A Fields Medal for Martin Hairer

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Before starting to work on KPZ and regularity structures, Martin's papers covered a wide range of important fields: Stochastic PDE's in infinite dimensions [3, 4, 7], uniqueness of the invariant measure for 2D Navier Stokes [5], transport properties of very singular heat conduction problems [6]. And it is perhaps of interest to point to another field where Martin excels:

The Swiss army knife of sound editing!

(He was raised in Switzerland.) Indeed, Martin's programming skills are known to many Disk-Jockeys through his award winning program "Amadeus" (for the Mac). And of course, these skills also show through in much of his mathematical work, and through wavelets the two are not entirely unrelated. One should note that his work on Amadeus started well before he entered university.

But, of course, this report is about *noise* not *music*; we want to explain what he got the Fields medal for, which is his spectacular recent work on non-linear stochastic partial differential equations. A good representative example is the one dimensional Kardar-Parisi-Zhang (KPZ) equation, which serves as a canonical physical model for the motion of an interface between a stable state expanding into a metastable state in two dimensional systems (one space, one time).

If both states are stable, the interface, which we assume is given by a height function $h(t,x)$, is driven by two main effects, relaxation, modeled by a heat flow, and a driving noise $\xi(t,x)$ which is idealized to be uncorrelated in space and time, *i.e.*, Gaussian white noise. The equation of motion is then the Langevin equation,

$$\partial_t h = \partial_x^2 h + \xi \,. \tag{1}$$

This is the *Edwards-Anderson model*, and although the driving noise is very singular, it is rather easy to make sense of it. If P is the heat kernel, the solution to (1) could really be nothing but

$$h = P \star \mathbf{1}_{t>0}\xi + Ph_0. \tag{2}$$

Given the initial condition, this is just a Gaussian variable with mean zero and covariance $\langle h(t,x); h(t',x')\rangle = \int_0^{\min t,t'} \int_{-\infty}^{\infty} P(t-s,x-y)P(t'-s,x'-y)\mathrm{d}y\mathrm{d}s$, and from this it is not hard to see that for fixed $t > 0$ it is locally Brownian in space. Furthermore, the result is stable in the sense that if we took smooth approximations $\xi_\varepsilon(t,x)$ to our space-time white noise, perhaps by convolving with a kernel of width ε, then the resulting solutions $h_\varepsilon(t,x)$ to the approximating equation, $\partial_t h_\varepsilon = \partial_x^2 h_\varepsilon + \xi_\varepsilon$, would converge to our $h(t,x)$. This remains true for any reasonable regularization. Equation (1) is self-similar under the $-1:2:4$ rescaling

$$h(t,x) \mapsto N^{-1}h(N^2x, N^4t), \tag{3}$$

which can be interpreted as saying that an initially nice interface will develop fluctuations of size $t^{1/4}$ by time t.

When, instead, one of the states is metastable, one modifies (1) by adding a nonlinear drift, which reasonably should only depend on $\partial_x h$ instead of the frame h:

$$\partial_t h = F(\partial_x h) + \partial_x^2 h + \xi\,. \tag{4}$$

Expanding

$$F(\partial_x h) = F(0) + F'(0)\partial_x h + \tfrac{1}{2}F''(0)(\partial_x h)^2 + \cdots, \tag{5}$$

the first two terms can be removed by simple changes of coordinates, and this leads to the celebrated KPZ equation

$$\partial_t h = \lambda(\partial_x h)^2 + \partial_x^2 h + \xi\,. \tag{6}$$

It is here that the puzzling difficulty appears which was solved by Hairer. For each $t > 0$, $h(t,x)$ is locally Brownian in the x variable, but we are asked to differentiate it (locally white noise) and then square the result! So it is a real challenge to make sense of (6), as well as of (5).

It would not matter so much if (6), as well as its cousins, such as the continuous parabolic Anderson model in 2 and 3 dimensions

$$\partial_t u = \Delta u + u\eta \,, \tag{7}$$

where η is white noise in space only, were not so important. Another example is the dynamic Φ^4 model in 3 dimensions,

$$\partial_t \Phi = \Delta \Phi + \Phi^3 + \xi \,. \tag{8}$$

Each of these equations is a model for non-Gaussian fluctuations for its own, large, universality class (for (8), think of the fluctuation field of the Glauber dynamics for the critical Ising model). However, all these problems sat well outside the existing theory of stochastic partial differential equations.

The one dimensional KPZ has, in addition, deep connections to integrable systems, which have led to a series of remarkable exact solutions [2]. Unlike (1), it is not scale invariant. Instead, it converges to (1) for large N in the $-1:2:4$ scaling (3) but also to a non-trivial *KPZ fixed point* in the $-1:2:3$ scaling

$$h(t,x) \mapsto N^{-1} h(N^2 x, N^3 t). \tag{9}$$

Now the interface at time t has locally Brownian fluctuations of size $t^{1/3}$. About the fixed point rather little is known: What we have, from exactly solvable models in the universality class, are several self-similar solutions, the so-called *Airy processes*. The KPZ equation also arises as universal *weakly asymmetric* or *intermediate disorder limits* of models which have a non-linearity or noise of size N^{-1} in the $-1:2:4$ large N scaling limit (3). This scaling steers them towards the EW fixed point (1), but at the last second they bifurcate to follow the KPZ equation (6). Proofs are available in special cases, by exponentiating, and showing convergence towards the solution of the well-posed multiplicative stochastic heat equation,

$$\partial_t Z = \partial_x^2 Z + \xi Z. \tag{10}$$

The *Hopf-Cole solution*,

$$h(t,x) = \log Z(t,x) \tag{11}$$

is the true solution of (6). Unfortunately, the trick works only when the exponentiated height function happens to satisfy a nice discrete version of (10).

Now suppose we want to make some sense of the passage from (5) to (6) as a weakly asymmetric limit. We start with

$$\partial_t h = N^{-1} F(\partial_x h) + \nu \partial_x^2 h + \zeta \,. \tag{12}$$

where F is an arbitrary even function, and ζ is a smooth Gaussian random field in space and time, with finite range correlations. After the $-1:4:2$ scaling (3), it should converge to the KPZ equation (6). But it is clear that in this case the approximation to (10) will be horrible to deal with, and this emphasizes why one really needs to give intrinsic meaning to the KPZ equation (6) itself. For other non-linear stochastic equations, one does not have analogous tricks.

We start with the integral form of the equation, which in the case of KPZ (6) would be

$$h = P \star \mathbf{1}_{t>0}((\partial_x h)^2 + \xi) + P h_0 \,. \tag{13}$$

Of course, the problem is to find a consistent way to make sense of a non-linear function of a distribution. For technical reasons, one works on a finite interval with periodic boundary conditions. Now of course, there is a classical solution map $(\xi, h_0) \mapsto h(t,x)$ which, unfortunately, is only defined for *nice* inputs ξ and h_0. We would like to take the first to be a smooth approximation ξ_ε to our *very not nice* noise ξ, and somehow take a limit of the resulting solution $h_\varepsilon(t,x)$. This turns out to be too much to ask. However, it is true that the solution $h_\varepsilon(t,x)$ to an appropriate renormalized equation does converge. In the KPZ case, it is achieved simply by subtracting an appropriate constant C_ε;

$$h_\varepsilon = P \star \mathbf{1}_{t>0}((\partial_x h_\varepsilon)^2 - C_\varepsilon + \xi_\varepsilon) + P h_0. \tag{14}$$

Hairer's main result is that there is a (random) $h(t,x)$ such that $h_\varepsilon \to h$, in probability, locally uniformly as continuous functions, that it coincides with the Hopf-Cole solution (11), and that it would be the same if we used any other reasonable regularization procedure. The result is analogous for other stochastic partial differential equations such as (7) and (8), except that one may have to act on the equation by a renormalization with a finite number of parameters, the *renormalization group*, $\mathfrak{R}$. The key condition is *local subcriticality*, which roughly means that when you zoom into the equation, the non-linearities go away.

His basic tool is the notion of *regularity structures*. This is an abstract vector space with enough information to provide a local description of the solution. It should contain abstract polynomials in the time and space variables, as well as a symbol Ξ representing the input noise, rules for multiplication, and abstract versions $\mathscr{P}$ of convolution with the heat kernel and $\mathscr{D}$ of differentiation, and transfer rules for moving from a description at one point, to another. The symbols all come with homogeneities, which in the case of monomials is just their degree. But in parabolic space-time, and because the homogeneity of Ξ is something just below $-3/2$, there have to be terms of negative homogeneity. The abstract fixed point problem is

$$H = \mathscr{P}((\mathscr{D}H)^2 + \Xi) + Ph_0. \tag{15}$$

But this is too abstract: A *model* on the regularity structure is a map to distributions which turns these homogeneities into concrete analytic behavior. And it is *admissible* if it plays nicely with the multiplication rules as well as the operators $\mathscr{P}$ and $\mathscr{D}$. It is in the latter that the highly non-trivial structure arises.

Once one has an admissible model, one can put a metric on the functions which take values in the regularity structure, generalizing the classical Hölder spaces, and solve the fixed point problem in the resulting space. Furthermore, one can build a *reconstruction operator* which pastes together the expansions at different points, so that the solution is realized as a genuine distribution, which, if the model $\Pi^{(\varepsilon)}$ is the one we built from a smooth function ξ_ε, is just the classical solution to the original equation. A version of the renormalization group $\mathfrak{R}$ acts on the set of admissible models, so if we can find $M_\varepsilon \in \mathfrak{R}$ so that

$\Pi^{(\varepsilon)} M_\varepsilon$ converges, the limit is our answer. And the renormalized equations are just what we get by translating back through the M_ε. Now it turns out that in order to prove convergence one only has to check it on the terms of negative homogeneity, and local sub-criticality means that there are only a finite number of these multilinear transformations of the noise (super-renormalizability).

Thus the entire problem is reduced to finding the correct renormalization (usually by educated guesswork) and then proving the corresponding finite collection of multilinear transformations of the noise converges.

If the smooth approximating noises are Gaussian, we are in a situation of Gaussian chaoses and there are easy criteria for convergence based on L^2 norms. These can be very complicated generalized convolutions of the various kernels in the problem, and convergence depends on a careful counting of singularities. This convergence problem is highly reminiscent of earlier calculations in constructive quantum field theory, and a similar diagrammatic method is used to keep track of the cumbersome computations.

Up to this point we have only described some well-posedness results. But the method gives an analogue of the Itô calculus (the Hairer calculus!) for such equations, and an extremely powerful tool for approximations. For example, by adding appropriate operators to the KPZ regularity structure, one can prove [10] that for any even polynomial F, the $-1:4:2$ scaling (3) of (12) converges to the standard quadratic KPZ equation (6), but with a λ which is *not* $\frac{1}{2}F''(0)$, but

$$\lambda = \tfrac{1}{2}\partial_\varrho^2 \bar{F}|_{\varrho=0} \tag{16}$$

where $\bar{F}(\varrho)$ is the expectation of F with respect to a Gaussian with mean ρ.

The KPZ equation has a rich mathematical structure which has now led to a number of deep mathematical works. Besides Hairer's regularity structures, one should also mention the MacDonald Processes of Borodin and Corwin [1], part of a large scale attempt to get at the source of the integrability of the equation, and explain how the random matrix distributions are coming in. One looks forward to similar developments in the case of (7) and (8). The KPZ fixed point and stochastic Navier-Stokes equations also stand as highly non-trivial field theories awaiting our efforts.

Our heartfelt congratulations to Martin! We are already looking forward to your coming successes.