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Hypercontractivity for Anharmonic Oscillators

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with an appendix by

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For suitable $V(x)$ the operator $-\Delta + V(x)$ on $L^2(\mathbb{R}^n, dx)$ is equivalent to the Dirichlet form $(\text{grad}\,\cdot\,, \text{grad}\,\cdot\,)_{\mathscr{H}}$ on $\mathscr{H} = L^2(\mathbb{R}^n, w^2\,dx)$ where w is the ground state wavefunction of $-\Delta + V$. The operator associated with this form is shown to be the generator of a hypercontractive semigroup on the spaces $L^p(\mathbb{R}^n, w^2\,dx)$.

We consider on $L^2(\mathbb{R}^n, dx)$ the quadratic form H_Q defined by $-\Delta + V$, where V is multiplication by a real function $V(x)$ satisfying suitable conditions which will be given below. The form H_Q is shown to give rise to a unique selfadjoint operator H which is bounded below. It has a unique ground state, whose energy is E, and whose eigenspace is spanned by a positive wave function w. Multiplication by w^{-1} is a unitary map U from $L^2(\mathbb{R}^n, dx)$ to $L^2(\mathbb{R}^n, w^2\,dx)$ and one can show by partial integration (cf. Section 3) that UHU^* is equal to an operator $G + E1$ on $L^2(\mathbb{R}^n, w^2\,dx)$ whose quadratic form equals

$$(f, Gg)_{L^2(\mathbb{R}^n, w^2 dx)} \qquad (\text{grad}\,f, \text{grad}\,g)_{L^2(\mathbb{R}^n, w^2 dx)} = (fw, H - E)\,gw)_{L^2(\mathbb{R}^n, dx)}\,.$$

The aim of this paper is to show that the semigroup $\exp(-tG)$, $t \geqslant 0$, is hypercontractive on the spaces $L^p(\mathbb{R}^n, w^2\,dx)$; i.e., for every q, $1 < p \leqslant q < \infty$, there exists a finite T such that for all $t \geqslant T$, $\exp(-tG)$ is a bounded map from $L^p(\mathbb{R}^n, w^2\,dx)$ to $L^q(\mathbb{R}^n, w^2\,dx)$.

The property for a selfadjoint operator to be the generator of a

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hypercontractive semigroup has been first observed for the operator $N = -\Delta + x^2/4 - n/2$ on $L^2(\mathbb{R}^n, dx)$, which has the ground state $w = (2\pi)^{-n/4} \exp(-x^2/4)$. In the limit where $n \to \infty$, this operator is equivalent to the number operator in Fock space. Proofs of the hypercontractivity of $\exp(-tN)$, $t \geqslant 0$, have been given by Nelson [8], Glimm [4], Segal [10], and Simon and Hoegh-Krohn [11], and this result has been extended to more general second quantized operators in [10] and [11]. A new proof with better bounds has then been given by Nelson [9]. These different proofs have in common that they either use the explicit form of the kernel of $\exp(-tN)$ or the explicit form of the eigenfunctions of N.

Recently, Gross showed in [5] that the projection B onto the complement of the constant functions in $L^2(\{1, -1\}, d\mu)$ is the generator of a hypercontractive semigroup on the spaces $L^p(\{1, -1\}, d\mu)$. Here, μ is the measure which assigns weight $\frac{1}{2}$ to each of the two points. The operator B is the analog of the number operator N for a one degree of freedom Fermion system (B counts the total spin). Combining a refined version of the results obtained in [5] with the central limit theorem and by showing an equivalence between "logarithmic Sobolev inequalities" for an operator and hypercontractivity of the corresponding semigroup, Gross was able to give a very elegant proof of the hypercontractivity of $\exp(-tN)$ with sharp bounds [6]. His results give the necessary insight to understand why potentials V which are much more general than $x^2/4$ give rise to an operator $-\Delta + V$ which is the generator of a hypercontractive semigroup in the representation in which the ground state of $-\Delta + V$ equals the constant function one. In fact, the connection between logarithmic Sobolev inequalities and hypercontractivity is the tool by which the hypercontractivity result of this paper is derived from a detailed study of the ground state of $-\Delta + V$.

Guerra, Rosen and Simon [7] have pointed out how hypercontractivity of the semigroup generated by the Hamiltonian of $P(\phi)_2$ field theories plus the Markov property implies a decay of correlations and hence a mass gap. The result of this paper can then also be viewed as a modest step towards showing that the perturbed measure in the $P(\phi)_2$ theory gives rise to hypercontractivity without using results on the mass gap.

This paper is organized as follows: In Section 1 we define the class of potentials for which the main result holds, and we collect the known results which show H_Q has the properties mentioned above. Section 2 contains a detailed analysis of the ground state which is the main technical input for the hypercontractivity proof of Section 3.

1. Self-Adjointness and the Ground State

This section is a collection of known results which we need to establish the existence of $-\Delta + V$ as a selfadjoint operator with a unique ground state, spanned by a positive function. We define now the class of potentials with which we work. We do not claim that this class is the most general for which the final Theorem 3.4 holds, but we think it is general enough to give an insight into what is essential for Theorem 3.4 to hold.

Definition 1.1. A function V on $\mathbb{R}^n/\{0\}$ is in $\mathscr{V}^{(n)}$ iff

(1) $V(x) = V_R(|x|)$;

(2) V_R is real, $V_R \in C^\infty((0, \infty))$;

(3) (Behavior at the origin)

- (a) The negative part[1] V_- of V is in $L^p(\mathbb{R}^n, dx)$ for some $p > n/2$ and $p > 1$,
- (b) $\lim_{r \to 0} \sup V_R(r) < \infty$,
- (c) For small enough r, V_R is monotone.

(4) (Behavior at infinity)

- (a) For large r, $V_R(r)$ is positive,
- (b) For large r, $V_R'(r)/V_R^{1/2}(r)$ is positive and bounded away from zero,
- (c) The quantity $V_R'(r)/V_R(r)$ is uniformly bounded for large r.

(5) V_R has a finite number of zeros in $(0, \infty)$.

$\mathscr{V}^{(n)}$ contains typically radially symmetric potential wells with a local singularity of at most $-|x|^{-1+\epsilon}$ (resp. $-|x|^{-2+\epsilon}$), $\epsilon > 0$ at the origin if $n = 1$ (resp. $n \geqslant 2$), and growing at least quadratically at infinity but not too fast.

Lemma 1.2. *If $V \in \mathscr{V}^{(n)}$, then the form sum $-\Delta + V$ is selfadjoint and bounded below as an operator on $L^2(\mathbb{R}^n, dx)$.*

Proof. $V \in \mathscr{V}^{(n)}$ is a special case of the assumption of the following theorem.

Theorem 1.3. (Faris [2], Prop. 6.5). *Let $\mathscr{H} = L^2(\mathbb{R}^n, dx)$. Let V be a real function on $\mathbb{R}^n$. Write $V = V_+ - V_-$, where $V_\pm \geqslant 0$.*

[1] $V_-(x) = V(x)$ if $V(x) \leqslant 0$, zero otherwise.

Assume that for $p > n/2$, $p > 1$, $V_- \in L^p(\mathbb{R}^n)$ and V_+ is locally in L_1. Then the form sum $-\Delta + V$ is self-adjoint and bounded below.

Lemma 1.2 follows now at once since the condition on V_- is (3a) of Definition 1.1, and V_+ is locally in L_1 since it is a continuous function on $\mathbb{R}^n$ by (1) and (2). We next show the existence of a ground state.

LEMMA 1.4. *If $V \in \mathscr{V}^{(n)}$, the operator $-\Delta + V$ has a ground state in $L^2(\mathbb{R}^n, dx)$.*

Proof. It is wellknown (cf. Appendix) that if $H = -\Delta + V$ has a ground state, it is rotationally symmetric. It suffices therefore to look for a ground state of H restricted to the rotationally symmetric subspace R of $L^2(\mathbb{R}^n, dx)$. In fact we shall show that $H|_R$ has discrete spectrum so that the assertion follows because H is lower bounded. Now $H|_R$ is unitarily equivalent to a multiple of

$$H_R = -\left(\frac{d}{dr}\right)^2 - \frac{n-1}{r}\frac{d}{dr} + V_R \text{ on } L^2([0, \infty), r^{n-1}\,dr)$$

and to a multiple of

$$H_R^1 = -\left(\frac{d}{dr}\right)^2 + \frac{n-1}{2}\frac{n-3}{2}\cdot\frac{1}{r^2} + V_R \text{ on } L^2([0, \infty), dr), \quad (1)$$

and we show that H_R^1 has discrete spectrum. We use Dunford and Schwartz [1] as a reference. By Theorem XIII.7.4 the essential spectrum of H_R^1 is equal to the union of the essential spectra of H_R^1 restricted to the intervals[2] where

$$V_R + \frac{n-1}{2}\frac{n-3}{2}\frac{1}{r^2} = W \text{ is } >0 \quad \text{and} \quad <0.$$

By (5) of Definition 1.1, there are only finitely many such intervals. Let $I_1, \ldots, I_m$ be these intervals $I_1 = (0, a_1]$, $I_k = [a_{k-1}, a_k]$, $k = 2, \ldots, m-1$, $I_m = [a_{m-1}, \infty)$. If $m = 1$ we divide the interval into two pieces to obtain two intervals $I_1 = (0, 1]$, $I_2 = [1, \infty)$. Now $H_R^1|_{I_k}$, $k = 2, \ldots, m-1$ has no essential spectrum, by XIII.7.17(b) of [1]. $H_R^1|_{I_m}$ has no essential spectrum by XIII.7.16(a). The case $H_R^1|_{I_1}$ is divided into two subcases according to whether $n \leqslant 3$ or not. If $n = 1, 2, 3$, then by assumption on V_R, W stays finite or goes to $-\infty$ no faster that $\mathcal{O}(1)\, r^{-2}$ as $r \to 0$. Furthermore, by assumption, W is monotone near $r = 0$ and therefore $H_R^1|_{I_1}$ has no essential spectrum by XIII.10.C22. If $n \geqslant 4$, then W goes to $+\infty$ like $\mathcal{O}(1)\, r^{-2}$

[2] i.e., H_R^1 acting on $L^2(I, dr)$, where I is the interval, we write $H_R^1|_I$.

and in this case $H_R^1 |_{I_1}$ has no essential spectrum by XIII.7.17. This proves the lemma.

LEMMA 1.5. *Let* $V \in \mathscr{V}^{(n)}$. *Then* $-\Delta + V \geqslant c$, *c is a simple eigenvalue whose eigenspace is spanned by a function which is positive almost everywhere.*

Proof. The existence of a ground state is established by Lemma 1.2 and 1.4. The remainder of the Lemma is a rewording of results of Faris [2]. The form domains[3] of $-\Delta$ and V have a dense intersection, namely the C^∞ functions with compact support. We claim that $Q(V_-) \supseteq Q(-\Delta)$. By assumption $V_- \in L^p(\mathbb{R}^n, d^n x)$ and $(-\Delta + c^2)^{-1/2}$, $c > 0$ has as its Fourier transform the function $(k^2 + c^2)^{-1/2}$ which is in $L^{2p}(\mathbb{R}^n, d^n k)$ if $p > n/2$. Therefore, by the Hölder inequality and the Young inequality we have for $f \in L^2(\mathbb{R}^n, dx)$, $\tilde{f}$ its Fourier transform and for large enough c

$$\begin{aligned} \|f\|_2 = \|\tilde{f}\|_2 &> \|V_-\|_p^{1/2} \|(k^2 + c^2)^{-1/2}\|_{2p} \|\tilde{f}\|_2 \\ &\geqslant \| |V_-|^{1/2} \|_{2p} \|(k^2 + c^2)^{-1/2} \tilde{f}\|_{2p/(p+1)} \\ &\geqslant \| |V_-|^{1/2} \|_{2p} \|(-\Delta + c^2)^{-1/2} f\|_{2p/(p-1)} \\ &\geqslant \| |V_-|^{1/2} (-\Delta + c^2)^{-1/2} f\|_2 . \end{aligned}$$

Therefore $Q(V_-) \supseteq Q(-\Delta)$.

We have thus established the hypotheses of Theorem 2.1 in [2], and this shows how the bounds of our Theorem 1.3 [2, Proposition 6.5] come about. We now want to apply Corollary 5.1 of [2]. Its hypotheses are the hypotheses of Theorem 2.1 in [2], which we have verified above, plus the fact that $\exp(t\Delta)$ is positivity preserving for $t \geqslant 0$, and ergodic for $t > 0$ (see [3] for definitions). But these latter facts are well known [3]. Therefore Corollary 5.1 of [2] applies and Lemma 1.5 is proved.

2. AN ANALYSIS OF THE GROUND STATE

So far, we have seen that $-\Delta + V$, $V \in \mathscr{V}^{(n)}$ is a self-adjoint operator, bounded below, with a unique ground state whose wave function is positive a.e. If $V \in \mathscr{V}^{(n)}$, then $V + \text{const.} \in \mathscr{V}^{(n)}$ and

[3] The form domain $Q(A)$ of a self adjoint operator A is the set of all $f \in L^2$ for which $\| |A|^{1/2} f\|_2 < \infty$.

therefore we may assume that the ground state of $-\Delta + V$ has eigenvalue zero. We write the ground state function as

$$w(x) = \exp(-h(x)/4), \qquad \text{with} \qquad \int dx \exp(-h(x)/2) = 1, \quad h(x) \text{ real.} \tag{1}$$

Before we can state our next result, we note that if $V \in \mathscr{V}^{(n)}$, then for all $\lambda > 0$, $\lambda V \in \mathscr{V}^{(n)}$. Consider h as a multiplication operator on $L^2(\mathbb{R}^n, dx)$. The main result of this section and the main technical input of this paper is the following theorem.

THEOREM 2.1. *Given $V \in \mathscr{V}^{(n)}$, with h defined through* (1), *there are constants $\gamma > 1$, and $\delta < \infty$ such that as self-adjoint operators*

$$(\gamma - 1)\left(-\Delta + \frac{\gamma}{\gamma - 1} V\right) - \frac{h}{4} \geqslant \delta.$$

Note. h is associated with the ground state of $-\Delta + V$, *not* with the ground state of $(-\Delta + (\gamma/(\gamma - 1))V)$.

We prove this Theorem as a Corollary of the following two Lemmas.

LEMMA 2.2. *If $V \in \mathscr{V}^{(n)}$, then the ground state is spanned by a positive function which is locally almost everywhere bounded away from zero.*

LEMMA 2.3. *If $V \in \mathscr{V}^{(n)}$, and h is defined through* (1) *then there are constants $\alpha > 4$, r_0 and β such that for all x with $|x| \geqslant r_0$ one has*

$$\alpha V(x) - h(x) \geqslant \beta.$$

Lemma 2.3 describes the behavior of h at infinity and Lemma 2.2 describes its behavior locally. Lemma 2.3 can be intuitively understood via the WKB method, if e. g. $V = x^{2q}$, $q \geqslant 1$, then h is asymptotically given by $\int^x V^{1/2}(\xi)\, d\xi$, i.e., $h(x) \sim x^{q+1}$ and $V(x) = x^{2q}$ which makes the assertion plausible in this case. Lemma 2.2 is just a restatement of the Sturm "oscillation theory"; the ground state has no zeros, the first excited state has one zero, etc. (in one dimension).

Proof of Theorem 2.1. By Lemma 2.3, there exists an r_0, $0 < r_0 < \infty$, such that for all $x \geqslant r_0$, $(\alpha/4)\, V(x) - h(x)/4 \geqslant \beta/4$ with constants $\alpha > 4$, β. Choose $\gamma = \alpha/4$. By Lemma 2.2 h is bounded for $x \leqslant r_0$. Therefore $W = \gamma V - h/4 - \beta/4$ has a negative part in $L^p(\mathbb{R}^n, dx)$ and a positive part locally in L^1 and therefore $-(\gamma - 1)\Delta + W \geqslant \beta'$

by Theorem 1.3, or $(\gamma - 1)(-\Delta + (\gamma/(\gamma - 1))V) - h/4 \geqslant \delta$ for some finite δ, which proves the assertion.

Proof of Lemma 2.2. It is almost obvious that for $x_0 \neq 0$, $w(x_0) > 0$. Indeed suppose for $x_0 \neq 0$, $w(x_0) = 0$, i.e., $w_R(|x_0|) = 0$; then, since $w_R(x) \geqslant 0$ we must have $(\partial/\partial r)\, w_R\,(|x_0|) = 0$. But with these initial conditions, the differential equation (cf. (1.1))

$$\left(-\partial_r^{\,2} + \left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)\frac{1}{r^2} + V_R\right) w_R = 0$$

has the unique solution $w_R(r) = 0$ for all $r \geqslant 0$, i.e., $w(x) = 0$ for all $x \in \mathbb{R}^n$, a contradiction to Lemma 1.5.

The situation at $r = 0$ is more delicate since V_R may be singular at that point. We handle separately the cases $n \neq 2$ and $n = 2$.

Case $n \neq 2$. We discuss the local behavior of the solution at the origin. Consider again $H_R^{\,1}$:

$$H_R^{\,1} u = -u'' + \frac{n-1}{2}\,\frac{n-3}{2}\,\frac{1}{r^2}\, u + V_R u. \tag{2}$$

If $V_R = 0$, the equation $H_R^{\,1} u = 0$ has the solution

$$u(r) = A u_1(r) + B u_2(r),$$

where $u_1(r) = r^{(n-1)/2}$, $u_2(r) = r^{-(n-3)/2}$. Therefore, if $V_R = 0$, the equation $H_R^{\,1} u + g = 0$ has the solution[4]

$$\begin{aligned} u(r) = {} & A u_1(r) + B u_2(r) + (n-2)^{-1} \\ & \times \left\{ u_1(r) \int_0^r d\rho\, u_2(\rho)\, g(\rho) - u_2(r) \int_0^r d\rho\, u_1(\rho)\, g(\rho) \right\}. \end{aligned} \tag{3}$$

By linearity, we may consider separately for $B = 0$ and $A = 0$ the integral equation resulting from (2) and (3) by substituting $g = V_R u$. The first equation is then

$$\begin{aligned} u(r) = {} & A u_1(r) + (n-2)^{-1} \\ & \times \left\{ u_1(r) \int_0^r d\rho\, u_2(\rho)\, u(\rho)\, V_R(\rho) - u_2(r) \int_0^r d\rho\, u_1(\rho)\, u(\rho)\, V_R(\rho) \right\}. \end{aligned} \tag{4}$$

[4] I thank D. Pearson for teaching me how to solve such problems.

Write $u(r) = r^{(n-1)/2}v(r)$, then the equation for v is

$$v(r) = A + (n-2)^{-1}\left\{\int_0^r d\rho\rho V_R(\rho)\,v(\rho) - r^{-(n-2)}\int_0^r d\rho\rho^{n-1}V_R(\rho)\,v(\rho)\right\}$$
$$= A + Kv. \tag{5}$$

LEMMA 2.4. *For small enough* $\epsilon' > 0$, $\| Kv \|_\infty \leqslant \frac{1}{2}\| v \|_\infty$ *on* $L^\infty((0, \epsilon'], dr)$.

We postpone the proof of this Lemma and continue the proof of Lemma 2.2. By Lemma 2.4, the iterated solution of Eq. (5) will converge and therefore there is a solution $u = u_A$ of (4) and hence of (2) which has the property that $\frac{1}{2}Au_1(r) \leqslant u_A(r) \leqslant 2Au_1(r)$ for for $0 \leqslant r \leqslant \epsilon$. It follows that the second solution ($A = 0$, $B \neq 0$), which can be written as $u_A \int^r d\rho\, u_A(\rho)^{-2}$, is of the form $\frac{1}{2}Bu_2(r) \leqslant u_B(r) \leqslant 2Bu_2(r)$ for small enough r.

Any solution w to the original equation $(-\Delta + V)w = 0$ has therefore (near zero) the asymptotic form $w(x) \sim A + B \mid x \mid^{-n+2}$. Therefore, if $n \geqslant 3$ and $w(0) = 0$, then $A = B = 0$, and w must vanish in a neighborhood of zero, by the inequalities on u_A and u_B. But this contradicts Lemma 1.5. So we have shown in this case that $w(0) \neq 0$. If $n = 1$ and $A = 0$ we could construct two ground states according to whether the function takes the same sign for $x > 0$ and $x < 0$, or not, and this is again in contradiction to Lemma 1.5. Therefore $w(0) \neq 0$ in this case.

Case $n = 2$. The basic solutions are now

$$u_1(r) = r^{1/2}, \qquad u_2(r) = r^{1/2}\ln r.$$

Repeat the argument of the case $n \neq 2$ and find $w(x) \sim A + B \ln \mid x \mid$. Therefore $w(0) \neq 0$ unless w vanishes in a neighborhood of 0. This completes the proof of the lemma.

Proof of Lemma 2.3. By definition of h, $(-\Delta + V)\exp(-h/4) = 0$, or, using radial coordinates, $h(x) = h_R(\mid x \mid)$,

$$V_R(r) = -(h_R/4)''(r) - (h_R/4)'(r)\frac{n-1}{r} + (h_R'/4)^2(r), \qquad 0 < r < \infty.$$

Write $p(r) = -(h_R'/4)(r)$. By Definition 1.1 (4a) and (4b), there is an r_1 such that for $r > r_1$, $V_R(r) > 1$. In this region, we write $p(r) = -V_R^{1/2}(r) + q(r)$. The equation for q is

$$q' = 2qV_R^{1/2} - q^2 + (V_R^{1/2})' - q\frac{n-1}{r} + \frac{n-1}{r}V_R^{1/2}. \tag{6}$$

We claim: There exists an $r_2 \geqslant r_1$ and a $b > 1$ such that if $q(r_3) < -b$ for some $r_3 \geqslant r_2$ then $\lim_{y \uparrow c} q(r_3 + y) = -\infty$ for some $0 < c \leqslant 1$.

Proof. Choose $r_2 \geqslant 2(n-1)$ so large that $r_2 \geqslant r_1$ and that $r \geqslant r_2$ is contained in the region in which the bounds of Definition 1.1. (4b) and (4c) apply. Observe that the equation $f' = -f^2$ has the property that if $f(r_0) = -a < 0$, then $f(r) = (r - r_0 - 1/a)^{-1}$ and hence $\lim_{y \uparrow 1/\alpha \epsilon} f(r + y) = -\infty$, and the divergence is like $-\epsilon^{-1}$ at $\epsilon = 0$. Coming back to Eq. (6), we rewrite it as

$$q' = -q^2 + \left\{\frac{q}{2} V_R^{1/2} + (V_R^{1/2})'\right\} + V_R^{1/2} \left\{\frac{q}{2} + \frac{n-1}{r}\right\} + \left\{q\left(V_R^{1/2} - \frac{n-1}{r}\right)\right\}.$$

By comparison with f above, the claim follows now by showing that each of the curly brackets is negative, so that $q(r) \leqslant (r - r_3 - b^{-1})^{-1}$. Now the second curly bracket is negative since $r \geqslant r_2 \geqslant 2(n-1)$ and $q < -1$. The third bracket is negative since $V_R > 1$ and $r \geqslant r_2 \geqslant 2(n-1)$. Finally, the first bracket is negative if b is so large that $V_R'(r)/V_R(r) < b$ for $r \geqslant r_2$, and such a b exists by Definition 1.1 (4c). This completes the proof of the claim.

We return to the proof of Lemma 2.3. If $q(r_3) < -b$ for some $r_3 \geqslant r_2$ then q diverges at least like $-\epsilon^{-1}$ at $\epsilon = 0$ to $-\infty$ for some finite r and hence h_R diverges at least logarithmically to $+\infty$ for some finite r, and this is in contradiction to Lemma 2.2. Therefore we must have that for all $r \geqslant r_2$, $p(r) > -b - V_R^{1/2}(r)$, or $h_R'(r) < 4\{b + V_R^{1/2}(r)\}$. But by Definition 1.1 (4b), there is an r_4 ($> r_2$) and a $c > 0$ such that $V_R'(r) > cV_R^{1/2}(r)$ for $r \geqslant r_4$. Therefore

$$h_R'(r) < 4(b+1)\, c^{-1} V_R'(r) \leqslant \max(5, 4(b+1)\, c^{-1})\, V_R'(r),$$

and hence

$$\alpha V_R(r) - h_R(r) \geqslant \alpha V_R(r_4) - h(r_4), \qquad r \geqslant r_4$$

with $\alpha = \max(5, 4(b+1)\, c^{-1})$, and this proves Lemma 2.3, since $h_R(r_4)$ is finite by Lemma 2.2, and $V_R(r_4)$ is finite by definition.

Proof of Lemma 2.4. We first show: If $V \in L^p(\mathbb{R}^n, dx)$, $p > n/2$, $p > 1$, $V(x) = V_R(|x|)$, then $\int_0^c dr\, |V_R(r)|\, r^{1-\epsilon} < \infty$ for $c < 1$, and some $\epsilon > 0$.

Proof. By hypothesis, $\int_0^c dr\, |V_R(r)|^p\, r^{n-1} < \infty$. The assertion is trivial for $n = 1$. If $n \geqslant 2$, let $\epsilon > 0$ so small that

$$\alpha = (n - 2 + \epsilon p)/(p - 1) < 2,$$

and let $B = \{r \in [0, c) \mid |V_R(r)|\, r^{\alpha-\epsilon} \geqslant 1\}$. Let χ be the characteristic function of B. Then

$$\begin{aligned}\int_0^c dr\,\chi(r)\,|V_R(r)|\,r^{1-\epsilon} &= \int_0^c dr\,\chi(r)\,|V_R(r)|\,r^{\alpha-\epsilon}r^{-\alpha+1}\\ &\leqslant \int_0^c dr\,\chi(r)\,|V_R(r)|^p\,r^{p(\alpha-\epsilon)}r^{-\alpha+1}\\ &= \int_0^c dr\,\chi(r)\,|V_R(r)|^p\,r^{n-1} < \infty.\end{aligned}$$

The assertion follows now by observing that

$$\int_0^c dr(1-\chi(r))\,|V_R(r)|\,r^{1-\epsilon} \leqslant \int_0^c dr\,r^{-\alpha+1-\epsilon} < \infty.$$

Now if $n \neq 2$, the first term in the curly bracket of Eq. (5) is obviously contractive on L^∞ by the above and Definition 1.1 (3a). In the second term, we bound

$$\left| r^{-n+2}\int_0^r d\rho\rho^{n-1}V_R(\rho)\,v(\rho)\right| \leqslant r^{-n+2}r^{n-2}\int_0^r d\rho\rho\,|V_R(\rho)|\,\|v\|_\infty,$$

and the lemma follows in this case.
If $n = 2$, the corresponding term is

$$1\cdot\left\{\int_0^r d\rho\rho\ln\rho v(\rho)\,V_R(\rho) - \ln r\int_0^r d\rho\rho v(\rho)\,V_R(\rho)\right\}.$$

The assertion follows now in the same way as before, by making use of the power $r^{-\epsilon}$ in $\int_0^c dr\,|V_R(r)|\,r^{1-\epsilon} < \infty$.

3. Hypercontractivity

We now show that logarithmic Sobolev inequalities [6] hold for the Dirichlet form on the space with probability measure $\exp(-h(x)/2)\,dx$. This is our main result and from it one can easily derive the hypercontractivity of the corresponding semigroup, using a slight variant of the powerful results of Gross [6].

THEOREM 3.1. *Let $V \in \mathscr{V}^{(n)}$ and let h be defined as above. Then* *there are constants $\gamma > 1$ and $\delta < \infty$ such that for all functions f with bounded second derivatives*

$$\gamma \int |\nabla f|^2 e^{-h/2} d^n x$$
$$\geqslant \int |f|^2 \ln |f| e^{-h/2} d^n x - \frac{1}{2} \int |f|^2 e^{-h/2} d^n x \cdot \ln \left(\int |f|^2 e^{-h/2} d^n x \right)$$
$$- \delta \int |f|^2 e^{-h/2} d^n x. \tag{1}$$

Proof. We use Gross' fundamental logarithmic Sobolev inequality [6, Corollary 4.2]

$$(2\pi)^{-n/2} \int |\nabla f|^2 e^{-x^2/2} d^n x$$
$$\geqslant (2\pi)^{-n/2} \int |f|^2 \ln |f| e^{-x^2/2} d^n x$$
$$- \tfrac{1}{2}(2\pi)^{-n/2} \int |f|^2 e^{-x^2/2} d^n x \cdot \ln \left((2\pi)^{-n/2} \int |f|^2 e^{-x^2/2} d^n x \right), \tag{2}$$

for bounded $f \in D$ (functions with bdd. second derivative). Put $f(x) = g(x) \exp(x^2/4)$, and use

$$\int |\nabla(g e^{x^2/4})|^2 e^{-x^2/2} d^n x$$
$$= \int |\nabla g|^2 d^n x + \int \frac{x}{2} \nabla |g|^2 d^n x + \int |g|^2 x^2/4 \, d^n x$$
$$= \int |\nabla g|^2 d^n x - \frac{n}{2} \int |g|^2 d^n x + \int |g|^2 x^2/4 \, d^n x.$$

Thus (2) is equivalent to

$$\int |\nabla g|^2 d^n x \geqslant \int |g|^2 \ln |g| \, d^n x - \frac{1}{2} \int |g|^2 d^n x \ln \left(\int |g|^2 d^n x \right)$$
$$+ \frac{n}{2}(1 + \tfrac{1}{2} \ln 2\pi) \int |g|^2 d^n x. \tag{3}$$

By a similar procedure, (1) can be transformed by substituting $f(x) = g(x) \exp(h(x)/4)$ and by using

$$\int |\nabla(g e^{h/4})|^2 e^{-h/2} d^n x$$
$$= \int |\nabla g|^2 d^n x + \frac{1}{4} \int (\nabla |g|^2) \cdot (\nabla h) \, d^n x + \frac{1}{16} \int |g|^2 (\nabla h)^2 d^n x$$
$$= \int |\nabla g|^2 d^n x - \frac{1}{4} \int |g|^2 (\Delta h) \, d^n x + \frac{1}{16} \int |g|^2 (\nabla h)^2 d^n x.$$

Therefore (1) is equivalent to showing

$$\gamma \int |\nabla g|^2 \, d^n x - \frac{\gamma}{4} \int |g|^2 (\Delta h) \, d^n x + \frac{\gamma}{16} \int |g|^2 (\nabla h)^2 \, d^n x$$
$$\geqslant \int |g|^2 \ln |g| \, d^n x - \frac{1}{2} \int |g|^2 \, d^n x \ln \left(\int |g|^2 \, d^n x \right)$$
$$+ 1/4 \int |g|^2 h \, d^n x - \delta \int |g|^2 \, d^n x.$$

In other words, it suffices to show, by (3), that

$$-(\gamma - 1) \int \bar{g}(\Delta g) \, d^n x - \gamma \int |g|^2 \left(\Delta \frac{h}{4} \right) d^n x + \gamma \int |g|^2 \left(\frac{\nabla h}{4} \right)^2 d^n x$$
$$+ \delta \int |g|^2 \, d^n x - \frac{1}{4} \int |g|^2 h \, d^n x \geqslant 0$$

or

$$-(\gamma - 1)\Delta - \gamma \left(\Delta \frac{h}{4} \right) + \gamma \left(\frac{\nabla h}{4} \right)^2 + \delta - \frac{h}{4} \geqslant 0.$$

But $(-\Delta(h/4) + (\nabla h/4)^2 = V$, so that we have to show the existence of γ and δ so that

$$+(\gamma - 1) \left(-\Delta + \frac{\gamma}{\gamma - 1} V \right) + \delta - h/4 \geqslant 0.$$

But this is true by Theorem 2.1. Theorem 3.1 is proven. We now proceed to the hypercontractivity proof.

We need the following slight generalization of the beautiful Theorem 6 in [6].

THEOREM 3.2. *Let μ be a probability measure on $\mathbb{R}^n$ and let G be a self-adjoint operator on $L^2(\mu)$. Suppose that the set D of twice continuously differentiable functions with bounded first and second derivatives is a core for G and that*

$$\langle Gf, g \rangle_{L^2(\mu)} = \int_{\mathbb{R}^n} \operatorname{grad} \bar{f}(x) \operatorname{grad} g(x) \, d\mu(x), f, g \in D.$$

If there exist constants $\gamma > 0$ and $\delta < \infty$ such that

$$\int_{\mathbb{R}^n} |f|^2 \ln |f| \, d\mu \leqslant \gamma \langle Gf, f \rangle + \|f\|_2^2 \ln \|f\|_2 + \delta \|f\|_2^2,$$

then

$$\| e^{-tG} \|_{p,1+(p-1)\exp(2t/\gamma)} \leqslant e^{td(p)} \qquad \text{with} \qquad d(p) = \delta \min(1, p-1).$$

(The generalization is allowing for a constant $\delta \neq 0$).

Proof. The proof is really only a copy of Gross' proof, with necessary modifications added.

LEMMA 3.3. Gross [6, Lemma 6.1]. *If μ is a probability measure on $\mathbb{R}^n$ and if for some $\gamma > 0$, $\delta < \infty$,*

$$\int_{\mathbb{R}^n} |f|^2 \ln |f| \, d\mu < \gamma \int_{\mathbb{R}^n} |\operatorname{grad} f|^2 \, d\mu + \int \|f\|_2^2 \ln \|f\|_2 + \delta \|f\|_2^2 \tag{4}$$

for all $f \in C^1$, then for all p, $1 < p < \infty$

$$\int_{\mathbb{R}^n} |f|^p \ln |f| \, d\mu \leqslant \frac{\gamma}{2} \frac{p}{p-1} \operatorname{Re} \int_{\mathbb{R}^n} (\overline{\operatorname{grad} f}) \cdot (\operatorname{grad} f_p) \, d\mu + \|f\|_p^p \ln \|f\|_p + d(p) \|f\|_p^p, \tag{5}$$

where $f_p = \operatorname{sign} f \cdot |f|^{p-1}$,

$$d(p) = \begin{cases} \delta & \text{if } p \geqslant 2, \\ (p-1)\delta & \text{if } p \leqslant 2. \end{cases}$$

Proof. Gross shows first for $p > 2$ ([5, Eq. (4.8)])

$$\int |\operatorname{grad} |f|^{p/2}|^2 \, d\mu \leqslant \left(\frac{p}{2}\right)^2 \frac{1}{p-1} \int \operatorname{Re} \operatorname{grad} f \cdot \operatorname{grad} \bar{f}_p \, d\mu,$$

$$\int |f|^p \ln |f|^{p/2} \, d\mu = \frac{p}{2} \int |f|^p \ln |f| \, d\mu.$$

and

$$\| |f|^{p/2} \|_2 = \|f\|_p^{p/2}.$$

Now replace f in (4) by $|f|^{p/2}$ (which is also in C^1 if f is), and (5) follows at once. If $1 < q < 2$, set $p = q/(q-1)$, and for $h \in C^1$ set $f = h_q$. Then $p > 2$, $f \in C^1$, $|f|^p = |h|^q$, $\|f\|_p^p = \|h\|_q^q$, $h = f_p$. Replace f by h_q in (5), we get

$$\begin{aligned} \frac{p}{q} \int |f|^p \ln |f| \, d\mu &= \int |h|^q \ln |h| \, d\mu \\ &\leqslant \frac{p}{q} \cdot \frac{\gamma}{2} \frac{p}{p-1} \operatorname{Re} \int \overline{\operatorname{grad} h} \cdot \operatorname{grad} h_q \, d\mu \\ &\quad + \frac{p}{q} \cdot \frac{q}{p} \|h\|_q^q \ln \|h\|_q + \frac{p}{q} \delta \|h\|_q^q. \end{aligned}$$

This proves Lemma 3.3.

The proof of Theorem 3.2 is now word by word the same as that of Theorem 6 in [6], with his H_0 replaced by $G + d(p)$.

We are now in a position to combine all these results into our main result.

Let $V \in \mathscr{V}^{(n)}$, let $H = -\Delta + V$. Then H is self-adjoint, has a unique ground state spanned by a wave function w which is locally bounded away form zero, $w(x) = e^{-h(x)/4}$, normalized, with energy E by results of Sections 1 and 2. H acting on $L^2(\mathbb{R}^n, dx)$ is unitarily equivalent to an operator $G + E1$ acting on $L^2(\mathbb{R}^n, w^2\, dx)$.

THEOREM 3.4. *The quadratic form associated with G is*

$$(f, Gg)_{L^2(\mathbb{R}^n, w^2 dx)} = \int_{\mathbb{R}^n} \overline{\operatorname{grad} f} \cdot \operatorname{grad} g\, w^2\, dx.$$

The semigroup $\exp(-tG)$, $t \leqslant 0$ *is hypercontractive on the spaces* $L^p(\mathbb{R}^n, w^2\, dx)$ *and there are constants* $0 < c < \infty$, $0 < d < \infty$ *such that*

$$\| \exp(-tG)\|_{p, 1+(p-1)\exp(2t/c)} \leqslant e^{t(d\cdot \min(1, p-1) - E)}.$$

for $t \geqslant 0$, $\infty > p > 1$. *Also* $\exp(-tG)$ *is a contraction on all* L^p, $1 \leqslant p \leqslant \infty$.

Proof. In the representation on $L^2(\mathbb{R}^n, e^{-h/2}\, dx)$, $H \sim G +$ const where G is defined in Theorem 3.2. This fact is already observed by Gross [6], but we repeat its easy proof for convenience (for $n = 1$, $E = 0$). Namely one has

$$\begin{aligned}
(fe^{-h/4}, (-\Delta + V) ge^{-h/4})_{L^2(dx)} &= (fe^{-h/4}, [-\Delta, g]\, e^{-h/4})_{L_2(dx)} \\
&= \int (-g'' + 2g'h'/4) \bar{f} e^{-h/2}\, dx \\
&= \int g'(\bar{f} e^{-h/2})' + g'(h'/2)\bar{f} e^{-h/2}\, dx \\
&= \int g' \bar{f}' e^{-h/2}\, dx.
\end{aligned}$$

We have to know that D, the set of C^2 functions with bounded derivatives, is a core for G, i.e., that the functions $De^{-h/4}$ are a core for H in order to be able to apply Theorem 3.2. This is shown by D. Pearson in the appendix. The inequality in the hypothesis of Theorem 3.2. holds by Theorem 3.1 and thus the conclusion to Theorem 3.2 holds in our case. This proves the Theorem up to the last affirmation, which is standard [11, Proposition 2.1 and 2.2].

APPENDIX BY D. PEARSON

LEMMA. *Let $V \in \mathscr{V}^{(n)}$, then C^∞ functions with compact support in $\mathbb{R}^n/\{0\}$, together with the ground state wave function, span a core for $-\Delta + V$.*

Proof. We have to show that there is no non-trivial element f in $L^2(\mathbb{R}^n, dx)$ such that $(-\Delta\Phi + V\Phi + i\Phi, f) = 0$ for all C^∞ functions Φ with compact support in $\mathbb{R}^n - \{0\}$ and such that

$$(-\Delta w + Vw + iw, f) = 0,$$

where w is the ground state function of $-\Delta + V$, and that the same result holds with $+i$ replaced by $-i$.

Now $-\Delta = -\partial_r^2 - ((n-1)/r)\,\partial_r - (1/r)\,\Delta_s$, where Δ_s is the spherical Laplacian acting in $L^2(S^{n-1})$.

Δ_s has eigenvalues $l(l+n-2)$, $l = 0, 1, 2,\ldots$ and has a sequence of orthogonal eigenfunctions which span $L^2(S^{n-1})$. (For $n = 3$ the eigenfunctions may be taken to be $Y_{lm}(\theta, \phi)$). Corresponding to each eigenfunction is a subspace of $L^2(\mathbb{R}^n)$ which reduces $-\Delta$, and such that the restriction of $-\Delta$ to this subspace is unitarily equivalent to

$$-\partial_r^2 - \frac{n-1}{r}\,\partial_r + \frac{l(l+n-2)}{r^2} \text{ on } L^2([0,\infty), r^{n-1}\,dr)$$

and to

$$-\partial_r^2 + \left(\frac{n^2}{4} - n + 3/4\right)\frac{1}{r^2} + l(l+n-2)\,\frac{1}{r^2} \text{ on } L^2([0,\infty), dr).$$

If we choose Φ to belong to one of these subspaces, and let $f_l(r) \in L^2([0,\infty), dr)$ be obtained by projecting f onto the same subspace, we have

$$\left(\left(-\partial_r^2 + \left(\frac{n^2}{4} - n + 3/4\right)\frac{1}{r^2} + l(l+n-2)\,\frac{1}{r^2} + V_R + i\right)\Phi, f_l\right) = 0.$$

for all C^∞ functions $\Phi(r)$ having compact support in $(0, \infty)$.

This means that for $r > 0$, $f_l(r)$ satisfies the equation

$$\frac{-d^2 f_l}{dr^2} + \left(\frac{n^2}{4} - n + 3/4 + l(l+n-2)\right) f_l/r^2 + V_R f_l - i f_l = 0. \qquad \text{(A1)}$$

(This equation is a priori satisfied in the sense of distributions, but since V_R is locally square integrable $-d^2 f_l/dr^2$ is a regular distribution and both f_l and (df_l/dr) are locally absolutely continuous.)

For $l > 0$ by estimating the behavior of solutions of (A1) near $r = 0$ as in the proof of Lemma 2.2 one finds that we have the limit point case at the origin (cf. [1, p. 1306]), and since we also have the limit point case at infinity, Eq. (A1) has no nontrivial solution in $L^2([0, \infty))$. Hence

$$f_l(r) = 0 \qquad \text{for} \quad l \neq 0.$$

For $l = 0$, one also has the limit point case at $r = 0$ for $n \geqslant 4$, in which case we have $f_0(r) \equiv 0$, so that $f(r) \equiv 0$ and the conclusion of the lemma follows. (So for $n \geqslant 4$ we have a core even without including the ground state wave function.)

For $n = 1, 3$ we have

$$((-\partial_r^2 + V_R + i)w, f_0) = 0. \tag{A2}$$

Now w satisfies at $r = 0$ a boundary condition of the form $aw(0) = bw'(0)$, (radial derivative), and if we use Eq. (A2) we find on integrating by parts that $\lim_{r\to 0} (w'f_0 - wf_0') = 0$, so that $f_0(r)$ satisfies the same boundary condition.

We also have on integrating by parts

$$((-\partial_r^2 + V_R)f_0, f_0) - (f_0, (-\partial_r^2 + V_R)f_0) = \lim_{r\to 0} (f_0'\bar{f}_0 - f_0\bar{f}_0') = 0$$

since a and b are real. Therefore $-2i(f_0, f_0) = 0$, so that $f_0 \equiv 0$. The same argument applies in the case $n = 2$ if we replace a and b by the constants determining the behavior of the ground state

$$w(r) \approx r^{1/2}(a + b \log r), \quad \text{near} \quad r = 0.$$

We have now shown in all cases that $f \equiv 0$, and this completes the proof of the Lemma.

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