Non-Vanishing Profiles for the Kuramoto-Sivashinsky Equation on the Infinite Line

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Abstract

We show that the Kuramoto-Sivashinsky equation on the infinite line has solutions with arbitrarily large limits at $x = \pm\infty$ and which vanish at the origin. This implies that an attractor for this equation cannot be defined in L^∞. We consider profiles with limits at $x = \pm\infty$, and show that initial conditions L^2-close to such profiles lead to solutions wich stay L^2-close to the profile for all times. The difference between these solutions and the initial profile tends to 0 pointwise as $x \to \pm\infty$. Analogous results hold for L^2-neighborhoods of periodic stationary solutions. This implies that profiles and periodic stationary solutions partition the phase space into mutually unattainable regions.

1 Introduction

The Kuramoto-Sivashinsky equation

$$\dot{\mu}(x,t) = -\mu''''(x,t) - \mu''(x,t) - \tfrac{1}{2}(\mu^2)'(x,t)\ , \qquad \mu(x,0) = \mu_0(x)\ , \tag{1.1}$$

is an interesting model for stabilization mechanisms of very indirect type. It can be considered on a finite interval of length L with periodic boundary conditions (KS_L) or on the infinite line (KS_∞). In both cases, one can formally multiply the equation with μ and integrate, leading to (after integration by parts)

$$\tfrac{1}{2}\partial_t \int \mathrm{d}x\, \mu^2 = \int \mathrm{d}x\ \left(\mu'\right)^2(x,t) - \int \mathrm{d}x\ \left(\mu''\right)^2(x,t) - \tfrac{1}{2}\int \mathrm{d}x\, \mu \cdot (\mu^2)'.$$

Note that the last term vanishes identically since it equals $\int (\mu^3)'/6$ and thus, surprisingly, the non-linearity does not contribute directly to the decay of initial conditions with large L^2 norm, in contrast to equations like the Ginzburg-Landau equation [12] which derive from a potential.[1]

[1]On the other hand, as we shall see, it is precisely this feature which allows for L^2 bounds which grow only exponentially in time.

Clearly, on the other hand, without the non-linear term, the equation is unstable. It has been shown [14, 7, 3, 5] that for finite L, the equation KS_L has an attractor in L^2 as well as in L^∞, whose radius is finite (known to be bounded by $L^{8/5}$, resp. $L^{48/25}$, see *e.g.*, [3, 4]). Numerical experiments seem to indicate that these bounds should in fact be extensive (with at worst $\mathcal{O}(\log L)$ corrections), *i.e.*, $L^{1/2}$ for the L^2 radius and L^0 for L^∞ . Here, we will show that this conjecture is wrong for initial conditions in $\mathrm{L}^\infty(\mathbf{R})$, since we shall construct initial data μ_Y which are antisymmetric, satisfying $\lim_{x\to\pm\infty}\mu_Y(x) = \pm Y$ and *for which*

$$\lim_{x\to\pm\infty}\mu(x,t) = \pm Y \tag{1.2}$$

for all $t > 0$.[2] These solutions correspond to the well known Bunsen flame fronts (see [11]).

The behavior of (1.2) is therefore similar to what happens for the diffusion equation, where initial conditions with different limits at $\pm\infty$ also maintain this property as time increases, see *e.g.*, [2]. What do we learn from this result? Basically, it shows, that if there is ever to be a definition of attractor for KS_∞ it must contain constraints on the initial condition which are much stronger than just being in L^∞ (plus arbitrary regularity). Rather, if there is any hope, it would have to be a condition which says that the initial condition looks everywhere "like" those well-known [9] patterns one encounters in numerical simulations. In the absence of a technique replacing localization as in [1, 13] this seems currently impossible to achieve. It *is* known [10] that periodic stationary solutions are universally bounded. The present paper obtains more information on the structure of the phase space, if not on an eventual attractor, by showing that the periodic stationary solutions divide naturally the phase space into mutually unattainable regions.

Our proof of (1.2) is based on the following simple idea. First of all, constants are clearly stationary solution of KS_∞. Furthermore, (1.1) has a one parameter family of explicit (albeit unbounded) solutions of the form

$$\mu(x,t) = \frac{bx}{1+bt}\ ,$$

with $b > 0$, showing that positive constant slopes are rotated clockwise. Our starting point consists in combining these two special solutions by taking as an initial condition the function

$$\psi_a(x) = a\arctan\left(x\right)\ ,$$

where $a = 2Y/\pi$. The 'middle' of this function is like the constant slope example (with $b = a$) while for large x it reaches very quickly $\pm Y$. It is therefore natural to assume that

$$\psi_a(x,t) = a\arctan\left(\frac{x}{1+at}\right)$$

is a good approximate profile for $t > 0$. In fact, while the definition of $\psi_a(x,t)$ is suggestive, as long as we only have bounds, and not small bounds in L^2, we may, and will, work with the *fixed* function $\psi_a = \psi_a(x)$ that is, with a profile which is not changing in time. Our main result

[2] Without the asymmetry, the question is ill posed, because, by the Galilean invariance of the equation, *e.g.*, $\mu_0(x) = Y$ is a solution for any Y.

is that *if the initial condition μ_0 is* L^2*-close to $\psi_a(x)$ and the difference decays at infinity then the solution stays* L^2*-close to $\psi_a(x)$, and the difference still decays at infinity for all $t > 0$.* This result also holds if ψ_a is replaced by ψ_{per}, with ψ_{per} a periodic and analytic stationary solution of (1.1) as constructed in [10]. Therefore, there exist solutions of KS_∞ which stay near $\pm Y$ at infinity for all times, and thus we have found a family of large initial conditions whose evolution does not get smaller in L^∞ as time goes to ∞.[3] On the other hand, initial conditions which behave asymptotically like periodic stationary solutions apart from L^2 corrections remain so for all times. Since the difference of two periodic functions with different periods is *not* in L^2, this shows that the phase space naturally splits into disconnected components. This last result is an extension of [6].

The discussion above suggests to consider the equation for $\nu(x,t) = \mu(x,t) - \psi(x)$, which reads

$$\dot{\nu} = -\nu'''' - \nu'' - \tfrac{1}{2}(\nu^2)' - (\nu\psi)' + \chi \,, \qquad \nu(x,0) = \nu_0(x) \tag{1.3}$$

where

$$\chi \equiv -\psi'''' - \psi'' - \psi\psi' \,,$$

and $\lim_{x\to\pm\infty} \nu_0(x) = 0$. We will consider (1.3) either with $\psi = \psi_a$ and corresponding χ_a, or with $\psi = \psi_{per}$ a periodic analytic stationary solution for which $\chi = \chi_{per} = 0$. Note that instead of ψ_a, we could have used the stationary profiles (*i.e.*, stationary solutions of (1.1)) constructed in [10]), or even the explicit one

$$\mu(x,t) = \frac{15}{361}\sqrt{209}\left(-9\tanh\left(\tfrac{\sqrt{209}}{38}x\right) + 11\tanh\left(\tfrac{\sqrt{209}}{38}x\right)^3\right)$$

found by Kuramoto [8]. The advantage of these choices would have been that $\chi_a = 0$, the disadvantage is the lack of explicit formulas, in particular for the Fourier transform of the profiles. While adding an inhomogeneous term to the equation, the choice of ψ_a retains the main properties of these stationary profiles, *e.g.* in terms of analyticity. As is easily seen, high frequency modes are strongly damped by (1.1) at the linear level. It is known (see *e.g.*, [4]) that solutions corresponding to periodic antisymmetric initial condition in $\mathrm{L}^2([-L/2, L/2])$ become analytic in finite time in a strip of finite width around the real axis. The error term χ_a of the equation (see (1.3)) and ψ_a are analytic in the strip $|\mathrm{Im}\ z| < 1$ and uniformly bounded in any smaller strip—these two facts are better seen in Fourier space, since the Fourier transform $\tilde{\psi}_a$ of ψ_a exists as a distribution and is given by

$$\tilde{\psi}_a(k) = a\frac{e^{-|k|}}{k} \,. \tag{1.4}$$

Definition 1.1 *Throughout, we denote by A the operator $A = \sqrt{-\partial_x^2}$.*

[3] Note that we do not claim (and it quite probably is not true) that $\mu(x,t) - \psi_a(x,t)$ stays bounded in L^2. We will rather see that it grows (quickly) in L^2. But the only thing which matters is that it remains in L^2 and decays at infinity.

Remark 1.2 *We fix $Y > 0$, and we tacitly admit that all constants occurring in the sequel may depend on Y.*

Theorem 1.3 *There are constants c and $\beta > 0$ such that the following holds. For any initial condition $\nu(\cdot, 0)$ with $\nu_0 \equiv \|\nu(\cdot, 0)\|_2 < \infty$, the solution of (1.3) exists for all $t > 0$ and*

$$\sup_{t \geq 0} e^{-\beta t} \|\nu(\cdot, t)\|_2 \leq \nu_0 + c \, .$$

Furthermore, the flow is regularizing in the sense that there exist constants $\delta_* > 0$, $\gamma \geq \beta$, and $C < \infty$ such that

$$\sup_{t \geq 0} \; e^{-\gamma t} \|e^{\min(\delta_*, t)A} \nu(\cdot, t)\|_2 \leq \nu_0 + C \, .$$

Corollary 1.4 *For every $m = 0, 1, \ldots$ there exists a constant C_m such that*

$$|\partial_x^m \nu(x, t)| \leq \frac{C_m}{\min(\delta_*,\, t)^m} e^{\gamma t} \;\; \text{and} \;\; \lim_{x \to \pm\infty} \partial_x^m \nu(x, t) = 0$$

for all $t > 0$.

Proof. By Theorem 1.3 there exists a C' such that $\|e^{\min(\delta_*, t)A} \nu(\cdot, t)\|_2 \leq C' e^{\gamma t}$. By the Schwarz inequality,

$$\|A^m \tilde{\nu}(\cdot, t)\|_1 \leq \left(\int \mathrm{d}k \; |k|^{2m} e^{-2 \min(\delta_*, t)|k|} \right)^{1/2} \|e^{\min(\delta_*, t)A} \nu(\cdot, t)\|_2 \, .$$

This immediately implies the first assertion since $\sup_x |\partial_x^m \nu(x, t)| \leq \|A^m \tilde{\nu}(\cdot, t)\|_1$. This bound also implies (by the Riemann-Lebesgue theorem) that $\lim_{x \to \pm\infty} \partial_x^m \nu(x, t) = 0$. ■

2 Functional Spaces, Estimates

In this section, we collect a few straightforward bounds on the function ψ_a and the operator $A = \sqrt{-\partial_x^2}$. We denote by $\mathcal{W}_{\tau,\delta}$ the (Banach) space obtained by completing $\mathcal{C}_0([\tau, \tau + \delta], \mathcal{C}_0^\infty(\mathbf{R}))$ in the norm $\sup_{t \in [\tau, \tau+\delta]} \|e^{(t-\tau)A} \cdot \|_2$, where, throughout, $\| \cdot \|_p$ is the L^p norm. We also denote by $\mathcal{B}_d \subset \mathcal{W}_{\tau,\delta}$ the open ball of radius d, centered on 0 in $\mathcal{W}_{\tau,\delta}$. The bounds of this section serve to control the non-linear and mixed terms in Eq.(1.3).

Lemma 2.1 *There is a $\delta_* > 0$ such that for all $t \in [\tau, \tau + \delta_*]$ one has*

$$\|e^{(t-\tau)A} \; \psi_a(\cdot) f(\cdot, t)\|_2 \leq 2\pi \; a \; \|\sqrt{1 + A^2} \; e^{(t-\tau)A} f(\cdot, t)\|_2 \, .$$

Remark 2.2 *One can choose $\delta_* = 1$ as will be seen from the proof. (This value is related to the domain of analyticity of ψ_a.)*

Proof. Define $F(x,t) = e^{(t-\tau)A}\,\psi_a(x) f(x,t)$. We write the Fourier transform of F as

$$\tilde{F}(k,t) = e^{(t-\tau)|k|} \int \mathrm{d}\ell\, \tilde{\psi}_a(k-\ell)\tilde{f}(\ell,t)\ .$$

Using (1.4), we find (using principal values)

$$\tilde{F}(k,t) = a\, e^{(t-\tau)|k|} \int \mathrm{d}\ell\, \frac{e^{-|k-\ell|}}{k-\ell}\tilde{f}(\ell,t)\ .$$

Denote $g(x,t) = e^{(t-\tau)A} f(x,t)$, so that $\tilde{f}(\ell,t) = e^{-(t-\tau)|\ell|}\tilde{g}(\ell,t)$. Rearranging exponentials, we get

$$\begin{aligned}
\tilde{F}(k,t) &= a \int \mathrm{d}\ell\, \frac{e^{-|k-\ell|}}{k-\ell} e^{(t-\tau)|k|} e^{-(t-\tau)|\ell|}\tilde{g}(\ell,t) \\
&= a \int \mathrm{d}\ell\, \frac{e^{-|k-\ell|(1-(t-\tau))}}{k-\ell} e^{(t-\tau)\big(|k|-|\ell|-|k-\ell|\big)}\tilde{g}(\ell,t)\ .
\end{aligned}$$

We decompose this as

$$\tilde{F}(k,t) = a \int \mathrm{d}\ell\, \frac{e^{-|k-\ell|(1-(t-\tau))}}{k-\ell}\ \tilde{g}(\ell,t) + a \int \mathrm{d}\ell\ \ G(k,\ell,t-\tau,(1-(t-\tau)))\ \tilde{g}(\ell,t)\ ,$$

where

$$G(k,\ell,\xi,\eta) = \frac{e^{-\eta|k-\ell|}}{k-\ell}\ \left(e^{\xi(|k|-|\ell|-|k-\ell|)} - 1\right)\ .$$

One checks easily, using the triangle inequality, that for $\xi \geq 0$,

$$|G(k,\ell,\xi,\eta)| \leq |G(k,\ell,\xi,0)| \leq \frac{\xi}{\sqrt{1+(\xi(k-\ell))^2}}\ .$$

Using $\int \mathrm{d}k(1+k^2)^{-1} = \pi$, we get

$$\begin{aligned}
\|F(\cdot,t)\|_2 &\leq \|\psi_a(\cdot)e^{(t-\tau)A}\ f(\cdot,t)\|_2 + a\sup_{\ell\in\mathbf{R}} \|G(\cdot,\ell,t-\tau,1+\tau)\|_2\ \|\tilde{g}(\cdot,t)\|_1 \\
&\leq \frac{a\pi}{2}\|e^{(t-\tau)A}\ f(\cdot,t)\|_2 + a\sup_{\ell\in\mathbf{R}} \|G(\cdot,\ell,t-\tau,0)\|_2\ \left\|\frac{\sqrt{1+A^2}}{\sqrt{1+A^2}}\ \tilde{g}(\cdot,t)\right\|_1 \\
&\leq \pi\ a\left(\tfrac{1}{2}+\sqrt{t-\tau}\right)\ \|\sqrt{1+A^2}\ e^{(t-\tau)A}\ f(\cdot,t)\|_2 \\
&\leq 2\pi\ a\|\sqrt{1+A^2}\ e^{(t-\tau)A}\ f(\cdot,t)\|_2\ ,
\end{aligned}$$

provided $\delta_* \leq 1$. The proof of Lemma 2.1 is complete. ■

Lemma 2.3 *Let ψ_{per} be periodic of period L, let $q = \frac{2\pi}{L}$ and assume that there exist constants c_{per}, δ such that $\sum_{m\in\mathbf{Z}} |\psi_{per,m}|e^{\delta q|m|} < c_{per}$, where $\psi_{per,m}$ denotes the m-th Fourier coefficient of ψ_{per}, then*

$$\|e^{(t-\tau)A}\psi_{per}(\cdot)f(\cdot,t)\|_2 \leq c_{per}\ \|e^{(t-\tau)A}f(\cdot,t)\|_2$$

for all $t \in [\tau,\tau+\delta]$.

Proof. As in Lemma 2.1, we define $F(x,t) = e^{(t-\tau)A}\ \psi_{per}(x) f(x,t)$. The Fourier transform of F satisfies

$$\tilde{F}(k,t) = e^{(t-\tau)|k|} \sum_{m\in\mathbf{Z}} \psi_{per,m} f(k-qm)\ ,$$

so that

$$\begin{aligned}\|F(\cdot,t)\|_2 &\le \sum_{m\in\mathbf{Z}} |\psi_{per,m}| \left(\int \mathrm{d}k\ e^{2(t-\tau)|k+qm|}\ |f(k)|^2\right)^{1/2}\\ &\le \|e^{(t-\tau)A} f(\cdot,t)\|_2 \sum_{m\in\mathbf{Z}} |\psi_{per,m}|\ e^{(t-\tau)q|m|}\ ,\end{aligned}$$

which completes the proof of the lemma since $0 \le t-\tau \le \delta$. ■

Lemma 2.4 *Let $\gamma > 0$ and $\|(1+A^2)e^{\gamma A}\ f\|_2 + \|e^{\gamma A}\ g\|_2 + \|e^{\gamma A}\ h\|_2 < \infty$. Then*

$$\left|\int \mathrm{d}x\, e^{\gamma A}\ f \cdot e^{\gamma A}(gh)'\right|\ \le\ \sqrt{\pi}\|(1+A^2)e^{\gamma A}\ f\|_2\ \|e^{\gamma A}\ g\|_2\ \|e^{\gamma A}\ h\|_2\ .$$

Proof. Set $F = e^{\gamma A}\ f$, $G = e^{\gamma A}\ g$ and $H = e^{\gamma A}\ h$. Since $|k| - |k-\ell| - |\ell| \le 0$ by the triangle inequality, we have

$$\begin{aligned}\left|\int \mathrm{d}x\ e^{\gamma A}\ f\cdot e^{\gamma A}(gh)'\right| &= \left|\int \mathrm{d}k\ ik\tilde{F}(k) \int \mathrm{d}\ell\ e^{\gamma(|k|-|k-\ell|-|\ell|)}\ \tilde{G}(k-\ell)\ \tilde{H}(\ell)\right|\\ &\le \left\|\frac{\sqrt{1+A^2}}{\sqrt{1+A^2}} A\tilde{F}\right\|_1 \|G\|_2\ \|H\|_2\\ &\le \sqrt{\pi}\|(1+A^2)F\|_2\ \|G\|_2\ \|H\|_2\ ,\end{aligned}$$

where we used again $\int \mathrm{d}k\,(1+k^2)^{-1} = \pi$. ■
The following proposition estimates how close ψ_a is to a solution of KS_∞.

Proposition 2.5 *Define $\psi_a(x) = a\arctan(x)$ and let $\chi_a = -\psi_a'''' - \psi_a'' - \psi_a\psi_a'$. Then, for $0 \le \delta \le \delta_* \equiv \frac{1}{2}$, one has*

$$\sup_{t\in[0,\delta]} \|e^{tA}\chi_a\|_2 \le B\ ,$$

for some B depending only on a.

Proof. The Fourier transform of $e^{tA}\psi_a''''$ is of the form $i^3 a e^{t|k|} k^3 e^{-|k|}$ so that for $t \le \delta$ we get

$$\|e^{tA}\psi_a''''\|_2^2 \le a^2 \int \mathrm{d}k\ e^{2(\delta-1)|k|} k^6 \le \mathcal{O}((1-\delta)^{-7})\ .$$

A similar bound holds for ψ_a''. The term $e^{tA}\psi_a\psi_a'$ is bounded using Lemma 2.1 with $\delta_* = \frac{1}{2}$ instead of 1 and $f = \psi_a'$. This yields a bound on the square of the L^2 norm which is of the form $a^2\mathcal{O}(1)\int \mathrm{d}k\,(1+k^2)e^{2\delta|k|}e^{-2|k|}$, and combining the bounds completes the proof. ■

3 The Local Cauchy Problem in L^2

In this section, we consider the local (in time) Cauchy problem

$$\dot{\nu} = -\nu'''' - \nu'' - \tfrac{1}{2}(\nu^2)' - (\nu\psi_a)' + \chi_a\ , \qquad \nu(x,\tau) = \nu_0(x)\ , \tag{3.1}$$

for (1.3) with $\nu_0 \in \mathrm{L}^2$. We will show, using a contraction argument, that it is well posed on any time interval $t \in [\tau, \tau+\delta]$ with $\delta \le \min(\delta_*, C_*\|\nu_0\|_2^{-2})$.[4] To this end, we construct the map $\rho \mapsto \mathcal{F}(\rho)$ defined by $\mathcal{F}(\rho) = \xi$, where ξ is the solution of

$$\dot{\xi} = -\xi'''' - \xi'' - \tfrac{1}{2}(\xi\rho)' - (\xi\psi_a)' + \chi_a\ , \qquad \xi(x,\tau) = \nu_0(x)\ , \tag{3.2}$$

and show that if δ is sufficiently small ($\delta \sim \|\nu_0\|_2^{-2}$) then $\mathcal{F}$ is a contraction in a ball of radius $> \|\nu_0\|_2$ in $\mathcal{W}_{\tau,\delta}$. Namely, let $f = e^{(t-\tau)A}\xi$ and $g = e^{(t-\tau)A}\rho$. Multiplying (3.1) with $fe^{(t-\tau)A}$, integrating over the space variable and using the results of the preceding section, we have

$$\begin{aligned}\tfrac{1}{2}\partial_t\|f\|_2^2 \le \|A^{1/2}f\|_2^2 - \|A^2 f\|_2^2 + \|Af\|_2^2 + \tfrac{\sqrt{\pi}}{2}\|(1+A^2)f\|_2\|f\|_2\|g\|_2 \\ + 2\pi a\|Af\|_2\|\sqrt{1+A^2}f\|_2 + B\|f\|_2\ .\end{aligned} \tag{3.3}$$

The first term on the r.h.s. comes from the time derivative of the exponential $e^{(t-\tau)A}$, the second and third from the space derivatives of ξ. The next term uses Lemma 2.4, while the last two use Lemma 2.1 and Proposition 2.5, respectively. Then, we use the inequalities

$$\|Af\|_2 \le \|\sqrt{1+A^2}f\|_2\ , \quad \|\sqrt{1+A^2}f\|_2^2 \le \|(1+A^2)f\|_2\|f\|_2$$

and get

$$\begin{aligned}\tfrac{1}{2}\partial_t\|f\|_2^2 \le \|A^{1/2}f\|_2^2 - \|A^2 f\|_2^2 + \|Af\|_2^2 + B\|f\|_2 \\ + \left(\tfrac{\sqrt{\pi}}{2}\|g\|_2 + 2\pi a\right)\|(1+A^2)f\|_2\|f\|_2\ .\end{aligned}$$

We also have

$$\begin{aligned}\|A^{1/2}f\|_2^2 \le \|f\|_2^{3/2}\|A^2 f\|_2^{1/2} &\le \frac{3}{4\varepsilon_1^{1/3}}\|f\|_2^2 + \frac{\varepsilon_1}{4}\|A^2 f\|_2^2\ , \\ \|Af\|_2^2 \le \|f\|_2\|A^2 f\|_2 &\le \frac{1}{2\varepsilon_2}\|f\|_2^2 + \frac{\varepsilon_2}{2}\|A^2 f\|_2^2\ , \\ X\|(1+A^2)f\|_2 &\le \frac{1}{2\varepsilon_3}X^2 + \frac{\varepsilon_3}{2}\Big(\|A^2 f\|_2^2 + 2\|Af\|_2^2 + \|f\|_2^2\Big)\ ,\end{aligned}$$

for all $X, \varepsilon_i > 0$, $i = 1,2,3$. Using these inequalities with sufficiently small ε_i shows that there is a positive constant c_* such that

$$\partial_t\|f\|_2^2 \le (c_* + \|g\|_2^2)\|f\|_2^2 + B^2\ ,$$

[4] Note that by our choice of δ_*, we have $t - \tau \le \frac{1}{2}$.

from which we get that for all ρ in the ball of radius d in $\mathcal{W}_{\tau,\delta}$, $\mathcal{F}(\rho)$ satisfies

$$\sup_{t\in[\tau,\tau+\delta]} \|e^{(t-\tau)A}\mathcal{F}(\rho)\|_2 \leq e^{\delta(c_*+d^2)/2}\sqrt{\|\nu_0\|_2^2+B^2}\ .$$

For all $d > \sqrt{\|\nu_0\|_2^2+2B^2}$, there exists a $\delta = \mathcal{O}(\|\nu_0\|_2^{-2})$ such that $\mathcal{F}$ maps $\mathcal{B}_d \subset \mathcal{W}_{\tau,\delta}$ strictly into itself. On the other hand $\varphi = \mathcal{F}(\rho_1) - \mathcal{F}(\rho_2)$ satisfies

$$\dot{\varphi} = -\varphi'''' - \varphi'' - \tfrac{1}{2}(\varphi\bar{\rho})' - \tfrac{1}{2}(\bar{\xi}(\rho_1-\rho_2))' - (\varphi\psi_a)'\ , \quad \varphi(x,\tau) = 0\ ,$$

where $\bar{\rho} = (\rho_1+\rho_2)/2$ and $\bar{\xi} = (\mathcal{F}(\rho_1)+\mathcal{F}(\rho_2))/2$. Since $\bar{\rho}$ and $\bar{\xi}$ are in $\mathcal{B}_d \subset \mathcal{W}_{\tau,\delta}$, similar arguments also show that for the same d and δ as above

$$\sup_{t\in[\tau,\tau+\delta]} \|e^{(t-\tau)A}(\mathcal{F}(\rho_1)-\mathcal{F}(\rho_2))\|_2 < \sup_{t\in[\tau,\tau+\delta]} \|e^{(t-\tau)A}(\rho_1-\rho_2)\|_2\ ,$$

so that $\mathcal{F}$ is a contraction in $\mathcal{B}_d \subset \mathcal{W}_{\tau,\delta}$. Thus the sequence of approximating solutions $\nu_{n+1} = \mathcal{F}(\nu_n)$ converges to a unique solution of (3.1) in $\mathcal{B}_d \subset \mathcal{W}_{\tau,\delta}$.

Note that the results of this section also hold for the equation

$$\dot{\nu} = -\nu'''' - \nu'' - \tfrac{1}{2}(\nu^2)' - (\nu\psi_{per})'\ , \quad \nu(x,\tau) = \nu_0(x)\ . \tag{3.4}$$

It follows easily from [10] (see also [6]) that periodic stationary solutions ψ_{per} of (1.1) satisfy the hypotheses of Lemma 2.3. The procedure is then exactly the same, *i.e.*, to show that the analogous of the map $\mathcal{F}$ is a contraction in $\mathcal{B}_d \subset \mathcal{W}_{\tau,\delta}$ for some d, τ and δ. Using obvious notations, we find that (3.3) is replaced by

$$\tfrac{1}{2}\partial_t\|f\|_2^2 \leq \|A^{1/2}f\|_2^2 - \|A^2f\|_2^2 + \|Af\|_2^2 + \tfrac{\sqrt{\pi}}{2}\|(1+A^2)f\|_2\|f\|_2\|g\|_2 + c_{per}\|Af\|_2\|f\|_2\ ,$$

from which it follows that

$$\partial_t\|f\|_2^2 \leq (c_* + \|g\|_2^2)\|f\|_2^2\ ,$$

for some positive c_*. The remainder of the proof is straighforward.

4 Proof of Theorem 1.3

Let now $\tau_0 = 0$ and $\nu(x,\tau_0) = \nu_0(x)$ with $\nu_0 \in \mathrm{L}^2$, and define $\sigma_0 = \|\nu_0\|_2$. From the results of the preceding section, we know that there exists a unique solution of (3.1) in $\mathcal{B}_{d_0} \subset \mathcal{W}_{\tau_0,\delta_0}$ for $d_0 = D_*\sigma_0$ and $\delta_0 = C_*\sigma_0^{-2}$, with C_* so small that $\delta_0 < \delta_*$. Let $\tau_1 = \tau_0 + \delta_0$. By the definition of $\mathcal{W}_{\tau,\delta}$ (see also Corollary 1.4), ν and all its derivatives tend to 0 as $x \to \pm\infty$ for all $t \in [\tau_0, \tau_1]$. *In particular, the trilinear form* $\int \nu(\nu^2)'$ *satisfies* $\int \nu(\nu^2)' = -\frac{1}{3}\int(\nu^3)' = 0$. Hence, multiplying (1.3) with ν and integrating over the space variable, we get:

$$\begin{aligned}\tfrac{1}{2}\partial_t\int\nu^2 &= -\int\nu\nu'''' - \int\nu\nu'' - \tfrac{1}{2}\int\nu(\nu^2)' - \int\nu(\nu\psi_a)' - \int\nu\chi_a \\ &= -\int(\nu'')^2 + \int(\nu')^2 - \tfrac{1}{2}\int\nu^2\psi_a' - \int\nu\chi_a\ .\end{aligned}$$

Note that because the trilinear form vanishes, we get only quadratic (and linear) terms in ν, and therefore it is natural to find an exponential bound in time for the evolution of the L^2 norm; this is the main explanation for the bounds which follow below. Using $-k^4 + k^2 \le \frac{1}{4}$, we have the inequalities

$$
\begin{aligned}
-\int (\nu'')^2 + \int (\nu')^2 &\le \tfrac{1}{4} \int \nu^2 \ , \\
|\psi_a'| &\le a \ , \\
\left| \int \nu \chi_a \right| &\le \tfrac{1}{2} \int \nu^2 + \tfrac{1}{2} \int \chi_a^2 \ .
\end{aligned}
$$

We find for, $t \in [\tau_0, \tau_1]$, with $\beta \equiv 2(\frac{3}{4} + a)$,

$$
\frac{1}{2}\partial_t \int \nu^2 \ \le\ \frac{\beta}{2} \int \nu^2 + \frac{1}{2} \int \chi_a^2 \ \le\ \frac{\beta}{2} \int \nu^2 + \frac{B^2}{2} \ . \tag{4.1}
$$

This differential inequality is valid for all $t \in [\tau_0, \tau_1]$, and implies that

$$
\|\nu(\cdot, \tau_1)\|_2 \le e^{\beta\tau_1/2} \sqrt{\sigma_0^2 + B^2} \equiv \sigma_1 \ .
$$

Again, from the results of the preceding section, we now see that there exists a unique solution of (3.1) in $\mathcal{B}_{d_1} \subset \mathcal{W}_{\tau_1,\delta_1}$ for $d_1 = D_*\sigma_1$ and $\delta_1 = C_*\sigma_1^{-2}$. Thus (4.1) is valid for all $t \in [\tau_0, \tau_2]$ with $\tau_2 = \tau_1 + \delta_1$, and we get

$$
\|\nu(\cdot, \tau_2)\|_2 \le e^{\beta\tau_2/2} \sqrt{\sigma_0^2 + B^2} \equiv \sigma_2 \ .
$$

Continuing by induction, we find

$$
\delta_n = C_*\sigma_n^{-2} = \frac{e^{-\beta\tau_n}}{\sigma_0^2 + B^2} \ ,
$$

so that

$$
\tau_{n+1} = \tau_n + E_* e^{-\beta\tau_n} \ ,
$$

with $E_* = (\sigma_0^2 + B^2)^{-1}$. This implies that $\lim_{n\to\infty} \tau_n = \infty$, and therefore (4.1) is valid for all $t > 0$. This completes the proof of Theorem 1.3 for the profile case. The periodic case follows along the same lines.

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