Non-linear Stability of Modulated Fronts for the Swift-Hohenberg Equation

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Abstract. We consider front solutions of the Swift-Hohenberg equation $\partial_t u = -(1 + \partial_x^2)^2 u + \varepsilon^2 u - u^3$. These are traveling waves which leave in their wake a periodic pattern in the laboratory frame. Using renormalization techniques and a decomposition into Bloch waves, we show the non-linear stability of these solutions. It turns out that this problem is closely related to the question of stability of the trivial solution for the model problem $\partial_t u(x,t) = \partial_x^2 u(x,t) + (1 + \tanh(x - ct))u(x,t) + u(x,t)^p$ with $p > 3$. In particular, we show that the instability of the perturbation ahead of the front is entirely compensated by a diffusive stabilization which sets in once the perturbation has hit the bulk behind the front.

Contents.

1. Statement of the problem

We consider the Swift-Hohenberg equation

$$\partial_t u \;=\; -(1+\partial_x^2)^2 u + \varepsilon^2 u - u^3 \;, \tag{1.1}$$

with $u(x,t) \in \mathbf{R}$, $x \in \mathbf{R}$, $t \geq 0$ and $0 < \varepsilon \ll 1$ a small bifurcation parameter. It has been shown some time ago that a 2-parameter family of (small) spatially periodic solutions exists which are independent of t. These solutions correspond to a periodic pattern which exists in the laboratory frame. These solutions are of the form

$$U_{q,a}(x) \;=\; A_q \cos\bigl((1+\varepsilon q)x + a\bigr) + \mathcal{O}(\varepsilon^2) \;,$$

which bifurcate from the solution $u \equiv 0$. Here,

$$A_q \;=\; 2\varepsilon(1-4q^2)^{1/2} \;.$$

It is furthermore well-known and proved in [CE90a] that these solutions are marginally stable for $4|q|^2 \leq \frac{1}{3}$, the so-called Eckhaus stability range ([Eck65]), and that the spectrum of the linearization about these solutions is all of $\mathbf{R}^-$. Finally, after a long time it was shown in [Schn96] that these solutions are also non-linearly stable, and this proof was the presented in a slightly different form in [EWW97].

In another direction, in earlier work of [CE86] and [EW91] traveling wave solutions of a special kind leaving a *fixed* pattern in the laboratory space were shown to exist, and their linear stability was studied in [CE87]. Our present paper is concerned with a first proof of the *non-linear* stability of these traveling solutions.

We first describe the traveling solutions. One way to view them is to write

$$U_{q,a}(x) \;=\; \tfrac{1}{2} \sum_{n \in 2\mathbf{Z}+1} A_{q,n} e^{in((1+\varepsilon q)x+a)} \;,$$

where $A_{q,1} = A_q$ as defined above and $A_{q,-n} = \bar{A}_{q,n}$, with $\bar{x}$ the complex conjugate of x. Here, the $A_{q,n}$ are in fact $\mathcal{O}(\varepsilon^{|n|})$, and furthermore $U_{q,a}$ extends to an analytic function. The modulated front solutions are then of the form

$$u(x,t) \;=\; F_{c,q,a}(x-ct, x) \;,$$

with

$$F_{c,q,a}(\xi, x) \;=\; \tfrac{1}{2} \sum_{n \in 2\mathbf{Z}+1} W_{c,q,n}(\xi) e^{in((1+\varepsilon q)x+a)} \;. \tag{1.2}$$

Note that these are *not* classical traveling waves of the form $u(x-ct)$, and note furthermore that $F_{c,q,a}$ is periodic in its second argument (with period $2\pi/(1+\varepsilon q)$). The modulated front solutions satisfy [CE86, EW91], when $c > 0$:

$$\lim_{\xi \to -\infty} W_{c,q,n}(\xi) \;=\; A_{q,n} \;, \quad \lim_{\xi \to \infty} W_{c,q,n}(\xi) \;=\; 0 \;.$$

These modulated front solutions are constructed with the help of a center manifold reduction, where all $W_{c,q,n}$ are determined by the central modes $W_{c,q,\pm1}$. In the reduced four-dimensional system for $W_{c,q,\pm1} = W_{c,q,\pm1}(\xi)$ there is a heteroclinic connection lying in the intersection of a four-dimensional stable manifold of the origin and a two-dimensional unstable manifold of an equilibrium corresponding to $U_{q,a}$. Since this is a very robust situation these solutions can be constructed by some perturbation analysis from the ones for $q = 0$. For small ε and $q = 0$ the solution $W_{c,0,1}$ of the amplitude equation on the center manifold is close to the real-valued front solution $W_{c,0,1}(\xi) = \varepsilon B(\varepsilon\xi) = \varepsilon B(\zeta)$ of the equation

$$4\partial_\zeta^2 B + c_B \partial_\zeta B + B - 3B|B|^2 = 0 ,$$

connecting $W_{c,0,1} = 0$ at $\zeta = +\infty$ with $W_{c,0,1} = A_0$ at $\zeta = -\infty$. The constant c_B is given by $c_B = \varepsilon^{-1}c = \mathcal{O}(1)$. Our paper deals with the question: *Under which conditions does the solution of (1.1) with initial data $F_{c,q,a}(x,x) + v(x)$ converge to $F_{c,q,a}(x - ct, x)$ as $t \to \infty$?*

We will show our results for the case $q = 0$ and $a = 0$ only, to keep the notation on a reasonable level. The extension to arbitrary a is trivial by translating the origin, while the extension to arbitrary q satisfying $4|q|^2 < \frac{1}{3}$ necessitates some notational work and leads to bounds which depend on q. Thus, we will write the periodic solution as

$$U_*(x) = A\cos x + \mathcal{O}(\varepsilon^2) , \tag{1.3}$$

with $A = 2\varepsilon$, and the modulated front (moving with speed $c = \mathcal{O}(\varepsilon)$) as

$$F_c(\xi, x) = \tfrac{1}{2} \sum_{n\in 2\mathbf{Z}+1} W_c(\xi) e^{inx} .$$

We describe next the nature of the stability problem. Consider an initial condition $u_0(x) = F_c(x,x) + v_0(x)$, and let $u(x,t)$ denote the solution of (1.1) with that initial condition. Since F_c solves (1.1), we find for the evolution of $v(x,t) \equiv u(x,t) - F_c(x - ct, x)$:

$$\partial_t v(x,t) = \big(Lv\big)(x,t) - 3F_c(x-ct,x)^2 v(x,t) - 3F_c(x-ct,x)v(x,t)^2 - v(x,t)^3 . \tag{1.4}$$

Here, $L = -(1+\partial_x^2)^2 + \varepsilon^2$. We define the translation operator τ_{ct} by $(\tau_{ct}f)(x) = f(x - ct, x)$, so that (1.4) can be written as

$$\partial_t v = Lv - 3(\tau_{ct}F_c)^2 v - 3(\tau_{ct}F_c)v^2 - v^3 . \tag{1.5}$$

Introduce now K_{ct} (the difference between the modulated front and the periodic solution) by

$$K_{ct}(x) = \big(\tau_{ct}F_c\big)(x) - U_*(x) = F_c(x-ct,x) - U_*(x) . \tag{1.6}$$

Note that $K_{ct}(x)$ vanishes as $x \to -\infty$, and approaches $-U_*(x)$ as $x \to \infty$. With these notations we can rewrite (1.5) as

$$\begin{aligned} \partial_t v &= Lv - 3U_*^2 v - 6U_* K_{ct} v - 3K_{ct}^2 v - 3U_* v^2 - v^3 - 3K_{ct} v^2 \\ &= \mathcal{M}v + \mathcal{M}_{\mathbf{i}} v + \mathcal{N}(v) + \mathcal{N}_{\mathbf{i}}(v) , \end{aligned} \tag{1.7}$$

where

$$\begin{aligned} \mathcal{M}v &= Lv - 3U_*^2 v \ , \\ \mathcal{M}_{\mathbf{i}}v &= -6U_* K_{ct}v - 3K_{ct}^2 v \ , \\ \mathcal{N}(v) &= -3U_* v^2 - v^3 \ , \\ \mathcal{N}_{\mathbf{i}}(v) &= -3K_{ct}v^2 \ . \end{aligned} \tag{1.8}$$

The variables with index **i** vanish with some exponential rate for fixed $x \in \mathbf{R}$ in the laboratory frame. They will be seen to be exponentially "irrelevant" in terms of a renormalization group analysis. In order to explain this renormalization problem, we will study, in the next section the model problem

$$\partial_t u(x,t) \ = \ \partial_x^2 u(x,t) + a(x-ct)u(x,t) + u(x,t)^p \ ,$$

with $a(\xi) = \frac{1}{2}(1 + \tanh \xi)$, and $p > 3$. This problem is nice in its own right. The similitude will come from the correspondence of $\mathcal{M}$ with ∂_x^2, and of $\mathcal{M}_{\mathbf{i}}v$ with the term $a(x-ct)u(x,t)$. Indeed:

- the first term will be seen to be diffusive in the laboratory frame,
- the second term will be seen to be irrelevant in the laboratory frame, but the first together with the second term will be exponentially damping in a suitable space of exponentially decaying functions in a frame moving with a speed close to c.

As in previous work [Sa77, BK94, Ga94, EW94] our analysis will be based on an interplay of estimates obtained in these two topologies.

Our main results are stated in Theorem 4.1 for the simplified problem and in Theorem 7.1 for the Swift-Hohenberg problem. We not only show convergence to the front, but give also precise first order estimates in both cases. As far as possible, the treatment of the two problems is done in analogous fashion, so that the reader who has followed the proof of the simplified problem should have no difficulty in reading the proof for the full, more complicated, problem.

Remark. An ideal treatment of this problem would necessitate a norm in a frame moving with the *same* speed as the front. Such a space is needed to study the stability of so-called critical fronts (moving at the minimal possible speed where they are linearly stable). Achieving this aim seems to be a necessary step in solving the long-standing problem of "front selection" [DL83], in a case where the maximum principle [AW78] is not available.

Remark. The method also applies to more complicated systems, like hydrodynamic stability problems. A typical example are the fronts connecting the Taylor vortices with the Couette flow in the Taylor-Couette problem. These fronts have been constructed in [HS99]. The stability of the spatially periodic Taylor vortices has been shown in [Schn98].

Notation. Throughout this paper many different constants are denoted with the same symbol C.

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Part I. A simplified problem

2. The model equation

Let

$$a(\xi) = \tfrac{1}{2}(1 + \tanh \xi) . \tag{2.1}$$

We want to study the equation

$$\partial_t u(x,t) = \partial_x^2 u(x,t) + a(x - ct)u(x,t) + u(x,t)^p , \tag{2.2}$$

with $c > 0$ and $p > 3$. For notational simplicity we assume $p \in \mathbf{N}$.

To understand the dynamics of (2.2) it might be useful to consider the following simplified problem

$$\partial_t v(x,t) = \partial_x^2 v(x,t) + \vartheta(x - ct)v(x,t) , \tag{2.3}$$

where $\vartheta(z) = 1$ when $z > 0$ and $\vartheta(z) = 0$ when $z < 0$. If we go to the moving frame $\xi = x - ct$ and let $w(\xi, t) = v(\xi + ct, t) = v(x, t)$, then the equation for w becomes

$$\partial_t w(\xi,t) = \partial_\xi^2 w(\xi,t) + c\partial_\xi w(\xi,t) + \vartheta(\xi)w(\xi,t) . \tag{2.4}$$

For $x > 0$, we have $\vartheta(x) = 1$ and hence the corresponding characteristic polynomial for (2.4) (in momentum space) is

$$-k^2 + ick + 1 ,$$

while for $x < 0$, we have $\vartheta(x) = 0$ with its corresponding polynomial

$$-k^2 + ick .$$

Thus, we expect the solution to be exponentially unstable ahead of the front, *i.e.*, for $x > 0$, and diffusively stable behind the front. If we consider an initial condition $v_0(\xi)$ localized near $\xi = \xi_0 > 0$, and of amplitude A, then we expect the amplitude to grow like $e^t A$ until $t = t_* = \xi_0/c$, when this perturbation "hits" the back of the front (in the moving frame), or, in other words, when the back of the front hits the perturbation (in the laboratory frame). Thus, the perturbation does not grow larger than $Ae^{\xi_0/c}$. We use this in the following way. Assume that the amplitude at $\xi > 0$ is bounded by $Ae^{-\beta\xi}$. Then, ignoring diffusion, we find that the contribution to the amplitude at the origin at time $t = \xi_0/c$ is bounded by

$$\int_0^{\xi_0} \mathrm{d}\xi \, Ae^{\xi(1-\beta c)/c} .$$

Clearly, if $\beta c > 1$, the initial perturbations are sufficiently small for the total effect at the origin (in the moving frame) to be small.

Once this has happened, a second epoch starts where the perturbation is behind the front. Then, due to the diffusive behavior, the amplitude will go down as

$$\frac{C}{(t - t_* + 1)^{1/2}} .$$

These considerations will be used in the choice of topology below.

2.1. Function spaces and Fourier transform

We start the precise analysis and will work in Fourier space and revert to the x-variables only at the end of the discussion. We define the Fourier transform by

$$\big(\mathcal{F}f\big)(k) \;=\; \frac{1}{2\pi}\int \mathrm{d}x\, f(x)e^{-ikx}\;.$$

Notation. If f denotes a function, then $\tilde{f}$ is defined by $\tilde{f} = \mathcal{F}f$, and if $\mathcal{A}$ is an operator, then $\tilde{\mathcal{A}}$ is defined by $\tilde{\mathcal{A}} = \mathcal{F}\mathcal{A}\mathcal{F}^{-1}$. We also use the notation $\tilde{f} * \tilde{g}$ for the convolution product $\big(\widetilde{fg}\big)(k) = \big(\tilde{f} * \tilde{g}\big)(k) = \int \mathrm{d}\ell \tilde{f}(k-\ell)\tilde{g}(\ell)$. Finally, $\widetilde{T}_\zeta$ denotes the conjugate of translation:

$$(\widetilde{T}_\zeta \tilde{f})(k) \;=\; e^{-i\zeta k}\tilde{f}(k)\;, \tag{2.5}$$

so that the Fourier transform of $\big(T_\zeta f\big)(x) = f(x-\zeta)$ is

$$\mathcal{F}T_\zeta f \;=\; \widetilde{T}_\zeta \mathcal{F}f\;.$$

The relation ([Ta97])

$$k^\alpha \partial_k^\beta (\mathcal{F}f)(k) = (-i)^{\alpha+\beta}\mathcal{F}(\partial_x^\alpha x^\beta f)(k)$$

motivates the introduction of the following norms: We fix a small $\delta > 0$ and define

$$\|\tilde{f}\|_{\tilde{\mathrm{H}}_n^{m,\delta}} \;=\; \left(\sum_{\ell=0}^{n}\sum_{j=0}^{m}\delta^{2(\ell+j)}\int \mathrm{d}k\,|\partial_k^j(k^\ell \tilde{f}(k))|^2\right)^{1/2}\;. \tag{2.6}$$

The dual norm to this is

$$\|f\|_{\mathrm{H}^n_{m,\delta}} \;=\; \left(\sum_{\ell=0}^{n}\sum_{j=0}^{m}\delta^{2(\ell+j)}\int \mathrm{d}x\,|\partial_x^\ell f(x)|^2 x^{2j}\right)^{1/2}\;. \tag{2.7}$$

Parseval's identity immediately leads to:

$$\|f\|_{\mathrm{H}^n_{m,\delta}} \;=\; \|\mathcal{F}f\|_{\tilde{\mathrm{H}}_n^{m,\delta}}\;.$$

In the following we mainly work with the spaces to $m = n = 2$ and $m = 0$, $n = 2$. For some constant C independent of $1 \geq \delta > 0$,

$$\begin{aligned} \|fg\|_{\mathrm{H}^2_{2,\delta}} \;&\leq\; C\|f\|_{\mathrm{H}^2_{2,\delta}}\|g\|_{\mathrm{H}^2_{2,\delta}}\;,\\ \|\tilde{f} * \tilde{g}\|_{\tilde{\mathrm{H}}_2^{2,\delta}} \;&\leq\; C\|\tilde{f}\|_{\tilde{\mathrm{H}}_2^{2,\delta}}\|\tilde{g}\|_{\tilde{\mathrm{H}}_2^{2,\delta}}\;, \end{aligned} \tag{2.8}$$

or in a stronger version

$$
\begin{aligned}
\|fg\|_{\mathrm{H}^2_{2,\delta}} &\le C\|f\|_{\mathrm{H}^2_{2,\delta}}\|g\|_{\mathrm{H}^2_{0,\delta}} ,\\
\|\tilde f * \tilde g\|_{\tilde{\mathrm{H}}_2^{2,\delta}} &\le C\|\tilde f\|_{\tilde{\mathrm{H}}_2^{2,\delta}}\|\tilde g\|_{\tilde{\mathrm{H}}_2^{0,\delta}} .
\end{aligned}
\tag{2.9}
$$

Finally, we shall also need the inequality

$$
\|\tilde f * \tilde g\|_{\tilde{\mathrm{H}}_2^{0,\delta}} \le \|f\|_{\mathcal{C}^2_{\mathrm{b},\delta}}\|\tilde g\|_{\tilde{\mathrm{H}}_2^{0,\delta}} ,
\tag{2.10}
$$

where

$$
\|f\|_{\mathcal{C}^2_{\mathrm{b},\delta}} = \sum_{j=0}^{2} \delta^j \sup_{x\in\mathbf{R}} |\partial_x^j f(x)| .
\tag{2.11}
$$

This follows from

$$
\|\tilde f * \tilde g\|_{\tilde{\mathrm{H}}_2^{0,\delta}} = \|f\cdot g\|_{\mathrm{H}^2_{0,\delta}} \le \|f\|_{\mathcal{C}^2_{\mathrm{b},\delta}}\|g\|_{\mathrm{H}^2_{0,\delta}} = \|f\|_{\mathcal{C}^2_{\mathrm{b},\delta}}\|\tilde g\|_{\tilde{\mathrm{H}}_2^{0,\delta}} ,
$$

where the inequality above is a direct consequence of the definition of $\tilde{\mathrm{H}}_2^{0,\delta}$.

We define the map $\mathcal{W}_{\beta,\hat c t}$ by

$$
(\mathcal{W}_{\beta,\hat c t} f)(\xi) = f(\xi+\hat c t)e^{\beta\xi} ,
\tag{2.12}
$$

where $\beta \in (0,\beta_*)$ and $\hat c \in (0,c)$ will be fixed later. The Fourier conjugate of this operator then satisfies

$$
\big(\widetilde{\mathcal{W}}_{\beta,\hat c t}\tilde f\big)(k) \equiv \big(\mathcal{F}\mathcal{W}_{\beta,\hat c t}\mathcal{F}^{-1}\tilde f\big)(k) = e^{i(k+i\beta)\hat c t}\tilde f(k+i\beta) ,
\tag{2.13}
$$

as one sees from the following equalities:

$$
\begin{aligned}
2\pi(\widetilde{\mathcal{W}}_{\beta,\hat c t}\tilde f)(k) &= \int \mathrm{d}\xi\, e^{-ik\xi}\big(\mathcal{W}_{\beta,\hat c t} f\big)(\xi)\\
&= \int \mathrm{d}\xi\, e^{-ik\xi} f(\xi+\hat c t)e^{\beta\xi}\\
&= \int \mathrm{d}\xi\, e^{-i(k+i\beta)\xi} f(\xi+\hat c t)\\
&= \int \mathrm{d}\xi\, e^{-i(k+i\beta)(\xi-\hat c t)} f(\xi)\\
&= 2\pi e^{i(k+i\beta)\hat c t}\tilde f(k+i\beta) .
\end{aligned}
$$

This calculation also shows that *if* $f(\xi)e^{\beta_*\xi} \in \mathrm{H}^2_{0,\delta}$ *for* $f \in \mathcal{C}^2_{\mathrm{b},\delta}$*, then* $\widetilde{\mathcal{W}}_{\beta,\hat c t}\tilde f$ *extends to an analytic function in* $\{0 < \mathrm{Im}\, k < \beta_*\}$ *and* $(\widetilde{\mathcal{W}}_{\beta,\hat c t}\tilde f)(\cdot - i\beta) \in \tilde{\mathrm{H}}_2^{0,\delta}$ *for all* $\beta \in [0,\beta_*)$.

Remark. Since the norms for different δ are equivalent, all theorems throughout this paper can also be formulated in a version with $\delta = 1$.

3. The linear simplified problem

In this section we study the linearization of equation (2.2):

$$\partial_t U(x,t) \;=\; \partial_x^2 U(x,t) + a(x-ct)U(x,t)\ . \tag{3.1}$$

The function a is given as

$$a(\xi) \;=\; \tfrac{1}{2}(1+\tanh\xi)\ , \tag{3.2}$$

but our methods will work for many other functions. The crucial property we need is the existence of a $\beta_* > 0$ such that $a(\xi)e^{-\beta\xi}$ satisfies

$$\|\xi \mapsto a(\xi)e^{-\beta\xi}\|_{\mathrm{H}^2_{2,\delta}} \;\le\; C\ , \tag{3.3}$$

for all $\beta \in (0,\beta_*)$. For the case of (3.2) we can take $\beta_* = 2$. The Fourier transform $\tilde a$ of a is therefore a tempered distribution which is the boundary value of a function (again called $\tilde a$) which is analytic in the strip $\{z \mid 0 > \mathrm{Im}\, z > -\beta_*\}$. Furthermore, there is a K such that, for all $\delta \in (0,1]$,

$$\|a\|_{\mathcal{C}^2_{\mathrm{b},\delta}} \;\le\; 1 + K\delta\ , \tag{3.4}$$

since

$$\sup_{x\in\mathbf{R}} |a(x)| \;\le\; 1\ . \tag{3.5}$$

The bound (3.5) will be tacitly used later.

The next proposition describes how solutions of (3.1) tend to 0 as $t \to \infty$. We write $U_t(x)$ for $U(x,t)$ and use similar notation for other functions of space and time.

Proposition 3.1. *Assume that there are a β and a $\hat c \in (0,c)$ such that $\beta^2 - \beta\hat c + 1 \equiv -2\gamma < 0$. Then there exists a $\delta \in (0,1]$ such that the following holds. Assume that $U_0 \in \mathrm{H}^2_{2,\delta}$ and that* $W_0(\xi) = \big(\mathcal{W}_{\beta,0}U_0\big)(\xi) = U_0(\xi)e^{\beta\xi} \in \mathrm{H}^2_{0,\delta}$. (These conditions are independent of $\delta > 0$.) *Then the solution $U_t(x) = U(x,t)$ of (3.1) with initial data U_0 exists for all $t > 0$ and with $\tilde\psi(k) = e^{-k^2}$ the rescaled solution $\tilde V(k,t) = \tilde U(kt^{-1/2},t)$ satisfies*

$$\|\tilde V_t - \tilde U_0(0)\tilde\psi\|_{\tilde{\mathrm{H}}_2^{2,\delta}} \;\le\; \frac{C}{(1+t)^{1/2}}\|\tilde U_0\|_{\tilde{\mathrm{H}}_2^{2,\delta}}\ . \tag{3.6}$$

The function $\widetilde W_t = \widetilde{\mathcal{W}}_{\beta,\hat c t}\tilde U_t$ satisfies

$$\|\widetilde W_t\|_{\tilde{\mathrm{H}}_2^{0,\delta}} \;\le\; Ce^{-3\gamma t/2}\|\widetilde W_0\|_{\tilde{\mathrm{H}}_2^{0,\delta}}\ . \tag{3.7}$$

The constant C does not depend on U_0.

Remark. Note that it is optimal to choose $\hat c$ arbitrarily close to c.

Proof. First of all, we rewrite the equation (3.1) for U_t in terms of $\widetilde{U}_t$ and $\widetilde{W}_t$: The equation for $W_t = \mathcal{W}_{\beta,\hat{c}t} U_t$ is

$$\begin{aligned}\partial_t W(\xi,t) &= \partial_\xi^2 W(\xi,t) + (\hat{c}-2\beta)\partial_\xi W(\xi,t)\\ &\quad + a(\xi-(c-\hat{c})t)W(\xi,t) + (\beta^2-\beta\hat{c})W(\xi,t)\,.\end{aligned} \tag{3.8}$$

Taking Fourier transforms we then find, omitting the argument k and using the notation of (2.5):

$$\partial_t \widetilde{U}_t = -k^2\widetilde{U}_t + (\widetilde{T}_{ct}\tilde{a}) * \widetilde{U}_t\,, \tag{3.9}$$

$$\partial_t \widetilde{W}_t = \big(\beta^2-\beta\hat{c}-k^2+ik(\hat{c}-2\beta)\big)\widetilde{W}_t + (\widetilde{T}_{(c-\hat{c})t}\tilde{a}) * \widetilde{W}_t\,. \tag{3.10}$$

It is at this point that the simultaneous choice of two representations for the solution and their associated topologies is crucial.

We first show that $\widetilde{W}_t$ converges to 0, *i.e.*, we show (3.7). We find from (2.10):

$$\|(\widetilde{T}_\zeta\tilde{a}) * \tilde{f}\|_{\tilde{\mathrm{H}}_2^{0,\delta}} \le \|a(\cdot-\zeta)\|_{\mathcal{C}^2_{\mathrm{b},\delta}} \cdot \|\tilde{f}\|_{\tilde{\mathrm{H}}_2^{0,\delta}} = \|a\|_{\mathcal{C}^2_{\mathrm{b},\delta}} \cdot \|\tilde{f}\|_{\tilde{\mathrm{H}}_2^{0,\delta}}\,. \tag{3.11}$$

Therefore, (3.4) implies

$$\|(\widetilde{T}_{(c-\hat{c})t}\tilde{a}) * \widetilde{W}_t\|_{\tilde{\mathrm{H}}_2^{0,\delta}} \le (1+K\delta)\|\widetilde{W}_t\|_{\tilde{\mathrm{H}}_2^{0,\delta}}\,,$$

and we get from (3.10) the bound

$$\tfrac{1}{2}\partial_t\|\widetilde{W}_t\|^2_{\tilde{\mathrm{H}}_2^{0,\delta}} \le (\beta^2-\beta\hat{c}+1+K\delta)\|\widetilde{W}_t\|^2_{\tilde{\mathrm{H}}_2^{0,\delta}}\,.$$

We choose $\delta > 0$ so small that

$$\beta^2-\beta\hat{c}+1+K\delta \le -3\gamma/2\,.$$

Integrating over t we get from the choice of β, δ, and $\hat{c}$:

$$\|\widetilde{W}_t\|_{\tilde{\mathrm{H}}_2^{0,\delta}} \le e^{-3\gamma t/2}\|\widetilde{W}_0\|_{\tilde{\mathrm{H}}_2^{0,\delta}}\,. \tag{3.12}$$

Thus, we have shown Eq.(3.7).

Next, we study $\widetilde{U}$. From (2.13) and deforming the contour of integration, we get

$$\begin{aligned}\big((\widetilde{T}_\zeta\tilde{a}) * \tilde{f}\big)(k) &= \int \mathrm{d}\ell\, e^{-i\zeta(k-\ell)}\tilde{a}(k-\ell)\tilde{f}(\ell)\\ &= \int \mathrm{d}\ell\, e^{-i\zeta(k-\ell)}\tilde{a}(k-\ell)\big(\widetilde{\mathcal{W}_{\beta,\hat{c}t}f}\,\big)(\ell-i\beta)e^{-i\ell\hat{c}t}\\ &= \int \mathrm{d}\ell\, e^{-i\zeta(k-\ell-i\beta)}\tilde{a}(k-\ell-i\beta)\big(\widetilde{\mathcal{W}_{\beta,\hat{c}t}f}\,\big)(\ell)e^{-i(\ell+i\beta)\hat{c}t}\\ &= e^{-\beta(\zeta-\hat{c}t)}\int \mathrm{d}\ell\, e^{-i\zeta(k-\ell)}\tilde{a}(k-\ell-i\beta)\big(\widetilde{\mathcal{W}_{\beta,\hat{c}t}f}\,\big)(\ell)e^{-i\ell\hat{c}t}\,.\end{aligned} \tag{3.13}$$

Let $\tilde{h}(k) = e^{-ictk}\tilde{a}(k - i\beta)$ and $\tilde{g}(k) = e^{-ik\hat{c}t}\big(\widetilde{\mathcal{W}_{\beta,\hat{c}t}U_t}\big)(k) = e^{-ik\hat{c}t}\widetilde{W}_t(k)$. Then (3.13) implies

$$\big(\widetilde{T}_{ct}\tilde{a}\big) * \widetilde{U}_t \;=\; e^{-\beta(c-\hat{c})t}\tilde{h} * \tilde{g}\;.$$

From this and (3.3) we conclude that

$$\begin{aligned} \|\big(\widetilde{T}_{ct}\tilde{a}\big) * \widetilde{U}_t\|_{\tilde{\mathrm{H}}_2^{2,\delta}} \;&=\; e^{-\beta(c-\hat{c})t}\|\tilde{h} * \tilde{g}\|_{\tilde{\mathrm{H}}_2^{2,\delta}} \\ &\le\; Ce^{-\beta(c-\hat{c})t}\|\tilde{h}\|_{\tilde{\mathrm{H}}_2^{2,\delta}}\;\|\tilde{g}\|_{\tilde{\mathrm{H}}_2^{0,\delta}} \\ &\le\; C(1+tc)^2e^{-\beta(c-\hat{c})t}\|\widetilde{W}_t\|_{\tilde{\mathrm{H}}_2^{0,\delta}}\;. \end{aligned} \tag{3.14}$$

On the other hand, from (3.7) we know that $\|\widetilde{W}_t\|_{\tilde{\mathrm{H}}_2^{0,\delta}}$ stays bounded (it actually decays exponentially), and thus the evolution equation for $\widetilde{U}_t$ is of the form

$$\partial_t\widetilde{U}_t(k) \;=\; -k^2\widetilde{U}_t(k) + \tilde{f}(k,t)(1+tc)^2e^{-\beta(c-\hat{c})t}\;,$$

with $\|\tilde{f}(\cdot,t)\|_{\tilde{\mathrm{H}}_2^{2,\delta}}$ uniformly bounded in t. Since, by construction, $\hat{c} < c$, we conclude that (3.6) holds, using well-known arguments which will be made explicit in the proof of Theorem 4.1. The proof of Proposition 3.1 is complete. □

4. The renormalization approach for the simplified problem

We consider now the non-linear problem (2.2) and its related version for $\tilde{w}_t = \widetilde{\mathcal{W}}_{\beta,\hat{c}t}\tilde{u}_t = \mathcal{F}\mathcal{W}_{\beta,\hat{c}t}u_t$ in Fourier space. It takes the form

$$\begin{aligned} \partial_t\tilde{u}_t \;&=\; -k^2\tilde{u}_t + \big(\widetilde{T}_{ct}\tilde{a}\big) * \tilde{u}_t + \tilde{u}_t^{*p}\;, \\ \partial_t\tilde{w}_t \;&=\; \big(\beta^2 - \beta\hat{c} - k^2 + ik(\hat{c} - 2\beta)\big)\tilde{w}_t + \big(\widetilde{T}_{(c-\hat{c})t}\tilde{a}\big) * \tilde{w}_t + (\widetilde{T}_{-\hat{c}t}\tilde{u}_t)^{*(p-1)} * \tilde{w}_t\;. \end{aligned} \tag{4.1}$$

Let M_β be the operator of multiplication: $(M_\beta f)(x) = e^{\beta x}f(x)$. Choose the constants $\hat{c}$, and β such that they satisfy as before

$$0 \;>\; -2\gamma \;=\; \beta^2 - \beta\hat{c} + 1\;,$$

and fix them henceforth. Our main result for the simplified problem is:

Theorem 4.1. *For all $\vartheta \in (0, 1/2)$ there are positive constants R, C and $\delta \in (0, 1]$ such that the following holds: Assume $\|u_0\|_{\mathrm{H}^2_{2,\delta}} + \|M_\beta u_0\|_{\mathrm{H}^2_{0,\delta}} \le R$. Then the solution u_t of (2.2) with initial condition u_0 converges to a Gaussian in the sense that there is a constant $A_* = A_*(u_0)$ such that with $\tilde{\psi}(k) = e^{-k^2}$ the rescaled solution $\tilde{v}(k,t) = \tilde{u}(kt^{-1/2}, t)$ satisfies*

$$\|\tilde{v}_t - A_*\tilde{\psi}\|_{\tilde{\mathrm{H}}_2^{2,\delta}} \;\le\; \frac{CR}{(t+1)^{1/2-\vartheta}}\;. \tag{4.2}$$

Furthermore,

$$\|\tilde{w}_t\|_{\tilde{\mathrm{H}}_2^{0,\delta}} \;=\; \|\mathcal{F}\mathcal{W}_{\beta,\hat{c}t}u_t\|_{\tilde{\mathrm{H}}_2^{0,\delta}} \;\le\; CRe^{-\gamma t}\;.$$

We shall use the renormalization technique of [BK92] to show that $\tilde{u}_t$ and $\tilde{w}_t$ behave (as $t \to \infty$) essentially in the same way as their linear counterparts $\widetilde{U}_t$ and $\widetilde{W}_t$ from the previous section. This technique consists, see [CEE92], in pushing forward the solution for some time and then rescaling it. This process makes the effective non-linearity smaller at each step, so that in the end the convergence properties of the linearized problem are obtained.

We fix $0 < \sigma \le 1$ and introduce:

$$\big(\tilde{\mathfrak{L}}\tilde{f}\big)(\varkappa) \;=\; \tilde{f}(\sigma\varkappa)\;. \tag{4.3}$$

This is a linear change of coordinates in function space. Definition (2.6) and (4.3) imply that

$$\|\tilde{\mathfrak{L}}\tilde{f}\|^2_{\tilde{\mathrm{H}}_2^{2,\delta}} \;=\; \sigma^{-1}\int \mathrm{d}(\sigma\varkappa)\sum_{j,\ell=0}^{2}\delta^{2\ell}\sigma^{-2\ell}\sigma^{2j}\big|(\partial^j\tilde{f})(\sigma\varkappa)\big|^2(\sigma\varkappa)^{2\ell}\;.$$

From this we conclude immediately that for $0 < \sigma < 1$:

$$\|\tilde{\mathfrak{L}}\tilde{f}\|_{\tilde{\mathrm{H}}_2^{2,\delta}} \;\le\; \sigma^{-5/2}\|\tilde{f}\|_{\tilde{\mathrm{H}}_2^{2,\delta}} \quad\text{and}\quad \|\tilde{\mathfrak{L}}^{-1}\tilde{f}\|_{\tilde{\mathrm{H}}_2^{2,\delta}} \;\le\; \sigma^{-3/2}\|\tilde{f}\|_{\tilde{\mathrm{H}}_2^{2,\delta}}\;. \tag{4.4}$$

Similarly

$$\|\tilde{\mathfrak{L}}\tilde{f}\|_{\tilde{\mathrm{H}}_2^{0,\delta}} \;\le\; \sigma^{-1/2}\|\tilde{f}\|_{\tilde{\mathrm{H}}_2^{0,\delta}} \quad\text{and}\quad \|\tilde{\mathfrak{L}}^{-1}\tilde{f}\|_{\tilde{\mathrm{H}}_2^{0,\delta}} \;\le\; \sigma^{1/2}\|\tilde{f}\|_{\tilde{\mathrm{H}}_2^{0,\delta}}\;. \tag{4.5}$$

Note also that

$$\begin{aligned}\big(\tilde{\mathfrak{L}}(\tilde{f}*\tilde{g})\big)(\varkappa) \;&=\; \int \mathrm{d}\varkappa'\,\tilde{f}(\sigma\varkappa-\varkappa')\tilde{g}(\varkappa')\\ &=\; \sigma\int \mathrm{d}(\sigma^{-1}\varkappa')\,\tilde{f}(\sigma\varkappa-\sigma\sigma^{-1}\varkappa')\tilde{g}(\sigma\sigma^{-1}\varkappa')\\ &=\; \sigma\big((\tilde{\mathfrak{L}}\tilde{f})*(\tilde{\mathfrak{L}}\tilde{g})\big)(\varkappa)\;.\end{aligned} \tag{4.6}$$

Furthermore,

$$\big(\tilde{\mathfrak{L}}(\widetilde{T}_\zeta\tilde{a})\big)(\varkappa) \;=\; e^{i\zeta\sigma\varkappa}\tilde{a}(\sigma\varkappa) \;=\; \big(\widetilde{T}_{\sigma\zeta}(\tilde{\mathfrak{L}}\tilde{a})\big)(\varkappa)\;,$$

and therefore we have

$$\tilde{\mathfrak{L}}\big((\widetilde{T}_\zeta\tilde{a})*\tilde{f}\big) \;=\; \sigma(\widetilde{T}_{\sigma\zeta}\tilde{\mathfrak{L}}\tilde{a})*(\tilde{\mathfrak{L}}\tilde{f})\;. \tag{4.7}$$

We next define

$$\begin{aligned}\tilde{u}_{n,\tau}(\varkappa) \;&=\; \big(\tilde{\mathfrak{L}}^n\tilde{u}\big)(\varkappa,\sigma^{-2n}\tau) \;=\; \tilde{u}(\sigma^n\varkappa,\sigma^{-2n}\tau)\;,\\ \tilde{w}_{n,\tau}(\varkappa) \;&=\; e^{\gamma\sigma^{-2n}\tau}\tilde{w}(\varkappa,\sigma^{-2n}\tau)\;,\end{aligned}$$

so that this corresponds to an additional rescaling of the time axis. Note that

$$\tilde{u}_{n,\sigma^2}(\varkappa) = \tilde{u}_{n-1,1}(\sigma\varkappa)\;,$$

and

$$\tilde w_{n,\sigma^2}(\varkappa) = e^{\gamma\sigma^{-2n}\sigma^2}\tilde w(\varkappa,\sigma^{-2n}\sigma^2) = \tilde w_{n-1,1}(\varkappa)\ ,$$

i.e., the exponentially damped variable w is not scaled in space. We also let $\tilde a_n = \tilde{\mathfrak{L}}^n \tilde a$. From (4.6), (4.7), and $\partial_\tau = \sigma^{-2n}\partial_t$ we find easily that (4.1) transforms to the system (omitting the argument $\varkappa$):

$$\partial_\tau \tilde u_{n,\tau} = -\varkappa^2 \tilde u_{n,\tau} + \sigma^{-n}(\widetilde T_{c\sigma^{-n}\tau}\tilde a_n) * \tilde u_{n,\tau} + \sigma^{n(p-3)}\tilde u^{*p}_{n,\tau}\ , \tag{4.8}$$

$$\begin{aligned}\sigma^{2n}\partial_\tau \tilde w_{n,\tau} &= \big((\beta^2-\beta\hat c+\gamma) - \varkappa^2 + i\varkappa(\hat c - 2\beta)\big)\tilde w_{n,\tau} \\ &\quad + (\widetilde T_{(c-\hat c)\sigma^{-2n}\tau}\tilde a) * \tilde w_{n,\tau} + (\widetilde T_{-\hat c\sigma^{-2n}\tau}(\tilde{\mathfrak{L}}^{-n}\tilde u_{n,\tau}))^{*(p-1)} * \tilde w_{n,\tau}\ .\end{aligned} \tag{4.9}$$

We see that under these rescalings the coefficients of the non-linear terms in the first equation go to 0 as $n\to\infty$. We will now put this observation into more mathematical form.

The equation (4.1) is of the form $\partial_t X_t = L\big(X_t\big) + \mathcal{N}\big(X_t\big)$, where L contains the linear parts with the exception of those depending on $\tilde a_n$ and $\mathcal{N}$ denotes the other terms. We can write the solution as

$$X_t = e^{(t-t_0)L}X_{t_0} + \int_{t_0}^t \mathrm{d}s\, e^{(t-s)L}\mathcal{N}(X_s)\ .$$

Going to the rescaled variables $X_{n,\tau}$, and taking $t_0 = \sigma^{-2(n-1)}$ and $t = \sigma^{-2n}\tau$, we can express this (for the $\tilde u$) as follows. The equation (4.8) leads to

$$\begin{aligned}\tilde u_{n,\tau}(\varkappa) &= e^{-\varkappa^2(\tau-\sigma^2)}\tilde u_{n,\sigma^2}(\varkappa) \\ &\quad + \int_{\sigma^2}^\tau \mathrm{d}\tau'\, e^{-\varkappa^2(\tau-\tau')}\Big(\sigma^{-n}(\widetilde T_{c\sigma^{-n}\tau'}\tilde a_n) * \tilde u_{n,\tau'} + \sigma^{n(p-3)}\tilde u^{*p}_{n,\tau'}\Big)(\varkappa)\ .\end{aligned} \tag{4.10}$$

Similarly, we rewrite (4.9) as

$$\partial_\tau \tilde w_{n,\tau} = \tilde G_{n,\tau}\tilde w_{n,\tau} + \sigma^{-2n}\big((\widetilde T_{-\hat c\sigma^{-2n}\tau}(\tilde{\mathfrak{L}}^{-n}\tilde u_{n,\tau}))^{*(p-1)} * \tilde w_{n,\tau}\big)\ ,$$

where $\tilde G_{n,\tau}$ is defined, *cf.* (4.9), by

$$\sigma^{2n}\big(\tilde G_{n,\tau}\tilde f\big)(\varkappa) = \big((\beta^2-\beta\hat c+\gamma) - \varkappa^2 + i\varkappa(\hat c-2\beta)\big)\tilde f(\varkappa) + \big((\widetilde T_{(c-\hat c)\sigma^{-2n}\tau}\tilde a) * \tilde f\big)(\varkappa)\ .$$

The solution of the linear evolution equation $\partial_\tau \tilde f_{n,\tau} = \tilde G_{n,\tau}\tilde f_{n,\tau}$ is nothing but (3.10) in a new coordinate system. We write the solution as $\tilde f_{n,\tau} = \tilde S_{n,\tau,\tau'}\tilde f_{n,\tau'}$. Then, in analogy to (4.10) we get

$$\begin{aligned}\tilde w_{n,\tau}(\varkappa) &= \big(\tilde S_{n,\tau,\sigma^2}\tilde w_{n,\sigma^2}\big)(\varkappa) \\ &\quad + \sigma^{-2n}\int_{\sigma^2}^\tau \mathrm{d}\tau'\,\Big(\tilde S_{n,\tau,\tau'}\big((\widetilde T_{-\hat c\sigma^{-2n}\tau'}(\tilde{\mathfrak{L}}^{-n}\tilde u_{n,\tau'}))^{*(p-1)} * \tilde w_{n,\tau'}\big)\Big)(\varkappa)\ .\end{aligned} \tag{4.11}$$

Remark. The proof of Theorem 4.1 is divided into several steps: In Lemma 4.2 below, we give the inequalities for the exponentially damped part in scaled variables. Then in Lemma 4.4 a priori estimates for the solutions of (4.10) and (4.11) are established. With these a priori bounds we show Proposition 4.5. From these results, Theorem 4.1 will follow rather simply by a contraction argument.

4.1. The weighted linear problem

We bound $\tilde{S}_{n,\tau,\tau'}$. Recall that we are assuming $\beta^2 - \beta\hat{c} + 1 = -2\gamma < 0$.

Lemma 4.2. *There exists a $C > 0$ such that for $1 > \tau > \tau' \geq 0$ one has*

$$\|\tilde{S}_{n,\tau,\tau'}\tilde{f}\|_{\tilde{\mathrm{H}}_2^{0,\delta}} \leq Ce^{-\gamma\sigma^{-2n}(\tau-\tau')/2}\|\tilde{f}\|_{\tilde{\mathrm{H}}_2^{0,\delta}} , \tag{4.12}$$

for all $n \in \mathbf{N}$.

Proof. We consider the equation $\partial_\tau \tilde{f}_\tau = \tilde{G}_{n,\tau}\tilde{f}_\tau$, whose solution is $\tilde{f}_\tau = \tilde{S}_{n,\tau,\tau'}\tilde{f}_{\tau'}$:

$$\partial_\tau \tilde{f}_\tau = \lambda_n \tilde{f}_\tau + \sigma^{-2n}(\widetilde{T}_{(c-\hat{c})\sigma^{-2n}\tau}\tilde{a}) * \tilde{f}_\tau , \tag{4.13}$$

where λ_n is the operator of multiplication by

$$\lambda_n(\varkappa) = \big((\beta^2 - \beta\hat{c} + \gamma) - \varkappa^2 + i\varkappa(\hat{c} - 2\beta)\big)\sigma^{-2n} .$$

The variation of constant formula yields

$$\tilde{f}_\tau = e^{\lambda_n(\tau-\tau')}\tilde{f}_{\tau'} + \int_{\tau'}^{\tau} \mathrm{d}s\, e^{\lambda_n(\tau-s)}\sigma^{-2n}(\widetilde{T}_{(c-\hat{c})\sigma^{-2n}s}\tilde{a}) * \tilde{f}_s .$$

We use

$$\|e^{\lambda_n\tau}\tilde{f}\|_{\tilde{\mathrm{H}}_2^{0,\delta}} \leq \|e^{\lambda_n\tau}\|_{\mathcal{C}_\mathrm{b}^0}\|\tilde{f}\|_{\tilde{\mathrm{H}}_2^{0,\delta}} ,$$

and

$$\|e^{\lambda_n\tau}\|_{\mathcal{C}_\mathrm{b}^0} \leq e^{(\beta^2-\beta\hat{c}+\gamma)\sigma^{-2n}\tau} .$$

We find

$$\begin{aligned}\|\tilde{f}_\tau\|_{\tilde{\mathrm{H}}_2^{0,\delta}} \leq\ & e^{(\beta^2-\beta\hat{c}+\gamma)\sigma^{-2n}(\tau-\tau')}\|\tilde{f}_{\tau'}\|_{\tilde{\mathrm{H}}_2^{0,\delta}} \\ &+ \int_{\tau'}^{\tau} \mathrm{d}s\, e^{(\beta^2-\beta\hat{c}+\gamma)\sigma^{-2n}(\tau-s)}\sigma^{-2n}\|a\|_{\mathcal{C}_{\mathrm{b},\delta}^2}\|\tilde{f}_s\|_{\tilde{\mathrm{H}}_2^{0,\delta}} ,\end{aligned}$$

since

$$\begin{aligned}\|(\tilde{T}_\zeta \mathcal{F}a) * \tilde{f}\|_{\tilde{\mathrm{H}}_2^{0,\delta}} &= \|a(\cdot - \zeta)\mathcal{F}^{-1}\tilde{f}\|_{\mathrm{H}_{0,\delta}^2} \\ &\leq \|a(\cdot - \zeta)\|_{\mathcal{C}_{\mathrm{b},\delta}^2}\|\mathcal{F}^{-1}\tilde{f}\|_{\mathrm{H}_{0,\delta}^2} = \|a\|_{\mathcal{C}_{\mathrm{b},\delta}^2}\|\tilde{f}\|_{\tilde{\mathrm{H}}_2^{0,\delta}} .\end{aligned}$$

Using $\|a\|_{\mathcal{C}_{\mathrm{b},\delta}^2} = 1 + K\delta$ and applying Gronwall's inequality to $e^{-(\beta^2-\beta\hat{c}+\gamma)\sigma^{-2n}\tau}\|\tilde{f}_\tau\|_{\tilde{\mathrm{H}}_2^{0,\delta}}$ we get

$$e^{-(\beta^2-\beta\hat{c}+\gamma)\sigma^{-2n}(\tau-\tau')}\|\tilde{f}_\tau\|_{\tilde{\mathrm{H}}_2^{0,\delta}} \leq e^{\sigma^{-2n}(1+K\delta)(\tau-\tau')}\|\tilde{f}_{\tau'}\|_{\tilde{\mathrm{H}}_2^{0,\delta}} ,$$

or equivalently,

$$\|\tilde{f}_\tau\|_{\tilde{\mathrm{H}}_2^{0,\delta}} \leq \|\tilde{f}_{\tau'}\|_{\tilde{\mathrm{H}}_0^{0,\delta}} e^{(\beta^2-\beta\hat{c}+\gamma+1+K\delta)\sigma^{-2n}(\tau-\tau')} . \tag{4.14}$$

Choosing $\delta \in (0, 1]$ so small that $K\delta < \gamma/2$ completes the proof of Lemma 4.2. □

4.2. An a priori bound on the non-linear problem

We now state and prove a priori bounds on the solution of (4.10) and (4.11). Finally these solutions will be controlled by proving inequalities for the elements of the following sequences.

Definition 4.3. *For all n, we define*

$$\rho_n^u = \|\tilde{u}_{n,1}\|_{\tilde{\mathrm{H}}_2^{2,\delta}} \quad \text{and} \quad \rho_n^w = \|\tilde{w}_{n,1}\|_{\tilde{\mathrm{H}}_2^{0,\delta}} \ .$$

Moreover, we define

$$R_n^u = \sup_{\tau\in[\sigma^2,1]} \|\tilde{u}_{n,\tau}\|_{\tilde{\mathrm{H}}_2^{2,\delta}} \quad \text{and} \quad R_n^w = \sup_{\tau\in[\sigma^2,1]} \|\tilde{w}_{n,\tau}\|_{\tilde{\mathrm{H}}_2^{0,\delta}} \ . \tag{4.15}$$

Lemma 4.4. *For all $n \in \mathbf{N}$ there is a constant $\eta_n > 0$ such that the following holds: If ρ_{n-1}^u, ρ_{n-1}^w, and $\sigma > 0$ are smaller than η_n, the solutions of (4.10) and (4.11) exist for all $\tau \in [\sigma^2, 1]$. Moreover, we have the estimates*

$$R_n^u \ \le \ C\sigma^{-5/2}\rho_{n-1}^u + Ce^{-C\sigma^{-n}} R_n^w + C\sigma^{n(p-3)}(R_n^u)^p \ , \tag{4.16}$$

and

$$R_n^w \ \le \ C\rho_{n-1}^w + C\sigma^{n(p-3/2)}(R_n^u)^{p-1} R_n^w \ , \tag{4.17}$$

with a constant C independent of σ and n.

Remark. There is no need for a detailed expression for $\eta = \eta_n$ since the existence of the solutions is guaranteed if we can show $R_n^u < \infty$ and $R_n^w < \infty$. With (4.16) and (4.17) we have detailed control of these quantities in terms of the norm of the initial conditions and σ.

Proof. We start with (4.11). We bound the first term of (4.11) by

$$C\rho_{n-1}^w \ . \tag{4.18}$$

For the second term in (4.11), we get with (4.5) and (4.6),

$$\begin{aligned}
&\|(\widetilde{T}_{-\hat{c}\sigma^{-2n}\tau'}(\tilde{\mathfrak{L}}^{-n}\tilde{u}_{n,\tau'}))^{*(p-1)} * \tilde{w}_{n,\tau'}\|_{\tilde{\mathrm{H}}_2^{0,\delta}} \\
\le\, &\|(\tilde{\mathfrak{L}}^{-n}(\widetilde{T}_{-\hat{c}\sigma^{-n}\tau'}\tilde{u}_{n,\tau'}))^{*(p-1)} * \tilde{w}_{n,\tau'}\|_{\tilde{\mathrm{H}}_2^{0,\delta}} \\
\le\, &\sigma^{n/2}\sigma^{n(p-2)}\|\tilde{u}_{n,\tau'}\|_{\tilde{\mathrm{H}}_2^{0,\delta}}^{p-1}\|\tilde{w}_{n,\tau'}\|_{\tilde{\mathrm{H}}_2^{0,\delta}} \\
\le\, &\sigma^{n(p-3/2)}(R_n^u)^{p-1}R_n^w
\end{aligned}$$

a bound

$$C\sigma^{n(p-7/2)}\int_{\sigma^2}^{\tau} \mathrm{d}\tau'\, e^{-\gamma\sigma^{-2n}(\tau-\tau')/2}(R_n^u)^{p-1}R_n^w \ \le \ C\sigma^{n(p-3/2)}(R_n^u)^{p-1}R_n^w \ .$$

We next consider (4.10). The first term is bounded by

$$
\begin{aligned}
\|\varkappa \mapsto & e^{-\varkappa^2(\tau-\sigma^2)} \tilde{u}_{n-1,1}(\sigma\varkappa)\|_{\tilde{\mathrm{H}}_2^{2,\delta}} \\
& \leq \|\varkappa \mapsto e^{-\varkappa^2(\tau-\sigma^2)}\|_{\mathcal{C}^2_{\mathrm{b},\delta}} \|\varkappa \mapsto \tilde{u}_{n-1,1}(\sigma\varkappa)\|_{\tilde{\mathrm{H}}_2^{2,\delta}} \\
& \leq C\sigma^{-5/2}\rho^u_{n-1} \,,
\end{aligned}
\tag{4.19}
$$

using (4.4). We use (4.7) and recall $\tilde{a}_n = \tilde{\mathfrak{L}}^n \tilde{a}$ to rewrite the second term of (4.10) into

$$
\begin{aligned}
\sigma^{-2n} & \int_{\sigma^2}^{\tau} \mathrm{d}\tau' \, e^{-\varkappa^2(\tau-\tau')} \big((\widetilde{T}_{c\sigma^{-2n}\tau'} \tilde{a}) * (\tilde{\mathfrak{L}}^{-n} \tilde{u}_{n,\tau'}) \big)(\varkappa) \\
& = \sigma^{-2n} \int_{\sigma^2}^{\tau} \mathrm{d}\tau' \, e^{-\varkappa^2(\tau-\tau')} \big((\widetilde{T}_{c\sigma^{-2n}\tau'} \tilde{a}) * \tilde{u}_{\sigma^{-2n}\tau'} \big)(\varkappa) \\
& = \sigma^{-2n} \int_{\sigma^2}^{\tau} \mathrm{d}\tau' \, e^{-\beta\sigma^{-2n}\tau'(c-\hat{c})} e^{-\varkappa^2(\tau-\tau')} \\
& \quad \times \int \mathrm{d}\ell \, e^{i(\varkappa-\ell)c\sigma^{-2n}\tau'} \tilde{a}(\varkappa-\ell-i\beta) \, \tilde{w}(\ell,\sigma^{-2n}\tau') e^{-i\ell\hat{c}\sigma^{-2n}\tau'} e^{-\gamma\sigma^{-2n}\tau'} \,,
\end{aligned}
$$

where (3.13) is used for the last equality. Using this identity, we get from the techniques leading to (3.14):

$$
\begin{aligned}
\sigma^{-n} & \|\varkappa \mapsto \int_{\sigma^2}^{\tau} \mathrm{d}\tau' \, e^{-\varkappa^2(\tau-\tau')} \big((\widetilde{T}_{c\sigma^{-n}\tau'} \tilde{a}_n) * \tilde{u}_{n,\tau'} \big)(\varkappa)\|_{\tilde{\mathrm{H}}_2^{2,\delta}} \\
& \leq \sigma^{-n} \int_{\sigma^2}^{\tau} \mathrm{d}\tau' \|(\widetilde{T}_{c\sigma^{-n}\tau'} \tilde{a}_n) * \tilde{u}_{n,\tau'}\|_{\tilde{\mathrm{H}}_2^{2,\delta}} \\
& \leq \sigma^{-2n} \int_{\sigma^2}^{\tau} \mathrm{d}\tau' \|\tilde{\mathfrak{L}}^n \big((\widetilde{T}_{c\sigma^{-2n}\tau'} \tilde{a}) * \tilde{u}_{\sigma^{-2n}\tau'} \big)\|_{\tilde{\mathrm{H}}_2^{2,\delta}} \\
& \leq \sigma^{-9n/2} \int_{\sigma^2}^{\tau} \mathrm{d}\tau' \|(\widetilde{T}_{c\sigma^{-2n}\tau'} \tilde{a}) * \tilde{u}_{\sigma^{-2n}\tau'}\|_{\tilde{\mathrm{H}}_2^{2,\delta}} \\
& \leq C\sigma^{-9n/2} \int_{\sigma^2}^{\tau} \mathrm{d}\tau' (1+c\sigma^{-2n}\tau')^2 e^{-(\beta(c-\hat{c})+\gamma)\sigma^{-2n}\tau'} R^w_n \\
& \leq C\sigma^{-17n/2} e^{-(\beta(c-\hat{c})+\gamma)\sigma^{-2(n-1)}} R^w_n \;\leq\; Ce^{-(\beta(c-\hat{c})+\gamma)\sigma^{-n}} R^w_n \,.
\end{aligned}
\tag{4.20}
$$

For the last term in (4.10) we get a bound

$$
C\sigma^{n(p-3)} \int_{\sigma^2}^{\tau} \mathrm{d}\tau' (R^u_n)^p \;\leq\; C\sigma^{n(p-3)} (R^u_n)^p \,.
\tag{4.21}
$$

The proof of Lemma 4.4 now follows by applying the contraction mapping principle to (4.10) and (4.11). For ρ^u_{n-1}, ρ^w_{n-1} and $\sigma > 0$ sufficiently small the Lipschitz constant for the right hand side of (4.10) and (4.11) in $\mathcal{C}([\sigma^2, 1], \tilde{\mathrm{H}}_2^{2,\delta} \times \tilde{\mathrm{H}}_2^{0,\delta})$ is smaller than 1. An application of a classical fixed point argument completes the proof of Lemma 4.4. □

4.3. The iteration process

We next decompose the solution $\tilde{u}_{n,\tau}$ for $\tau = 1$ into a Gaussian part and a remainder. Let $\tilde{\psi}(\varkappa) = e^{-\varkappa^2}$ and write

$$\tilde{u}_{n,1}(\varkappa) = A_n \tilde{\psi}(\varkappa) + \tilde{r}_n(\varkappa) ,$$

where $\tilde{r}_n(0) = 0$, and the amplitude A_n is in **R**. We also define $\widetilde{\Pi} : \tilde{\mathrm{H}}_2^{2,\delta} \to \mathbf{R}$ by

$$\widetilde{\Pi}\tilde{f} = \tilde{f}\big|_{\varkappa=0} . \tag{4.22}$$

Then (4.10) can be decomposed accordingly and takes the form

$$\begin{aligned} A_n &= A_{n-1} + \widetilde{\Pi}\biggl(\int_{\sigma^2}^1 \mathrm{d}\tau'\, e^{-\varkappa^2(1-\tau')}\Bigl(\sigma^{-n}(\widetilde{T}_{c\sigma^{-n}\tau'}\tilde{a}_n) * \tilde{u}_{n,\tau'} + \sigma^{n(p-3)}\tilde{u}_{n,\tau'}^{*p}\Bigr)(\varkappa)\biggr) , \\ \tilde{r}_n(\varkappa) &= e^{-\varkappa^2(1-\sigma^2)}\tilde{r}_{n-1}(\sigma\varkappa) & (4.23) \\ &\quad + \int_{\sigma^2}^1 \mathrm{d}\tau'\, e^{-\varkappa^2(1-\tau')}\Bigl(\sigma^{-n}(\widetilde{T}_{c\sigma^{-n}\tau'}\tilde{a}_n) * \tilde{u}_{n,\tau'} + \sigma^{n(p-3)}\tilde{u}_{n,\tau'}^{*p}\Bigr)(\varkappa) \\ &\quad + e^{-\varkappa^2(1-\sigma^2)}A_{n-1}\tilde{\psi}(\sigma\varkappa) - A_n\tilde{\psi}(\varkappa) . & (4.24) \end{aligned}$$

Then we define $\rho_n^r = \|\tilde{r}_n\|_{\tilde{\mathrm{H}}_2^{2,\delta}}$ and so $\rho_n^u \le C(|A_n| + \rho_n^r)$. Our main estimate is now

Proposition 4.5. *There is a constant $C > 0$ such that for $\sigma > 0$ sufficiently small the solution $\tilde{u}$ of (2.2) satisfies for all $n \in \mathbf{N}$:*

$$\begin{aligned} |A_n - A_{n-1}| &\le Ce^{-C\sigma^{-n}}R_n^w + C\sigma^{n(p-3)}(R_n^u)^p , & (4.25) \\ \rho_n^r &\le C\sigma\rho_{n-1}^r + Ce^{-C\sigma^{-n}}R_n^w + C\sigma^{n(p-3)}(R_n^u)^p , \\ \rho_n^w &\le Ce^{-C\sigma^{-2n}}\rho_{n-1}^w + C\sigma^{n(p-3/2)}(R_n^u)^{p-1}R_n^w . & (4.26) \end{aligned}$$

Proof. We begin by bounding the difference $A_n - A_{n-1}$ using (4.23). Observe that since we work in $\tilde{\mathrm{H}}_2^{2,\delta}$, we have

$$|\widetilde{\Pi}\tilde{f}| \le C\|\tilde{f}\|_{\tilde{\mathrm{H}}_2^{2,\delta}} , \tag{4.27}$$

with C independent of δ. Thus, it suffices to bound the norm of the integral in (4.23). The first term in (4.23) is the one containing the translated term $\tilde{a}_n$ and was already bounded in (4.20) while the second was bounded in (4.21). Combining these bounds with (4.27), we find (4.25).

We next bound $\tilde{r}_n$ in terms of $\tilde{r}_{n-1}$, using (4.24). The first term is the one where the projection is crucial: For $\sigma > 0$ sufficiently small, $\tilde{f} \in \tilde{\mathrm{H}}_2^{2,\delta}$ with $\tilde{f}(0) = 0$ one has

$$\|\varkappa \mapsto e^{-\varkappa^2(1-\sigma^2)}\tilde{f}(\sigma\varkappa)\|_{\tilde{\mathrm{H}}_2^{2,\delta}} \le C\sigma\|\tilde{f}\|_{\tilde{\mathrm{H}}_2^{2,\delta}} . \tag{4.28}$$

Indeed, writing out the definition (2.6) of $\tilde{\mathrm{H}}_2^{2,\delta}$, one gets for the term with $j = \ell = 0$:

$$\int \mathrm{d}\varkappa\, e^{-2\varkappa^2(1-\sigma^2)} |\tilde{f}(\sigma\varkappa)|^2 = \int \mathrm{d}\varkappa\, e^{-2\varkappa^2(1-\sigma^2)} (\sigma\varkappa)^2 \left| \frac{\tilde{f}(\sigma\varkappa) - \tilde{f}(0)}{\sigma\varkappa} \right|^2 .$$

Clearly, a bound of the type of (4.28) follows for this term by the assumptions on $\tilde{f}$. The derivatives are handled similarly, except that there is no need to divide and multiply by powers of $\sigma\varkappa$ since each derivative produces a factor σ.

We now bound the other terms in (4.24). The first term is bounded using (4.28) and yields a bound (in $\tilde{\mathrm{H}}_2^{2,\delta}$) of

$$C\sigma\rho_{n-1}^r . \tag{4.29}$$

The second and third terms have been bounded in (4.20) and (4.21):

$$Ce^{-C\sigma^{-n}} R_n^w + C\sigma^{n(p-3)} (R_n^u)^p . \tag{4.30}$$

Finally, the last term in (4.24) can be written as

$$\tilde{X}_n \equiv A_{n-1}(e^{-\varkappa^2(1-\sigma^2)} e^{-\varkappa^2\sigma^2} - e^{-\varkappa^2}) + (A_{n-1} - A_n)e^{-\varkappa^2} .$$

The first expression vanishes and we get a bound (in $\tilde{\mathrm{H}}_2^{2,\delta}$):

$$\|\tilde{X}_n\|_{\tilde{\mathrm{H}}_2^{2,\delta}} \leq Ce^{-C\sigma^{-n}} R_n^w + C\sigma^{n(p-3)} (R_n^u)^p . \tag{4.31}$$

Collecting the bounds (4.29)–(4.31), the assertion (4.26) for $\tilde{r}_n$ follows. Finally, the bounds on ρ_n^w follow as those in Lemma 4.4. The proof of Proposition 4.5 is complete. □

Proof of Theorem 4.1. The proof is an induction argument, using repeatedly the above estimates. Again we write C for (positive) constants which can be chosen independent of σ and n. Assume that $R = \sup_{n\in\mathbf{N}} R_n^u < \infty$ exists. From Lemma 4.4 we observe for $\sigma > 0$ sufficiently small

$$\begin{aligned} R_n^w &\leq \frac{C\rho_{n-1}^w}{1 - C\sigma^{n(p-3/2)} R^{p-1}} \leq C\,\rho_{n-1}^w , \\ R_n^u &\leq \frac{C\sigma^{-5/2}\rho_{n-1}^u + Ce^{-C\sigma^{-n}} R_n^w}{1 - C\sigma^{n(p-3)} R^{p-1}} \\ &\leq C\sigma^{-5/2}\rho_{n-1}^u + Ce^{-C\sigma^{-n}} \rho_{n-1}^w , \end{aligned} \tag{4.32}$$

with a constant C which can be chosen independent of R. Using Proposition 4.5 we find

$$\begin{aligned} |A_n - A_{n-1}| &\leq Ce^{-C\sigma^{-n}} \rho_{n-1}^w + C\sigma^{n(p-3)} \sigma^{-5/2} \rho_{n-1}^u , \\ \rho_n^r &\leq C\sigma\rho_{n-1}^r + Ce^{-C\sigma^{-n}} \rho_{n-1}^w + C\sigma^{n(p-3)} \sigma^{-5/2} \rho_{n-1}^u , \\ \rho_n^u &\leq C(|A_n| + \rho_n^r) , \\ \rho_n^w &\leq Ce^{-C\sigma^{-2n}} \rho_{n-1}^w + C\,\sigma^{n(p-3/2)} \rho_{n-1}^w . \end{aligned} \tag{4.33}$$

Therefore, we can choose $\sigma > 0$ so small that for $n > 3$: (recall $p > 3$ and $p \in \mathbf{N}$)

$$\begin{aligned} |A_n - A_{n-1}| &\leq \rho^w_{n-1}/10 + \sigma^{n-3}(|A_{n-1}| + \rho^r_{n-1}) , \\ \rho^r_n &\leq 3\rho^r_{n-1}/4 + \rho^w_{n-1}/10 + \sigma^{n-3}|A_{n-1}| , \\ \rho^w_n &\leq \rho^w_{n-1}/10 . \end{aligned}$$

Thus, the sequence of A_n converges geometrically to a finite limit A_*. Furthermore, we find that $\lim_{n\to\infty} \rho^r_n = 0$, and $\lim_{n\to\infty} \rho^w_n = 0$. Since the quantities $|A_n|$, ρ^r_n, ρ^w_n increase only for at most three steps the term CR^{p-1} in (4.32) stays less than $1/2$ if we choose $|A_1|$, ρ^r_1, $\rho^w_1 = \mathcal{O}(\sigma^m)$, for an $m > 0$ sufficiently large. We then deduce from (4.32) the existence of a finite constant $R = \sup_{n\in\mathbf{N}} R^u_n$. Going back to (4.33) for given $\vartheta > 0$ we can choose $\sigma > 0$ so small that

$$|A_n - A_{n-1}| + \rho^r_n \leq C\sigma^{(1-2\vartheta)n}$$

which implies the associated convergence rate stated in Theorem 4.1. This holds since $\rho_n \leq C\sigma\rho_{n-1}$ implies $\rho|_{t=\sigma^{-2n}} = \rho_n \leq (C\sigma)^n\rho_0$ and $\rho(t) \leq \rho_0 t^{-1/2} t^{\ln C/\ln\sigma^{-2}} \leq \rho_0 t^{-1/2+\vartheta}$ for $\sigma > 0$ sufficiently small. Finally, the scaling of $\tilde{w}_{n,\tau}$ implies the exponential decay of $\tilde{w}_t$. The proof of Theorem 4.1 is complete. □

Part II. The Swift-Hohenberg equation

5. Bloch waves

Since the problem we consider takes place in a setting with a *periodic* background provided by the stationary solution of the Swift-Hohenberg, it is natural to work with the Bloch representation of the functions. For additional informations see [RS72].

The starting point of Bloch wave analysis in case of a 2π–periodic underlying pattern is the following relation

$$\begin{aligned} u(x) \;&=\; \int \mathrm{d}k\, e^{ikx}\tilde{u}(k) \;=\; \sum_{n\in\mathbf{Z}} \int_{-1/2}^{1/2} \mathrm{d}\ell\, e^{i(n+\ell)x}\tilde{u}(n+\ell) \\ &=\; \int_{-1/2}^{1/2} \mathrm{d}\ell \sum_{n\in\mathbf{Z}} e^{i(n+\ell)x}\tilde{u}(n+\ell) \;=\; \int_{-1/2}^{1/2} \mathrm{d}\ell\, e^{i\ell x}\hat{u}(\ell,x)\;, \end{aligned} \tag{5.1}$$

where we define

$$\big(\mathcal{T}u\big)(\ell,x) \;\equiv\; \hat{u}(\ell,x) \;=\; \sum_{n\in\mathbf{Z}} e^{inx}\tilde{u}(n+\ell)\;. \tag{5.2}$$

The operator $\mathcal{T}$ will play a rôle analogous to that played by the Fourier transform $\mathcal{F}$ for the simplified problem of Part I. We will use analogous notation:

Notation. If f denotes a function, then $\hat{f}$ is defined by $\hat{f} = \mathcal{T}f$, and if $\mathcal{A}$ is an operator, then $\hat{\mathcal{A}}$ is defined by $\hat{\mathcal{A}} = \mathcal{T}\mathcal{A}\mathcal{T}^{-1}$.

Note that

$$\int_{\mathbf{R}} \mathrm{d}x\, |u(x)|^2 \;=\; 2\pi \int_{-1/2}^{1/2} \mathrm{d}\ell \int_0^{2\pi} \mathrm{d}x\, |\hat{u}(\ell,x)|^2\;. \tag{5.3}$$

This is easily seen from Parseval's identity:

$$\begin{aligned} \int_{\mathbf{R}} \mathrm{d}x\, |u(x)|^2 \;&=\; 2\pi \int_{\mathbf{R}} \mathrm{d}k |\tilde{u}(k)|^2 \\ &=\; 2\pi \sum_{n\in\mathbf{Z}} \int_{-1/2}^{1/2} \mathrm{d}\ell\, |\tilde{u}(n+\ell)|^2 \\ &=\; 2\pi \int_{-1/2}^{1/2} \mathrm{d}\ell \sum_{n\in\mathbf{Z}} |\tilde{u}(n+\ell)|^2 \\ &=\; 2\pi \int_{-1/2}^{1/2} \mathrm{d}\ell \int_0^{2\pi} \mathrm{d}x\, |\hat{u}(\ell,x)|^2\;. \end{aligned}$$

The sum and the integral can be interchanged in (5.1) due to Fubini's theorem when u is in the Schwartz space $\mathcal{S}$.

We shall use frequently the following fundamental properties (which follow at once from (5.2)):

$$\begin{aligned}\hat{u}(\ell, x) &= e^{ix}\hat{u}(\ell+1, x)\,,\\ \hat{u}(\ell, x) &= \hat{u}(\ell, x+2\pi)\,,\\ \hat{u}(\ell, x) &= \overline{\hat{u}}(-\ell, x) \text{ for real–valued } u\,.\end{aligned} \tag{5.4}$$

Finally $\widehat{T}_\zeta$ denotes the conjugate of translation, so that the Bloch transform of $\big(T_\zeta f\big)(x) = f(x-\zeta)$ is

$$\mathcal{T}T_\zeta f = \widehat{T}_\zeta \mathcal{T} f\,.$$

Multiplication in position space corresponds to a modified convolution operation for the Bloch-functions:

$$\big(\widehat{u \cdot v}\big)(\ell, x) = \int_{-1/2}^{1/2} \mathrm{d}\ell'\, \hat{u}(\ell-\ell', x)\hat{v}(\ell', x) \equiv \big(\hat{u} \circledast \hat{v}\big)(\ell, x)\,.$$

This follows from (5.4) and the identities:

$$\begin{aligned}\big(\widehat{u \cdot v}\big)(\ell, x) &= \sum_{m\in\mathbf{Z}} \int_{\mathbf{R}} \mathrm{d}k\, \tilde{u}(\ell+m-k)\tilde{v}(k)e^{imx}\\ &= \int_{-1/2}^{1/2} \mathrm{d}\ell' \sum_{m,n\in\mathbf{Z}} \tilde{u}(\ell+m-\ell'-n)\tilde{v}(\ell'+n)e^{i(m-n)x}e^{inx}\,.\end{aligned}$$

Recalling the norm

$$\|f\|_{\mathrm{H}^n_{s,\delta}} = \left(\sum_{j=0}^{s}\sum_{m=0}^{n} \delta^{2(m+j)} \int \mathrm{d}x\, |\partial_x^m f(x)|^2 x^{2j}\right)^{1/2}$$

we now introduce

$$\|\hat{f}\|_{\hat{\mathrm{H}}^{s,\delta}_n} = \left(\sum_{j=0}^{n}\sum_{m=0}^{s} \delta^{2(m+j)} \int_{1/2}^{1/2} \mathrm{d}\ell \int_0^{2\pi} \mathrm{d}x\, |\partial_x^j \partial_\ell^m \hat{f}(\ell, x)|^2\right)^{1/2}\,.$$

We get from Parseval's equality

$$C^{-1}\|u\|_{\mathrm{H}^n_{s,\delta}} \le \|\hat{u}\|_{\hat{\mathrm{H}}^{s,\delta}_n} \le C\|u\|_{\mathrm{H}^n_{s,\delta}}\,,$$

for some C independent of $\delta \in (0, 1)$.

As before, in the following we mainly work with the spaces to $s = n = 2$ and $s = 0$, $n = 2$. Similarly, in analogy to (2.8), we also have

$$\|\widehat{u\,v}\|_{\hat{\mathrm{H}}^{2,\delta}_2} = \|\hat{u} \circledast \hat{v}\|_{\hat{\mathrm{H}}^{2,\delta}_2} \le C\|\hat{u}\|_{\hat{\mathrm{H}}^{2,\delta}_2}\, \|\hat{v}\|_{\hat{\mathrm{H}}^{2,\delta}_2}\,, \tag{5.5}$$

$$\|\big(\widehat{u\,v}\big)(\cdot - i\beta, \cdot)\|_{\hat{\mathrm{H}}^{0,\delta}_2} = \|\big(\hat{u} \circledast \hat{v}\big)(\cdot - i\beta, \cdot)\|_{\hat{\mathrm{H}}^{0,\delta}_2} \le C\|\hat{u}\|_{\hat{\mathrm{H}}^{0,\delta}_2}\, \|\hat{v}(\cdot - i\beta, \cdot)\|_{\hat{\mathrm{H}}^{0,\delta}_2}\,, \tag{5.6}$$

or in a stronger version

$$\|\widehat{u\,v}\|_{\hat{\mathrm{H}}_2^{2,\delta}} = \|\hat{u} \circledast \hat{v}\|_{\hat{\mathrm{H}}_2^{2,\delta}} \le C\|\hat{u}\|_{\hat{\mathrm{H}}_2^{0,\delta}} \|\hat{v}\|_{\hat{\mathrm{H}}_2^{2,\delta}} , \tag{5.7}$$

$$\|\big(\widehat{u\,v}\big)(\cdot - i\beta, \cdot)\|_{\hat{\mathrm{H}}_2^{2,\delta}} = \|\big(\hat{u} \circledast \hat{v}\big)(\cdot - i\beta, \cdot)\|_{\hat{\mathrm{H}}_2^{2,\delta}} \le C\|\hat{u}\|_{\hat{\mathrm{H}}_2^{2,\delta}} \|\hat{v}(\cdot - i\beta, \cdot)\|_{\hat{\mathrm{H}}_2^{0,\delta}} .$$

Finally, suppose f is a function in $\mathcal{C}^2_{\mathrm{b},\delta}$ (see (2.11) for the definition): Then,

$$\|\widehat{f\,v}\|_{\hat{\mathrm{H}}_2^{0,\delta}} = \|\hat{f} \circledast \hat{v}\|_{\hat{\mathrm{H}}_2^{0,\delta}} \le C\|f\|_{\mathcal{C}^2_{\mathrm{b},\delta}} \|\hat{v}\|_{\hat{\mathrm{H}}_2^{0,\delta}} , \tag{5.8}$$

$$\|\big(\hat{f} \circledast \hat{v}\big)(\cdot - i\beta, \cdot)\|_{\hat{\mathrm{H}}_2^{0,\delta}} \le C\|f\|_{\mathcal{C}^2_{\mathrm{b},\delta}} \|\hat{v}(\cdot - i\beta, \cdot)\|_{\hat{\mathrm{H}}_2^{0,\delta}} . \tag{5.9}$$

Thus, apart from notational differences, we can work in the Bloch spaces with much the same bounds as in the spaces used for the model problem of the previous sections.

6. The linearized problem

We discuss here again the behavior of the linearized problem as in Section 3, but now for the Swift-Hohenberg equation. The discussion will again be split in an aspect behind the front and one ahead of the front. In Section 3, the behavior of the problem in the bulk behind the traveling front was diffusive by construction, and the only difficulty was to understand the rôle of the decay of a to 0 (as $e^{-\beta|x|}$) as $x \to -\infty$. For the problem of the Swift-Hohenberg equation, the situation is similar, leading again to diffusive behavior. However, this observation is not obvious. Therefore, the first problem consists in showing the diffusive behavior. In order to obtain optimal results for the analysis ahead of the front, *i.e.*, for the variable in the weighted representation, we use our approximate knowledge of the shape of the front.

6.1. The unweighted representation

In analogy with the simplified example, the linearized problem would be now

$$\partial_t v = \mathcal{M}v + \mathcal{M}_{\mathbf{i}}v , \tag{6.1}$$

where $\mathcal{M}$ and $\mathcal{M}_{\mathbf{i}}$ have been defined in Eqs.(1.7) and (1.8). By the analysis for the model problem we expect that the term $\mathcal{M}_{\mathbf{i}}v$ will be irrelevant for the dynamics in the bulk with some exponential rate. Therefore, it will be considered in the sequel together with the non-linear terms. As a consequence, the linear equation dominating the behavior behind the front is given by

$$\partial_t v = \mathcal{M}v . \tag{6.2}$$

We recall those features of the proof of diffusive stability of [Schn96, Schn98] which are relevant to the study of (6.2).

In order to do this, we need to localize the spectrum of $\mathcal{M}$. Since this is well-documented, we just summarize the results. As the linearized problem has periodic coefficients, the operator

$\hat{\mathcal{M}} = \mathcal{T}\mathcal{M}\mathcal{T}^{-1}$ equals a direct integral $\int^{\oplus} \mathrm{d}\ell\, \mathcal{M}_\ell$, where each $\mathcal{M}_\ell$ acts on the subspace with fixed quasi-momentum ℓ in $\hat{\mathrm{H}}_2^{2,\delta}$. The eigenfunctions of $\mathcal{M}_\ell$ are given by Bloch waves of the form $e^{i\ell x} w_{\ell,n}$ with 2π-periodic $w_{\ell,n}$. The index $n \in \mathbf{N}$ counts various eigenvalues for fixed ℓ. For each $\ell \in \mathbf{R}$ (or rather in the Brillouin zone $[-\frac{1}{2}, \frac{1}{2}]$) they are solutions of the eigenvalue equation

$$\big(\mathcal{M}_\ell w_\ell\big)(x) \;\equiv\; -\big(1 + (i\ell + \partial_x)^2\big)^2 w_\ell(x) + \varepsilon^2 w_\ell(x) - 3U_*^2(x) w_\ell(x) \;=\; \mu_\ell w_\ell(x) \;.$$

The spectrum takes the familiar form of a curve $\mu_1(\ell)$ with an expansion

$$\mu_1(\ell) \;=\; -c_1 \ell^2 + \mathcal{O}(\ell^3) \;,$$

and $c_1 > 0$ and the remainder of the spectrum negative and bounded away from 0. The eigenfunction associated with $\mu_1(0)$ is $\partial_x U_*(x)$, reflecting the translation invariance of the original problem (1.1). There is an $\ell_0 > 0$ such that for fixed $\ell \in (-\ell_0, \ell_0)$ the eigenfunction $\varphi_\ell(x) = w_{\ell,1}(x)$ of the main branch $\mu_1(\ell)$ is well defined (and a continuation of $\partial_x U_*(x)$) as ℓ is varied away from 0. Corresponding to this we define the central projections $\hat{P}_{\mathrm{c}}(\ell)$ by

$$\hat{P}_{\mathrm{c}}(\ell) f \;=\; \langle \bar{\varphi}_\ell, f \rangle \varphi_\ell \;,$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathrm{L}^2([0, 2\pi])$ and $\bar{\varphi}_\ell$ the associated eigenfunction of the adjoint problem. We will need a smooth version of the projection in $\hat{\mathrm{H}}_2^{2,\delta}$. We fix once and for all a non-negative smooth cutoff function χ with support in $[-\ell_0/2, \ell_0/2]$ which equals 1 on $[-\ell_0/4, \ell_0/4]$. Then we define the operators $\hat{E}_{\mathrm{c}}$ and $\hat{E}_{\mathrm{s}}$ by:

$$\hat{E}_{\mathrm{c}}(\ell) \;=\; \chi(\ell) \hat{P}_{\mathrm{c}}(\ell) \;, \quad \hat{E}_{\mathrm{s}}(\ell) \;=\; \mathbf{1}(\ell) - \hat{E}_{\mathrm{c}}(\ell) \;.$$

It will be useful to define auxiliary "mode filters" $\hat{E}_{\mathrm{c}}^{\mathrm{h}}$ and $\hat{E}_{\mathrm{s}}^{\mathrm{h}}$ by

$$\hat{E}_{\mathrm{c}}^{\mathrm{h}}(\ell) \;=\; \chi(\ell/2) \hat{P}_{\mathrm{c}}(\ell) \;, \quad \hat{E}_{\mathrm{s}}^{\mathrm{h}}(\ell) \;=\; \mathbf{1}(\ell) - \chi(2\ell) \hat{P}_{\mathrm{c}}(\ell) \;.$$

These definitions are made in such a way that

$$\hat{E}_{\mathrm{c}}^{\mathrm{h}} \hat{E}_{\mathrm{c}} \;=\; \hat{E}_{\mathrm{c}} \;, \quad \hat{E}_{\mathrm{s}}^{\mathrm{h}} \hat{E}_{\mathrm{s}} \;=\; \hat{E}_{\mathrm{s}} \;,$$

which will be used to replace the (missing) projection property of $\hat{E}_{\mathrm{c}}$ and $\hat{E}_{\mathrm{s}}$.

We next extend the definitions (4.3) of Section 4 to the Bloch spaces. To avoid cumbersome notation, we shall use mostly the same symbols as in that section. Thus, with $\sigma < 1$ as before, we let now

$$\big(\widehat{\mathcal{L}} \hat{u}\big)(\varkappa, x) \;=\; \hat{u}(\sigma \varkappa, x) \;.$$

Note that here, and elsewhere, the scaling does not act on the x variable, only on the quasi-momentum $\varkappa$. The novelty of renormalization in Bloch space here is that since the integration region over the ℓ variable is finite it will change with the scaling. Therefore, we introduce (for fixed $\delta > 0$),

$$\mathcal{K}_{\sigma,\rho} = \{\hat{u} \mid \|\hat{u}\|_{\mathcal{K}_{\sigma,\rho}} < \infty\} , \tag{6.3}$$

where

$$\|\hat{u}\|^2_{\mathcal{K}_{\sigma,\rho}} \equiv \sum_{n,n'=0}^{2} \int_{-1/(2\sigma)}^{1/(2\sigma)} \mathrm{d}\ell \int_0^{2\pi} \mathrm{d}x \; \delta^{2(n+n')} |\partial_\ell^n \partial_x^{n'} \hat{u}(\ell,x)|^2 (1+\ell^2)^{2\rho} .$$

For technical reasons we introduced a weight in the Bloch variable ℓ. It turns out that an appropriate choice for the critical part is $\mathcal{K}_\sigma^c = \mathcal{K}_{\sigma,3/2}$ and for the stable part $\mathcal{K}_\sigma^s = \mathcal{K}_{\sigma,1}$. Note that $\mathcal{T}$, as defined in (5.2) is an isomorphism between the space $\mathrm{H}^2_{2,\delta}$ and the space $\mathcal{K}_{\sigma,\rho}$ by (5.3) and the definition (6.3). As before we have

$$\|\widehat{\mathcal{L}} \hat{f}\|_{\mathcal{K}_{\sigma^n,\rho}} \leq \sigma^{-5/2-2\rho} \|\hat{f}\|_{\mathcal{K}_{\sigma^{n-1},\rho}} , \tag{6.4}$$

for $0 < \sigma \leq 1$ where the additional factor $\sigma^{-2\rho}$ is due the weight in the ℓ-variable. Moreover, as before, we will not scale the weighted variable and so we fix

$$\mathcal{K}^w = \hat{\mathrm{H}}_2^{0,\delta} .$$

Consider again the eigenfunctions $\varphi_\ell(x)$. The function

$$\hat{v}_t(\ell,x) = e^{\mu_1(\ell)t} \varphi_\ell(x) ,$$

solves the equation

$$\partial_t \hat{v}_t(\ell,\cdot) = \mathcal{M}_\ell(\hat{v}_t(\ell,\cdot)) .$$

Because of the nature of the spectrum $\mu_1(\ell)$, this solution satisfies

$$\hat{v}_t(\ell t^{-1/2},x) = e^{-c_1\ell^2} \hat{v}_0(0,x) + \mathcal{O}(t^{-1/2}) .$$

Using this observation and the fact that the $\hat{E}_{\mathrm{s}}$-part is exponentially damped, the result will be

Proposition 6.1. *The solution $\widehat{V}_t$ of the problem (6.2) with initial data $\widehat{V}_0$ satisfies:*

$$\|(\ell,x) \mapsto \widehat{V}_t(\ell t^{-1/2},x) - e^{-c_1\ell^2} \hat{P}_{\mathrm{c}}(0) \widehat{V}_0(0,x)\|_{\mathcal{K}_{1/\sqrt{t},1}} \leq \frac{C}{t^{1/2}} \|\widehat{V}_0\|_{\hat{\mathrm{H}}_2^{2,\delta}} , \tag{6.5}$$

for a constant $C > 0$ and all $t \geq 1$. Moreover, there is a constant $\gamma_- > 0$ such that

$$\|(\ell,x) \mapsto \big(\hat{E}_{\mathrm{s}} \widehat{V}_t\big)(\ell t^{-1/2},x)\|_{\mathcal{K}_{1/\sqrt{t},1}} \leq C e^{-\gamma_- t} \|\widehat{V}_0\|_{\hat{\mathrm{H}}_2^{2,\delta}} , \tag{6.6}$$

for all $t \geq 1$.

6.2. The weighted representation

The weighted representation will be obtained by translating the effect of the transformation $\mathcal{W}_{\beta,\hat{c}t}$ defined in (2.12) to the language of the Bloch waves. In accordance with our notational conventions, we set

$$\widehat{\mathcal{W}}_{\beta,\hat{c}t} = \mathcal{T}\mathcal{W}_{\beta,\hat{c}t}\mathcal{T}^{-1} ,$$

and we get now, in analogy to (2.13),

$$\big(\widehat{\mathcal{W}}_{\beta,\hat{c}t}\hat{f}\big)(\ell,x) = e^{i\hat{c}(\ell+i\beta)t}\hat{f}(\ell+i\beta,x+\hat{c}t) .$$

The equation (6.1), expressed in terms of $\widehat{\mathcal{W}}_{\beta,\hat{c}t}\hat{v}$, then takes the form

$$\partial_t\big(\widehat{\mathcal{W}}_{\beta,\hat{c}t}\hat{v}\big) = \widehat{\mathcal{M}}_{\beta,\hat{c}t}\big(\widehat{\mathcal{W}}_{\beta,\hat{c}t}\hat{v}\big) + \widehat{\mathcal{M}}_{\mathbf{i},\beta,\hat{c}t}\big(\widehat{\mathcal{W}}_{\beta,\hat{c}t}\hat{v}\big) , \tag{6.7}$$

with

$$\begin{aligned}\big(\widehat{\mathcal{M}}_{\beta,\hat{c}t}\hat{f}\big)(\ell,x) &= \big(\hat{L}_{i\beta}\hat{f}\big)(\ell,x) - 3U_*^2(x+\hat{c}t)\hat{f}(\ell,x) + \hat{c}(i(\ell+i\beta)+\partial_x)\hat{f}(\ell,x) ,\\ \big(\widehat{\mathcal{M}}_{\mathbf{i},\beta,\hat{c}t}\hat{f}\big)(\ell,x) &= -6U_*(x+\hat{c}t)\big(\widehat{T}_{-\hat{c}t}\widehat{K}_{ct}\circledast\hat{f}\big)(\ell,x) - 3(\widehat{T}_{-\hat{c}t}\widehat{K}_{ct}\circledast\widehat{T}_{-\hat{c}t}\widehat{K}_{ct}\circledast\hat{f}\big)(\ell,x) .\end{aligned}$$

Some explanations are in order: $\hat{L}_{i\beta}$ is the operator $-(1+(\partial_x+i\ell-\beta)^2)^2+\varepsilon^2$. The functions U_* are just multiplications in the Bloch representation because they are periodic. More precisely, one has $\widehat{U}_*(\ell,x) = U_*(x)\delta(\ell)$ in the sense of distributions. The functions $\widehat{K}_{ct}$ are derived from K_{ct} of Eq.(1.6) and are seen to be given by

$$\widehat{K}_{ct}(\ell,x) \equiv \big(\mathcal{T}K_{ct}\big)(\ell,x) = e^{-i\ell ct}\widehat{F}_c(\ell,x-ct,x) - U_*(x)\delta(\ell) ,$$

where the Bloch transform is taken in the first (non-periodic) variable of F_c.

In order to obtain optimal results for the analysis ahead of the front, *i.e.*, for the variable in the weighted representation, we recall some facts from the construction [CE86, EW91] of the fronts.

For small $\varepsilon > 0$ the bifurcating solutions u of the Swift-Hohenberg equation can be approximated by

$$\tilde{\psi}(x,t,\varepsilon) = \varepsilon A(\varepsilon x,\varepsilon^2 t)e^{ix} + \text{c.c.} ,$$

up to an error $\mathcal{O}(\varepsilon^2)$, where A satisfies the Ginzburg-Landau equation

$$\partial_T A = 4\partial_X^2 A + A - 3A|A|^2 ,$$

with $X \in \mathbf{R}$, $T \geq 0$ and $A(X,T) \in \mathbf{C}$. See [CE90b, vH91, KSM92, Schn94]. This equation possesses a real-valued front $A_{\mathrm{f}}(X,T) = B(X - c_B T)$, where $\xi \mapsto B(\xi)$ satisfies the ordinary differential equation

$$4B'' + c_B B' + B - 3B|B|^2 = 0 .$$

For $|c_B| \geq 4$ the real–valued fronts of this equation are monotonic. These fronts and the trivial solution $A = 0$ can be stabilized by introducing a weight $e^{\beta_A x}$ satisfying the stability condition

$$\varrho_A(c_B, \beta_A) = 4\beta_A^2 - \beta_A c_B + 1 < 0 ,$$

see [BK92].

Remark. Since $B(\xi)$ converges at a faster rate to $1/\sqrt{3}$ for $\xi \to -\infty$ than to 0 for $\xi \to \infty$ there will be no additional restriction such as (3.3) on β_A.

Remark. Our result will be optimal in the sense that each modulated front F_c which corresponds to a front of the associated amplitude equation satisfying $\varrho_A(c_B, \beta_A) < 0$ is stable. The connection between the quantities of the Ginzburg-Landau equation and the associated Swift-Hohenberg equation is as follows. We have $c = \varepsilon c_B + \mathcal{O}(\varepsilon^2)$, and $\beta = \varepsilon\beta_A + \mathcal{O}(\varepsilon^2)$.

In order to prove this remark we write the modulated front F_c as defined in (1.2) as a sum of the Ginzburg-Landau part and a remainder

$$F_c(\xi, x) = 2\varepsilon B(\varepsilon\xi)\cos(x) + \varepsilon^2 F_{\mathrm{r}}(\xi, x) ,$$

where F_{r} satisfies

$$\sup_{y\in\mathbf{R}} \|F_{\mathrm{r}}(\cdot + y, \cdot)\|_{\mathcal{C}^2_{\mathrm{b},\delta}} \leq C ,$$

for a constant C independent of $\varepsilon \in (0, 1)$ and $\delta \in (0, 1)$. Then we consider (6.7) which we write without decomposition as

$$\partial_t \widehat{W} = \big(\widehat{L}_{i\beta}\widehat{W}\big) - 3\widehat{T}_{-\hat{c}t}(\widehat{\tau_{ct}F_c}) \circledast \widehat{T}_{-\hat{c}t}(\widehat{\tau_{ct}F}) \circledast \widehat{W} + \hat{c}(i(\ell + i\beta) + \partial_x)\widehat{W} . \qquad (6.8)$$

In order to control these solutions we use that the linearized system (6.7) evolves in such a way that during times of order $\mathcal{O}(1/\varepsilon^2)$ it can be approximated by the associated linearized Ginzburg-Landau equation

$$\partial_\tau A = 4(\partial_X - \beta_A)^2 A + c_B(\partial_X - \beta_A)A + A - B^2(2A + \bar{A}) . \qquad (6.9)$$

Theorem 6.2. *For all $C_0 > 0$, and $\tau_1 > 0$ there exist positive constants ε_0, C_1, C_2, and τ_0 such that for all $\varepsilon \in (0, \varepsilon_0]$ the following is true: For all initial conditions $\widehat{W}_0$ with $\|\widehat{W}_0\|_{\hat{\mathrm{H}}^{0,\delta}_2} \leq C_0\varepsilon$ there are a solution $\widehat{W}_t$ of (6.8) and a solution A_τ of (6.9) with $\|A_0\|_{\tilde{\mathrm{H}}^{0,\delta}_2} \leq C_1$ such that the function A_τ approximates $\widehat{W}_t$ in the sense that*

$$\|\widehat{W}_t - \varepsilon\mathcal{T}(A_{\varepsilon^2 t - \tau_0}(x)e^{ix} + \text{c.c.})\|_{\hat{\mathrm{H}}^{0,\delta}_2} \leq C_2\varepsilon^2 ,$$

for all $t \in [\tau_0/\varepsilon^2, (\tau_0 + \tau_1)/\varepsilon^2]$. Here $\mathcal{T}$ again denotes the map of Eq.(5.2) from a function f of x to its Bloch representation $\hat{f}(\ell, x)$.

Proof. The proof of this is very similar to the case of the (non-linear) Swift-Hohenberg equation which was discussed in the literature [CE90b, vH91, KSM92, Schn94]. Our (linear) problem is in fact easier and the proof is left to the reader. □

For the system (6.9) we have the estimate [BK92]

$$\|A_\tau\|_{\mathrm{H}^2_{0,\delta}} \leq Ce^{\varrho_A(c_B,\beta_A,\delta)\tau}\|A_0\|_{\mathrm{H}^2_{0,\delta}} ,$$

with $\lim_{\delta\to 0}\varrho_A(c_B,\beta_A,\delta) = \varrho_A(c_B,\beta_A)$. The deviation of $\varrho_A(c_B,\beta_A,\delta)$ from $\varrho_A(c_B,\beta_A)$ comes again from the derivatives of B. As a consequence of this estimate and of Theorem 6.2 we conclude that

$$\|\widehat{W}_t\|_{\hat{\mathrm{H}}^{0,\delta}_2} \leq Ce^{\varrho(c,\beta,\varepsilon,\delta)(t-t')}\|\widehat{W}_{t'}\|_{\hat{\mathrm{H}}^{0,\delta}_2} , \tag{6.10}$$

for a constant C and a coefficient $\varrho = \varrho(c,\beta,\varepsilon,\delta)$. We can (and will) choose this constant ϱ in such a way that (for $\varepsilon \to 0$):

$$\varrho(c,\beta,\varepsilon,\delta) = \varepsilon^2(\varrho_A(c_B,\beta_A,\delta) + o(1)) . \tag{6.11}$$

We define $\varrho(c,\beta,\varepsilon) = \lim_{\delta\to 0}\varrho(c,\beta,\varepsilon,\delta)$.

Remark. The choice of a sufficiently small $\delta > 0$ and $\varepsilon > 0$ will allow us to prove the stability of all fronts which are predicted to be stable by the associated amplitude equation since $\lim_{(\varepsilon,\delta)\to 0}\varepsilon^{-2}\varrho(c,\beta,\varepsilon,\delta) = \varrho_A(c_B,\beta_A)$.

In the following we consider a modulated front with velocity c and a given (sufficiently small) bifurcation parameter $\varepsilon > 0$ for which there are a β and a $\hat{c} \in (0,c)$ which satisfy:

$$\varrho(\hat{c},\beta,\varepsilon) = -2\gamma < 0 . \tag{6.12}$$

Proposition 6.3. *Suppose that the above stability condition (6.12) is satisfied. Then there is a* $\delta \in (0,1]$ *such that: There is a* $C < \infty$ *for which the functions* $\widehat{W}_t = \widehat{\mathcal{W}}_{\beta,\hat{c}t}\widehat{V}_t$ *obey the bounds*

$$\|\widehat{W}_t\|_{\hat{\mathrm{H}}^{0,\delta}_2} \leq Ce^{-3\gamma(t-s)/2}\|\widehat{W}_s\|_{\hat{\mathrm{H}}^{0,\delta}_2} . \tag{6.13}$$

As in the previous sections this result will have to be improved for the non-linear problem. Therefore, we skip at this point the proof, and will only deal with the improved version later.

Thus, the linear problems (6.2) and (6.7) are the analogs of (3.9) and (3.10) and can be studied pretty much as in the case of the simplified problem, yielding inequalities similar to (3.6) and (3.7).

7. The renormalization process for the full problem

We assume throughout this section that the stability condition (6.12) is satisfied. We prove here our main

Theorem 7.1. *There are a $\delta > 0$ and positive constants R and C such that the following holds: Assume $\|v_0\|_{\mathrm{H}^2_{2,\delta}} + \|M_\beta v_0\|_{\mathrm{H}^2_{2,\delta}} \le R$ and denote by v_t the solution of (1.4) with initial condition v_0. Let $\tilde{\psi}(\ell) = \exp(-c_1\ell^2)$. There is a constant $A_* = A_*(v_0)$ such that the rescaled solution $\hat{v}^{\mathrm{r}}_t(\ell, x) = \hat{v}_t(\ell t^{-1/2}, x)$ satisfies*

$$\|\hat{v}^{\mathrm{r}}_t - A_*\tilde{\psi}\partial_x U_*\|_{\mathcal{K}_{1/\sqrt{t},1}} \le \frac{CR}{(t+1)^{1/4}} . \tag{7.1}$$

Furthermore,

$$\|\widehat{w}_t\|_{\mathcal{K}^w} = \|\widehat{\mathcal{W}}_{\beta,\hat{c}t}\hat{v}_t\|_{\mathcal{K}^w} \le CRe^{-\gamma t} . \tag{7.2}$$

Remarks.

- The inequality (7.1) really says that the difference

  $$\hat{v}_t(\ell t^{-1/2}, x) - A_* e^{-c_1\ell^2}\partial_x U_*(x)$$

  is small, where U_* is the periodic solution (see Eq.(1.3)) of the Swift-Hohenberg equation. Expressed in the laboratory frame, this means that *an initial perturbation $v_0(x)$ will go to 0 like*

  $$v_t(x) \approx A_*(v_0)\sqrt{\frac{\pi}{c_1 t}}\exp\Big(\frac{-x^2}{4c_1 t}\Big)\,\partial_x U_*(x) ,$$

  when $t \to \infty$, uniformly for $x \in \mathbf{R}$. See [Schn96]. In particular, this means that near the extrema of U_* the convergence is faster than $\mathcal{O}(t^{-1/2})$ since at those points $\partial_x U_*$ vanishes.
- The inequality (7.2) gives some more precise bound on the growth of a perturbation ahead of the front, because it says that this perturbation decays exponentially in the weighted norm. More explicitly, we have at least a bound

  $$|v_t(x + ct)| \le Ce^{\beta x - \gamma' t} ,$$

  with γ' slightly smaller than γ
- The decay $(t+1)^{-1/4}$ in (7.1) can be improved easily to $(t+1)^{-1/2+\vartheta}$ for any $\vartheta > 0$. We have chosen $\vartheta = 1/4$ to keep the notation at a reasonable level.

Proof. As we explained before, the proof is similar to the one in Section 3 except that now the function behind the front is split into a diffusive part $\hat{v}_{\mathrm{c}}$ and into an exponentially damped part $\hat{v}_{\mathrm{s}}$, and correspondingly there will be a few more equations.

In Bloch space the initial conditions satisfy $\|\hat{v}_0\|_{\hat{\mathrm{H}}^{2,\delta}_2} + \|\hat{v}_0(\cdot + i\beta, \cdot)\|_{\hat{\mathrm{H}}^{0,\delta}_2} \le R$. The system for the variables $\hat{v}_{\mathrm{c}}$ and $\hat{v}_{\mathrm{s}}$ with initial conditions $\hat{v}_{\mathrm{c}}|_{t=0} = \hat{E}_{\mathrm{c}}\hat{v}|_{t=0}$, $\hat{v}_{\mathrm{s}}|_{t=0} = \hat{E}_{\mathrm{s}}\hat{v}|_{t=0}$, and for the variable $\widehat{w} = \widehat{\mathcal{W}}_{\beta,\hat{c}t}\hat{v}$ with initial conditions $\widehat{w}|_{t=0} = \widehat{\mathcal{W}}_{\beta,0}\hat{v}|_{t=0}$ is given in Bloch space by

$$\begin{aligned}
\partial_t \hat{v}_{\mathrm{c}} &= \widehat{\mathcal{M}}\hat{v}_{\mathrm{c}} + \hat{E}_{\mathrm{c}}\hat{\mathcal{H}}(\hat{v}_{\mathrm{c}}, \hat{v}_{\mathrm{s}}) + \hat{E}_{\mathrm{c}}\widehat{\mathcal{N}}(\hat{v}_{\mathrm{c}}, \hat{v}_{\mathrm{s}}) \,, \\
\partial_t \hat{v}_{\mathrm{s}} &= \widehat{\mathcal{M}}\hat{v}_{\mathrm{s}} + \hat{E}_{\mathrm{s}}\hat{\mathcal{H}}(\hat{v}_{\mathrm{c}}, \hat{v}_{\mathrm{s}}) + \hat{E}_{\mathrm{s}}\widehat{\mathcal{N}}(\hat{v}_{\mathrm{c}}, \hat{v}_{\mathrm{s}}) \,, \\
\partial_t \widehat{w} &= \widehat{\mathcal{M}}_{\mathrm{w}}\widehat{w} + \widehat{\mathcal{N}}_{\mathrm{w}}(\hat{v}_{\mathrm{c}}, \hat{v}_{\mathrm{s}}, \widehat{w}) \,,
\end{aligned} \tag{7.3}$$

where, see (1.8) and (6.7), with $\hat{v} = \hat{v}_{\mathrm{c}} + \hat{v}_{\mathrm{s}}$,

$$\begin{aligned}
\widehat{\mathcal{M}} &= \mathcal{T}\mathcal{M}\mathcal{T}^{-1} \,, \\
\hat{\mathcal{H}}(\hat{v}_{\mathrm{c}}, \hat{v}_{\mathrm{s}}) &= \mathcal{T}\mathcal{M}_{\mathbf{i}}\mathcal{T}^{-1}\hat{v} + \mathcal{T}\mathcal{N}_{\mathbf{i}}(\mathcal{T}^{-1}\hat{v}) \,, \\
\widehat{\mathcal{N}}(\hat{v}_{\mathrm{c}}, \hat{v}_{\mathrm{s}}) &= \mathcal{T}\mathcal{N}(\mathcal{T}^{-1}\hat{v}) \,, \\
\widehat{\mathcal{M}}_{\mathrm{w}} &= \widehat{\mathcal{M}}_{\beta,\hat{c}t} + \widehat{\mathcal{M}}_{\mathbf{i},\beta,\hat{c}t} \,, \\
\widehat{\mathcal{N}}_{\mathrm{w}}(\hat{v}_{\mathrm{c}}, \hat{v}_{\mathrm{s}}, \widehat{w}) &= -3\widehat{T}_{-\hat{c}t}U_* \cdot \widehat{T}_{-\hat{c}t}\hat{v} \circledast \widehat{w} - 3\widehat{T}_{-\hat{c}t}\widehat{K}_{ct} \circledast \widehat{T}_{-\hat{c}t}\hat{v} \circledast \widehat{w} - \widehat{T}_{-\hat{c}t}\hat{v} \circledast \widehat{T}_{-\hat{c}t}\hat{v} \circledast \widehat{w} \,.
\end{aligned}$$

It is useful to modify this system by introducing the coordinates $(\hat{u}_{\mathrm{c}}, \hat{u}_{\mathrm{s}})$ by

$$\hat{u}_{\mathrm{c}} = \hat{v}_{\mathrm{c}}, \quad \hat{u}_{\mathrm{s}} = -\widehat{\mathcal{M}}^{-1}\hat{E}_{\mathrm{s}}(3U_* \cdot \hat{v}_{\mathrm{c}} \circledast \hat{v}_{\mathrm{c}}) + \hat{v}_{\mathrm{s}} \,. \tag{7.4}$$

This coordinate transform takes care of the fact that asymptotically $\hat{v}_s$ can be expressed by $\hat{v}_c$. Under the scaling used below the new variable $\hat{u}_s$ converges to zero, while the old variable $\hat{v}_s$ converges to a nontrivial expression.

Under this transform (7.3) becomes

$$\begin{aligned}
\partial_t \hat{u}_{\mathrm{c}} &= \widehat{\mathcal{M}}\hat{u}_{\mathrm{c}} + \widehat{\mathcal{N}}_{\mathrm{c},\mathbf{i}}(\hat{u}_{\mathrm{c}}, \hat{u}_{\mathrm{s}}) + \widehat{\mathcal{N}}_{\mathrm{c}}(\hat{u}_{\mathrm{c}}, \hat{u}_{\mathrm{s}}), \\
\partial_t \hat{u}_{\mathrm{s}} &= \widehat{\mathcal{M}}\hat{u}_{\mathrm{s}} + \widehat{\mathcal{N}}_{\mathrm{s},\mathbf{i}}(\hat{u}_{\mathrm{c}}, \hat{u}_{\mathrm{s}}) + \widehat{\mathcal{N}}_{\mathrm{s}}(\hat{u}_{\mathrm{c}}, \hat{u}_{\mathrm{s}}), \\
\partial_t \widehat{w} &= \widehat{\mathcal{M}}_{\mathrm{w}}\widehat{w} + \widehat{\mathcal{N}}'_{\mathrm{w}}(\hat{u}_{\mathrm{c}}, \hat{u}_{\mathrm{s}}, \widehat{w}) \,,
\end{aligned} \tag{7.5}$$

where

$$\begin{aligned}
\widehat{\mathcal{N}}_{\mathrm{c},\mathbf{i}}(\hat{u}_{\mathrm{c}}, \hat{u}_{\mathrm{s}}) &= \hat{E}_{\mathrm{c}}\hat{\mathcal{H}}(\hat{u}_{\mathrm{c}}, \widehat{\mathcal{M}}^{-1}\hat{E}_{\mathrm{s}}(3U_* \cdot \hat{u}_{\mathrm{c}} \circledast \hat{u}_{\mathrm{c}}) + \hat{u}_{\mathrm{s}}) \,, \\
\widehat{\mathcal{N}}_{\mathrm{s},\mathbf{i}}(\hat{u}_{\mathrm{c}}, \hat{u}_{\mathrm{s}}) &= \hat{E}_{\mathrm{s}}\hat{\mathcal{H}}(\hat{u}_{\mathrm{c}}, \widehat{\mathcal{M}}^{-1}\hat{E}_{\mathrm{s}}(3U_* \cdot \hat{u}_{\mathrm{c}} \circledast \hat{u}_{\mathrm{c}}) + \hat{u}_{\mathrm{s}}) \,, \\
\widehat{\mathcal{N}}_{\mathrm{c}}(\hat{u}_{\mathrm{c}}, \hat{u}_{\mathrm{s}}) &= \hat{E}_{\mathrm{c}}\widehat{\mathcal{N}}(\hat{u}_{\mathrm{c}}, \widehat{\mathcal{M}}^{-1}\hat{E}_{\mathrm{s}}(3U_* \cdot \hat{u}_{\mathrm{c}} \circledast \hat{u}_{\mathrm{c}}) + \hat{u}_{\mathrm{s}}) \,, \\
\widehat{\mathcal{N}}_{\mathrm{s}}(\hat{u}_{\mathrm{c}}, \hat{u}_{\mathrm{s}}) &= \hat{E}_{\mathrm{s}}\widehat{\mathcal{N}}(\hat{u}_{\mathrm{c}}, \widehat{\mathcal{M}}^{-1}\hat{E}_{\mathrm{s}}(3U_* \cdot \hat{u}_{\mathrm{c}} \circledast \hat{u}_{\mathrm{c}}) + \hat{u}_{\mathrm{s}}) - \partial_t[\widehat{\mathcal{M}}^{-1}\hat{E}_{\mathrm{s}}(3U_* \cdot \hat{u}_{\mathrm{c}} \circledast \hat{u}_{\mathrm{c}})] \,, \\
\widehat{\mathcal{N}}'_{\mathrm{w}}(\hat{u}_{\mathrm{c}}, \hat{u}_{\mathrm{s}}, \widehat{w}) &= \widehat{\mathcal{N}}_{\mathrm{w}}(\hat{v}_{\mathrm{c}}, \hat{v}_{\mathrm{s}}, \widehat{w}) \,.
\end{aligned}$$

We follow the lines of Section 4 and start with the renormalization process by introducing the scalings

$$\begin{aligned}
\hat{v}_{\mathrm{c},n}(\varkappa, x, \tau) &= \hat{u}_{\mathrm{c}}(\sigma^n\varkappa, x, \sigma^{-2n}\tau) \,, \\
\hat{v}_{\mathrm{s},n}(\varkappa, x, \tau) &= \sigma^{-3n/2}\hat{u}_{\mathrm{s}}(\sigma^n\varkappa, x, \sigma^{-2n}\tau) \,, \\
\widehat{w}_n(\varkappa, x, \tau) &= e^{\gamma\sigma^{-2n}\tau}\widehat{w}(\varkappa, x, \sigma^{-2n}\tau) \,.
\end{aligned}$$

(The 3rd argument is the time, and the function w has here another meaning than in Section 4.) Note again that only the Bloch variable is rescaled, but x is left untouched. As before the Bloch variable is not scaled in the weighted representation w.

Under these scalings the functions $\hat{v}_{\mathrm{s},n}$ and $\widehat{w}_n$ still converge to 0 as $n \to \infty$. The variation of constant formula yields now

$$\begin{aligned}
\hat{v}_{\mathrm{c},n}(\varkappa, x, \tau) &= e^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{c},n}(\tau-\sigma^2)}\hat{v}_{\mathrm{c},n-1}(\sigma\varkappa, x, 1) \\
&\quad + \sigma^{-2n}\int_{\sigma^2}^{\tau}\mathrm{d}\tau'\, e^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{c},n}(\tau-\tau')}\big(\widehat{\mathcal{N}}_{\mathrm{c},\mathbf{i},n}(\hat{v}_{\mathrm{c},n}, \hat{v}_{\mathrm{s},n})\big)(\varkappa, x, \tau') \\
&\quad + \sigma^{-2n}\int_{\sigma^2}^{\tau}\mathrm{d}\tau'\, e^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{c},n}(\tau-\tau')}\big(\widehat{\mathcal{N}}_{\mathrm{c},n}(\hat{v}_{\mathrm{c},n}, \hat{v}_{\mathrm{s},n})\big)(\varkappa, x, \tau')\;, \qquad (7.6) \\
\hat{v}_{\mathrm{s},n}(\varkappa, x, \tau) &= e^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{s},n}(\tau-\sigma^2)}\sigma^{-3/2}\hat{v}_{\mathrm{s},n-1}(\sigma\varkappa, x, 1) \\
&\quad + \sigma^{-7n/2}\int_{\sigma^2}^{\tau}\mathrm{d}\tau'\, e^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{s},n}(\tau-\tau')}\big(\widehat{\mathcal{N}}_{\mathrm{s},\mathbf{i},n}(\hat{v}_{\mathrm{c},n}, \hat{v}_{\mathrm{s},n}))\big)(\varkappa, x, \tau') \\
&\quad + \sigma^{-7n/2}\int_{\sigma^2}^{\tau}\mathrm{d}\tau'\, e^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{s},n}(\tau-\tau')}\big(\widehat{\mathcal{N}}_{\mathrm{s},n}(\hat{v}_{\mathrm{c},n}, \hat{v}_{\mathrm{s},n})\big)(\varkappa, x, \tau')\;, \qquad (7.7) \\
\widehat{w}_n(\varkappa, x, \tau) &= \widehat{S}_n(\tau, \sigma^2)\widehat{w}_{n-1}(\varkappa, x, 1) \\
&\quad + \sigma^{-2n}\int_{\sigma^2}^{\tau}\mathrm{d}\tau'\, \widehat{S}_n(\tau, \tau')\big(\widehat{\mathcal{N}}_{\mathrm{w},n}(\hat{v}_{\mathrm{c},n}, \hat{v}_{\mathrm{s},n}, \widehat{w}_n)\big)(\varkappa, x, \tau')\;, \qquad (7.8)
\end{aligned}$$

with

$$\begin{aligned}
\widehat{\mathcal{M}}_{\mathrm{c},n} &= \widehat{\mathcal{L}}^n \hat{E}^{\mathrm{h}}_{\mathrm{c}}\widehat{\mathcal{M}}\widehat{\mathcal{L}}^{-n}\;, \\
\widehat{\mathcal{M}}_{\mathrm{s},n} &= \widehat{\mathcal{L}}^n \hat{E}^{\mathrm{h}}_{\mathrm{s}}\widehat{\mathcal{M}}\widehat{\mathcal{L}}^{-n}\;, \\
\widehat{\mathcal{N}}_{\mathrm{c},\mathbf{i},n}(\hat{v}_{\mathrm{c},n}, \hat{v}_{\mathrm{s},n}) &= \widehat{\mathcal{L}}^n\widehat{\mathcal{N}}_{\mathrm{c},\mathbf{i}}(\widehat{\mathcal{L}}^{-n}\hat{v}_{\mathrm{c},n}, \sigma^{3n/2}\widehat{\mathcal{L}}^{-n}\hat{v}_{\mathrm{s},n})\;, \\
\widehat{\mathcal{N}}_{\mathrm{s},\mathbf{i},n}(\hat{v}_{\mathrm{c},n}, \hat{v}_{\mathrm{s},n}) &= \widehat{\mathcal{L}}^n\widehat{\mathcal{N}}_{\mathrm{s},\mathbf{i}}(\widehat{\mathcal{L}}^{-n}\hat{v}_{\mathrm{c},n}, \sigma^{3n/2}\widehat{\mathcal{L}}^{-n}\hat{v}_{\mathrm{s},n})\;, \\
\widehat{\mathcal{N}}_{\mathrm{c},n}(\hat{v}_{\mathrm{c},n}, \hat{v}_{\mathrm{s},n}) &= \widehat{\mathcal{L}}^n\widehat{\mathcal{N}}_{\mathrm{c}}(\widehat{\mathcal{L}}^{-n}\hat{v}_{\mathrm{c},n}, \sigma^{3n/2}\widehat{\mathcal{L}}^{-n}\hat{v}_{\mathrm{s},n})\;, \\
\widehat{\mathcal{N}}_{\mathrm{s},n}(\hat{v}_{\mathrm{c},n}, \hat{v}_{\mathrm{s},n}) &= \widehat{\mathcal{L}}^n\widehat{\mathcal{N}}_{\mathrm{s}}(\widehat{\mathcal{L}}^{-n}\hat{v}_{\mathrm{c},n}, \sigma^{3n/2}\widehat{\mathcal{L}}^{-n}\hat{v}_{\mathrm{s},n})\;, \\
\widehat{\mathcal{N}}_{\mathrm{w},n}(\hat{v}_{\mathrm{c},n}, \hat{v}_{\mathrm{s},n}, \widehat{w}_n) &= \widehat{\mathcal{N}}'_{\mathrm{w}}(\widehat{\mathcal{L}}^{-n}\hat{v}_{\mathrm{c},n}, \sigma^{3n/2}\widehat{\mathcal{L}}^{-n}\hat{v}_{\mathrm{s},n}, \widehat{w}_n)\;,
\end{aligned}$$

where we recall the definition

$$\big(\widehat{\mathcal{L}}\hat{f}\big)(\ell, x) \equiv \hat{f}(\sigma\ell, x)\;,$$

and where $\widehat{S}_n(\tau, \tau')$ is now the evolution operator associated with the equation

$$\partial_\tau \hat{f}_\tau = \sigma^{-2n}(\widehat{\mathcal{M}}_{\mathrm{w}} + \gamma)\hat{f}_\tau\;. \qquad (7.9)$$

Again, the exponential scaling of $\widehat{w}_n$ with respect to time does not affect the definition of $\widehat{\mathcal{N}}_{\mathrm{w}}$ due to the fact that $\widehat{w}_n$ only appears linearly.

All this is quite analogous to the developments in Eqs.(4.10) and (4.11).

7.1. The scaled linear evolution operators

First we bound the linear evolution operators generated by $\widehat{\mathcal{M}}_{\mathrm{c},n}$ and $\widehat{\mathcal{M}}_{\mathrm{s},n}$.

Lemma 7.2. *For all $\rho_1 \geq \rho_2 \geq 0$ there exist $C_{\rho_1,\rho_2} > 0$ and $\gamma_- > 0$ such that for $1 \geq \tau > \tau' \geq \sigma^2$ and all $\sigma \in (0,1)$ one has*

$$\begin{aligned}
\|e^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{c},n}(\tau-\tau')}\widehat{\mathcal{L}}^n\hat{E}^{\mathrm{h}}_{\mathrm{c}}\widehat{\mathcal{L}}^{-n}\hat{g}\|_{\mathcal{K}_{\sigma^n,\rho_1}} &\leq C(\tau-\tau')^{\rho_2-\rho_1}\|\hat{g}\|_{\mathcal{K}_{\sigma^n,\rho_2}}\,, \\
\|e^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{s},n}(\tau-\tau')}\widehat{\mathcal{L}}^n\hat{E}^{\mathrm{h}}_{\mathrm{s}}\widehat{\mathcal{L}}^{-n}\hat{g}\|_{\mathcal{K}_{\sigma^n,\rho_1}} &\leq Ce^{-\gamma_-\sigma^{-2n}(\tau-\tau')}(\tau-\tau')^{\rho_2-\rho_1}\|\hat{g}\|_{\mathcal{K}_{\sigma^n,\rho_2}}\,,
\end{aligned}$$

for all $n \in \mathbf{N}$.

Proof. The first estimate follows directly from the fact that

$$\widehat{\mathcal{M}}_{\mathrm{c},n}(\ell)f = \mu_1(\ell)\hat{P}_{\mathrm{c}}(\ell)f = -c_1\ell^2\hat{P}_{\mathrm{c}}(\ell)f + \mathcal{O}(\ell^3)\,.$$

The second estimate follows from the fact that the real part of the spectrum of $\widehat{\mathcal{M}}_{\mathrm{s},n}(\ell)$ as a function of ℓ can be bounded from above by a strictly negative parabola. □

Next, we bound $\widehat{S}_n(\tau,\tau')$ as defined through (7.9) and state the analog of Lemma 4.2.

Lemma 7.3. *Suppose that the stability condition (6.12) is satisfied. Then there is a $\delta \in (0,1]$ and a $C > 0$ such that for $1 > \tau > \tau' \geq 0$ and all $\sigma \in (0,1]$ one has*

$$\|\widehat{S}_n(\tau,\tau')\widehat{w}\|_{\mathcal{K}^w} \;\leq\; Ce^{-\gamma\sigma^{-2n}(\tau-\tau')/2}\|\widehat{w}\|_{\mathcal{K}^w}\,, \tag{7.10}$$

for all $n \in \mathbf{N}$.

The proof of Lemma 7.3 follows closely the one of Lemma 4.2 in Section 4.1. Therefore, it will be omitted here.

7.2. The scaled non-linear terms

Next we estimate the scaled non-linear terms in $\mathcal{N}_{\mathrm{c},n}$, $\mathcal{N}_{\mathrm{s},n}$, and $\mathcal{N}_{\mathrm{w},n}$. In order to estimate the time derivatives on the right hand side of (7.5) coming from the coordinate transform (7.4) we need to choose $\hat{v}_{\mathrm{c},n}$ in the better space $\mathcal{K}^c_{\sigma^n} = \mathcal{K}_{\sigma^n,3/2}$ instead of only being in $\mathcal{K}_{\sigma^n,1}$.

Lemma 7.4. *Suppose* $\max\{\|\hat{v}_{\mathrm{c},n}\|_{\mathcal{K}^c_{\sigma^n}}, \|\hat{v}_{\mathrm{s},n}\|_{\mathcal{K}^s_{\sigma^n}}, \|\widehat{w}_n\|_{\mathcal{K}^w}\} \leq 1$. *Then there exists a $C_1 > 0$ such that for all $\sigma \in (0,1]$ one has*

$$\begin{aligned}
\|\widehat{\mathcal{N}}_{\mathrm{c},n}\|_{\mathcal{K}_{\sigma^n,3/4}} &\leq C_1\sigma^{5n/2}(\|\hat{v}_{\mathrm{c},n}\|_{\mathcal{K}^c_{\sigma^n}} + \|\hat{v}_{\mathrm{s},n}\|_{\mathcal{K}^s_{\sigma^n}})^2 \\
\|\widehat{\mathcal{N}}_{\mathrm{s},n}\|_{\mathcal{K}^s_{\sigma^n}} &\leq C_1\sigma^{2n}(\|\hat{v}_{\mathrm{c},n}\|_{\mathcal{K}^c_{\sigma^n}} + \|\hat{v}_{\mathrm{s},n}\|_{\mathcal{K}^s_{\sigma^n}})^2 \\
\|\widehat{\mathcal{N}}_{\mathrm{w},n}\|_{\mathcal{K}^w} &\leq C_1\sigma^{n/2}(\|\hat{v}_{\mathrm{c},n}\|_{\mathcal{K}^c_{\sigma^n}} + \|\hat{v}_{\mathrm{s},n}\|_{\mathcal{K}^s_{\sigma^n}})\|\widehat{w}_n\|_{\mathcal{K}^w}\,.
\end{aligned}$$

Proof. Throughout the proof we use

$$\big(\widehat{\mathcal{L}}(\hat{f} \circledast \hat{g})\big)(\varkappa) \;=\; \sigma\big((\widehat{\mathcal{L}}\hat{f}) \circledast (\widehat{\mathcal{L}}\hat{g})\big)(\varkappa)\,. \tag{7.11}$$

i) We start with the estimates for $\widehat{\mathcal{N}}_{\mathrm{w},n}$. The most dangerous term in $\widehat{\mathcal{N}}_{\mathrm{w},n}$ coming from $\widehat{\mathcal{N}}_{\mathrm{w}}(\hat{v}_{\mathrm{c}}, \hat{v}_{\mathrm{s}}, \widehat{w})$ is

$$3\widehat{T}_{-\hat{c}\sigma^{-2n}\tau}\widehat{K}_{ct} \circledast \widehat{T}_{-\hat{c}\sigma^{-2n}\tau}(\widehat{\mathcal{L}}^{-n}\hat{v}_{\mathrm{c},n}) \circledast \widehat{w}_n\,.$$

From $\|\widehat{\mathcal{L}}^{-n}\hat{v}_{\mathrm{c},n}\|_{\hat{\mathrm{H}}_2^{0,\delta}} \leq C\sigma^{n/2}\|\hat{v}_{\mathrm{c},n}\|_{\mathcal{K}^{\mathrm{c}}_{\sigma^n}}$ and (5.6) we obtain

$$\begin{aligned}
&\|\widehat{T}_{-\hat{c}\sigma^{-2n}\tau}\widehat{K}_{ct} \circledast \widehat{T}_{-\hat{c}\sigma^{-2n}\tau}(\widehat{\mathcal{L}}^{-n}\hat{v}_{\mathrm{c},n}) \circledast \widehat{w}_n\|_{\hat{\mathrm{H}}_2^{0,\delta}}\\
&\leq \|\widehat{T}_{-\hat{c}\sigma^{-2n}\tau}\widehat{K}_{ct}\|_{\mathcal{C}^2_{\mathrm{b},\delta}}\; \|\widehat{\mathcal{L}}^{-n}(\widehat{T}_{-\hat{c}\sigma^{-n}\tau}\hat{v}_{\mathrm{c},n}) \circledast \widehat{w}_n\|_{\hat{\mathrm{H}}_2^{0,\delta}}\\
&\leq C\sigma^{n/2}\|\hat{v}_{\mathrm{c},n}\|_{\mathcal{K}^{\mathrm{c}}_{\sigma^n}}\|\widehat{w}_n\|_{\mathcal{K}^w}\,.
\end{aligned}$$

ii) We use (7.11) to obtain the estimates for $\widehat{\mathcal{N}}_{\mathrm{s},n}$. The only difficulty stems from the term

$$\partial_t[\widehat{\mathcal{M}}^{-1}\hat{E}_{\mathrm{s}}(3U_* \cdot \hat{u}_{\mathrm{c}} \circledast \hat{u}_{\mathrm{c}})] = \widehat{\mathcal{M}}^{-1}\hat{E}_{\mathrm{s}}(6U_* \cdot \hat{u}_{\mathrm{c}} \circledast \partial_t\hat{u}_{\mathrm{c}})$$

coming from the change of coordinates (7.4). This can be estimated in the required way by expressing $\partial_t\hat{u}_{\mathrm{c}}$ by the right hand side of (7.5), by using then the points ii.1)–ii.3) and the fact we already have a factor σ^n by $\hat{u}_{\mathrm{c}} \circledast \partial_t\hat{u}_{\mathrm{c}}$ using again (7.11).

ii.1) The first bound for the terms on the right hand side of (7.5) is

$$\|\widehat{\mathcal{M}}_{c,n}\hat{v}_{\mathrm{c},n}\|_{\mathcal{K}_{\sigma^n,1}} \leq C\sigma^n\|\hat{v}_{\mathrm{c},n}\|_{\mathcal{K}_{\sigma^n,3/2}}\,,$$

which follows from the form of $\mu_1(\ell)$ by using the following lemma.

Lemma 7.5. *Let* $\mu \in \mathcal{C}^2_{\mathrm{per}}([-1/2,1/2), \mathcal{C}^2((0,2\pi),\mathbf{C}))$ *with* $\|\mu(\ell,\cdot)\|_{\mathcal{C}^2((0,2\pi),\mathbf{C})} \leq C|\ell|^{2(\rho_1-\rho_2)}$ *for a* $\rho_1 \geq \rho_2 \geq 0$*. Then, there exists a* $C>0$ *such that for all* $\sigma \in (0,1]$ *we have*

$$\|(\widehat{\mathcal{L}}_\sigma\mu)\hat{u}\|_{\mathcal{K}_{\sigma,\rho_2}} \leq C\sigma^{2(\rho_1-\rho_2)}\|\mu\|_{\mathcal{C}^2_{\mathrm{per}}([-1/2,1/2),\mathcal{C}^2((0,2\pi),\mathbf{C}))}\|\hat{u}\|_{\mathcal{K}_{\sigma,\rho_1}}\,. \tag{7.12}$$

Proof. This follows since

$$\sup_{\ell\in\mathbf{R}}\Big|\frac{\ell^{2(\rho_1-\rho_2)}\sigma^{2(\rho_1-\rho_2)}}{(1+\ell^2)^{(\rho_1-\rho_2)}}\Big| < C\sigma^{2(\rho_1-\rho_2)}\,.$$

□

ii.2) By Lemma 7.8 below the term $\widehat{\mathcal{N}}_{\mathrm{c},\mathbf{i},n}$ is exponentially small in terms of σ.

ii.3) From (7.11) we easily obtain

$$\|\widehat{\mathcal{N}}_{c,n}\|_{\mathcal{K}_{\sigma^n,1}} \leq \sigma^n(\|\hat{v}_{c,n}\|_{\mathcal{K}^c_{\sigma^n}} + \|\hat{v}_{s,n}\|_{\mathcal{K}^s_{\sigma^n}})^2 .$$

iii) From [Schn96] we recall the estimates for the $\widehat{\mathcal{N}}_{c,n}$ part. Note that $\widehat{\mathcal{N}}_{c,n}$ can be written as

$$\widehat{\mathcal{N}}_{c,n} = \hat{s}_1 + \hat{s}_2 + \widehat{\mathcal{N}}_{c,n,r} ,$$

where

$$\begin{aligned}
\hat{s}_1 &= -3\sigma^n \widehat{\mathcal{L}}^n \hat{E}_c \widehat{\mathcal{L}}^{-n}(U_* \cdot \hat{v}_{c,n} \circledast \hat{v}_{c,n}) , \\
\hat{s}_2 &= -6\sigma^{2n} \widehat{\mathcal{L}}^n \hat{E}_c \widehat{\mathcal{L}}^{-n}(U_* \cdot \hat{v}_{c,n} \circledast (\widehat{\mathcal{M}}_{s,n})^{-1}(3U_* \cdot \hat{v}_{c,n} \circledast \hat{v}_{c,n})) \\
&\quad - \sigma^{2n} \widehat{\mathcal{L}}^n \hat{E}_c \widehat{\mathcal{L}}^{-n}(\hat{v}_{c,n} \circledast \hat{v}_{c,n} \circledast \hat{v}_{c,n}) , \\
\|\widehat{\mathcal{N}}_{c,n,r}\|_{\mathcal{K}_{\sigma^n,1}} &= \mathcal{O}(\sigma^{5n/2}(\|\hat{v}_{c,n}\|_{\mathcal{K}^c_{\sigma^n}} + \|\hat{v}_{s,n}\|_{\mathcal{K}^s_{\sigma^n}})^2) .
\end{aligned}$$

The estimate for $\widehat{\mathcal{N}}_{c,n,r}$ follows easily by applying again (7.11).

It remains to estimate $\hat{s}_1$ and $\hat{s}_2$. These estimates have been obtained in [Schn96]. For completeness we recall some of the arguments. Introducing $a_n(\ell) \in \mathbf{C}$ by $\hat{v}_{c,n}(\ell, x) = a_n(\ell)\varphi_{\sigma^n\ell}(x)$ shows that the terms $\hat{s}_1$ and $\hat{s}_2$ are of the form

$$\begin{aligned}
\hat{s}_2(\ell, x) \;&=\; \left(\sigma^{2n} \int \mathrm{d}m \int \mathrm{d}k\, K_2(\sigma^n\ell, \sigma^n(\ell-m), \sigma^n(m-k), \sigma^n k) \right. \\
&\qquad \left. \times\, a_n(\ell-m)a_n(m-k)a_n(k) \right) \varphi_{\sigma^n\ell}(x) , \\
\hat{s}_1(\ell, x) \;&=\; \left(\sigma^n \int \mathrm{d}m\, K_1(\sigma^n\ell, \sigma^n(\ell-m), \sigma^n m)\, a_n(\ell-m)\, a_n(m)\right) \varphi_{\sigma^n\ell}(x) ,
\end{aligned}$$

with $K_j : \mathbf{R}^{2+j} \to \mathbf{C}$ the kernel of an integral operator. The detailed expression for K_1 is given in (7.13) below.

The case $n = m = k = \ell = 0$ corresponds to the spatially periodic case. In the spatially periodic case there exists a center manifold

$$\Gamma = \{u = U_{0,a} \mid a \in \mathbf{R}\} ,$$

consisting of the spatially periodic fixed points related to each other by the translation invariance of the original Swift-Hohenberg equation. By a formal calculation it turns out that the flow of the one-dimensional center manifold Γ is determined by the ordinary differential equation

$$\frac{d}{dt}a = 0 \cdot a + K_1(0,0,0)a^2 + K_2(0,0,0,0)a^3 + \mathcal{O}(a^4) .$$

Since the center manifold consists of fixed points the flow $a = a(t)$ is trivial, *i.e.*, $\frac{d}{dt}a = 0$. Consequently, we obtain $K_1(0,0,0) = K_2(0,0,0,0) = 0$. Therefore,

$$|K_2(\ell, \ell-m, m-k, k)| \;\leq\; C(|\ell| + |\ell-m| + |m-k| + |k|) ,$$

and so (7.11) and (7.12) imply

$$\|\hat{s}_2\|_{\mathcal{K}_{\sigma^n,1}} \leq C\sigma^{3n}(\|\hat{v}_{\mathrm{c},n}\|_{\mathcal{K}^c_{\sigma^n}})^2 .$$

Interestingly it turned out that the first derivatives of K_1 vanish as well. Since the eigenvalue problem $\mathcal{M}_\ell \varphi_\ell = \mu_1(\ell)\varphi_\ell$ is self-adjoint, the projection $\hat{P}_\mathrm{c}(\ell)$ is orthogonal in $\mathrm{L}^2(0, 2\pi)$ and is given by $\hat{P}_\mathrm{c}(\ell)u = (\int \overline{\varphi_\ell(x)}u(\ell, x)dx)\varphi_\ell(\cdot)$. Thus

$$K_1(\ell, \ell - m, m) = 3 \int \mathrm{d}x\, \overline{\varphi_\ell(x)}\varphi_{\ell-m}(x)\varphi_m(x)U(x) . \tag{7.13}$$

Expanding $\varphi_\ell(x) = \partial_x U(x) + i\ell g(x) + \mathcal{O}(\ell^2)$, with $g(x) \in \mathbf{R}$ yields

$$\begin{aligned} K_1(\ell, \ell - m, m) =& 3 \int \mathrm{d}x \Big((\partial_x U(x))^3 U(x) \\ & - i\ell g(x)(\partial_x U(x))^2 U(x) + i(\ell - m)g(x)(\partial_x U(x))^2 U(x) \\ & + (\partial_x U(x))^2 i m g(x)U(x) + \mathcal{O}(\ell^2 + (\ell - m)^2 + m^2) \Big) . \end{aligned}$$

Note that $U(x)$ is an even function, so $\partial_x U$ is odd, which proves again $K_1(0, 0, 0) = 0$. Since, in addition, the first order terms cancel we have

$$|K_1(\ell, \ell - m, m)| \leq C|\ell^2 + (\ell - m)^2 + m^2| ,$$

and so from (7.11) and (7.12)

$$\|\hat{s}_1\|_{\mathcal{K}_{\sigma^n,3/4}} \leq C\sigma^{5n/2}(\|\hat{v}_{\mathrm{c},n}\|_{\mathcal{K}^c_{\sigma^n}})^2 .$$

Summing the estimates shows the assertion. □

7.3. Bounds on the integrals

Here we estimate the integrals in the variation of constant formula in terms of the following quantities.

Definition 7.6. *For all n, we define*

$$R^u_{\mathrm{cs},n} = \sup_{\tau\in[\sigma^2,1]} \|\hat{v}_{\mathrm{c},n}(\tau)\|_{\mathcal{K}^c_{\sigma^n}} + \sup_{\tau\in[\sigma^2,1]} \|\hat{v}_{\mathrm{s},n}(\tau)\|_{\mathcal{K}^s_{\sigma^n}} , \quad \textit{and} \quad R^w_n = \sup_{\tau\in[\sigma^2,1]} \|\widehat{w}_n(\tau)\|_{\mathcal{K}^w} .$$

In the following two lemmas we estimate the integrals appearing in (7.6)–(7.8).

Lemma 7.7. *Assume* $R^u_{\mathrm{cs},n} + R^w_n \leq 1$*. Then for all* $1 \geq \tau \geq \sigma^2$ *and all* $\sigma \in (0,1]$ *one has*

$$\|\sigma^{-2n}\int_{\sigma^2}^{\tau} \mathrm{d}\tau'\, e^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{c},n}(\tau-\tau')}\big(\widehat{\mathcal{N}}_{\mathrm{c},n}(\hat{v}_{\mathrm{c},n},\hat{v}_{\mathrm{s},n})\big)(\cdot,\cdot,\tau')\|_{\mathcal{K}^c_{\sigma^n}} \leq C\sigma^{n/2}(R^u_{\mathrm{cs},n})^2\,,$$

$$\|\sigma^{-7n/2}\int_{\sigma^2}^{\tau} \mathrm{d}\tau'\, e^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{s},n}(\tau-\tau')}\big(\widehat{\mathcal{N}}_{\mathrm{s},n}(\hat{v}_{\mathrm{c},n},\hat{v}_{\mathrm{s},n})\big)(\cdot,\cdot,\tau')\|_{\mathcal{K}^s_{\sigma^n}} \leq C\sigma^{n/2}(R^u_{\mathrm{cs},n})^2\,,$$

$$\|\sigma^{-2n}\int_{\sigma^2}^{\tau} \mathrm{d}\tau'\, \widehat{S}_n(t,\tau')\big(\widehat{\mathcal{N}}_{\mathrm{w},n}(\hat{v}_{\mathrm{c},n},\hat{v}_{\mathrm{s},n},\hat{w}_n)\big)(\cdot,\cdot,\tau')\|_{\mathcal{K}^w} \leq\ C\sigma^{n/2}R^u_{\mathrm{cs},n}R^w_n\,.$$

Proof. We first use Lemma 7.2 and Lemma 7.4. For the second integral in (7.6) we get a bound

$$\begin{aligned}
&\sup_{\tau\in[\sigma^2,1]}\|\sigma^{-2n}\int_{\sigma^2}^{\tau}\mathrm{d}\tau' e^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{c},n}(\tau-\tau')}\big(\widehat{\mathcal{N}}_{\mathrm{c},n}(\hat{v}_{\mathrm{c},n},\hat{v}_{\mathrm{s},n})\big)(\cdot,\cdot,\tau')\|_{\mathcal{K}^c_{\sigma^n}}\\
&\leq\ C\sigma^{-2n}(R^u_{\mathrm{cs},n})^2\sigma^{5n/2}\int_{\sigma^2}^{1}\mathrm{d}\tau'(1-\tau')^{-3/4}\\
&\leq\ C\sigma^{n/2}(R^u_{\mathrm{cs},n})^2\,.
\end{aligned}$$

For the second integral in (7.7) we find similarly

$$\begin{aligned}
&\sup_{\tau\in[\sigma^2,1]}\|\sigma^{-7n/2}\int_{\sigma^2}^{\tau}\mathrm{d}\tau'\, e^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{s},n}(\tau-\tau')}\big(\widehat{\mathcal{N}}_{\mathrm{s},n}(\hat{v}_{\mathrm{c},n},\hat{v}_{\mathrm{s},n})\big)(\cdot,\cdot,\tau')\|_{\mathcal{K}^s_{\sigma^n}}\\
&\qquad\leq\ C(R^u_{\mathrm{cs},n})^2\sigma^{-3n/2}\int_{\sigma^2}^{1}\mathrm{d}\tau'\, e^{-C\sigma^{-2n}(1-\tau')}\\
&\qquad\leq\ C\sigma^{n/2}(R^u_{\mathrm{cs},n})^2\,.
\end{aligned}$$

For the integral in (7.8) we find, using now Lemma 7.3 and Lemma 7.4, a bound

$$\begin{aligned}
C\sigma^{-2n}&\int_{\sigma^2}^{\tau}\mathrm{d}\tau'\, e^{-\gamma\sigma^{-2n}(\tau-\tau')/2}(\sigma^{n/2}R^u_{\mathrm{cs},n}R^w_n)\\
&\leq\ C\sigma^{n/2}R^u_{\mathrm{cs},n}R^w_n\,.
\end{aligned}$$

□

Lemma 7.8. *Assume* $R^u_{\mathrm{cs},n} + R^w_n \leq 1$*. Then for all* $1 \geq \tau \geq \sigma^2$ *and all* $\sigma \in (0,1)$ *one has*

$$\|\sigma^{-2n}\int_{\sigma^2}^{\tau}\mathrm{d}\tau'\, e^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{c},n}(\tau-\tau')}\big(\widehat{\mathcal{N}}_{\mathrm{c},\mathbf{i},n}(\hat{v}_{\mathrm{c},n},\hat{v}_{\mathrm{s},n})\big)(\cdot,\cdot,\tau')\|_{\mathcal{K}^c_{\sigma^n}} \leq\ Ce^{-(\beta(c-\hat{c})+\gamma)\sigma^{-n}}R^w_n\,,$$

$$\|\sigma^{-7n/2}\int_{\sigma^2}^{\tau}\mathrm{d}\tau'\, e^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{s},n}(\tau-\tau')}\big(\widehat{\mathcal{N}}_{\mathrm{s},\mathbf{i},n}(\hat{v}_{\mathrm{c},n},\hat{v}_{\mathrm{s},n})\big)(\cdot,\cdot,\tau')\|_{\mathcal{K}^s_{\sigma^n}} \leq\ Ce^{-(\beta(c-\hat{c})+\gamma)\sigma^{-n}}R^w_n\,.$$

Proof. We restrict ourselves to the linear part $\mathcal{M}_{\mathbf{i}}$. A typical term of (7.6)—the first in the definition of $\mathcal{M}_{\mathbf{i}}$ in (1.8)—can be rewritten as

$$
\begin{aligned}
&\sigma^{-2n}\left(\int_{\sigma^2}^{\tau} \mathrm{d}\tau'\, e^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{c},n}(\tau-\tau')}\widehat{\mathcal{L}}^n\big(\widehat{K}_{c\sigma^{-2n}\tau'} \circledast (\widehat{\mathcal{L}}^{-n}\hat{v}_{\mathrm{c},n}(\tau'))\big)\right)(\varkappa,x)\,U(x)\\
&\quad = \sigma^{-2n}\left(\int_{\sigma^2}^{\tau} \mathrm{d}\tau'\, e^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{c},n}(\tau-\tau')}\widehat{\mathcal{L}}^n\big(\widehat{K}_{c\sigma^{-2n}\tau'} \circledast \hat{u}_{\sigma^{-2n}\tau'}\big)\right)(\varkappa,x)\,U(x)\,.
\end{aligned}
$$

Since $K_{ct}(x)$ vanishes as $x \to -\infty$ with some exponential rate, its Bloch wave transform $\widehat{K}_{c\sigma^{-2n}\tau'}$ can be extended into a strip in the complex plane such that

$$
\begin{aligned}
&\big(\widehat{K}_{c\sigma^{-2n}\tau'} \circledast \hat{u}_{\sigma^{-2n}\tau'}\big)(\varkappa,x)\\
&\quad = \int \mathrm{d}\ell\,\widehat{K}_{c\sigma^{-2n}\tau'}(\varkappa-\ell-i\beta,x)\,\widehat{w}(\ell,x,\sigma^{-2n}\tau')e^{-i\ell\hat{c}\sigma^{-2n}\tau'}e^{-\gamma\sigma^{-2n}\tau'}\\
&\quad \times e^{-\beta(c-\hat{c})\sigma^{-2n}\tau'}e^{i(\varkappa-\ell)c\sigma^{-2n}\tau'}\,.
\end{aligned}
$$

Using this identity, we get, as in (4.20)—because $\exp(\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{c},n}(\tau-\tau'))$ is bounded—from $\|\hat{\mathcal{L}}^n\hat{u}\|_{\mathcal{K}_{\sigma^n,3/2}} \le C\sigma^{-11n/2}\|\hat{u}\|_{\hat{\mathrm{H}}_2^{2,\delta}}$ and (5.7), and recalling $\mathcal{K}^w = \hat{\mathrm{H}}_2^{0,\delta}$:

$$
\begin{aligned}
&\sigma^{-2n}\,\Big\|\int_{\sigma^2}^{\tau}\mathrm{d}\tau'\,e^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{c},n}(\tau-\tau')}\widehat{\mathcal{L}}^n\big(\widehat{K}_{c\sigma^{-2n}\tau'}\circledast\hat{u}_{n,\tau'}\big)\Big\|_{\mathcal{K}_{\sigma^n,3/2}}\\
&\quad\le C\sigma^{-2n}\int_{\sigma^2}^{\tau}\mathrm{d}\tau'\|\widehat{\mathcal{L}}^n(\widehat{K}_{c\sigma^{-2n}\tau'}\circledast\hat{u}_{n,\tau'})\|_{\mathcal{K}_{\sigma^n,3/2}}\\
&\quad\le C\sigma^{-2n}\int_{\sigma^2}^{\tau}\mathrm{d}\tau' e^{-\beta(c-\hat{c})\sigma^{-2n}\tau'}\sigma^{-11n/2}(1+c\sigma^{-2n}\tau')^2\\
&\qquad\qquad\times\ \|(\varkappa,x)\mapsto e^{-ic\sigma^{-2n}\tau'\varkappa}\widehat{K}_{c\sigma^{-2n}\tau'}(\varkappa-i\beta,x)\|_{\hat{\mathrm{H}}_2^{2,\delta}}\\
&\qquad\qquad\times\ \|(\varkappa,x)\mapsto e^{-i\varkappa\hat{c}\sigma^{-2n}\tau'}\widehat{w}_{n,\tau'}(\varkappa,x)\|_{\mathcal{K}^w}e^{-\gamma\sigma^{-2n}\tau'}\\
&\quad\le C\sigma^{-15n/2}\int_{\sigma^2}^{\tau}\mathrm{d}\tau'(1+c\sigma^{-2n}\tau')^2e^{-\beta(c-\hat{c})\sigma^{-2n}\tau'}e^{-\gamma\sigma^{-2n}\tau'}R_n^w\\
&\quad\le C\sigma^{-23n/2}e^{-(\beta(c-\hat{c})+\gamma)\sigma^{-2(n-1)}}R_n^w\ \le\ Ce^{-(\beta(c-\hat{c})+\gamma)\sigma^{-n}}R_n^w\,.
\end{aligned}
\tag{7.14}
$$

The non-linear terms coming from $\mathcal{N}_{\mathbf{i}}$ can be handled in exactly the same way and yield similar bounds. The same is true for the terms with $\mathcal{N}_{\mathrm{s},\mathbf{i},n}$ in (7.7). □

7.4. Bounds on the initial condition

Here, we estimate the first terms on the right hand side of the variation of constant formulae (7.6)–(7.8).

Lemma 7.9. *For all* $1 \geq \tau \geq \sigma^2$ *and all* $\sigma \in (0, 1]$ *we have*

$$\begin{aligned} \|e^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{c},n}(\tau-\sigma^2)}\widehat{\mathfrak{L}}^n \hat{E}_{\mathrm{c}}^{\mathrm{h}}\widehat{\mathfrak{L}}^{-n}\widehat{\mathfrak{L}}\hat{g}\|_{\mathcal{K}^c_{\sigma^n}} &\leq C\sigma^{-11/2}\|\hat{g}\|_{\mathcal{K}^c_{\sigma^{n-1}}} , \\ \|e^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{s},n}(\tau-\sigma^2)}\widehat{\mathfrak{L}}^n \hat{E}_{\mathrm{s}}^{\mathrm{h}}\widehat{\mathfrak{L}}^{-n}\sigma^{-3/2}\widehat{\mathfrak{L}}\hat{g}\|_{\mathcal{K}^s_{\sigma^n}} &\leq C\sigma^{-6}e^{-C\sigma^{-2n}(\tau-\sigma^2)}\|\hat{g}\|_{\mathcal{K}^s_{\sigma^{n-1}}} , \\ \|\widehat{S}_n(\tau, \sigma^2)\hat{g}\|_{\mathcal{K}^w} &\leq Ce^{-\gamma\sigma^{-2n}(\tau-\sigma^2)/2}\|\hat{g}\|_{\mathcal{K}^w} . \end{aligned}$$

Proof. The first two bounds of Lemma 7.9 follow immediately from Lemma 7.2 and (6.4). The third inequality is an immediate consequence of Lemma 7.3. □

7.5. A priori bounds on the non-linear problem

This section follows closely Section 4.2. We need a priori bounds on the solution of (7.6)–(7.8). We (re)define now quantities analogous to those of Definition 4.3.

Definition 7.10. *For all* $n \in \mathbf{N}$, *we define*

$$\rho^u_{\mathrm{cs},n} = \|\hat{v}_{\mathrm{c},n}|_{\tau=1}\|_{\mathcal{K}^c_{\sigma^n}} + \|\hat{v}_{\mathrm{s},n}|_{\tau=1}\|_{\mathcal{K}^s_{\sigma^n}} , \quad \text{and} \quad \rho^w_n = \|\widehat{w}_n|_{\tau=1}\|_{\mathcal{K}^w} .$$

Lemma 7.11. *For all* $n \in \mathbf{N}$ *there is a constant* $\eta_n > 0$ *such that the following holds: If* $\rho^u_{\mathrm{cs},n-1}$, ρ^w_{n-1}, *and* $\sigma > 0$ *are smaller than* η_n, *the solutions of (7.6)–(7.8) exist for all* $\tau \in [\sigma^2, 1]$. *Moreover, we have the estimates*

$$R^u_{\mathrm{cs},n} \leq C\sigma^{-6}\rho^u_{\mathrm{cs},n-1} + Ce^{-C\sigma^{-n}}R^w_n + C\sigma^{n/2}(R^u_{\mathrm{cs},n})^2 , \tag{7.15}$$

and

$$R^w_n \leq C\rho^w_{n-1} + C\sigma^{n/2}R^u_{\mathrm{cs},n}R^w_n , \tag{7.16}$$

with a constant C *independent of* σ *and* n.

Remark. We remark again that there is no need for a detailed expression for η_n since the existence of the solutions is guaranteed if we can show $R^u_{\mathrm{cs},n} < \infty$ and $R^w_n < \infty$. By (7.15) and (7.16) we have detailed control of these quantities in terms of the norms of the initial conditions and σ.

Proof. For the derivation of the estimates we assume in the sequel, without loss of generality, that $R^u_{\mathrm{cs},n} + R^w_n \le 1$. For the first term in (7.8) we obtained in Lemma 7.9 a bound

$$C\rho^w_{n-1}\,. \tag{7.17}$$

For the second term in (7.8), we obtained in Lemma 7.7 a bound $C\sigma^{n/2}R^u_{\mathrm{cs},n}R^w_n$.

We now discuss in detail (7.7). Using Lemma 7.9 the first term is bounded by $C\sigma^{-6}\rho^u_{\mathrm{cs},n-1}$. Lemma 7.7 and Lemma 7.8 yield for the second and third terms a bound $C\sigma^{n/2}(R^u_{\mathrm{cs},n})^2 + Ce^{-C\sigma^{-n}}R^w_n$ for a $C>0$ independent of $\sigma \in (0,1]$ and $n \in \mathbf{N}$.

Finally, we come to the bounds for (7.6). Using Lemma 7.9 the first term is bounded by $C\sigma^{-11/2}\rho^u_{\mathrm{cs},n-1}$. Lemma 7.7 and Lemma 7.8 yield for the second and third terms a bound $C\sigma^{n/2}(R^u_{\mathrm{cs},n})^2 + Ce^{-C\sigma^{-n}}R^w_n$ for a $C>0$ independent of $\sigma \in (0,1]$ and $n \in \mathbf{N}$.

The proof of Lemma 7.11 now follows by applying the contraction mapping principle to the system consisting of (7.6), (7.7), and (7.8).

Then for $\rho^u_{\mathrm{cs},n-1}$, ρ^w_{n-1} and $\sigma > 0$ sufficiently small the Lipschitz constant on the right hand side of (7.6) to (7.8) in $\mathcal{C}([\sigma^2,1], \mathcal{K}^c_{\sigma^n} \times \mathcal{K}^s_{\sigma^n} \times \mathcal{K}^w)$ is smaller than 1. An application of a classical fixed point argument completes the proof of Lemma 7.11. $\square$

7.6. The iteration process

As in the case of the simplified problem, we decompose the solution $\hat{v}_{\mathrm{c},n}(\cdot,\cdot,\tau)$ for $\tau = 1$ into a Gaussian part and a remainder. Let $\tilde{\psi}(\varkappa) = e^{-c_1\varkappa^2}$ and write

$$\hat{v}_{\mathrm{c},n}(\varkappa,x,1) = A_n\tilde{\psi}(\varkappa)\varphi_{\sigma^{-n}\varkappa}(x) + \hat{r}_n(\varkappa,x)\,,$$

where $\hat{r}_n(0,x) = 0$, and the amplitude A_n is in $\mathbf{C}$. We also define $\widehat{\Pi} : \mathcal{K}^c_{\sigma^n} \to \mathbf{C}$ by

$$(\widehat{\Pi}f)\varphi_0 = \hat{P}_{\mathrm{c}}(0)f\big|_{\varkappa=0}\,. \tag{7.18}$$

Then (7.6) can be decomposed accordingly and takes the form

$$A_n = A_{n-1} + \widehat{\Pi}\left(\int_{\sigma^2}^1 \mathrm{d}\tau'\, e^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{c},n}(1-\tau')}\big(\sigma^{-2n}(\widehat{\mathcal{N}}_{\mathrm{c},\mathbf{i},n} + \widehat{\mathcal{N}}_{\mathrm{c},n})\big)\right), \tag{7.19}$$

$$\begin{aligned}\hat{r}_n(\varkappa,x) &= e^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{c},n}(1-\sigma^2)}\hat{r}_{n-1}(\sigma\varkappa,x)\\ &\quad + \sigma^{-2n}\int_{\sigma^2}^1 \mathrm{d}\tau'\,\left(e^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{c},n}(1-\tau')}(\widehat{\mathcal{N}}_{\mathrm{c},\mathbf{i},n} + \widehat{\mathcal{N}}_{\mathrm{c},n})\right)(\varkappa,x) \qquad (7.20)\\ &\quad + e^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{c},n}(1-\sigma^2)}A_{n-1}\tilde{\psi}(\sigma\varkappa)\varphi_{\sigma^{-n}\varkappa}(x) - A_n\tilde{\psi}(\varkappa)\varphi_{\sigma^{-n}\varkappa}(x)\,.\end{aligned}$$

If we define next $\rho^r_n = \|\hat{r}_n\|_{\mathcal{K}^c_{\sigma^n}} + \|\hat{v}_{\mathrm{s},n}|_{\tau=1}\|_{\mathcal{K}^s_{\sigma^n}}$ then the above construction implies $\rho^u_{\mathrm{cs},n} \le C(|A_n| + \rho^r_n)$.

Our main estimate is now

Proposition 7.12. *There is a constant $C > 0$ such that for sufficiently small $\sigma > 0$ the solution $(v_{\mathrm{c},n}, v_{\mathrm{s},n}, w_n)$ of (7.6)–(7.8) satisfies for all $n \in \mathbf{N}$:*

$$|A_n - A_{n-1}| \le Ce^{-C\sigma^{-n}} R_n^w + C\sigma^{n/2}(R^u_{\mathrm{cs},n})^2 , \tag{7.21}$$

$$\rho_n^r \le C\sigma\rho_{n-1}^r + Ce^{-C\sigma^{-n}} R_n^w + C\sigma^{n/2}(R^u_{\mathrm{cs},n})^2 + C\sigma^n R^u_{\mathrm{cs},n} , \tag{7.22}$$

$$\rho_n^w \le Ce^{-C\sigma^{-2n}} \rho_{n-1}^w + C\sigma^{n/2} R^u_{\mathrm{cs},n} R_n^w . \tag{7.23}$$

Proof. We begin by bounding the difference $A_n - A_{n-1}$ using (7.19). Since $\hat f$ is in H^2 as a function of ℓ we obviously have

$$|\widehat{\Pi} \hat f| \le C \|\hat f\|_{\mathcal{K}^c_{\sigma^n}} . \tag{7.24}$$

Thus, it suffices to bound the norm of the integral in (7.19), but this has already been done in the proof of Lemma 7.7 and Lemma 7.8.

We next bound $\hat r_n$ in terms of $\hat r_{n-1}$, using (7.20). The first term is the one where the projection is crucial: For $\sigma > 0$ sufficiently small, $\hat r_{n-1} \in \mathcal{K}^c_{\sigma^{n-1}}$ with $\hat r_{n-1}(0) = 0$ one has

$$\|(\varkappa, x) \mapsto e^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{c},n}(1-\sigma^2)} \hat r_{n-1}(\sigma\varkappa, x)\|_{\mathcal{K}^c_{\sigma^n}} \le C\sigma \|\hat r_{n-1}\|_{\mathcal{K}^c_{\sigma^{n-1}}} , \tag{7.25}$$

as in the proof of Proposition 4.5. This leads for the first term in (7.20) to a bound (in $\mathcal{K}^c_{\sigma^n}$)

$$C\sigma\rho^r_{n-1} . \tag{7.26}$$

The second and third term have been bounded in the proof of Lemma 7.7 and Lemma 7.8 by

$$Ce^{-C\sigma^{-n}} R^u_n + C\sigma^{n/2}(R^u_n)^2 . \tag{7.27}$$

Finally, the last term

$$\widehat{X}_n(\varkappa, x) \equiv e^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{c},n}(1-\sigma^2)} A_{n-1}\tilde\psi(\sigma\varkappa)\varphi_{\sigma^{-n}\varkappa}(x) - A_n\tilde\psi(\varkappa)\varphi_{\sigma^{-n}\varkappa}(x) ,$$

in (7.20) leads to a bound (in $\mathcal{K}^c_{\sigma^n}$):

$$\|\widehat{X}_n\| \le Ce^{-C\sigma^{-n}} R^w_{n-1} + C\sigma^{n/2}(R^u_{\mathrm{cs},n})^2 + C\sigma^n R^u_{\mathrm{cs},n} , \tag{7.28}$$

where the last term is due to $\mu_1(\ell) = -c_1\ell^2 + \mathcal{O}(\ell^3)$ not being exactly a parabola. For details see [Schn96]. Collecting the bounds, the assertion (7.22) for $\hat r_n$ follows. Finally, the bounds on ρ_n^w follow the in the same way as those in Lemma 7.11. The proof of Proposition 7.12 is complete. □

Proof of Theorem 7.1. As before the proof is just an induction argument, using repeatedly the above estimates. Again we write C for constants which can be chosen independent of σ and n. Assume that $R = \sup_{n\in\mathbf{N}} R^u_{\mathrm{cs},n} < \infty$ exists. From Lemma 7.11 we observe for $\sigma > 0$ sufficiently small,

$$\begin{aligned} R^w_n &\leq \frac{C\rho^w_{n-1}}{1 - C\sigma^{n/2}R} \leq C\rho^w_{n-1}\,, \\ R^u_{\mathrm{cs},n} &\leq \frac{C\sigma^{-6}\rho^u_{\mathrm{cs},n-1} + Ce^{-C\sigma^{-n}}R^w_n}{1 - C\sigma^{n/2}R} \\ &\leq C\sigma^{-6}\rho^u_{\mathrm{cs},n-1} + Ce^{-C\sigma^{-n}}\rho^w_{n-1}\,, \end{aligned} \tag{7.29}$$

with a constant C which can be chosen independent of R. Using Proposition 7.12 we find

$$\begin{aligned} |A_n - A_{n-1}| &\leq Ce^{-C\sigma^{-n}}\rho^w_{n-1} + C\sigma^{n/2}\sigma^{-6}\rho^u_{\mathrm{cs},n-1}\,, \\ \rho^r_n &\leq C\sigma\rho^r_{n-1} + Ce^{-C\sigma^{-n}}\rho^w_{n-1} + C\sigma^{n/2}\sigma^{-6}\rho^u_{\mathrm{cs},n-1}\,, \\ \rho^u_{\mathrm{cs},n} &\leq C(|A_n| + \rho^r_n)\,, \\ \rho^w_n &\leq Ce^{-C\sigma^{-2n}}\rho^w_{n-1} + C\sigma^{n/2}\rho^w_{n-1}\,. \end{aligned} \tag{7.30}$$

Therefore, we can choose $\sigma > 0$ so small that for $n > 13$:

$$\begin{aligned} |A_n - A_{n-1}| &\leq \rho^w_{n-1}/10 + \sigma^{(n-13)/2}(|A_{n-1}| + \rho^r_{n-1}), \\ \rho^r_n &\leq 3\rho^r_{n-1}/4 + \rho^w_{n-1}/10 + \sigma^{(n-13)/2}|A_n|, \\ \rho^w_n &\leq \rho^w_{n-1}/10\,. \end{aligned}$$

Thus, the sequence of A_n converges geometrically to a finite limit A_*. Furthermore, we find that $\lim_{n\to\infty}\rho^r_n = 0$, and $\lim_{n\to\infty}\rho^w_n = 0$. Since the quantities $|A_n|$, ρ^r_n, ρ^w_n increase only for at most 13 steps the term CR in (7.29) stays less than $1/2$ if we choose $|A_1|$, ρ^r_1, $\rho^w_1 = \mathcal{O}(\sigma^m)$, for a sufficiently large $m > 0$. From (7.29) the existence of a finite constant $R = \sup_{n\in\mathbf{N}} R^u_{\mathrm{cs},n}$ follows . Going back to (7.30) we can choose $\sigma > 0$ so small that

$$|A_n - A_{n-1}| + \rho^r_n \leq C\sigma^{n/2}$$

which implies the associated convergence rate stated in Theorem 7.1. Finally, the scaling of $w_n(\cdot,\cdot,\tau)$ implies the exponential decay of $w(t)$. The proof of Theorem 7.1 is complete. □

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