbb{R}$ *and* $0 < \delta < \tau$*, there exists a way to inject drivers from the left entrance* $\partial\Gamma_{\rm L}^{(1)}$ *at time* 0 *such that at time* δ *the* ℓ*th disc has angular velocity* $\hat{\omega}_\ell$ *if* $\ell \neq j$ *and* ω_j *if* $\ell = j$ *and all the drivers have left the system (through* $\partial\Gamma_{\rm L}^{(1)}$*). The same holds for* $\partial\Gamma_{\rm R}^{(N)}$*.*

Proof. The proof is by induction over the subscript $j = 1, \ldots, N$. The case $j = 1$ has already been treated in the preceding section (lemma 4.3). Assume now that $j > 1$ and that one can control discs 1 to $j - 1$. We shall show that there exists a way to control disc j. Since, by the inductive hypothesis, one can set the angular velocities of the discs $1, \ldots, j - 1$ to any values in an arbitrarily short time, one can assume, without loss of generality, that these discs are initially at rest, i.e. $\hat{\omega}_1 = \ldots = \hat{\omega}_{j-1} = 0$ (see also remark 4.5).

As in the proof of lemma 4.3 we shall construct a class of admissible paths γ_j, with parameters $(\hat{\omega}_j, \omega_j, \delta)$, starting from the left bath $\partial\Gamma_{\rm L}^{(1)}$, going to disc j and then returning to the left bath. We shall denote by $\gamma_{\rm in}$ the incoming path linking the left bath to disc j and by $\gamma_{\rm out}$ the outgoing path from disc j to the left bath; thus $\gamma_j = \gamma_{\rm in} \cup \gamma_{\rm out}$ (see figure 7).

Consider figure 8. We first choose an open segment Δ centred at ϑ_0 such that for every $\vartheta_{\rm in} \in \Delta$ the line emerging from $\vartheta_{\rm in}$ and intersecting disc j at the horizontal broken line does not cross a wall (i.e. the boundary $\partial\Gamma_{\rm box}\backslash(\partial\Gamma_{\rm L} \cup \partial\Gamma_{\rm R})$). For every angle $\vartheta_{\rm in} \in \Delta$ we choose an admissible path from $\partial\Gamma_{\rm L}^{(1)}$ to $\vartheta_{\rm in}$, which exists by lemma 5.3. This specifies the incoming path $\gamma_{\rm in}$ (see figures 7 and 8). We next drive a particle (called the *controller*) along the incoming path, *where it will play the role of a driver for disc* j. Given the inductive hypothesis and proposition 4.7, there is clearly a set of drivers which will drive the controller along this path. We now scale the initial velocities of the controller and of all the drivers by a common factor λ and scale the injection times by $1/\lambda$. Note that this scaling preserves the trajectories of the controller and of the drivers.

Similarly, given $\gamma_{\rm in}$, λ and $\hat{\omega}_j$, there are an associated admissible outgoing path $\gamma_{\rm out}$ (specified by an angle $\vartheta_{\rm out} \in \Delta$) and a corresponding sequence of drivers so that the controller will be driven back to the left bath after it has collided with disc j (provided λ is large enough, see below). A typical scenario is shown in figure 7.

It is clear that one can choose the families of paths $\{\gamma_{\rm in}\}_{\vartheta_{\rm in} \in \Delta}$ and $\{\gamma_{\rm out}\}_{\vartheta_{\rm out} \in \Delta}$ such that the following properties hold:

1. The length of the full paths $\gamma_j = \gamma_{\rm in} \cup \gamma_{\rm out}$ is bounded uniformly in $\vartheta_{\rm in}, \vartheta_{\rm out} \in \Delta$.
2. For each λ, the incoming speed $|v_{\rm in}|$ varies continuously with $\vartheta_{\rm in}$.
3. For each $\vartheta_{\rm in} \in \Delta$, the speed $|v_{\rm in}|$ is an increasing and continuous function of λ.

Step 1. Let $0 < \delta < \tau$ and $\hat{\omega}_j \in \mathbb{R}$ be fixed. By property 1 there is a finite threshold λ_1 so that, for every $\lambda > \lambda_1$ and every $\vartheta_{\rm in} \in \Delta$, the controller will travel through $\gamma_{\rm in}$, collide with

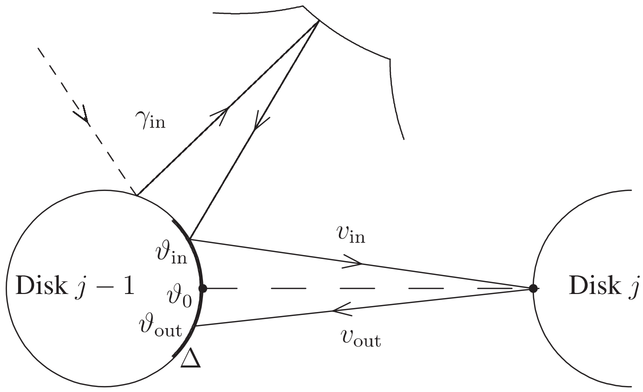

Figure 8. Some parameters.

disc j and return to the left bath through γ_{out} in a time shorter than $\delta/2$. Note that, if the initial angular speed $|\hat{\omega}_j|$ of disc j is big, then λ has to be large enough so that the controller will not meet a wall when returning to disc $j-1$ after its collision with disc j.

To obtain the above statement, one can proceed as follows. First define

$$T_{\text{in}}(\lambda) = \sup_{\vartheta_{\text{in}}\in\Delta} \{\text{Time the controller takes to complete } \gamma_{\text{in}} \text{ starting with speed } \lambda\},$$

$$T_{\text{out}}(\lambda) = \sup_{\vartheta_{\text{out}}\in\Delta} \{\text{Time the controller takes to complete } \gamma_{\text{out}} \text{ starting with speed } v^*(\lambda)\},$$

where $v^*(\lambda) = \inf_{\vartheta_{\text{in}}\in\Delta}\{|v_{\text{out}}(\vartheta_{\text{in}}, \lambda, \hat{\omega}_j)|\}$ ($\hat{\omega}_j$ is fixed) if there is a return $\vartheta_{\text{out}} \in \Delta$ associated with each $\vartheta_{\text{in}} \in \Delta$, and $v^*(\lambda) = 0$ otherwise. Then, by property 1, there is a threshold $0 < \lambda_1 < \infty$ such that the times $T_{\text{in}}(\lambda)$ and $T_{\text{out}}(\lambda)$ are finite for all $\lambda > \lambda_1$. Moreover, these travelling times decrease with λ. Note finally that for each $\vartheta_{\text{in}} \in \Delta$ the travelling time of the controller along the full path $\gamma_j = \gamma_{\text{in}} \cup \gamma_{\text{out}}$ is bounded by $T_{\text{in}}(\lambda) + T_{\text{out}}(\lambda)$.

Step 2. Let $\omega_j \in \mathbb{R}$ be given. From the properties 2 and 3 it follows that one can choose $\lambda > \lambda_1$ and the angle $\vartheta_{\text{in}} \in \Delta$ so that disc j will have the required angular velocity after the controller has collided with it. Note that if one wants to give a very small angular velocity to disc j, it suffices to choose ϑ_{in} sufficiently close to ϑ_0.

Step 3. In the remaining time $\delta/2$ we stop the discs 1 to $j-1$.

Therefore, by choosing λ sufficiently large and the angle ϑ_{in} correctly, the disc j will have any required angular velocity at time δ, the controller (and all drivers) will have left the system and all the perturbed discs (with subscript smaller than j) will have been restored to their initial state. □

Finally, using proposition 5.4, one obtains by inspection of the proof of theorem 4.17 the main result:

Theorem 5.5. *Assume the system to be* 1*-controllable. Then, for every admissible initial state* $\xi_0 \in \Omega^N_{n_1,\ldots,n_N}$ *there exists an admissible movie with* $\gamma_i(0) = q_{0,i}$ *and* $\overline{v}_i(0) = v_{0,i}$ *for* $i = 1, \ldots, n$*. In particular, for almost every initial state* $\xi_0 \in \Omega^N$ *(with respect to Liouville) there exist a finite time* $T > 0$ *and an open set* $\mathcal{B}([0, T])$ *of realizations of the injection process in the time interval* $[0, T]$ *such that* $\Phi^T(\xi_0, \mathcal{I}) = \xi_g$ *for all* $\mathcal{I} \in \mathcal{B}([0, T])$.

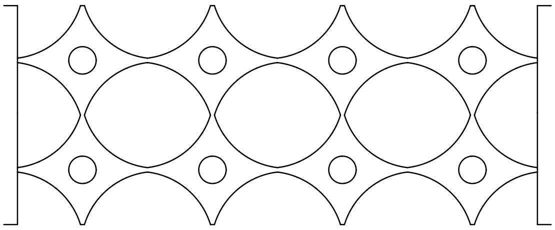

Figure 9. A 1-controllable system in 2D.

Remark 5.6. In theorem 5.5, we used the notion of admissible movie to show that the system can be emptied of any particle in a finite time. There is another way to obtain this result. Assume that one can control all discs as stated in proposition 5.4. Then, one can control them so that the particles make *specular reflections* with the discs (see also remark 4.10). Since such a system is ergodic [6, 7], there must be a finite time at which the system will be empty. Note that if one can show that the N-cell system, *with rotating discs*, is ergodic then one obtains controllability as an immediate consequence.

Remark 5.7. First note that the particles and the discs evolve under deterministic rules and thus the considered systems constitute Markov processes. If one can prove that for a 1-controllable system (composed of N cells) there exists an invariant measure on Ω^N and that this invariant measure is sufficiently regular, then it follows from controllability (theorem 5.5) that it is unique and therefore ergodic. (Time-reversibility and theorem 5.5 imply that for almost every state $\xi \in \Omega^N$ (with respect to Liouville) and any open set $A \subset \Omega^N$ there is a finite time $T > 0$ such that the probability for the system initially in the state ξ to be inside A after time T is positive: $P_T(\xi, A) > 0$.)

6. Concluding remarks

We have shown that every chain of 1-controllable identical chaotic cells is controllable with respect to generic baths. As a consequence, one obtains the existence of, at most, one *regular* invariant measure. The 1-controllable property, introduced through the notion of illumination, allows for a large class of cells and is a rather simple geometrical criterion to check. For the sake of convenience, we have made some simplifying assumptions on the outer boundary $\partial\Gamma_{\text{box}}$ of the cell (i.e. conditions 1 to 3 in section 3.1). These assumptions are clearly not optimal to obtain controllability. For example, one can handle systems in which there are some intersection points between $\partial\Gamma_{\text{box}}$ and ∂D and in which there are more than one intersection point between the segment $[c, z]$ and $\partial\Gamma_{\text{box}}$. However, such a gain of generality was not of interest to us. One could also consider chains of non-identical 1-controllable cells, change the position of the disc or replace it by a some kind of 'potato' or a needle. One should then be able to control these dynamical scatterers and thus obtain controllability. Note that the present results also prove the controllability for some class of 2D models; see for example figure 9.

Some readers may wonder why we do not prove ergodicity, and what is missing for such a proof. The basic problem is that we do not know if an invariant measure exists, and even less what its regularity properties would be. This problem appears regularly in non-equilibrium systems, and is related to the potential divergence of the energy in the system, which could be heated up arbitrarily by the heat baths. So this is a problem of non-compactness of phase space, which only disappears in such trivial cases as linear systems [3, 8].

For truly non-linear systems the difficulty can in general only be eliminated in two ways: (i) relatively strong coupling of the system to the baths, (ii) addition of dissipation and noise. Examples of the first case are, for example, present in [9] and are worked out in a particularly illuminating way in [10], where it is shown that when the system reaches high energy, it *must* dissipate energy back to the baths. The second case, which has a longer history, consists in building a mechanism which avoids the heating-up of the system. For the model discussed in this paper, this could perhaps be achieved by adding a small dissipative term to the equations of motion of the turning discs, so that at very high energy, they are slowed down. An interesting bound with quite minimal assumptions occurs in the case of the Kuramoto–Sivashinsky equation [11].

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