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Nonlinear Stability of Bifurcating Front Solutions for the Taylor-Couette Problem

We consider the Taylor-Couette problem in an infinitely extended cylindrical domain. There exist modulated front solutions which describe the spreading of the stable Taylor vortices into the region of the unstable Couette flow. These transient solutions have the form of a front-like envelope advancing in the laboratory frame and leaving behind the stationary, spatially periodic Taylor vortices. We prove the nonlinear stability of these solutions with respect to small spatially localized perturbations.

Keywords: modulated fronts, diffusive behavior

MSC (1991): 35Q30, 76E99

1 Introduction

Ludwig Prandtl's boundary layer theory of 1904 was a breakthrough in modern fluid mechanics. It allows to distinguish two domains for small viscosity flows. In a narrow neighborhood of the boundaries viscous effects play a role while outside this boundary layer the flow can be considered as a potential flow. The Taylor-Couette problem we consider here is a hydrodynamic problem where the onset of instability is due to the motion of the boundaries. While Prandtl's ideas are still relevant, they do not apply as such in our case because the infinite length of the boundaries will produce an infinite amount of energy.

The Taylor-Couette problem consists in describing the flow of a viscous, incompressible fluid in the domain between two (counter-) rotating cylinders. More precisely, we are interested in the velocity field of a viscous incompressible fluid filling the domain $\Omega = \mathbb{R} \times \Sigma$ between two concentric rotating infinite cylinders, where $\Sigma \subset \mathbb{R}^2$ denotes the bounded cross section. The flow between the rotating cylinders is described by the Navier-Stokes equations in Ω with no-slip boundary conditions. For simplicity we assume throughout this paper the outer cylinder to be at rest.

When the rotational velocity of the inner cylinder is small, a stationary flow appears, called the Couette flow U_{Cou} with associated pressure field p_{Cou}. It is homogeneous along the cylinders and the streamlines are given by concentric circles. For small Reynolds number $\mathcal{R}$ (which is proportional to the rotational velocity of the inner cylinder) these solutions are exponentially stable.

A perturbation (U, p) of the Couette flow satisfies the Navier-Stokes equations

$$\begin{aligned} \partial_t U &= \Delta U - \mathcal{R}[(U_{\text{Cou}} \cdot \nabla)U + (U \cdot \nabla)U_{\text{Cou}} + (U \cdot \nabla)U] - \nabla p \ , & (1)\\ \nabla \cdot U &= 0 \ , & (2) \end{aligned}$$

with Dirichlet boundary conditions

$$U|_{\mathbb{R}\times\partial\Sigma} = 0 \ . \tag{3}$$

This problem has a unique solution U and ∇p if we add the vanishing mean flux condition,

$$[U_{(\mathrm{x})}]_\Sigma(x) = \frac{1}{|\Sigma|}\int_{z\in\Sigma} U_{(\mathrm{x})}(x,z)dz = 0 \ , \quad \text{for all } x \in \mathbb{R} \ , \tag{4}$$

where $U_{(\mathrm{x})}$ stands for the velocity component along the infinite x–direction. See [5].

The trivial branch of solutions of the system (1)–(4) is $U(x, z) \equiv 0$, which corresponds to the Couette flow. It becomes unstable when the Reynolds number $\mathcal{R}$ goes beyond a certain threshold $\mathcal{R}_c$. We define $\varepsilon > 0$ by $\varepsilon^2 = \mathcal{R} - \mathcal{R}_c$ and assume throughout $\varepsilon \ll 1$.

For $\varepsilon > 0$ a family of small spatially periodic equilibria (Taylor vortices) bifurcates from the trivial solution $U \equiv 0$:

$$U_{\mathrm{TV},[q,\varepsilon,\vartheta]}(x,z) = \varepsilon\, A\sqrt{1-c_1q^2}\, U_{k_c}(z)e^{i(k_c+\varepsilon q)(x+\vartheta)} + \text{c.c.} + \mathcal{O}(\varepsilon^2), \qquad q \in [0, 1/\sqrt{c_1}),\ \vartheta \in \mathbb{R}\ . \tag{5}$$

In (5), c_1 is a positive constant, $k_c > 0$ is a critical wavenumber, and $U_{k_c}(z) \in \mathbb{C}^3$. We will henceforth assume $\vartheta = 0$ and omit it from the parameters of U_{TV}. Such solutions have been constructed in [25, 33, 26]. It turned out that the linearization around the stationary solutions $U_{\mathrm{TV},[q,\varepsilon]}$ possesses continuous spectrum at least up to the imaginary axis. These solutions are linearly stable for $c_1 q^2 \le \frac{1}{3}$ and so-called sideband- or Eckhaus unstable for $c_1 q^2 > \frac{1}{3}$. See [14] or [8].

In actual experiments, the transition from the Couette flow to the Taylor vortices usually begins from perturbations at the ends of the (finite) cylinders. A first vortex forms near the end of the cylinder, and from this seed new vortices form away from the ends, and an advancing front seems to move through the cylinder, leaving stationary, spatially periodic vortices behind. Finally the cylinder is filled with Taylor vortices.

These advancing fronts cannot just be described as $U(x,t) = U_{\mathrm{f}}(x-ct)$, because of the pattern they leave in the laboratory frame. Rather they are of the form described in

Theorem 1. *[22]. For each q with $c_1 q^2 < \frac{1}{3}$ the following holds. For sufficiently small $\varepsilon > 0$ and for $c > 0$ the system* (1)–(4) *possesses modulated front solutions*

$$U(x,z,t) = U_{\mathrm{mf},[q,\varepsilon,c]}(x-ct,x,z) \ ,$$

with

$$U_{\mathrm{mf},[q,\varepsilon,c]}(\xi,x,z) = U_{\mathrm{mf},[q,\varepsilon,c]}(\xi, x + \frac{2\pi}{k_{\mathrm{c}} + \varepsilon q}, z) \ ,$$

and the boundary conditions

$$\begin{aligned} \lim_{\xi\to-\infty} U_{\mathrm{mf},[q,\varepsilon,c]}(\xi,x,z) &= U_{\mathrm{TV},[q,\varepsilon]}(x,z) \ , \qquad \textit{the Taylor vortices} \ , \\ \lim_{\xi\to\infty} U_{\mathrm{mf},[q,\varepsilon,c]}(\xi,x,z) &\equiv 0 \ , \qquad \textit{the Couette flow} \ . \end{aligned}$$

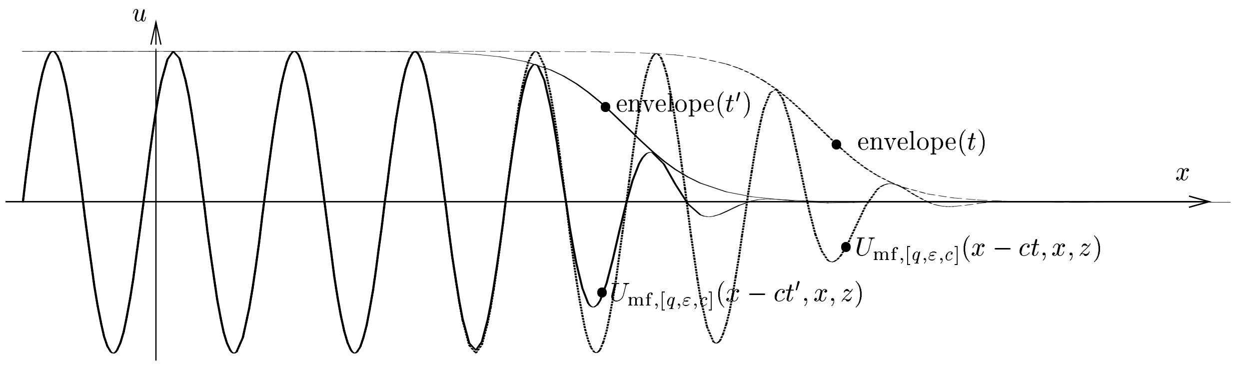

Figure 1: Sketch of a modulated front, advancing from left to right at times $t' < t$, and the advancing envelopes.

In the present paper we prove the nonlinear stability—under spatially localized perturbations—of these bifurcated front solutions $U_{\mathrm{mf},[q,\varepsilon,c]}$. In particular, this means that a small perturbation cannot destroy the growing regular pattern in the bulk, nor make it move. This is not an obvious result since there could be a soft mode in the system, due to translation invariance. But, although the Taylor vortices are only diffusively stable, their positions are pinned. The precise statement of the result can be found in Theorem 6.

Our proof is an adaptation of our previous paper [19] where we dealt with the stability of front solutions for the Swift-Hohenberg equation

$$\partial_t u = -(1+\partial_x^2)^2 u + \varepsilon^2 u - u^3 \ , \qquad u(x,t) \in \mathbb{R}, \ x \in \mathbb{R}, \ t \ge 0$$

connecting the stable periodic equilibria

$$u_{\mathrm{per},[q,\varepsilon]} = \varepsilon \sqrt{\frac{1-4q^2}{3}} e^{i(1+\varepsilon q)x} + \mathrm{c.c.} + \mathcal{O}(\varepsilon^2) \ ,$$

for $4q^2 < 1/3$ with the unstable trivial solution $u = 0$.

The stability proof is based on the following idea. Consider a spatially localized perturbation ahead of the front. In this unstable region any perturbation will grow exponentially because 0 is an unstable solution. But, after a *finite* time the front will have reached the perturbation, or equivalently, the perturbation has entered the stable region behind the front. This region is *diffusively stable* and thus perturbations decay as $1/\sqrt{t}$. Therefore, one expects the front to be stable under sufficiently small spatially localized perturbations which decay exponentially with the distance ahead of the front. In particular, the front is stable for sufficiently small perturbations with compact support.

In order to make these ideas clearer, we describe first two easier model problems. In Section 2 we explain for a very simplified model how weighted norms can be used to stabilize the a priori unstable region ahead of the front. In this very simplified model the equilibrium in the bulk will be assumed to be stable with some *exponential* rate. In Section 3 we give up the exponential stability of the equilibrium and replace it by a diffusively stable equilibrium, i.e., the continuous spectrum of the linearization now reaches the imaginary axis and is no longer separated from it. As an example we consider the amplitude equation associated with the Taylor-Couette problem, namely the Ginzburg-Landau equation. For this equation it will be more or less obvious that spatially localized perturbations of the stable equilibrium decay as a solution of a linear diffusion equation. For the Taylor-Couette problem, this is no longer obvious for the solutions we are interested in and so the main difficulty for the fronts U_{mf} is to recover the diffusive behavior behind the front.

Remark 1. Since the internal wavelength of the spatially periodic flow is in general much smaller than the length of the cylinders and due to the fact that the ends of the cylinders in a laboratory experiment do not influence the form of the pattern in the interior, we consider the Taylor-Couette problem with cylinders of infinite length. As explained, up to a certain extent this idealization is a good description of Nature.

Remark 2. The Taylor-Couette problem is one of several examples where our theory applies. Other examples are Bénard's problem and reaction-diffusion systems. Loosely speaking our theory applies whenever a spatially homogeneous state becomes unstable via a supercritical stationary bifurcation at a non-zero Fourier wavenumber in a translationally invariant system.

Remark 3. The existence proof for modulated fronts $U(x,z,t) = U_{\mathrm{mf},[q,\varepsilon,c]}(x-ct,x,z)$ connecting the stable Taylor vortices with the Couette flow is based on center manifold theory for elliptic problems on unbounded cylindrical domains [27]. It is complicated here by an infinite number of eigenvalues on the imaginary axis for $\mathcal{R} = \mathcal{R}_c$. The related result for the Swift-Hohenberg equation can be found in [6, 16].

2 Basic ideas

The ordinary differential equation $\partial_t u = u$ with $u(t) \in \mathbb{R}$ is used to model the exponential growth of a certain quantity u, for instance a chemical concentration or a population of bacteria. Usually nonlinear saturation effects give some bound on the possible size of u. This can be taken into account by adding a nonlinear term, for example $-u^2$, to the model, leading to

$$\partial_t u = u - u^2 \ . \tag{6}$$

This nonlinear model possesses in addition to the unstable equilibrium $u = 0$ also a stable equilibrium $u = 1$, and so every solution with initial condition in the interval $(0,1)$ converges towards $u = 1$ for $t \to \infty$. This model is refined further by taking into account the spatial structure of the solutions: Chemical reactions or the growth of a population in general do not happen in a spatially homogeneous manner and so the above model is extended by adding a diffusion term $\partial_x^2 u$ to the system, i.e.,

$$\partial_t u = \partial_x^2 u + u - u^2 \ , \tag{7}$$

with $u(x,t) \in \mathbb{R}$. For solutions with small positive spatially localized initial conditions the dynamics is very simple. Solutions with such initial conditions develop in a universal way. By the reaction term $u - u^2$ at fixed $x \in \mathbb{R}$ the solutions are drawn to the stable equilibrium $u = 1$. Then, by the diffusion term the part of the solution which is close to the stable phase $u = 1$ starts to spread into the region of the unstable phase $u = 0$ forming two fronts with velocities $c = \pm 2$, one to left and one to the right.

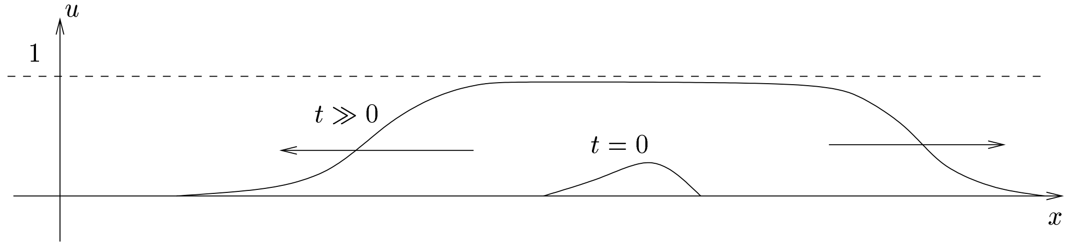

Figure 2: Two fronts for (7), generated by localized initial data.

Such front solutions $u(x,t) = v_{\mathrm{f},[c]}(x-ct) = v_{\mathrm{f},[c]}(\xi)$ connecting the stable phase $u=1$ for $\xi \to -\infty$ with the unstable phase $u=0$ for $\xi \to \infty$ can be found by analyzing the second order ordinary differential equation satisfied by $v = v_{\mathrm{f},[c]}$ in the (v,v')–phase plane. The partial differential equation (7) can be considered in the moving frame $\xi = x - ct$, where it takes the form

$$\partial_t v = \partial_\xi^2 v + c\partial_\xi v + v - v^2 \ . \tag{8}$$

A front $v_{\mathrm{f},[c]}$ then satisfies $v''_{\mathrm{f},[c]} + cv'_{\mathrm{f},[c]} + v_{\mathrm{f},[c]} - v^2_{\mathrm{f},[c]} = 0$. It can be found by showing that there exists a heteroclinic connection between $(v,v') = (1,0)$ and $(v,v') = (0,0)$, where the latter fixed point has real eigenvalues when $|c| \geq 2$. As a consequence, when $|c| \geq 2$ the fronts satisfy $v_{\mathrm{f},[c]}(\xi) \in (0,1)$.

Thinking of a chemical reaction front it is obvious that the stability of these fronts is a relevant question. We recall that an equilibrium $y=0$ of an ordinary differential equation $\dot{y} = Ay + \mathcal{O}(y^2)$ is called (Lyapunov) stable in $\mathbb{R}^d$ if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|y(0)\|_{\mathbb{R}^d} \leq \delta$ implies $\|y(t)\|_{\mathbb{R}^d} \leq \varepsilon$ for all $t \geq 0$.

A good starting point is to consider stability under perturbations in the space $\mathcal{C}_b^0$ of uniformly continuous bounded functions equipped with the norm $\|u\|_{\mathcal{C}_b^0} = \sup_{x\in\mathbb{R}} |u(x)|$. Since the fixed point $u=0$ is already unstable for the ordinary differential equation (6) it is also unstable under the dynamics of the partial differential equation (7). On the other hand, perturbations of $u=1$ in $\mathcal{C}_b^0$ can be bounded by the maximum principle using spatially constant functions, and so the stability of this equilibrium follows again from the stability of $u=1$ in the ordinary differential equation (6).

A drawback of this topology is that the fronts $v_{\mathrm{f},[c]}$ are actually unstable with respect to perturbations in $\mathcal{C}_b^0$. In order to see this consider the initial condition $w(\xi,0) = \max(v(\xi),\delta)$. Then, as we see in Fig.2,

$$\sup_{\xi\in\mathbb{R}} |w(\xi,0) - v(\xi)| < \delta, \ \text{but} \ \lim_{t\to\infty} \sup_{\xi\in\mathbb{R}} |w(\xi,t) - v(\xi)| = 1 \ .$$

However, we shall now argue that these fronts are stable with respect to spatially *localized* perturbations. Suppose that a perturbation is localized near $\xi = \xi_0 > 0$. Then the perturbation has only a finite time $t = \xi_0/c$ to grow exponentially before it is hit by the bulk of the front, where for (7) perturbations are damped exponentially.

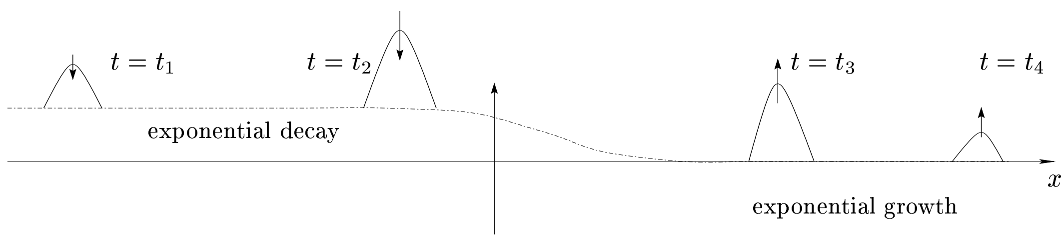

Figure 3: Stability mechanism for fronts of (7).

Next we explain how this heuristic argument can be made rigorous with the help of exponentially weighted norms. In order to do so we consider the stability of the zero solution $u \equiv 0$ for the equation $\partial_t u = \partial_\xi^2 u + c\partial_\xi u + u$. Let $w(\xi) = u(\xi)e^{\beta\xi}$ with a $\beta > 0$. Then the function w satisfies

$$\partial_t w = \partial_\xi^2 w + (c - 2\beta)\partial_\xi w + (\beta^2 - \beta c + 1)w \ .$$

If $\beta^2 - \beta c + 1 < 0$ (which we assume henceforth) the maximum principle implies the stability of $w=0$ in $\mathcal{C}_b^0$.

However, this reasoning does not carry over to the nonlinear problem since the product of two functions $u_j(x) = w_j(x)e^{-\beta x}$, $(j=1,2)$ with $w_j \in \mathcal{C}_b^0$ leads to $u_1u_2e^{\beta x} = w_1w_2e^{-2\beta x}e^{\beta x}$, which is in general not bounded for $x \to -\infty$. Therefore, the exponential stability of $w=0$ cannot be used directly.

Only an interplay of both norms, i.e., the consideration of u and w simultaneously, and the presence of a front will suffice to show the stability. We consider thus a front $v_{\mathrm{f},[c]}$ traveling with speed c, which is a solution of (7) (resp. (8)). The deviations $r = u - v_{\mathrm{f},[c]}$ and $w(\xi) = r(\xi)e^{\beta\xi}$ from the front $v_{\mathrm{f},[c]}$ satisfy

$$\begin{aligned}
\partial_t r &= \partial_\xi^2 r + c\partial_\xi r + r - 2v_{\mathrm{f},[c]}r - r^2 \ , \\
\partial_t w &= \partial_\xi^2 w + (c-2\beta)\partial_\xi w + (\beta^2 - \beta c + 1)w - 2v_{\mathrm{f},[c]}w - rw \ .
\end{aligned}$$

At first sight this seems to bring no advantage, but the term $-2v_{\mathrm{f},[c]}r$ in the equation for r can be expressed as

$$-2v_{\mathrm{f},[c]}r = -2r + 2(1 - v_{\mathrm{f},[c]})r = -2r + 2(1 - v_{\mathrm{f},[c]})e^{-\beta\xi}e^{\beta\xi}r = -2r + 2(1 - v_{\mathrm{f},[c]})e^{-\beta\xi}w \ .$$

Thus, the equation for r is transformed to:

$$\partial_t r = \partial_\xi^2 r + c\partial_\xi r - r + 2(1 - v_{\mathrm{f},[c]})e^{-\beta\xi} w - r^2 \ .$$

The term $2(1 - v_{\mathrm{f},[c]})e^{-\beta\xi}$ is *bounded* when $\beta^2 - \beta c + 1 < 0$, since $1 - v_{\mathrm{f},[c]}(\xi) = \mathcal{O}(e^{-\gamma|\xi|})$ as $\xi \to -\infty$ with $\gamma > 0$ solving $\gamma^2 - \gamma c - 1 = 0$. Because of the term $-r$ in the equation for r and because we have $-2v_{\mathrm{f},[c]} < 0$ in the equation for w the linearization of the system (r, w) around $(r, w) = (0, 0)$ is now *exponentially stable* and so this obviously also holds for the nonlinear system.

Theorem 2. *Assume β and c satisfy $\beta^2 - \beta c + 1 < 0$. Then for all $\varepsilon > 0$ there exist $C, \delta > 0$ such that*

$$\sup_{\xi\in\mathbb{R}}(|r(\xi, t)| + |w(\xi, t)|) < Ce^{(\max(-1,\beta^2-\beta c+1)+\varepsilon)t} \ ,$$

provided

$$\sup_{\xi\in\mathbb{R}} |r(\xi, 0)\,(1 + e^{\beta\xi})| < \delta \ .$$

The discussion of this equation goes back to [20, 29]. The above result can be found for instance in [35].

Remark 4. The discussion of the marginally stable case $\beta^2 - c\beta + 1 = 0$ is much more delicate, but covers the important case of minimal velocity $|c| = 2$. For these solutions global stability results have been obtained in [2]. The exact asymptotic decay of the perturbations in the sense of the next section has been investigated in [21].

3 The Ginzburg-Landau equation

In this section we consider a model which comes even closer to the Taylor-Couette problem, namely the Ginzburg-Landau equation:

$$\partial_t A = \partial_x^2 A + A - |A|^2 A \ , \tag{9}$$

with $A = A(x, t) \in \mathbb{C}$, $t \geq 0$, and $x \in \mathbb{R}$. This equation has a two-parameter family of nontrivial equilibria

$$A_{\mathrm{eq},[q,\vartheta]}(x) = \sqrt{1 - q^2} e^{i(qx+\vartheta)} \ .$$

In his pioneering work, Eckhaus [14] showed that the equilibria with $q^2 < 1/3$ are linearly stable while those with $q^2 > 1/3$ are linearly unstable. In contrast to the equilibrium $u = 1$ of the previous problem the stable equilibria $A_{\mathrm{eq},[q,\vartheta]}$ are no longer exponentially stable. Indeed, the linearization always possesses continuous spectrum $(-\infty, 0]$ up to the imaginary axis, i.e., at best polynomial decay rates can be expected.

The following discussion can be carried out for all $q^2 < 1/3$. Since our aim is to explain the basic ideas, we will restrict ourselves to the stability of the front $A(x, t) = B(x - ct)$ connecting the stable equilibrium $A_{\mathrm{eq},[0,0]} = 1$ with the unstable equilibrium $A = 0$ which corresponds to the case $q = 0$, $\vartheta = 0$.

The real-valued front $B = B_{\mathrm{f},[c]}(\xi)$ satisfies the ordinary differential equation $B'' + cB' + B - B^3 = 0$. It can be found by analyzing this second order ordinary differential equation in the (B, B')–phase plane. There exists a heteroclinic connection between $(B, B') = (1, 0)$ and $(B, B') = (0, 0)$, where the latter fixed point has real eigenvalues for $|c| \geq 2$. As a consequence, when $|c| \geq 2$ the fronts satisfy $B(\xi) \in (0, 1)$.

We introduce the deviation v from the front by writing $A(x, t) = B_{\mathrm{f},[c]}(x - ct) + v(x, t)$, and if A satisfies (9) then v solves the equation

$$\partial_t v = \partial_x^2 v + v - B_{\mathrm{f},[c]}^2(2v + \overline{v}) - v^2 B_{\mathrm{f},[c]} - 2|v|^2 B_{\mathrm{f},[c]} - v|v|^2 \ . \tag{10}$$

Again, the stability of the front will follow only by an interplay of the usual norm and the spatially exponentially weighted norm. Therefore, we introduce $w(\xi) = v(\xi + ct)e^{\beta\xi}$ and the equation for w is

$$\begin{aligned}\partial_t w \ &= \ \partial_\xi^2 w + (c - 2\beta)\partial_\xi w + (\beta^2 - c\beta + 1)w \\ &\quad -B_{\mathrm{f},[c]}^2(2w + \overline{w}) - B_{\mathrm{f},[c]} v w - 2B_{\mathrm{f},[c]}\overline{v} w - |v|^2 w \ ,\end{aligned}$$

where $\xi = x - ct$. When $\beta^2 - c\beta + 1 = -2\gamma < 0$, the maximum principle applied to the system written in polar coordinates $w = |w|e^{i\varphi}$ implies the exponential stability of $w = 0$ and we get a bound $w = \mathcal{O}(\exp(-\gamma t))$ as $t \to \infty$

provided v stays bounded. To exploit the effect of the bulk, we write the difference between $B_{\mathrm{f},[c]}$ and the stable equilibrium $A_{\mathrm{eq},[0,0]} = 1$ as $g = B_{\mathrm{f},[c]} - 1$. Using g, (10) can be written as

$$\begin{aligned}\partial_t v &= \partial_x^2 v + v - (2v+\overline{v}) - v^2 - 2|v|^2 - v|v|^2 \\ &\quad -(2g+g^2)(2v+\overline{v}) - v^2 g - 2|v|^2 g \ .\end{aligned}$$

All terms in the second line vanish at some exponential rate as $x - ct \to -\infty$. We use this as above by expressing v in terms of w, for instance, $gv = (e^{-\beta\xi}g)(e^{\beta\xi}v) = (e^{-\beta\xi}g)w$, and so

$$\partial_t v = \partial_x^2 v + v - (2v+\overline{v}) - v^2 - 2|v|^2 - v|v|^2 + \mathcal{O}(w) \ .$$

Writing A in polar coordinates $A = (1+r)e^{i\varphi}$ we then find

$$\partial_t r = \partial_x^2 r - 2r - (\partial_x\varphi)^2(1+r) - 3r^2 - r^3 + \mathcal{O}(w) \ , \tag{11}$$

$$\partial_t \varphi = \partial_x^2 \varphi + 2\frac{(\partial_x\varphi)(\partial_x r)}{1+r} + \mathcal{O}(w) \ . \tag{12}$$

Linearizing this system, ignoring the terms $\mathcal{O}(w)$ which decay at some exponential rate, gives

$$\partial_t r = \partial_x^2 r - 2r \quad \text{and} \quad \partial_t\varphi = \partial_x^2\varphi \ .$$

This shows that the variable r decays with some exponential rate. However, the variable φ shows some diffusive behavior. Therefore, the variable r can be expressed asymptotically by φ, that is, $r = -(\partial_x\varphi)^2/2 + \text{h.o.t.}$. Inserting this into the equation for φ yields

$$\partial_t\varphi = \partial_x^2\varphi - (\partial_x\varphi)\partial_x(\partial_x\varphi)^2 + \text{h.o.t.} \ .$$

We explain now the arguments which are needed to analyze this equation. In order to do so we go back to the linearized system. In Fourier space φ satisfies $\partial_t\tilde{\varphi}(k,t) = -k^2\tilde{\varphi}(k,t)$, that is $\tilde{\varphi}(k,t) = \exp(-k^2 t)\tilde{\varphi}(k,0)$. Thus, the Fourier modes of $\tilde{\varphi}$ are concentrated around $k = 0$ as time evolves and so

$$\tilde{\varphi}(k/\sqrt{t},t) = \exp(-k^2)\tilde{\varphi}(k/\sqrt{t},0) = \exp(-k^2)\tilde{\varphi}(0,0) + \mathcal{O}(1/\sqrt{t}) \ ,$$

for sufficiently smooth $\tilde{\varphi}(k,0)$. Since smoothness in Fourier space corresponds to decay rates for $|x| \to \infty$ this explains the well known fact that solutions to spatially localized initial conditions decay to zero in a universal manner,

$$\varphi(x,t) = \frac{\varphi_*}{\sqrt{t}}\exp(\frac{-x^2}{4t}) + \mathcal{O}(t^{-1}) \ ,$$

with some constant $\varphi_* \in \mathbb{R}$ depending only on the initial conditions. As a consequence, we have

$$\partial_t\varphi = \mathcal{O}(t^{-3/2}), \ \partial_x^2\varphi = \mathcal{O}(t^{-3/2}) \ ,$$

but under these asymptotics,

$$-(\partial_x\varphi)\partial_x(\partial_x\varphi)^2 = \mathcal{O}(t^{-7/2}) \ .$$

Therefore, all nonlinear terms in the equations for (r,φ) vanish much faster than those which are part of the linear diffusion equation. In general, nonlinear terms $\varphi^{p_1}(\partial_x\varphi)^{p_2}(\partial_x^2\varphi)^{p_3}$ are therefore called irrelevant if $p_1 + 2p_2 + 3p_3 > 3$.

With the help of renormalization theory these heuristic arguments have been made rigorous and it has been shown that the nonlinear system possesses the same asymptotics [10, 3]. Thus spatially localized perturbations of the equilibrium $A_{\mathrm{eq},[0,0]} = 1$ vanish asymptotically like solutions of a linear diffusion problem.

Notation. Let $\|u\|_{\mathrm{L}^2}^2 = \int |u(x)|^2 dx$. We define the usual Sobolev norm by $\|u\|_{\mathrm{H}^2} = \|u\|_{\mathrm{L}^2} + \|\partial_x u\|_{\mathrm{L}^2} + \|\partial_x^2 u\|_{\mathrm{L}^2}$. Moreover, we set $\|u\|_{\mathrm{H}^2_2} = \|u\rho\|_{\mathrm{H}^2}$, where $\rho(x) = (1+x^2)^{1/2}$.

Theorem 3. *There exist positive constants δ, and C, such that for all v_0 with $\|v_0\|_{\mathrm{H}^2_2} < \delta$ the following holds. Let $A = A_{\mathrm{eq},[0,0]} + v$ be the solution of the Ginzburg-Landau equation (7) with initial condition $x \mapsto A_{\mathrm{eq},[0,0]}(x) + v_0(x) = 1 + v_0(x)$. Then v exists for all times $t \geq 0$ and there exists a constant $v_* \in \mathbb{R}$ depending only on the initial condition v_0 such that for all $t \geq 1$ one has*

$$\sup_{x\in\mathbb{R}} |v(x,t) - \frac{v_*}{\sqrt{t}}\exp(\frac{-x^2}{4t})| \leq \frac{C}{t} \ .$$

Furthermore, spatially localized perturbations of the front $A(x,t) = B_{\mathrm{f},[c]}(x-ct)$ vanish asymptotically as a solution of a linear diffusion equation [4, 17].

Theorem 4. *Let $B_{\mathrm{f},[c]}$ be a front with velocity $c > 2$. Let $\beta > 0$ be such that $\beta^2 - c\beta + 1 = -2\gamma < 0$. Then there exist positive constants δ and C such that for all v_0 with $\|x \mapsto v_0(x)(1+e^{\beta x})\|_{\mathrm{H}^2_2} < \delta$ the following holds. Let $A(x,t) = B_{\mathrm{f},[c]}(x-ct) + v(x,t)$ be the solution of the Ginzburg-Landau equation (7) with initial condition $A_0(x) = B_{\mathrm{f},[c]}(x) + v_0(x)$. Then v exists for all $t \geq 0$ and there is a constant v_* depending only on v_0 such that for all $t \geq 1$ one has*

$$\sup_{x\in\mathbb{R}} |v(x,t) - \frac{v_*}{\sqrt{t}} \exp(\frac{-x^2}{4t})| \leq \frac{C}{t} ,$$

and

$$\sup_{\xi\in\mathbb{R}} |v(\xi+ct,t)e^{\beta\xi}| \leq Ce^{-\gamma t} .$$

The case of minimal velocity $c = 2$, which is more delicate, has been handled in [4, 17] with norms which take more details of $B_{\mathrm{f},[c]}$ into account.

Remark 5. These fronts are closely related to the modulated fronts of the Taylor-Couette problem. Indeed, using the multiple scaling ansatz [32, 13],

$$U = \psi_{\varepsilon,A} + \mathcal{O}(\varepsilon^2), \quad \text{where } \psi_{\varepsilon,A}(x,t) = \varepsilon \; A(\varepsilon x, \varepsilon^2 t) \; U_{k_c}(x) e^{ik_c x} + \text{c.c.} ,$$

one sees that the Taylor-Couette problem is approximated by the Ginzburg-Landau equation for the A. Mathematical theorems which show exact relations between the Ginzburg-Landau equation and the Taylor-Couette system can be found in [9, 44, 28, 38, 39, 41].

4 The Taylor-Couette problem

In line with the discussion of the previous sections the stability proof for the modulated fronts $U_{\mathrm{mf},[q,\varepsilon,c]}(x-ct,x,z)$ mainly consists of two parts: i) The introduction of an exponential weight to stabilize the unstable part ahead of the front and ii) A stability analysis of the stationary solution $U_{\mathrm{TV},[q,\varepsilon]}$ behind the front. As above we will restrict ourselves in this paper to the case $q = 0$ which lies in the Eckhaus-stable region.

Note that for this problem the stability analysis of the equilibrium is by no means obvious, but can be found in [41]. The equilibrium $U_{\mathrm{TV},[0,\varepsilon]}$ of Eqs.(1)–(4) possesses continuous spectrum up to the imaginary axis just like $A_{\mathrm{eq},[0,0]}$ of Eq.(9). Therefore, one can expect at best polynomial decay in time. Furthermore, it has been shown that the linearly stable Taylor vortices are also nonlinearly stable with respect to small spatially localized perturbations and that the perturbations vanish in a universal way like a solution of a linear diffusion equation.

Theorem 5. *[41]. Let $\varepsilon > 0$ sufficiently small. Then there exist $\delta > 0$, and constants $C_1 > 0$, C_2 such that for all V_0 with $\|V_0\|_{\mathrm{H}^2_2} < \delta$ the following holds. Let $U = U_{\mathrm{TV},[0,\varepsilon]} + V$ be the solution of the system (1)–(4) with initial condition $(x,z) \mapsto U_{\mathrm{TV},[0,\varepsilon]}(x,z) + V_0(x,z)$. Then V exists for all times $t \geq 0$, and there exists a constant $V_* \in \mathbb{R}$ depending only on the initial condition V_0 such that for all $t \geq 1$ one has*

$$\sup_{(x,z)\in\mathbb{R}\times\Sigma} |V(x,z,t) - \frac{1}{\sqrt{t}} V_* \exp(\frac{-x^2}{4C_1 t}) \partial_x U_{\mathrm{TV},[0,\varepsilon]}(x,z)| \leq \frac{C_2}{t^{3/4}} .$$

The corresponding result for the Swift-Hohenberg equation can be found in [40, 18].

After the stability analysis of the part behind the front it remains to stabilize the zero solution ahead of the front. We again use the interplay of two norms.

We introduce W by $W(\xi,z,t) = V(\xi+ct,z,t)e^{\beta\xi}$. If we insert this into the linearization around the Couette flow we obtain an estimate for the variable W, namely

$$\|W(\cdot,\cdot,t)\|_{\mathrm{H}^2} \leq Ce^{-\rho(c,\beta,\varepsilon)t} \|W(\cdot,\cdot,0)\|_{\mathrm{H}^2} .$$

For given $\varepsilon > 0$, sufficiently small, there is a minimal velocity $c_0 > 0$ such that for all $c > c_0$ there are β_l and β_u such that for all $\beta \in (\beta_l, \beta_u)$ one has $\rho(c,\beta,\varepsilon) > 0$. Given such an $\varepsilon > 0$ and a $c > c_0$ we choose now a β such that

$$\rho(c,\beta,\varepsilon) = 2\gamma > 0 .$$

Then we have:

Theorem 6. *Let $\varepsilon > 0$ sufficiently small. Then there exist a $\delta > 0$, and constants $C_1 > 0$, C_2 such that for all V_0 with $\|(x,z) \mapsto V_0(x,z)(1+e^{\beta x})\|_{\mathrm{H}^2_2} < \delta$ the following holds. Let $U(x,z,t) = U_{\mathrm{mf},[0,\varepsilon,c]}(x-ct,x,z) + V(x,z,t)$ be a solution of the system (1)–(4) with initial condition $(x,z) \mapsto U_{\mathrm{mf},[0,\varepsilon,c]}(x,x,z) + V_0(x,z)$. Then V exists for all times $t \geq 0$, and there exists a constant $V_* \in \mathbb{R}$ depending only on the initial condition V_0 such that for all $t \geq 1$ one has*

$$\sup_{(x,z)\in\mathbb{R}\times\Sigma} |V(x,z,t) - \frac{1}{\sqrt{t}} V_* \exp(\frac{-x^2}{4C_1 t}) \partial_x U_{\mathrm{TV},[0,\varepsilon]}(x,z)| \leq \frac{C_2}{t^{3/4}} \ , \tag{13}$$

and

$$\sup_{(\xi,z)\in\mathbb{R}\times\Sigma} |V(\xi+ct,z,t)e^{\beta\xi}| \leq C_2 e^{-\gamma t} \ .$$

Sketch of proof. We proceed as in the previous sections. First we write (1)–(4) as a dynamical system in the space of divergence free vector fields [23]. Then we consider the system for V and the exponentially weighted variable W. The proof is exactly the same as in [19] given for the fronts in the Swift-Hohenberg equation. The analysis behind the front and the complete functional analytic set-up can be found in [41]. Finally, in [42], one finds the estimate showing that the Taylor-Couette problem is approximated by the Ginzburg-Landau equation. This is used ahead of the front. □

Remark 6. One can interpret (13) as saying that the perturbations decay faster near the extrema of U_{TV}. This can also be understood in the sense that at large amplitude of the underlying periodic pattern (i.e., where $\partial_x U_{\mathrm{TV}}(x,z) = 0$), the restoring force is stronger.

Remark 7. The choice of a sufficiently small $\varepsilon > 0$ actually allows to prove the stability of *all* fronts which are predicted to be stable by the associated amplitude equation (the Ginzburg-Landau equation) since

$$\lim_{\varepsilon\to 0} \varepsilon^{-2} \varrho(\varepsilon c_B, \varepsilon\beta_A, \varepsilon) = \varrho_A(c_B, \beta_A) \ ,$$

where $\varrho_A(c_B, \beta_A)$ is the stability condition of the associated amplitude equation.

Remark 8. Note that the Taylor-Couette problem is structured by the periodic background of the stationary solution. In such a setting it is natural to study perturbations (which need not be periodic) in the so-called Bloch wave representation (see e.g. [34]). This means that one writes

$$u(x) = \int_{-1/2}^{1/2} e^{i\ell x} \hat{u}(\ell, x) \ ,$$

where

$$\hat{u}(\ell, x) = \sum_{n\in\mathbb{Z}} e^{inx} \tilde{u}(n+\ell) \ ,$$

with $\tilde{u}$ the Fourier transform of u. Since the linearized problem for the Taylor vortices is periodic, the corresponding linear operator is diagonal in the ℓ and thus the Bloch representation simplifies the analysis. Nonlinear terms can be written in Bloch space with the help of convolution integrals. Decay for $|x| \to \infty$ corresponds in Fourier and Bloch space to smoothness with respect to the Fourier wave number k and Bloch wave number ℓ, respectively. As a consequence, the important relation $\partial_x \cdot = \mathcal{O}(t^{-1/2})$ for the irrelevance of terms corresponds in Bloch space to $\ell \cdot = \mathcal{O}(t^{-1/2})$.

Remark 9. The diffusive behavior of solutions close to the Taylor vortices can be understood as follows: The eigenfunctions of the linearization of the Taylor vortices are given by Bloch waves $e^{i\ell x} v_n(\ell, x, z)$ with $v_n(\ell, x, z) = v_n(\ell, x + 2\pi/k_{\mathrm{c}}, z)$ having the same periodicity as the Taylor vortices. For each fixed Bloch wave number ℓ there is discrete spectrum, i.e., $n \in \mathbb{N}$. Therefore, there are smooth curves $\mu_n(\ell)$ of eigenvalues over the Bloch wave numbers.

There is one curve, μ_1, coming up to zero. This curve has approximately the form of a parabola, i.e., $\mu_1(\ell) = -c_1\ell^2 + \mathcal{O}(\ell^3)$ with $c_1 > 0$ in the Eckhaus stable region. Thus, the solution $\hat{v}_1$ rescaled in Bloch space satisfies

$$\lim_{t\to\infty} \hat{v}_1(\ell/\sqrt{t}, t) = v_* e^{-c_1\ell^2} \ ,$$

for smooth initial conditions in Bloch space, and shows the same asymptotic behavior as the linear diffusion equation $\partial_t v_1 = c_1 \partial_x^2 v_1$ for spatially localized initial conditions.

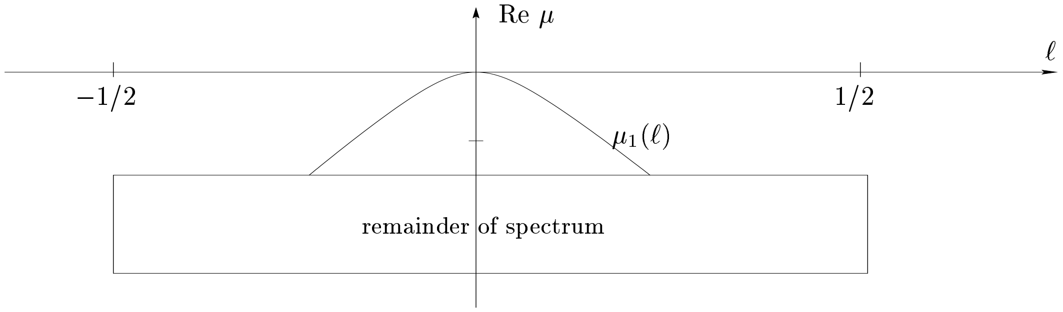

Figure 4: Spectrum of the linearization around the Taylor vortices as a function over the Bloch wave numbers.

Remark 10. As for the case of the Ginzburg-Landau equation we have, for the perturbations of the Taylor vortices, asymptotically irrelevant nonlinearities. This is not obvious and has been proved in [41] by deriving an effective equation for the variable corresponding to the curve of critical eigenvalues μ_1 and proving that the coefficients in front of the quadratic and cubic nonlinear terms vanish up to a certain order as the Bloch wave number ℓ goes to zero. These facts are a reflection of the translation invariance and of the existence of a circle of fixed points for the stationary problem.

Remark 11. The nonlinear stability of so-called critical fronts (moving at the minimal possible speed for which they are linearly stable) remains open. See [7] for the linear stability analysis of the fronts in the Swift-Hohenberg equation. Achieving this aim seems to be a necessary step in solving the long-standing problem of "front selection" [12], in a case where the maximum principle [1] is not available.

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