Space-Time Behavior in Problems of Hydrodynamic Type: A Case Study

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Abstract

We review results concerning normal forms, fronts, pattern formation, and convergence for certain 1-dimensional partial differential equations. The relationship between these results is put into perspective and is supplemented by a series of unpublished results on the convergence problem for parabolic PDE's. The paper does not require special prerequisites and we hope to interest the non-specialist for this nice subject.

CONTENTS

1. Introduction

Experiments on dynamical systems in finite spatial domains have recently received a lot of attention. They are described by differential equations ("dynamical systems") with only finitely many modes excited in phase space. However, when the size of the container becomes large with respect to the size of the spatial structure, the number of excited modes increases and the spectrum of the linearized evolution becomes denser until it is difficult to identify the various points of the spectrum. In other words, we reach the limit of an infinite domain which is known to produce a *continuous* spectrum. We want to study directly this infinite domain case with the continuous spectrum. This is very similar to the procedure in Statistical Mechanics where one can use the Thermodynamic Limit to describe a large system which is observed at a sufficient distance from the boundary. The advantage of this approach comes from the elimination of boundary problems; the disadvantage are technical complications. On the other hand, and this is the point to be made throughout, the conceptual rôle and relations between the various problems become much clearer.

In our present situation, this means that we want to study first partial differential equations in the infinite domain and then try to prove that far away boundary conditions do not change very much the behavior. We will concentrate here on the first question.

In order to simplify the discussion we will develop the ideas on the particular example of the one dimensional Swift-Hohenberg equation in its relation with the Ginzburg-Landau equation (see [Swift, Hohenberg, 1977], [Manneville, 1989] for a motivation of this equation from physical principles). The methods described below can of course be used in the study of a large class of equations. The Swift-Hohenberg equation is the partial differential equation, for $U : \mathbf{R} \times \mathbf{R}^+ \to \mathbf{R}$, $U = U(x,t)$, given by

$$\partial_t U \,=\, \big(\alpha - (1+\partial_x^2)^2\big)U - U^3 \,, \tag{1.1}$$

where U is a real function. The Ginzburg-Landau equation is

$$\partial_t u \,=\, (\partial_x^2 + 1)u - u|u|^2 \,, \tag{1.2}$$

where u is a complex function. Such equations are called quasi-linear equations, because the highest derivatives occur only in linear terms. In a bounded domain these equations should be supplemented by boundary conditions. However, as explained above, we want to study this equation as a dynamical system in an infinitely extended domain, here the real line. If we think of U or u as a local order parameter evolving in space and time (for example, a temperature, magnetization, velocity of a fluid etc.) it is natural to impose that U should be uniformly bounded, i.e., a function in $L^\infty(\mathbf{R})$. In addition, this natural property should (hopefully) be preserved under the time evolution. We will sketch later the relevant arguments why this is indeed the case. In other words, the SH-equation and the GL-equation define regular semi-flows in the space $L^\infty(\mathbf{R})$.

We can now start to analyze the dynamical system defined by the SH-equation. This analysis will of course try to follow the successful lines of thought used for the study of dynamical systems with only a finite number of excited modes. The first step is to notice that $U = 0$ is a time-independent solution of the equation, and it is natural to ask for its stability. Indeed, it is this (in)stability which is responsible for the interest in the SH-equation. The linearized equation around the zero solution is governed by the partial differential operator

$$L_\alpha = \partial_t - \alpha + (1 + \partial_x^2)^2 .$$

Since the coefficients are constant, it can be analyzed using Fourier analysis in the space variable. The (generalized) eigenfunctions are $e^{\sigma t} e^{iqx}$ with $\sigma = \alpha - (1 - q^2)^2$. One finds easily that for $\alpha < 0$ the solution $U = 0$ is stable and it is unstable for $\alpha > 0$. We therefore expect some sort of bifurcation at $\alpha = 0$.

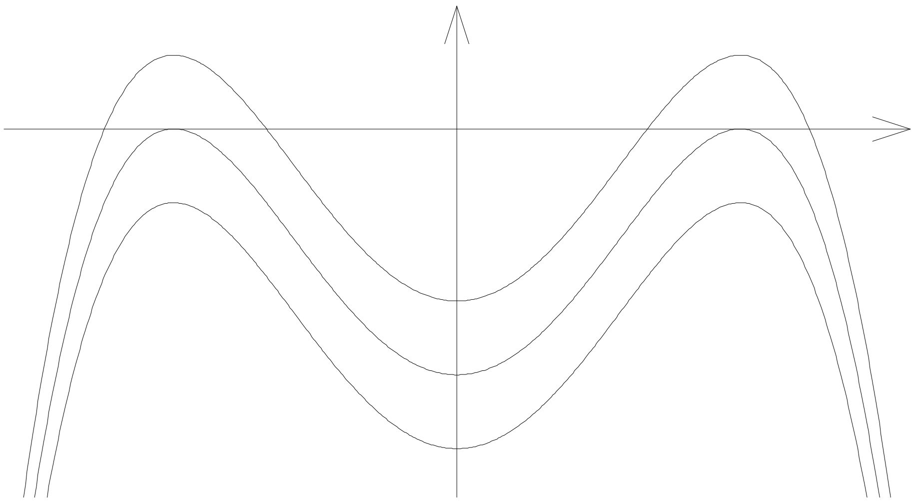

Fig. 1: The spectrum in the momentum-space variable q of the linear part for the SH-equation. Top to bottom curve: $\alpha = 0.3, 0.0, -0.3$.

There is, however, an important difference with the theory of bifurcations for dynamical systems with finitely many relevant modes. In the standard theory, the spectrum of the linearized vector field is discrete (the operator has a compact resolvent). Therefore, for a given parameter value, there can be only finitely many unstable (excited) modes. This idea can be made rigorous using the Center Manifold theorem [Carr, Muncaster, 1983a, 1983b], [Kirchgässner, 1988], [Mielke, 1988], [Eckmann, Wayne, 1991]. Here, however, the situation is very different because as soon as α is positive (no matter how small) there is a continuum of unstable modes. Thus, as soon as the bifurcation has taken place, there are infinitely many modes excited. As a consequence, one can show that many different solutions bifurcate simultaneously (space periodic solutions, fronts etc.), and they differ by their spatial structure. We refer to [Collet, Eckmann, 1990a] for a detailed study of some of the possibilities. In the next section, we want to relate the SH-equation to the GL-equation, and argue that, in some sense, the latter is a normal form of the former, near a bifurcation.

2. The GL-equation as a normal form of the SH-equation

The relation between the SH-equation and the GL-equation is commonly explained by viewing the latter as an "amplitude equation" for the former. More precisely, we view this procedure as a recentering of the parabola $(1-q^2)$ describing the linear part of the GL-equation in Fourier space under one of the maxima of the linear part of the SH-equation: $(\alpha-(1-q^2)^2)$. Clearly, since α is small, we will want to rescale variables so that the two curves fit near $q=1$, so that we really compare $(1-(1-q^2)^2)$ to $1-4(q-1)^2$, cf. Fig. 2.

Consider the Swift-Hohenberg equation, for $\alpha>0$, $\alpha=3\eta^2$:

$$\partial_t U(x,t) \;=\; \bigl(3\eta^2-(1+\partial_x^2)^2\bigr)U(x,t)-U^3(x,t)\;. \tag{2.1}$$

Here, U is a function $\mathbf{R}\times\mathbf{R}^+\to\mathbf{R}$. In order to perform the frequency shift to $q=1$, we rewrite the function $U(x,t)$ in the form

$$U(x,t) \;=\; \eta\,\mathrm{Re}\,\bigl(v(\beta\eta x,\gamma\eta^2 t)e^{ix}\bigr)\;. \tag{2.2}$$

In terms of v, the evolution equation (2.1) can be rewritten as

$$\gamma\partial_t v \;=\; (4\beta^2\partial_x^2+3-4i\beta^3\eta\partial_x^3-\beta^4\eta^2\partial_x^4)v-3v|v|^2-e^{2ix/\beta\eta}v^3\;. \tag{2.3}$$

Setting $\beta=\sqrt{3}/2$ and $\gamma=3$, and dividing by 3, we get

$$\partial_t v \;=\; (\partial_x^2+1-i\eta\tfrac{\sqrt{3}}{2}\partial_x^3-\eta^2\tfrac{3}{16}\partial_x^4)v-v|v|^2-\tfrac{1}{3}e^{4ix/\sqrt{3}\eta}v^3\;. \tag{2.4}$$

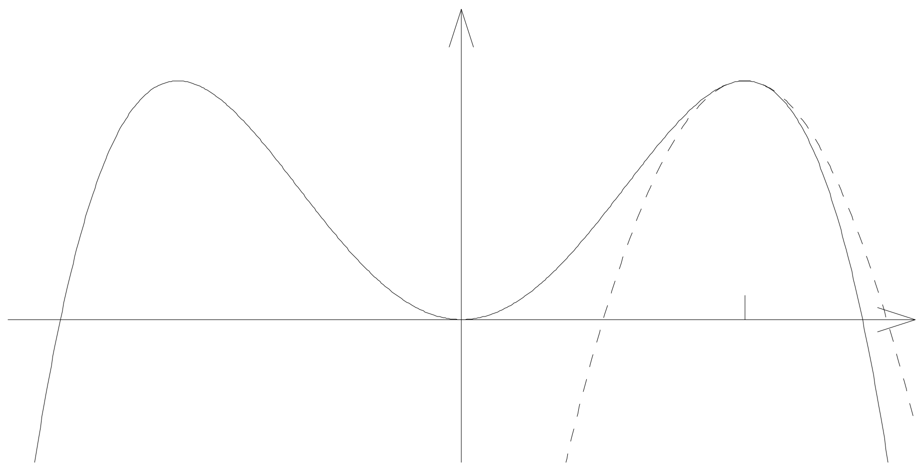

Fig. 2: The spectra of the linear parts of SH (solid line) and GL (dashed line), with a contact at $q_0=1$.

This is an infinite dimensional dynamical system in a suitable function space, and we define the corresponding time evolution operator $\mathcal{T}_{\eta,t}$ by

$$(\mathcal{T}_{\eta,t}v_0)(x) \;=\; v(x,t)\;,$$

where $v_0(x) = v(x,0)$ is given. When $\eta \to 0$, we find, on a very formal level, upon omitting all η-dependent terms in Eq.(2.4), the equation

$$\partial_t v \;=\; (\partial_x^2 + 1)v - v|v|^2\;. \tag{2.5}$$

This is the Ginzburg-Landau equation! Therefore, we see how the spectral arguments of Fig. 2 show up in the actual equation, Eq.(2.4). We call $\mathcal{S}_t$ the time evolution corresponding to (2.5). The general feeling is now that (2.5) is the “normal form” at bifurcation for the equation (2.1), and that any other equation whose spectrum near bifurcation is $\eta^2 - (k - k_0)^2$ would lead to a similar normal form. In the case of the SH-equation this idea can be made precise. One has the following relation between the evolution $\mathcal{T}_{\eta,t}$, defined by (2.4) and the evolution $\mathcal{S}_t$, defined by (2.5).

Theorem 2.1. [Collet, Eckmann, 1990b], [van Harten, 1992]. *Consider any function v_0 such that $\partial_x^k v_0$ in L^∞ for $k = 0,\ldots,4$. There are a $T > 0$, an $\eta_0 > 0$ and a $C < \infty$ such that for all $t \in (0,T]$ and all $\eta \in [0,\eta_0]$ one has,*

$$\|\mathcal{T}_{\eta,t}v_0 - \mathcal{S}_t v_0\|_\infty \;\le\; C\,\min(\eta t^{1/2}, \eta^{1/2}t^{3/4})\;. \tag{2.6}$$

Remarks. The constant C only depends on $\|\partial_x^k v_0\|_\infty$, for $k = 0,\ldots,4$. The solution itself can be shown to exist already when one only assumes v_0 in L^∞. However, the bounds will then show slight divergences near $t = 0$, because of the time which is needed until the evolution has smoothed the initial data.

It is instructive to translate Theorem 2.1 back to the original equation (2.1), and to the original time scale. Denote by $\hat{\mathcal{T}}_{\eta,t}$ the time evolution defined by (2.1), and let $v_0 \in L^\infty$ be given. Define

$$u_0(x) \;=\; (\mathcal{P}_\eta v_0)(x) \;\equiv\; \eta\,\mathrm{Re}\,v_0(\beta\eta x)e^{ix}\;. \tag{2.7}$$

With this definition, one has the relation

$$\hat{\mathcal{T}}_{\eta,t}\mathcal{P}_\eta \;=\; \mathcal{P}_\eta\mathcal{T}_{\eta,\gamma t\eta^2}\;.$$

Thus we find:

Corollary 2.2. *Consider any function v_0 such that $\partial_x^k v_0$ in L^∞ for $k = 0,\ldots,4$. There is a $T > 0$ and an $\eta_0 > 0$ such that for all $\eta \in [0,\eta_0]$and all $t \in (0,T/\eta^2]$ one has,*

$$|\hat{\mathcal{T}}_{\eta,t}u_0(x) - \eta\mathrm{Re}\,e^{ix}\big(\mathcal{S}_{t\gamma\eta^2}v_0\big)(\eta x)| \;\le\; \eta^3 C\,\min(t^{1/2}, t^{3/4})\;.$$

Note that if $\|v\|_\infty = \mathcal{O}(1)$, then $\|\mathcal{P}_\eta v\|_\infty = \mathcal{O}(\eta)$, so that Corollary 2.2 gives interesting bounds for solutions of size η. This is the size of stationary solutions of (2.1), see Section 4.

The difficulty in the proof of Theorem 2.1 is related to the fact that two seemingly contradictory arguments are used to eliminate the correction terms: For the differential operators one bounds the semigroup they generate while for the high frequency part $e^{2ix/\eta}$, one uses WKB-like methods [Hörmander, 1983], which need smoothness. But this smoothness will only appear after a short time and a careful study of local singularities is necessary. This is responsible for the bound (2.6), while the reader might have expected $\mathcal{O}(\eta t)$ instead.

The transformation $\mathcal{P}_\eta$ can be viewed as a zero order approximation to the problem of finding a normal form for the Swift-Hohenberg equation. The general problem can be formulated as follows: We are looking for a transformation $\mathcal{P}_\eta^*$ which satisfies approximately

$$D\mathcal{P}_\eta^* \cdot Y \;=\; X\mathcal{P}_\eta^* \;,$$

where

$$\begin{aligned} X(U)(x,t) &= (3\eta^2 - (1+\partial_x^2)^2)U(x,t) - U^3(x,t) \;, \\ Y(v)(x,t) &= (\partial_x^2 + 1)v(x,t) - v|v|^2(x,t) \;. \end{aligned}$$

The Equation (2.7) is really a lowest order approximation, $\mathcal{P}_\eta$, to $\mathcal{P}_\eta^*$.

3. Uniform bounds

As mentioned in the Introduction, our analysis is done in spaces of *bounded* functions, not in spaces of integrable functions, such as L^2. We believe that this is an important point in the infinite domain, which is the framework of interest to us. Indeed, the space of bounded functions will be seen to contain such things as fronts and the like, while L^p-spaces with $p < \infty$ do not. In this section, we show how "energy bounds" are converted to bounds in L^∞. Such bounds are not known for the Navier-Stokes equation in dimension 3.

It is well-known that the SH-equations have a Liapunov functional, i.e., a functional of the solution which is non-increasing in time. This functional is of the form

$$F \;=\; \int_{-\infty}^{\infty} dx \;\left[\tfrac{1}{2}(1-\alpha)U^2 - (\partial_x U)^2 + \tfrac{1}{2}(\partial_x^2 U)^2 + \tfrac{1}{4}U^4\right] \;.$$

Then, differentiating and integrating by parts, one gets

$$\begin{aligned} \partial_t F &= \int \partial_t U \cdot \left[(1-\alpha)U + 2\partial_x^2 U^2 + \partial_x^4 U + U^3\right] \\ &= -\int \left[(1-\alpha)U + 2\partial_x^2 U^2 + \partial_x^4 U + U^3\right]^2 \;\leq\; 0 \;. \end{aligned} \tag{3.1}$$

Thus, F cannot increase in time if U is a solution of the SH-equation. In a nutshell, this argument says that the dynamical system is a gradient field. The flaw with this argument is that both sides in (3.1) are infinite when U is only bounded. To get a finite energy F, one needs U to be at least in some suitable Sobolev space; in particular, U would have to vanish at infinity. This is in contradistinction with the principles we have set forth in our study. On the other hand, such energy estimates are extremely useful for problems in finite volume.

One can study another functional, namely $\int dx|U|^2$. A simple calculation leads to

$$\tfrac{1}{2}\partial_t \int_{-\infty}^{\infty} dx\,|U|^2 = \int_{-\infty}^{\infty} dx\, \Big(\alpha|U|^2 - \big|(1+\partial_x^2)U\big|^2 - |U|^4\Big)\ . \tag{3.2}$$

The r.h.s. of (3.2) does not have a definite sign when $\alpha > 0$, but one can argue that $\int U^2$ cannot grow without bounds. Indeed, if $|U|$ is very large, then $\alpha|U|^2 - |U|^4$ will be negative, and the r.h.s. of (3.2) looks negative. We call this an "energy bound." This argument has the same flaws as the one involving F, since we are talking not about individual values of U but some integrals, which in addition might be infinite.

However, the general idea of domination by the term U^4 can still be used, by a *localization* of the energy bounds (3.2). The argument is quite general, and so we consider equations of the form

$$\partial_t u_t = -Pu_t - u_t f(|u_t|^2)\ , \tag{3.3}$$

where P is a linear elliptic differential operator, or, more generally associated to a symbol [Hörmander, 1983],

$$P(x,\xi) = a\xi^{2k} + R(x,\xi)\ , \tag{3.4}$$

where $a > 0$ and where R is a polynomial of degree $< 2k$ in ξ with sufficiently smooth coefficients which are bounded together with their derivatives. In our case, $P(x,\xi) = (1-\xi^2)^2 - \alpha$, and $f(y) = y$.

We also assume that $f(\rho) \to \infty$ as $\rho \to \infty$ and $f'(\rho) \geq 0$, $f(0) \geq 0$. This last condition is somewhat too restrictive but makes the estimates easier. We denote by P the operator $P = P(x, -i\partial_x)$.

Theorem 3.1. [Collet, Eckmann, 1990]. *Assume* $\|u_0\|_\infty < \infty$ *and* $\|\partial_x u_0\|_\infty < \infty$. *Under the above assumptions on* P *and* f, *the solutions of (3.3) are bounded in* L^∞. *There is a* $C = C(\|u_0\|_\infty, \|\partial_x u_0\|_\infty) < \infty$ *such that*

$$\|u_t\|_\infty \leq C(\|u_0\|_\infty, \|\partial_x u_0\|_\infty)\ .$$

Remark. By standard regularity theorems, there is, for every p, a $T > 0$, and a $C > 0$, such that $\|\partial_x^p u_t\|_\infty < C$ for all $t \in (0,T)$, and for $p \leq p_0$. We may thus assume that the initial data are in $\mathcal{C}^{p_0}$, for any $p_0 < \infty$ we wish. The proof of Theorem 3.1 is given in Appendix A.

4. Time-independent solutions

If we consider the GL-equation, then time-independent solutions are solutions of the equation

$$0 = u'' + u - u|u|^2 , \tag{4.1}$$

where $u' = \partial_x u$. Similarly, for the SH-equation, stationary solutions are solutions of

$$0 = \big(\alpha - (1 + \partial_x^2)^2\big)U - U^3 . \tag{4.2}$$

Note that u and U depend only on the space variable x.

Important remark. It is here that we will use heavily the dimensionality of x-space. Because it is 1, the Eqs.(4.1), (4.2) are *ordinary* differential equations. Since the differential operator is a polynomial in ∂_x, we can write these equations as *finite dimensional dynamical systems.* For example, (4.1) can be written as a system of 2 coupled complex equations,

$$\begin{aligned} u' &= v , \\ v' &= -u + u|u|^2 , \end{aligned} \tag{4.3}$$

and similarly, (4.2) can be written as a system of four coupled real equations:

$$\begin{aligned} U' &= U_1 , \\ U_1' &= U_2 , \\ U_2' &= U_3 , \\ U_3' &= (\alpha - 1)U - 2U_2 - U^3 . \end{aligned} \tag{4.4}$$

If we interpret x as the "time," then Eqs.(4.3) and (4.4) can be viewed as a dynamical system *because the x-variable extends over all of* $\mathbf{R}$. The dynamical systems obtained in this way are a little special, when the original problem (such as (4.1) or (4.2)) is invariant under the symmetry $x \leftrightarrow -x$: It is "reversible" (see, e.g., [Moser, 1973]). This implies that the eigenvalues of the linearized problem come in pairs σ, $-\sigma$. For example, the matrices of the linearized problem in the GL case and the SH case are (in the case of GL we write 4 real components):

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 + |u|^2 & 2(\mathrm{Re}\, u)(\mathrm{Im}\, u) & 0 & 0 \\ 2(\mathrm{Re}\, u)(\mathrm{Im}\, u) & -1 + |u|^2 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \alpha - 1 - 3U^2 & 0 & -2 & 0 \end{pmatrix} .$$

A *constant solution* of (4.1) or (4.2) corresponds to a *fixed point* of the corresponding dynamical system. These are zeros of the vector field on the r.h.s. of (4.3). It is easy to see that these are $v = 0$, $u = 0$ or $u = e^{i\varphi}$ with some arbitrary phase φ. Similarly, for the SH-equation (4.4), one has constant solutions $U = 0$ and $U = \pm\sqrt{\alpha - 1}$. (Since we are interested in small α, the latter solutions are uninteresting for our study.)

A *periodic solution* corresponds to a periodic orbit of the corresponding dynamical system, such solutions exist for GL and SH. For GL, one can write them down explicitly:

$$s_{q,\varphi}(x) \;=\; \sqrt{1-q^2}e^{i(qx+\varphi)} \;, \tag{4.5}$$

with arbitrary parameters $q \in [-1,1]$ and $\varphi \in [0,2\pi]$. We shall call these solutions "spirals" (when $q \neq 0$). One can also show that the SH-equation has periodic solutions, but this needs a proof, since we cannot expect an explicit formula in this case:

Proposition 4.1. [Collet, Eckmann, 1990a]. *For sufficiently small $\alpha > 0$, the Eq.(1.1) has a two parameter family of periodic solutions which are of the form*

$$S_{\varepsilon,\varphi}(x) \;=\; \sum_{n\in\mathbf{Z}} S_n e^{iqnx} \;. \tag{4.6}$$

The real parameter ε must satisfy $3\varepsilon^2 < \alpha$. One has $|S_1| = \varepsilon$, and φ is the phase of the complex number S_1. Moreover, $S_n = \overline{S}_{-n}$. Finally, q is determined by ε through a relation which is approximately $3\varepsilon^2 + (1-q^2)^2 = \alpha$. One has the bounds, $|S_n| \leq \mathcal{O}(\varepsilon^{|n|})$, for $n \in \mathbf{Z}$.

Remark. The proof is essentially a perturbation argument.

Since the phase spaces of the dynamical systems described by Eqs.(4.3) and (4.4) are of dimension 4, one can expect a large variety of other stationary solutions, some of which may be chaotic in space [Eckmann, Procaccia, 1991, 1992].

5. Linear stability analysis

In line with the set of ideas from finite-dimensional dynamical systems, once we have found fixed points (or periodic orbits) we may inquire about their stability. This stability analysis is crucially dependent on the space of allowed perturbations. We are here interested in those aspects which come from the infinite domain. For both GL and SH, we found a two parameter continuous family of stationary solutions: One parameter is related to the translation invariance of the problem and the other to the non-linear coupling of the amplitude with the band of unstable modes. Differentiating the solution with respect to these parameters gives "soft" modes, i.e., eigenvectors with eigenvalue 0, *provided* these eigenvectors are in the space of allowed perturbations. The most interesting case here is the soft mode corresponding to a *change of wavelength* of the solution. This is known as the Eckhaus instability [Eckhaus, 1965]. The linearization of the GL-equation around a periodic solution $s_{q,\varphi}$ is

$$\partial_t v \;=\; \partial_x^2 v + v - 2v|s_{q,\varphi}|^2 - \overline{v}s_{q,\varphi}^2 \;\equiv\; L_q v \;. \tag{5.1}$$

One can view this as a linear problem in a periodic potential (the apparent complications of the expressions reflect the use of complex notation). The eigenfunctions of the above system are of the form

$$v(x,t) \;=\; e^{\sigma_q(\varkappa)t}\big(a_{q,\varphi}e^{i(q+\varkappa)x} + b_{q,\varphi}e^{-i(q+\varkappa)x}\big) \;, \tag{5.2}$$

and

$$\sigma_q(\varkappa) \;=\; -(1-q^2) - \varkappa^2 + \sqrt{(1-q^2)^2 + 4\varkappa^2 q^2} \;. \tag{5.3}$$

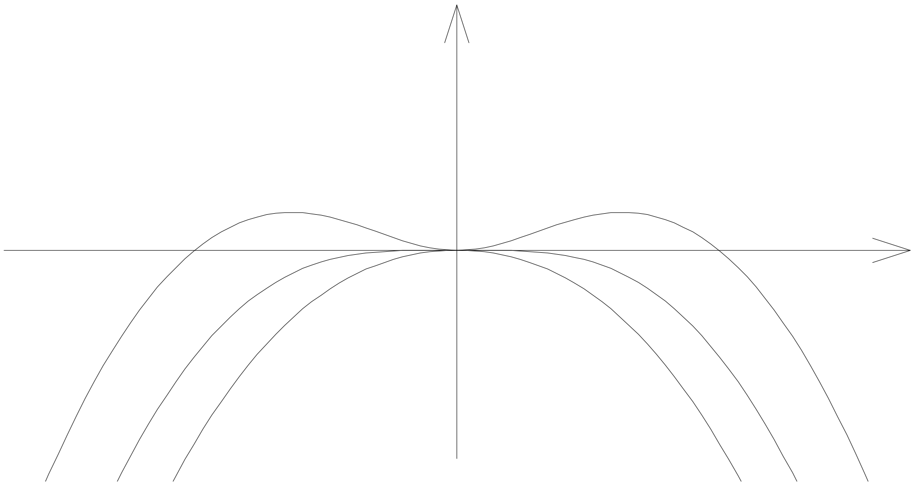

Fig. 3: The graph of the function σ_q of Eq.(5.3), for $q = \sqrt{1/3}+0.1$, $q = \sqrt{1/3}$, $q = \sqrt{1/3}-0.1$ (top to bottom curve).

The graphs of this function are shown for different values of q in Fig. 3. One can see that at $q^2 = \frac{1}{3}$ there is a change of stability: For $q^2 < \frac{1}{3}$, the function σ is non-positive, while for $q^2 > \frac{1}{3}$ it always has a segment above 0. Two remarks are in order:

i) Even when $q^2 < \frac{1}{3}$, the spectrum is only marginally stable, i.e., it always has an eigenvalue 0, due to the translation invariance.

ii) The Eckhaus instability can be understood as an instability towards a change of wavelength, cf. the form of the eigenfunctions in (5.2). The system is unstable against this change of wavelength when the amplitude is “small” (i.e., q is “large,” $q^2 > \frac{1}{3}$), but not when the amplitude is large.

In view of Eq.(5.3), we have the following

Theorem 5.1. [Eckhaus, 1965], [Collet, Eckmann, 1990a]. *The energy-momentum spectrum of the largest 0 eigenvalue of the operator $\partial_t - L_q$ of Eq.(5.1) in L^2 is of the form of Fig. 3.*

Calculations which are analogous to the ones just given can be done for the SH-equation [Collet, Eckmann, 1990a], [Eckmann, Procaccia, 1991, 1992]: The linear operator is given by

$$f \mapsto L_{S_{\varepsilon,\varphi}} f(x) \equiv \big(\alpha - (1+\partial_x^2)^2\big) f(x) - 3S_{\varepsilon,\varphi}^2(x) f(x) \,, \tag{5.4}$$

where $S_{\varepsilon,\varphi}$ is one of the solutions of (4.6). The eigenvectors v are essentially of the form

$$v(x) = \varepsilon q \mathrm{Re}\, e^{iqx} \left(a(\varkappa) e^{i\varkappa x} + b(\varkappa) e^{-i\varkappa x} \right) \,, \tag{5.5}$$

when $\varphi = 0$. Here, $a(0) = i, b(0) = 0$, since, by the translation invariance of the Eq.(1.1) we see that $S'_{\varepsilon,\varphi}(x)$ is an eigenvector with eigenvalue 0. The form (5.5) neglects terms of the form e^{inqx} with $|n| \neq 1$. These terms can be dealt with and one has the

Theorem 5.2. [Collet, Eckmann, 1990a]. *The spectrum of the largest eigenvalue of the operator $L_{S_{\varepsilon,\varphi}}$ of Eq.(5.4) in L^2 is of the form of Fig. 3. The Eckhaus instability occurs at $3\varepsilon^2 = \frac{2}{3}\alpha$.*

Remark. The proof is a calculation, augmented by perturbation theory for linear operators [Kato, 1970]. The calculation reduces to a 2×2 matrix problem: In the basis whose components are $e^{i(q+\varkappa)x}$ and $e^{i(-q+\varkappa)x}$, the operator $L_{S_{\varepsilon,\varphi}}$ is approximately equal to the 2×2-matrix

$$\begin{pmatrix} \alpha - (1-(q+\varkappa)^2)^2 - 6\varepsilon^2 & -3\varepsilon^2 \\ -3\varepsilon^2 & \alpha - (1-(-q+\varkappa)^2)^2 - 6\varepsilon^2 \end{pmatrix} .$$

For example, the term $-6\varepsilon^2$ is the coefficient of $e^{i(q+\varkappa)x}$ in the term $-3S_{\varepsilon,\varphi}^2 e^{i(-q+\varkappa)x}$. The form (5.5) is now just a transcription of the eigenvectors of this matrix in the basis $e^{i(q+\varkappa)x}$, $e^{i(-q+\varkappa)x}$.

Open problem. It should be possible to describe a general mechanism which is responsible for an Eckhaus-like instability in generic systems with pattern forming bifurcations.

6. Fronts

Now that we have identified stationary solutions of the GL and SH-equations, and made an analogy with low-dimensional dynamical systems, we can push this analysis further and define "front solutions." Front solutions interpolate between two (different) stationary solutions at $x = \pm\infty$; in addition, they move with a certain speed in the laboratory frame. The effect is then that one of the two stationary solutions seems to invade the other. For example, an explicit front solution is known for the GL-equation (1.2):

$$u(x,t) \;=\; \frac{1-\tanh(x-ct)}{2} , \quad c \;=\; \frac{3\sqrt{2}}{2} .$$

It connects the (stable) solution $u \equiv 1$ with the (unstable) solution $u \equiv 0$, and the former "invades" the latter from the left with speed c. More general solutions will be of the form

$$u(x,t) \;=\; \sum_{n\in\mathbf{Z}} a_n(x-ct) e^{iq_- x + i(q_+ - q_-)nx} , \tag{6.1}$$

for the GL-equation and

$$U(x,t) \;=\; \sum_{n\in\mathbf{Z}} A_n(x-ct) e^{iqnx} , \tag{6.2}$$

for the SH-equation. We shall see that such solutions exist. Before we state precise results, we want to give some details about the analogy with the low-dimensional case.

Consider a finite dimensional vector field with two fixed points, with their stable and unstable manifolds, and a saddle connection, as shown in Fig. 4.

A front for the GL-equation or the SH-equation is also a saddle connection, but now between those fixed points or periodic orbits which have the meaning of constant or periodic stationary solutions. For example, for the GL-equation, it is a solution $u(x,t)$ which, in the laboratory frame, looks like s_{q_-,φ_-}, for fixed t, as $x \to -\infty$, and like s_{q_+,φ_+}, when $x \to +\infty$. We can draw as sort of "phase diagram" in the parameters $q_\pm$, $\varphi_\pm$, and ask what will happen if an initial condition has such asymptotic behavior. One has the following result:

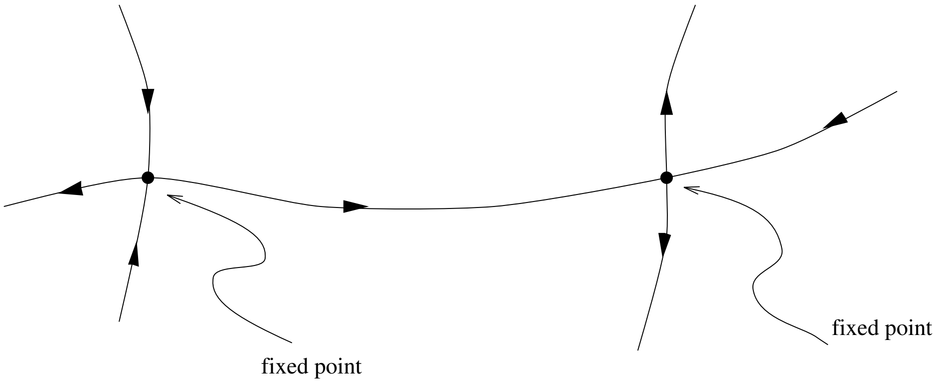

Fig. 4: Two fixed points, with their stable and unstable manifolds, and a saddle connection.

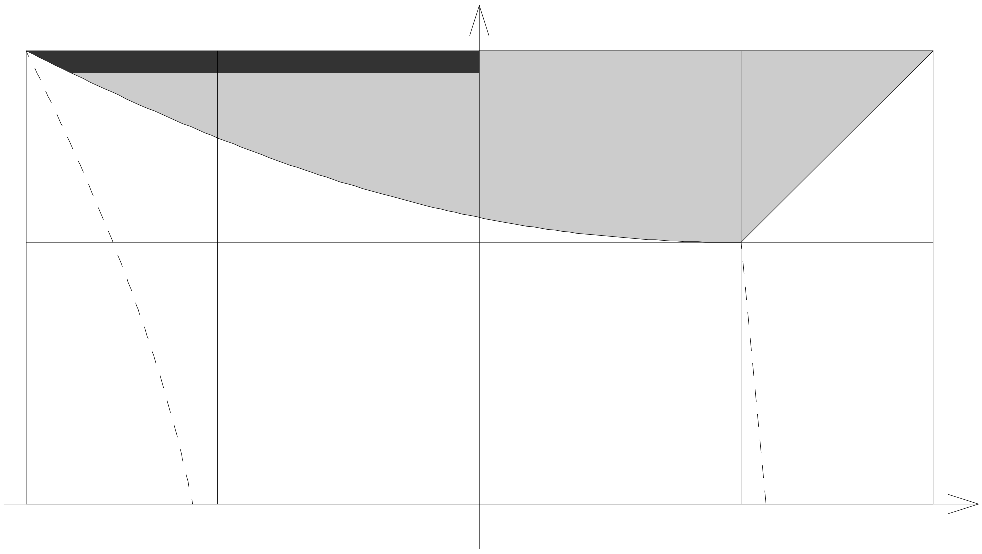

Fig. 5: The domains of existence for fronts for the GL-equation. For q_-, q_+ in the shaded region, it is believed that front solutions travelling to the right exist. This can be shown rigorously in the darker region. Fronts travelling in the opposite direction are expected in the regions bounded by dashed lines, by symmetry. The Eckhaus stable region occurs for the values of q satisfying $|q|^2 < q^2_{\text{Eckhaus}} = 1/3$.

Theorem 6.1. [Eckmann, Gallay, 1992], [Gallay, 1992]. *For the GL-equation, there exist fronts of the form of Eq.(6.1) for q_+, q_- in the shaded region of Fig. 5 and arbitrary φ_+, φ_- in $[0, 2\pi]$. These fronts have the property that*

$$\lim_{\xi\to-\infty} a_0(\xi) \;=\; \sqrt{1-q_-^2}\;,$$

$$\lim_{\xi\to\infty} a_1(\xi) \;=\; \sqrt{1-q_+^2}\;.$$

The limits for all other a_n, and the remaining limits for a_0, a_1 are 0, cf. Fig. 6. Such solutions exist for speeds c satisfying $c > c_0$ for some sufficiently large $c_0(q_+, q_-)$.

Fig. 6: The typical shape of the amplitudes a_n of Theorem 6.1.

Remark. It is not claimed that the fronts are *stable* solutions, i.e., that nearby initial data would converge to a front. However, if q_- is in the Eckhaus stable region, $q_-^2 < \frac{1}{3}$, and q_+ is in the unstable region, i.e., $q_+^2 > \frac{1}{3}$, then we conjecture that the front solution is stable, with the stable pattern, i.e., the q_--solution, "invading" the unstable one. A framework in which one could try to prove this is provided by perturbations with compact support.

Example. In Fig. 7, we show in three frames a typical time evolution for front solutions of the GL-equation. Note how the stable solution "invades" the unstable one.

Remark. (S. Angenent, private communication.) Since the GL-equation is of second order, it follows from the maximum principle that, for every fixed α, the number of zeroes of $\mathrm{Re}\, e^{i\alpha}u(x,t)$ is a decreasing function of time. This is then a precise constraint on the occurrence of "phase slips:" Nodes can only disappear, but not be spontaneously generated as time goes on.

A set of results similar to Theorem 6.1 is valid for the SH-equation, and one can find front solutions of the form Eq.(6.2), where $|q|$ is close to 1, and $\lim_{\xi\to-\infty} A_n(\xi) = S_n$, with S_n given by Proposition 4.1. Ahead of the front, we have $\lim_{\xi\to+\infty} A_n(\xi) = 0$. The amplitudes A_n are of order $\varepsilon^{|n|}$, and they are essentially the solution of a GL-equation: If $A(\xi) = A_1(\xi)/\varepsilon$, $c_0 = c/\varepsilon^2$, $\alpha_0 = \alpha/\varepsilon^2$, then A is approximately equal to the solution of

$$4A'' + c_0 A' + \alpha_0 A - A|A|^3 \;=\; 0\;. \tag{6.3}$$

In other terms, near the bifurcation $\alpha = 0$, the fronts are approximately equal to

$$U(x,t) \;\approx\; \varepsilon \mathrm{Re}\, e^{iqx} A(\varepsilon x - \varepsilon^2 c_0 t)\;,$$

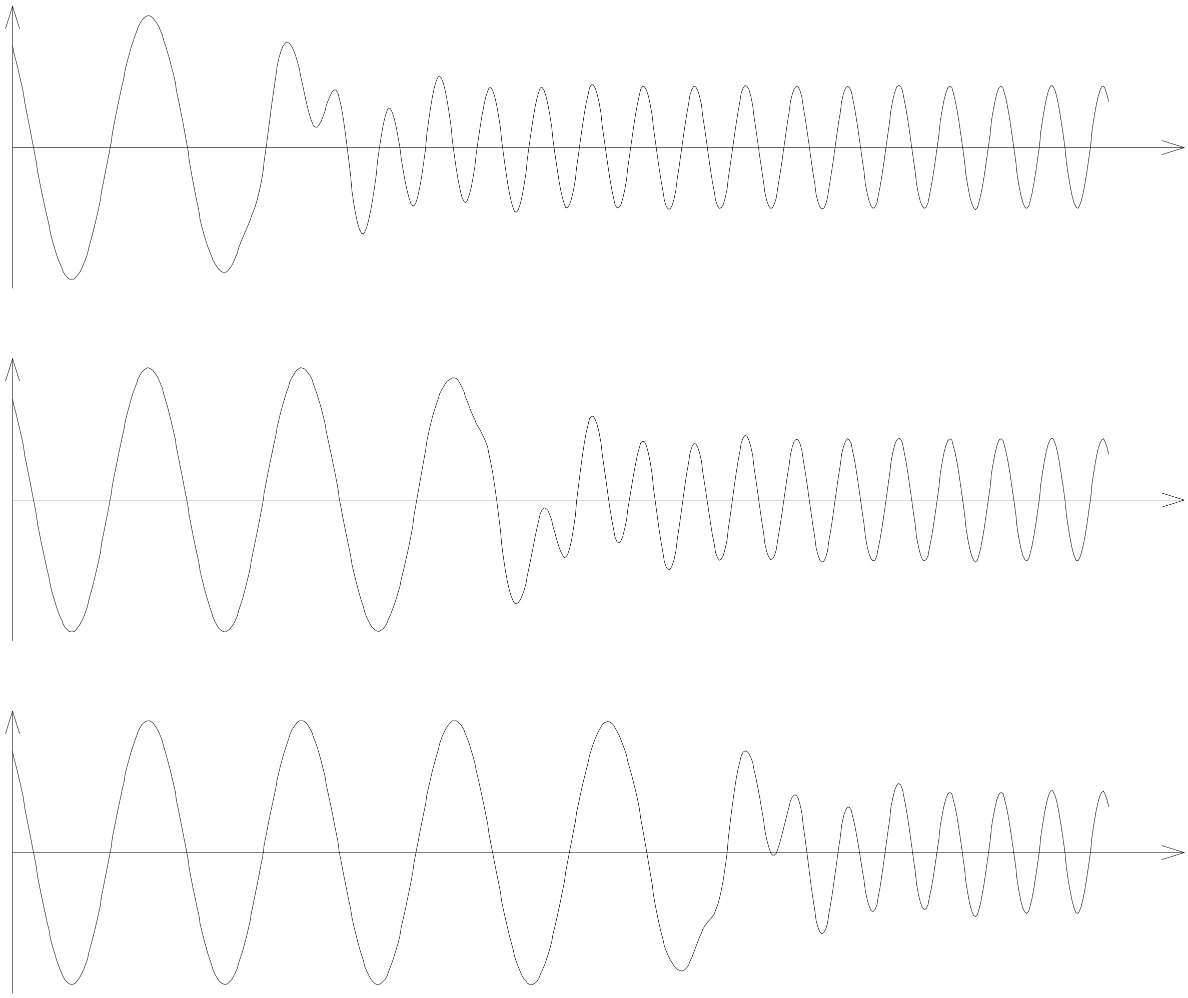

Fig. 7: Three time frames showing the real part of the solution of the GL-equation. The parameters are $q_{\_} = -0.3$, $q_{+} = 0.9$ and $c = 5$.

with $(1 - q^2)^2 + 3\varepsilon^2 = \alpha$. This should be compared with the result for the time dependent problem, cf. Eq.(2.2). The proofs of these results can be found in [Collet, Eckmann, 1990a], [Eckmann, Wayne, 1991]. Intuitively, the methods of proof for this result, as well as for Theorem 6.1 are all based on the presence of a small number of relevant variables, e.g., the 4-dimensional real dynamical system described by Eq.(6.3). The Center Manifold theorem, [Mielke, 1988], [Eckmann, Wayne, 1991], [Gallay, 1992], makes sure that the other variables, i.e., the amplitudes a_n, $|n| \neq 0, 1$ and the amplitudes A_n, $|n| \neq 1$ are "slaved" by the relevant amplitudes. The deep reason of why this is possible has to do with the ansatz we require for the solutions. We impose some sort of periodicity in (6.1) and (6.2). The spectrum of the linear part on the corresponding subspace is then *discrete*; in other words, we have *compactified* the problem. This is in sharp contrast with the methods leading to Theorem 2.1, where the continuous spectrum is visible in its full extent. Of course, the absence of compactness is also the reason of why much less is known if the requirement of periodicity is dropped. In the next two sections, we will again address a problem which has to do with the full partial differential equation. To recapitulate: The front problem, although it is a space-time problem, can, *in 1 space dimension,* be reduced to an "almost finite dimensional" problem, by the Center Manifold techniques, because the space periodicity *compactifies* the infinite line. We can say that the initial data, or the physical preparation of the sample is responsible for this compactification.

7. Space-time behavior

In this section, we would like to describe what we know about the global time-behavior of equations such as GL and SH. While there are extensive numerical simulations in this subject, not much is known rigorously, in the infinite domain. Also, of course, it is difficult to do numerical simulation in the infinite domain. Before stating results, we want to put these difficulties in a mathematical perspective. We have seen in the previous section that in the infinite domain, problems become slightly more tractable if some sort of periodicity is imposed—we have called this "compactification." Most numerical experiments dealing with so-called spectral code, i.e., working in Fourier space, will of course treat exactly this class of compactified problems. Is this the only class of tractable problems? The answer is no, and we can classify the known results into three categories:

i) Control by compactification,
ii) Control by the maximum principle,
iii) Control by convergence.

To these one should of course add results about global existence, such as Theorem 2.1, which are somehow precursors of iii) above. The control by the maximum principle is one of the most astonishing features of semi-linear parabolic equations. It essentially says, cf. [Aronson, Weinberger, 1978], [Bramson, 1983], [Collet, Eckmann, 1990a], that solutions are constrained ("sandwiched") between other solutions. Thus, if the initial data for the GL-equation are real and between two front solutions as in Fig. 8, then the solution will evolve forever between the two advancing fronts.

The control by convergence seems at first sight quite trivial. If we have a stable fixed point of a dynamical system, then by definition, all initial data sufficiently close to this fixed point will be attracted to it by the time evolution. In particular, this means that they stay bounded. This is what we mean by "control by convergence." A set of partial differential equations for which this kind of phenomenon has been well studied is:

$$\partial_t u \;=\; \partial_x^2 u + u^p \;, \tag{7.1}$$

where u is a real function on the real line (and of time). We denote by $\tilde{u}$ the Fourier transform of u and let $u_t(x) = u(x,t)$. Then one has

Proposition 7.1. [Collet, Eckmann, Epstein, 1992], [Bricmont, Kupiainen, Lin, 1992]. *For every $p > 3$, $p \in \mathbf{N}$, there is a constant $\varepsilon_p > 0$ such that for*

$$\|\tilde{u}_0\|_\infty < \varepsilon_p \;, \quad \|\tilde{u}_0\|_1 < \varepsilon_p \;,$$

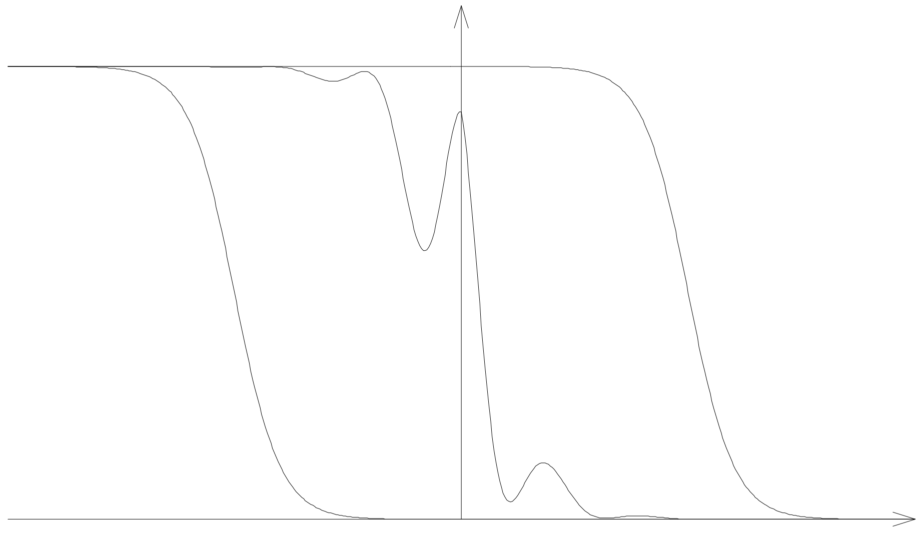

Fig. 8: Two front solutions for the GL-equation and some arbitrary initial data in between. As time goes on the solution will stay between the two front solutions which advance with the same speed to the right.

the solution of the initial value problem (7.1) with $u(x,0) = u_0$ tends to zero. In addition, one has the bound

$$\|\tilde{u}_t\|_1 \;\leq\; C_p(t+1)^{-1/2}\;,$$

for some finite constant C_p.

Thus, in this case, we have just (diffusive) convergence to zero: When $p > 3$, the diffusive "force" of the Laplacian in (7.1) is stronger than the buildup by the non-linearity. It is known that for $p \leq 3$ one has divergence of the solutions. See [Weissler, 1981], [Levine, 1990] for a review of the literature on this well-studied subject.

Here, we want to study cases where the limit state can be more interesting. We start with initial data for the GL-equation which look like stationary solutions at infinity:

$$\lim_{x\to-\infty} u_0(x) - s_{q_-,\varphi_-}(x) \;=\; 0\;, \quad \lim_{x\to\infty} u_0(x) - s_{q_+,\varphi_+}(x) \;=\; 0\;. \tag{7.2}$$

We shall specify below the nature of the convergence in (7.2). Note that we have already encountered such initial conditions in the study of fronts, where we allowed $q_\pm$ in the shaded domain of Fig. 5. Now we concentrate on another part of the phase diagram, namely the regions near the diagonal $q_+ = q_-$, and $\varphi_+ = \varphi_-$, with the additional requirement that q_+, q_- are in the Eckhaus stable domain.

It is useful to introduce polar coordinates in the GL-equation and to write

$$u(x,t) \;=\; r(x,t)e^{i\varphi(x,t)}\;,$$

where r and φ are real functions. (This representation is not unique when $r = 0$, but we shall avoid this problem through the choice of initial data). If the amplitude r goes through 0, one talks about a phase slip. The boundary conditions of Eq.(7.2) now take the form

$$\lim_{x\to\pm\infty} r(x,0) \;=\; \sqrt{1-q_\pm^2}\;, \quad \lim_{x\to\pm\infty} \varphi'(x,0) \;=\; q_\pm\;. \tag{7.3}$$

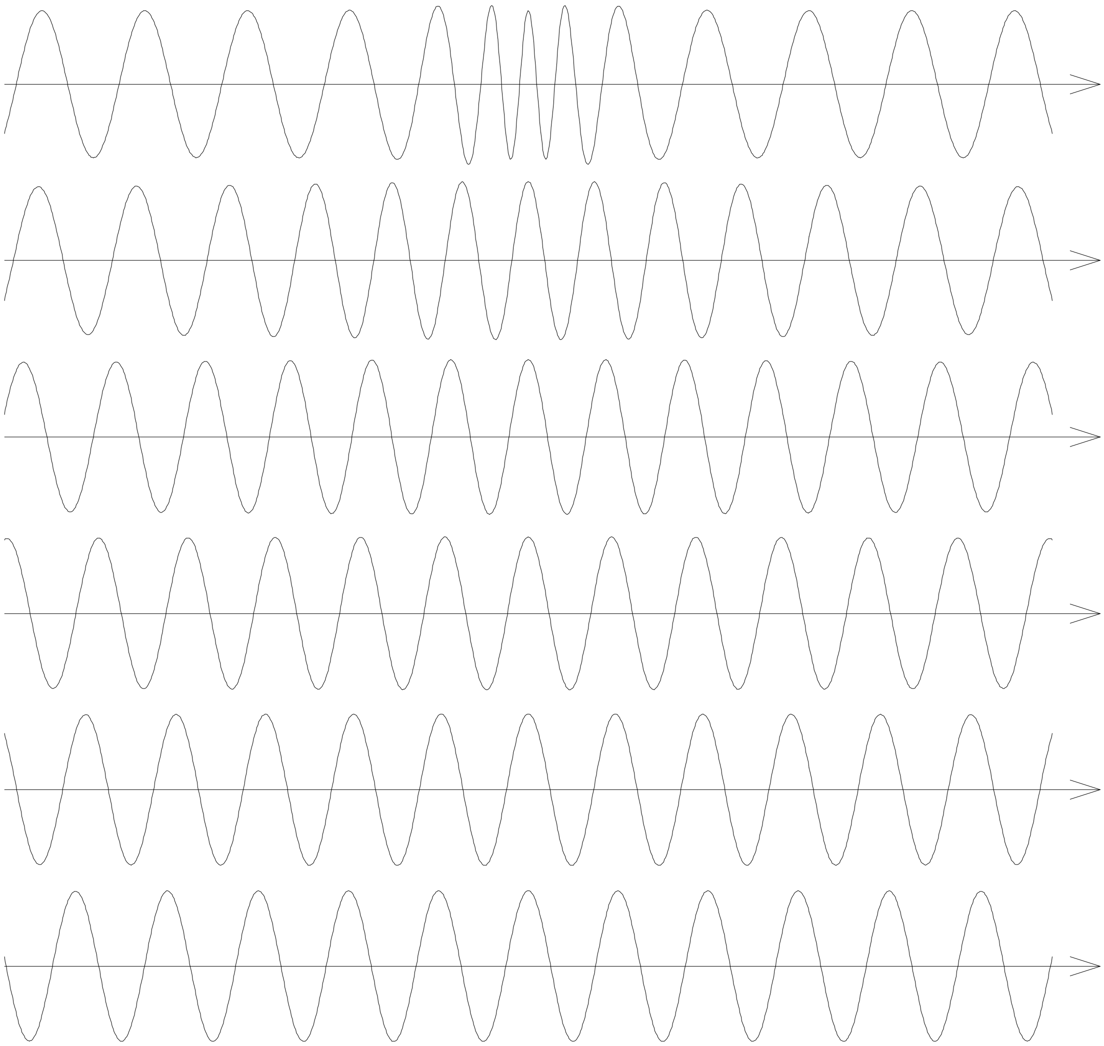

Fig. 9: A schematic illustration of the healing process for an initial condition with identical values for q_- and q_+. Note that there is a non-trivial overall phase shift, which gets repaired without a phase slip, purely by a rearrangement of the nodes. Time runs from top to bottom and we show the real part of u.

Note that even if $q_+ = q_- = q$, there may be a non-trivial global phase shift $\delta \neq 0$:

$$\delta \;=\; \int_{-\infty}^{\infty} dx \,\big(\varphi'(x) - q\big) \;.$$

In Fig. 9, we show an example of an evolution for initial data of this type.

We now want to explain how this happens, and we want to put this in relation with some field-theoretic methods. Consider the GL-equation Eq.(1.2) written in the polar coordinates, $u = r\, e^{i\varphi}$. One gets

$$\partial_t r \;=\; \partial_x^2 r + r - r^3 - r\zeta^2 \;, \tag{7.4}$$

$$\partial_t \zeta \;=\; \partial_x^2 \zeta + 2\partial_x \left(\frac{\partial_x r}{r}\zeta\right) \;, \tag{7.5}$$

with $\zeta = \partial_x \varphi$.

Note that the r.h.s. of (7.5) is a derivative, so that $\int (\zeta - q)$ is a conserved quantity, i.e., the total phase shift δ is conserved. On the other hand, $\zeta(x,t)$ converges to q for every fixed x as $t \to \infty$, (for suitable initial data.) So we see that the nature of the convergence is not very uniform. Such behavior is already encountered for the simple diffusion equation $\partial_t \zeta = \partial_x^2 \zeta$: We shall see below that it is at the origin of some very delicate questions about convergence for the GL-equation.

7.1. The case $q_+ = q_-$

We start our discussion for the case $q_+ = q_- = q$. Later we shall also deal with the general case $q_+ \neq q_-$ which is more intriguing.

Consider the vicinity of a stationary solution, $s_{q,0}$, cf. Eq.(4.5). We can write u as the difference from such a solution:

$$r(x,t) \;=\; \sqrt{1-q^2} + s(x,t)\;, \quad \zeta(x,t) \;=\; q + \eta(x,t)\;.$$

Expressed in the new variables s, η, the equations (7.4) and (7.5) read

$$\begin{aligned}
\partial_t s \;&=\; \partial_x^2 s + s - 3(1-q^2)s - 3\sqrt{1-q^2}\, s^2 - s^3 \\
&\quad - \sqrt{1-q^2}\,\eta^2 - 2\sqrt{1-q^2}\, q\eta - sq^2 - 2sq\eta - s\eta^2\;, \\
\partial_t \eta \;&=\; \partial_x^2 \eta + 2\partial_x \left(\frac{\partial_x s}{\sqrt{1-q^2}+s} \right) q + 2\partial_x \left(\frac{\partial_x s}{\sqrt{1-q^2}+s}\eta \right)\;.
\end{aligned} \tag{7.6}$$

The corresponding linear system is (linearized around $u = s_{q,0}$)

$$\begin{aligned}
\partial_t s \;&=\; \partial_x^2 s - 2(1-q^2)s - 2\sqrt{1-q^2}\, q\eta\;, \\
\partial_t \eta \;&=\; \partial_x^2 \eta + 2\partial_x^2 s \frac{2q}{\sqrt{1-q^2}}\;.
\end{aligned} \tag{7.7}$$

When $q = 0$, the problem (7.6) reduces to

$$\begin{aligned}
\partial_t s \;&=\; \partial_x^2 s - 2s - \eta^2 - 3s^2 - s^3 - s\eta^2\;, \\
\partial_t \eta \;&=\; \partial_x^2 \eta + 2\partial_x \left(\frac{\partial_x s}{1+s}\eta \right)\;.
\end{aligned} \tag{7.8}$$

So, we have a field theory with two fields, s and η. These two fields have a non-diagonal mass matrix and a calculation shows that one of the masses is positive when $q^2 < \frac{1}{3}$. (This is another manifestation of the Eckhaus instability.) When $q = 0$, the s-field is massive and the η-field is massless, and when $q \neq 0$, the masses and the eigendirections change continuously with q. For $q = 0$, in essence, the repair of the defect is now seen as follows: Due to the term $-2s$, the s variable relaxes quickly to its equilibrium value, and then the η variable diffusively tends to zero. The problem with this argument comes from the η^2 term in the first equation of (7.8). In fact, s does *not* relax (fast) to zero but rather to something like η^2, or some other η dependent function $F(\eta)$. Since η is diffusive, we expect $\eta = \mathcal{O}(1/t^{1/2})$. Substituting into the second equation, we get the slow equation

$$\partial_t \eta \;=\; \partial_x^2 \eta + 2\partial_x \left(\frac{DF(\eta)\partial_x \eta}{1+F(\eta)} \eta \right) . \tag{7.9}$$

Does this equation repair diffusively? In other words: If we start with reasonable s and η, i.e., close to a periodic solution, do these initial data evolve into a stationary solution? We will see that this is indeed the case. In the case $q \neq 0$, we see from Eq.(7.7) that the coupling of the two fields is not diagonal, due to the term $-2\sqrt{1-q^2}q\eta$. It is a general principle that in such cases one should diagonalize the problem, however, the difficulty here is that this diagonalizing transformation does depend on the momentum, k, and therefore does create new difficulties in the non-linear terms. These problems can be handled through a careful balancing and a suitable choice of the space of allowed initial data. The details are given in [Collet, Eckmann, Epstein, 1992] but here we just state the result with a few remarks.

Theorem 7.2. [Collet, Eckmann, Epstein, 1992]. *Let* $q^2 < \frac{1}{3}$. *There is an* $\varepsilon = \varepsilon_q > 0$ *such that the solutions* (s, η) *of the Eq.(7.8) tend to zero in* L^∞ *as* $t \to \infty$ *if the initial data satisfy*

$$\begin{gathered} \|k\tilde{s}_0\|_2 < \varepsilon \,, \quad \|\tilde{s}_0\|_\infty < \varepsilon \,, \\ \|\frac{k}{|k|+1}\tilde{\eta}_0\|_2 < \varepsilon \,, \quad \|\tilde{\eta}_0\|_1 < \varepsilon \,, \quad \|\tilde{\eta}_0\|_\infty < \varepsilon \,. \end{gathered}$$

Here, $\tilde{f}$ denotes the Fourier transform of f and $\|k\tilde{f}\|_2 = \left(\int dk\, k^2 |\tilde{f}(k)|^2\right)^{1/2}$.

Remark. The "funny" norms make sure that the diagonalization problem mentioned earlier can be solved. Note also that the conditions are not very strong, any small initial condition s, η which together with its derivatives is in L^2 and in L^1 will do. Recall that s and η are only the deviations from the stationary solution $s_{q,0}$ so that the situation of Fig. 9 applies. One can also show that the convergence has diffusive speed: $\|\tilde{s}_t\|_\infty + \|\tilde{\eta}_t\|_1 = \mathcal{O}(t^{-1/2})$. This also shows in which sense the solutions tend *uniformly* to the stationary limit in x-space.

7.2. The case $q_+ \neq q_-$

Here, we deal with the asymmetric case, when both q_+ and q_- are in the Eckhaus stable domain. In fact, we shall require that $|q_+|$, $|q_-|$, and $|q_+ - q_-|$ are small. Thus, we fill in some details in the "phase diagram" of Fig. 5, near the diagonal $q_+ = q_-$ and near the origin. As we shall see below, one cannot expect uniform convergence in this case. We can again write the GL-equation in its polar form, but now we subtract 1 from the solution. This leads to

$$\begin{aligned} \partial_t s \;&=\; \partial_x^2 s - 2s + 3s^2 - s^3 + \eta^2 - s\eta^2 \,, \\ \partial_t \eta \;&=\; \partial_x^2 \eta + 2\partial_x \left(\eta \frac{\partial_x s}{1-s} \right) \,, \end{aligned} \tag{7.10}$$

with $s = 1 - r$ and $\eta = \varphi'$. It is useful to define functions F and G and to rewrite (7.10) as

$$\begin{aligned} \dot{s} &= s'' - 2s - 2F(s,\eta) \,, \\ \dot{\eta} &= \eta'' + G(s,s',\eta)' \,, \end{aligned} \tag{7.11}$$

where $\dot{}$ denotes the time derivative and $'$ the space derivative. By the nature of the initial data, and of the time evolved solution, the r.h.s. of equation (7.11) is 0 at $x = \pm\infty$. Since the whole problem is of diffusive type, we expect, and it will follow from the solution of the problem, that *the values at infinity stay unchanged in time.* However, s and η themselves are *not* zero at infinity. We explain now how one deals with such a problem.

Before we can do so, we want to explain what it means that the initial data have different values at $-\infty$ and at $+\infty$. A nice example which illustrates this is the simple case of the pure diffusion equation $\dot{f} = f''$, with $\lim_{x\to-\infty} f(x,0) = a$, $\lim_{x\to+\infty} f(x,0) = b$. Here, and in the sequel, we use a so-called "scaling argument," i.e., we look for a solution of the form

$$f(x,t) = h^*(x/\sqrt{t+t_0}) \,,$$

with $t_0 \ge 0$. It is easy to check that $f(x,t)$ is a solution of the diffusion equation if $\partial_\xi h^*(\xi) = Ae^{-\xi^2/4}$. Therefore, integrating the Gaussian from $\xi = 0$, we see that f equals an error function, and $f(\pm\infty,t) = B \pm 2\sqrt{\pi}A$, so that f has different limits at $\pm\infty$. Note also that

$$\dot{f}(x,t) = -\tfrac{1}{2}(h^*)'\left(\frac{x}{\sqrt{t+t_0}}\right)\frac{x}{(t+t_0)^{3/2}} \,.$$

For fixed x this *is* integrable in t, but it is clear that the convergence is *not uniform* in x.

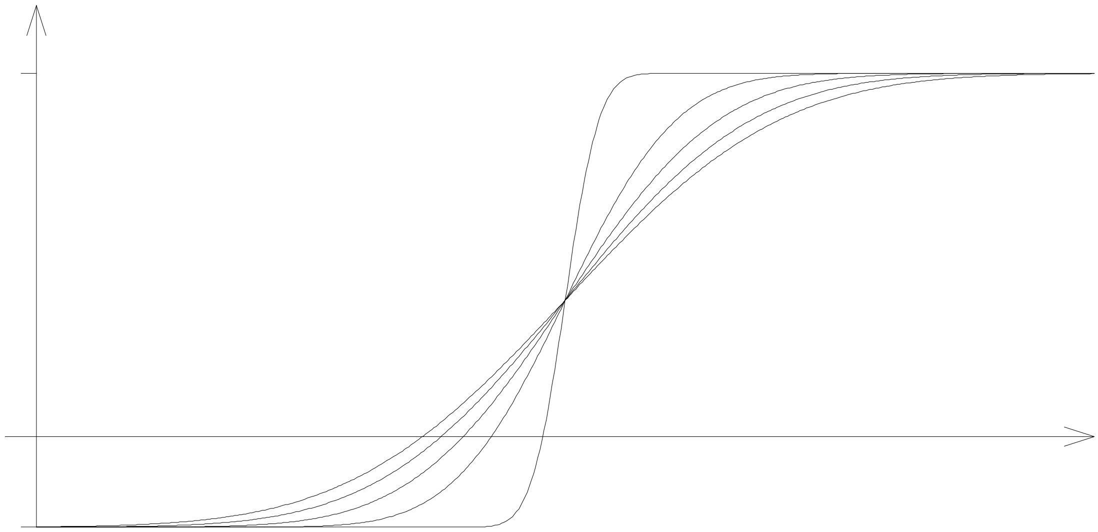

Fig. 10: The solution of the diffusion equation $\dot{f} = f''$, with $\lim_{x\to-\infty} f(x,0) = a$, $\lim_{x\to+\infty} f(x,0) = b$.

We choose now $a = B - 2\sqrt{\pi}A$, $b = B + 2\sqrt{\pi}A$. Then, on the one hand, if we fix a finite x, then the solution $f(x,t)$ converges to $(a+b)/2$ as $t \to \infty$. On the other hand, if we fix any finite time t then we have $\lim_{x\to-\infty} f(x,t) = a$, $\lim_{x\to+\infty} f(x,t) = b$, see Fig. 10. Thus, in diffusive problems with different boundary conditions at ∞, we find *convergence to equilibrium contrasted with stationarity at* ∞. If we think of f as the temperature in a problem involving heat diffusion, the interpretation of this phenomenon is that the temperature at $\pm\infty$ is fixed, and that there is a heat flux across. We shall elaborate in Section 8 how the availability of an infinite amount of "energy," allowing for this heat flux is the source of convergence problems. This problem is absent when $q_+ = q_-$ and is the reason for the simpler proofs one can obtain in that case. Coming back to the Eq.(7.11), the above discussion suggests to introduce a new independent variable $z : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ which *vanishes at* ∞ (s does not), and so we define

$$z = s + F(s,\eta) , \tag{7.12}$$

where F is the polynomial

$$2F(s,\eta) = -3s^2 + s^3 - \eta^2 + s\eta^2 .$$

In fact, our methods use a little less: One only needs to know that $F(\varepsilon x, \varepsilon y) = \mathcal{O}(\varepsilon^2)$ as $\varepsilon \to 0$, where $x, y \in \mathbf{R}$, and that F is $\mathcal{C}^2$. Clearly, there is a function f such that for sufficiently small z and η we have

$$s = z + f(z,\eta) .$$

Note that $f(\varepsilon z, \varepsilon\eta) = \mathcal{O}(\varepsilon^2)$. We will change variables in (7.10) from s to z. Denoting partial derivatives with appended subscripts, we have

$$\begin{aligned} s' &= (1+f_z)z' + f_\eta \eta' , \\ s'' &= (1+f_z)z'' + f_{zz}{z'}^2 + 2f_{z\eta}z'\eta' + f_{\eta\eta}{\eta'}^2 + f_\eta \eta'' . \end{aligned} \tag{7.13}$$

We introduce notation to simplify the sums of derivatives. D_1, D_2 denote the vectors

$$\begin{aligned} D_1 &= (z', \eta', 0, 0, 0) , \\ D_2 &= (z'', \eta'', {z'}^2, z'\eta', {\eta'}^2) , \end{aligned}$$

so that (7.13) becomes

$$\begin{aligned} s' &= z' + H_1 \cdot D_1 , \\ s'' &= z'' + H_2 \cdot D_2 , \end{aligned}$$

with

$$\begin{aligned} H_1 &= \bigl(f_z, f_\eta, 0, 0, 0\bigr) , \\ H_2 &= \bigl(f_z, f_\eta, f_{zz}, 2f_{z\eta}, f_{\eta\eta}\bigr) , \end{aligned}$$

and $H_i \cdot D_k$ denoting the scalar product. Note that if z and η are small in L^∞ then H_1 and the first two components of H_2 are small in L^∞. Furthermore, the last three components of H_2 are bounded. We call $\mathcal{H}$ the set of functions like H_1, H_2:

$$\begin{aligned}\mathcal{H} \;=\; \Big\{ \{h_j\}_{j=1,\ldots,5} : \mathbf{R}\times\mathbf{R}\to\mathbf{R} \mid & \|h_j(\varepsilon f, \varepsilon g)\|_\infty \le \mathcal{O}(\varepsilon)\;,\; j=1,2\;, \\ & \|h_j(\varepsilon f, \varepsilon g)\|_\infty \le \mathcal{O}(1)\;,\; j=3,4,5\;, \\ & \text{when } \|f\|_\infty,\; \|g\|_\infty \le 1 \Big\}\;.\end{aligned}$$

The term G in the second equation of (7.11) can be handled in a similar fashion. These calculations, cf. [Collet, Eckmann, 1992], which are a change of coordinates in function space from (s,η) to (z,η), finally lead to an equation of the form:

$$\begin{aligned}\dot z \;&=\; z'' - 2Az + H^{(z)}\cdot D_2\;,\\ \dot\eta \;&=\; \eta'' + H^{(G)}\cdot D_2\;,\end{aligned} \tag{7.14}$$

with $H^{(z)}$, $H^{(G)}$ in $\mathcal{H}$ and $A-1=\mathcal{O}(\varepsilon)$. We have now two "variables," the variable z *does* vanish at ∞, and is again seen to be a "massive" field, as in Eq.(7.8).

We now consider in more detail the second equation of (7.14). We encounter here a problem which is similar to the one found in the relation between s and z. Namely, since the limits of η at ∞ and $-\infty$ are not necessarily the same, we do not expect $\eta\in L^2$, which would be useful in view of an application of "energy bounds" as in Section 3. But we can now use the fact that the limits at infinity stay constant in time and express η as an approximate solution ζ which has the correct behavior at infinity and then bound the difference $y=\eta-\zeta$ in L^2. In other words, we perform a second change of variables in function space, from (z,η) to (z,y), with an additional *unknown function* ζ.

We first recall that $s=z+f(z,\eta)$, and therefore the r.h.s. of the second equation of (7.11) can be written as

$$\begin{aligned}&\eta'' + \left(\frac{z' + f_\eta(z,\eta)\eta' + f_z(z,\eta)z'}{1-z-f(z,\eta)}\eta\right)'\\ &\equiv\; \eta'' + \big(\eta\eta' Q_1(z,\eta)\big)' + \big(\eta z' Q_2(z,\eta)\big)'\;,\end{aligned}$$

where

$$\begin{aligned}Q_1 \;&=\; f_\eta(1-z-f)^{-1}\;,\quad Q_1 \;=\; \mathcal{O}(\varepsilon)\;,\\ Q_2 \;&=\; (1+f_z)(1-z-f)^{-1}\;,\quad Q_2 \;=\; \mathcal{O}(1)\;.\end{aligned}$$

With this notation, we have

$$\dot\eta \;=\; \eta'' + (\eta\eta' Q_1)' + (\eta z' Q_2)'\;. \tag{7.15}$$

We define the approximate quantity ζ as the solution of

$$\dot{\zeta} \;=\; \zeta'' + (\eta\zeta' Q_1)' \;, \tag{7.16}$$

with the boundary condition $\zeta(\pm\infty, t = 0) \;=\; \eta(\pm\infty, t = 0)$. This equation is of the type

$$\dot{\zeta} \;=\; \partial_x(1 + \eta Q_1)\partial_x \zeta \;. \tag{7.17}$$

We define $y = \eta - \zeta$, and the equation for y is

$$\dot{y} \;=\; y'' + (\eta y' Q_1)' + (\eta z' Q_2)' \;. \tag{7.18}$$

We now formulate some assumptions on z, η, ζ, and y.

Definition. We denote by $\mathcal{B}_\varepsilon$ the set of all functions z, η, ζ, and y, mapping $\mathbf{R} \to \mathbf{R}$, which satisfy

$$\int_{-\infty}^{\infty} dx \left| z^2 + {z'}^2 + {z''}^2 + {\eta'}^2 + {\eta''}^2 + y^2 + {y'}^2 \right| + \|\zeta\|_\infty^2 \;\le\; \varepsilon^2 \;. \tag{7.19}$$

Main Assumption. The initial data of the problem (1.2)—or equivalently, of (7.11), (7.14), (7.16), (7.18)—with $\zeta(t = 0, \cdot) = \eta(t = 0, \cdot)$ are in $\mathcal{B}_\varepsilon$.

Theorem 7.3. [Collet, Eckmann, 1992]. *There is an $\varepsilon_0 > 0$ such that if the initial data satisfy the Main Assumption with $\varepsilon < \varepsilon_0$, then the solution is, for all $t \ge 0$, in $\mathcal{B}_\varepsilon$.*

Remark. The conditions in the Main Assumption are somewhat redundant, since, e.g., the bound on z' follows from that on z, z'', and y, ζ, and η are not independent variables.

The interpretation of the result is as follows: Recall the transformation from the variables (r, φ) to (z, y, ζ), Eqs.(7.10), (7.12), $y = \eta - \zeta$. In particular, since z and z' are bounded in L^2 for all times, under the Main Assumption, we find that $\sup_{x\in\mathbf{R}} |z(x, t)|$ is bounded for all $t > 0$. Similarly ζ and η are bounded in L^∞ and therefore s is bounded in L^∞. All these bounds are of the order $\mathcal{O}(\varepsilon)$ and therefore, if ε is sufficiently small, we find that $r(x, t) = 1 - s(x, t) > 0$ for all x and t. *The amplitude is thus positive, there are no phase slips.* We can imagine therefore a "restoring" process whereby the initial condition with a phase derivative $\varphi' = q_\pm$ at $\pm\infty$ will "melt," on every compact interval, into a solution with a constant phase derivative q^* as time goes to ∞. An illustration is given in Fig. 11.

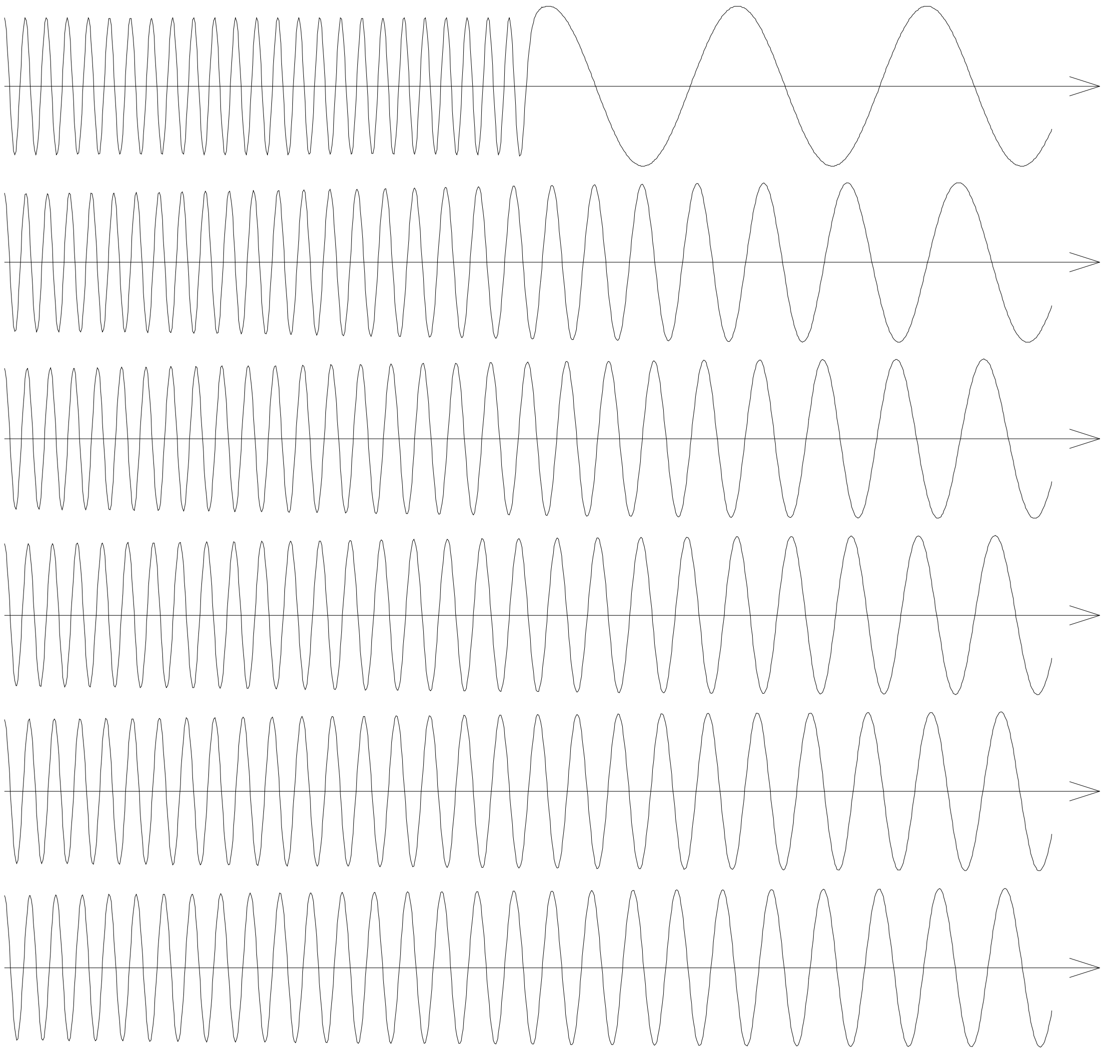

Fig. 11: A schematic rendering of the melting process when $q_+ \neq q_-$ is given. Time runs from top to bottom. Note that a new wavevector, q^*, is established as time goes on, *without* any phase slip. It should also be kept in mind that the wavevectors at $\pm\infty$ will stay forever at $q_\pm$. We show the real part of the solution.

7.2.1. Convergence

We finally address the question of convergence of the solution to one of the stationary solutions of Eq.(1.2), which has been solved in the nice papers [Bricmont, Kupiainen, Lin, 1992], [Bricmont, Kupiainen, 1992]. One has the following result:

Theorem 7.4. [Bricmont, Kupiainen, Lin, 1992], [Bricmont, Kupiainen, 1992]. *For every initial datum u sufficiently close to a scaling solution there are a stationary solution*

$$u^*(x) \;=\; \sqrt{1-q^{*2}}e^{iq^*x+i\varphi^*}\ , \tag{7.20}$$

and a generally non-trivial phase ϑ^**, such that for every finite interval* $I \subset \mathbf{R}$ *one has*

$$\lim_{t\to\infty} \sup_{x\in I} |u(x,t) - e^{i\vartheta^*\sqrt{t}}u^*(x)| \;=\; 0\;. \tag{7.21}$$

Note that the solution does *not* converge to a fixed function, but to a stationary solution *which is multiplied by a varying phase* whose speed of variation goes to 0. On the other hand, the derivative of the solution *does* converge. We shall sketch the main ideas of the proof and the nature of the scaling solutions in Section 8. Here, we continue with a somewhat more general discussion.

The convergence of the type of (7.21) holds true with $\vartheta^* = 0$ in the case when $q_+ = q_-$, and in fact convergence even takes place in suitable L^p spaces, as we have seen in Section 7.1. The case of $q_+ \neq q_-$ is more complex. As we have seen in Section 7.2, one cannot expect convergence of u in L^p, because the values at infinity are different. However, one can study the convergence of the *time derivative.* If, for example, $|\dot u(x,t)| \le C(x)t^{-3/2}$, then clearly the pointwise time-limit $\lim_{t\to\infty} u(x,t)$ will exist.

A first result is

Theorem 7.5. [Collet, Eckmann, 1992]. *There is an* $\varepsilon_0 > 0$ *such that for the initial data satisfying the Main Assumption with* $\varepsilon < \varepsilon_0$*, the solution converges in the following sense:*

$$\begin{aligned}\lim_{t\to\infty} \|z(\cdot,t)\|_\infty &= 0\;,\\ \lim_{t\to\infty} \|z'(\cdot,t)\|_\infty &= 0\;,\\ \lim_{t\to\infty} \|\eta'(\cdot,t)\|_\infty &= 0\;,\\ \lim_{t\to\infty} \|y(\cdot,t)\|_\infty &= 0\;.\end{aligned} \tag{7.22}$$

Remark. Note that we make no statement about the convergence of ζ which is the solution of Eq.(7.17). Indeed, (7.17) is a diffusion equation with a coefficient, $a(x,t)$, which is varying (slowly) in space *and* time, namely the function $a(x,t) = 1 + \eta Q_1$. Furthermore, the limiting values of ζ at $x = \pm\infty$ are *unequal.* It is the combination of these two problems which is responsible for the difficulty in proving the convergence of ζ as $t \to \infty$. It will be seen that the proof of Theorem 7.4 is crucially based on the observation that the coefficient $a(x,t)$ is a function of the solution, only.

On the other hand the Theorem 7.5 *does* imply convergence in a weaker sense. Write the solution u in polar coordinates: $u(x,t) = r(x,t)e^{i\varphi(x,t)}$. Then it follows at once from Theorem 7.5 that r and φ satisfy

$$\lim_{t\to\infty} \sup_{x\in\mathbf{R}} |r'(x,t)| \;=\; 0\;, \qquad \lim_{t\to\infty} \sup_{x\in\mathbf{R}} |\varphi''(x,t)| \;=\; 0\;. \tag{7.23}$$

This means that on every compact interval, the solutions looks at large times like a stationary solution $R(t)e^{iQ(t)x+i\Phi(t)}$.

8. Convergence in diffusive equations

This section describes what we know about problems of the type

$$\dot{u} = \partial_x a \partial_x u , \tag{8.1}$$

when a is strictly positive and of one of the forms

$$a = a(x) , \quad a = a(x,t) , \quad a = a(u(x,t), u'(x,t)) .$$

We discuss in particular the case when the initial data, $u(x,0) = u_0(x)$, have the property that

$$\lim_{x\to\pm\infty} u_0(x) = u_\pm ,$$

with $u_+ \neq u_-$. Such initial data are of course motivated by the phase equation discussed in Section 7.2. However, the present section is largely independent of the remainder of this paper. We shall see that the results are based on two basic principles:

i) A scaling argument for unequal limit values of u_0,

ii) A priori estimates on the fundamental solution of the operator $\partial_x a \partial_x$.

We start by discussing these general results in the next two subsections and will then proceed to increasingly complex examples of the convergence problem, ending in a sketch of the proof of Theorem 7.4.

8.1. The scaling solution for asymmetric problems

We consider the special case of a diffusion equation with non-constant coefficients (in space), which are step functions. Let

$$A(x) = \begin{cases} A_+, & \text{if } x > 0 , \\ A_-, & \text{if } x < 0 , \end{cases} \tag{8.2}$$

with $A_\pm > 0$. We consider the diffusion equation

$$\partial_t g(x,t) = \partial_x \big(A(x) \partial_x g(x,t) \big) , \tag{8.3}$$

with initial data $g(x,0) = u_+$ if $x > 0$ and $g(x,0) = u_-$ if $x < 0$. For $x \neq 0$, have the particular solutions

$$u_s(x,t) \equiv e^{isx/A^{1/2}(x) - s^2 t} ,$$

when $s \in \mathbf{R}$, so that we can write the general solution in the form

$$g(x,t) = \int_{-\infty}^{\infty} ds\, h_+(s)\theta(+x) u_s(x,t) + \int_{-\infty}^{\infty} ds\, h_-(s)\theta(-x) u_s(x,t) .$$

The continuity of g corresponds to the requirement

$$h_+(s) + h_+(-s) \;=\; h_-(s) + h_-(-s) \;, \tag{8.4}$$

and then the vanishing of the integral of (8.3) from $-\varepsilon$ to ε leads (in the limit $\varepsilon \to 0$) to the condition

$$A_+^{1/2}\big(h_+(s) - h_+(-s)\big) \;=\; A_-^{1/2}\big(h_-(s) - h_-(-s)\big) \;. \tag{8.5}$$

Thus, we find

$$h_\pm(s) \;=\; S(s) + \frac{1}{A_\pm} T(s) \;,$$

where S is an even function of s and T is odd. If we denote by $\tilde{S}$ and $\tilde{T}$ the Fourier transforms of S and T, then the initial data lead to an equation

$$u_+ \;=\; \tilde{S}(x/A_+^{1/2}) + \frac{1}{A_+^{1/2}}\tilde{T}(x/A_+^{1/2}) \;, \quad u_- \;=\; \tilde{S}(x/A_-^{1/2}) + \frac{1}{A_-^{1/2}}\tilde{T}(x/A_-^{1/2}) \;,$$

so that finally

$$\tilde{S}(x) \;=\; \frac{u_+ A_+^{1/2} + u_- A_-^{1/2}}{A_+^{1/2} + A_-^{1/2}} \;, \quad \tilde{T}(x) \;=\; (u_+ - u_-)\frac{A_+^{1/2} A_-^{1/2}}{A_+^{1/2} + A_-^{1/2}} \,\mathrm{sign}\, x \;.$$

From this, $g(x,t)$ is obtained at once:

$$\begin{aligned} g(x,t) \;&=\; \frac{1}{\sqrt{4\pi t}} \sum_{\sigma \in \{+,-\}} \theta(\sigma x) \int_{-\infty}^{\infty} dz\, e^{-(x/A_\sigma^{1/2} - z)^2/4t} \left(\tilde{S}(z) + \frac{1}{A_\sigma^{1/2}}\tilde{T}(z)\right) \\ &=\; \frac{u_+ A_+^{1/2} + u_- A_-^{1/2}}{A_+^{1/2} + A_-^{1/2}} \\ &+\; \frac{1}{\sqrt{4\pi t}} \frac{(u_+ - u_-) A_+^{1/2} A_-^{1/2}}{A_+^{1/2} + A_-^{1/2}} \sum_{\sigma \in \{+,-\}} \frac{\sigma \theta(\sigma x)}{A_\sigma^{1/2}} \int_{-\infty}^{\infty} dz e^{-(x/A_\sigma^{1/2} - z)^2/4t} \,\mathrm{sign}\, z \;. \end{aligned} \tag{8.6}$$

From this, we see that

$$u^* \;\equiv\; g(x,\infty) \;=\; \frac{u_+ A_+^{1/2} + u_- A_-^{1/2}}{A_+^{1/2} + A_-^{1/2}} \;. \tag{8.7}$$

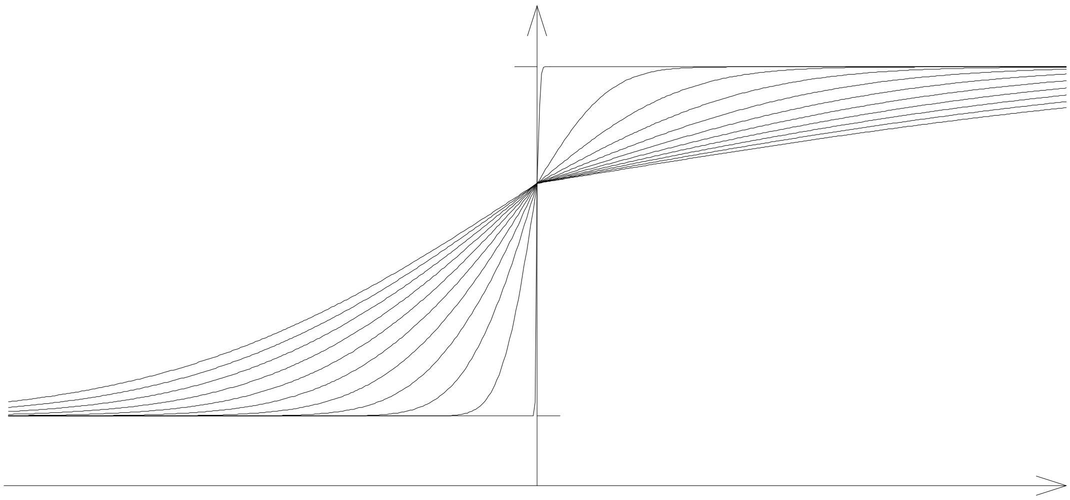

Fig. 12: The solution of the diffusion equation with two different diffusion constants for $x > 0$ and $x < 0$, and two different initial values, u_+ and u_-. Note the convergence to u^* in time. Also observe that the ratio of the derivatives at $x = 0$ is constant.

This calculation shows that the "equilibrium value," $u^* = g(x, \infty)$ is determined by the continuity of the heat flux across the discontinuity of the diffusion constant at $x = 0$. The ratio of the derivatives of g at $x = 0$ is constant in time. Note that the result (8.7) is not really obvious. Indeed, for any *finite* system, one would first see convergence towards u^* and then a crossover to an equilibrium temperature which can be expressed in terms of specific heats. In the infinite system, the total heat content is infinite. If the temperature is not constant, this infinite amount of heat can move around over large distances. This phenomenon is important for equations such as (7.16) in which, so to speak, the diffusion constant is a function of space and time *with different limits at* $\pm\infty$. Since we know the fundamental solution for (8.3) explicitly, we have an explicit integral representation for any initial data.

We can write the solution in another form, making directly a scaling ansatz. Set $g(x, t) = g^*(x/t^{1/2})$. Substituting, we get the equation, for $g^* = g^*(\xi)$:

$$-\tfrac{1}{2}\xi\partial_\xi g^* \;=\; \partial_\xi A \partial_\xi g^* \;.$$

This can be integrated, and the boundary conditions, together with the continuity requirement for $\partial_\xi A \partial_\xi g^*$ at $\xi = 0$ lead to

$$g^*(\xi) \;=\; \frac{u_+ A_+^{1/2} + u_- A_-^{1/2}}{A_+^{1/2} + A_-^{1/2}} + \frac{(u_+ - u_-)A_-^{1/2}}{A_+^{1/2}\sqrt{4\pi}(A_+^{1/2} + A_-^{1/2})} \int_0^\xi d\eta e^{-\eta^2/(4A_+)} \;, \qquad (8.8)$$

when $\xi > 0$ and similarly for $\xi < 0$.

8.2. A priori bounds on the fundamental solution. Harnack inequalities

Here, we recall the very nice bounds which are known for the case of the operator $L = \partial_x a(x,t)\partial_x$. We assume throughout that a is "strictly positive," i.e., there are constants $\lambda > 0$ and $\Lambda < \infty$ such that

$$\lambda \,<\, a(x,t) \,<\, \Lambda\,, \quad \text{for all } x,t\,. \tag{8.9}$$

Theorem 8.1. [Moser, 1973], [Fabes, Stroock, 1986]. *Assume that a is strictly positive. Then, the fundamental solution Γ for the operator $L = \partial_x a(x,t)\partial_x$ satisfies the bounds, for $t > s$,*

$$C_<^{-1}\frac{e^{-C_<(x-y)^2/(t-s)}}{(t-s)^{1/2}} \,\le\, \Gamma((x,t),(y,s)) \,\le\, C_>^{-1}\frac{e^{-C_>(x-y)^2/(t-s)}}{(t-s)^{1/2}}\,, \tag{8.10}$$

for two positive constants $C_<$, $C_>$.

Remark. The theorem in [Fabes, Stroock, 1986] is formulated for the multi-dimensional case. The theorem tells us that the diffusion operator L with non-constant coefficient a has the same kind of bound as the explicitly known Gaussian kernel for the operator ∂_x^2.

8.3. Convergence for $u_0 \in L^1$

We start by exploiting the results of the previous subsections and assume here that $u_0 \in L^1$. Later we shall generalize the results to the case when only $u_0' \in L^1$, (for $a = \mathrm{const.}$ or $a = a(x)$).

Lemma 8.2. *Let $u(x,0) = u_0(x)$ in L^1. Then the solution $u(x,t)$ of the equation $\partial_t u = \partial_x a \partial_x u$, with a satisfying the inequality (8.9) converges pointwise to zero.*

Proof. The solution is given by

$$u(x,t) \,=\, \int_{-\infty}^{\infty} dy\, \Gamma((x,t),(y,0))\, u_0(y)\,. \tag{8.11}$$

We can use the upper bound from (8.10) to get

$$|u(x,t)| \,\le\, \int dy\, |u_0(y)| \cdot \sup_{x,y} |\Gamma((x,t),(y,0))| \,\le\, \mathcal{O}(t^{-1/2})\|u_0\|_1\,. \tag{8.12}$$

Thus, $u(x,t)$ tends pointwise to 0 as $t \to \infty$. Note that, on the other hand, the L^1 norm $\|u_t\|_1 = \int dx |u(x,t)|$ does *not* decrease in time.

Remark. The lemma generalizes to the case when $u_0(x) - c \in L^1$. In particular, it implies that $u(x,t) = c + \mathcal{O}(t^{-1/2})$ as $t \to \infty$.

Remark. If a is of the form $a(x,t) = a(u(x,t), u'(x,t))$, convergence still follows from the general bounds of Eq.(8.10).

Remark. The result generalizes obviously to $u_0 \in L^p$, $p < \infty$, by using a Hölder inequality in Eq.(8.12).

8.4. The rôle of the unequal limit values

We now start our study of initial data u_0 which are not in L^1, but we assume throughout that they have limits as $x \to \pm\infty$. This corresponds to the case of the GL-equation, cf. Eq.(7.17), when $q_+ \neq q_-$.

8.4.1. Convergence for diffusion when $u_0' \in L^1$

We begin by considering the problem of a constant coefficient but with initial data which have different limits at $\pm\infty$. On the other hand, we need to require $\int dx |u_0'(x)| < \infty$.

Lemma 8.3. *Let u_0 be a function for which $u_0' \in L^1$. Consider the equation*

$$\dot{u} = u'' , \tag{8.13}$$

with initial data $u(x,0) = u_0(x)$. Then, for each x, $u(x,t)$ converges as $t \to \infty$. The limit is $u^* = \frac{1}{2}(u_0(+\infty) + u_0(-\infty))$.

Proof. We have

$$u(x,t) = \int dy \, \frac{e^{-y^2/4t}}{\sqrt{4\pi t}} u_0(x-y) . \tag{8.14}$$

Since (8.13) is linear and since constants are solutions, we may subtract from u_0 a constant and therefore we can assume without loss of generality that $u_0(-\infty) = 0$. Integrating by parts in (8.14), we get

$$u(x,t) = \int dy \int_{-\infty}^{y} dz \, \frac{e^{-z^2/4t}}{\sqrt{4\pi t}} u_0'(x-y) .$$

Let now $\mathrm{erf}(s) = \pi^{-1/2} \int_{-\infty}^{s} dx e^{-x^2}$. Then

$$u(x,t) = \int dy \, \mathrm{erf}(y/\sqrt{4t}) u_0'(x-y) , \tag{8.15}$$

and, clearly, $u(x,\infty) = \frac{1}{2} \int dy \, u_0'(x-y) = \frac{1}{2}(u_0(+\infty) + u_0(-\infty))$.

Fix now $x \in \mathbf{R}$ and assume $\varepsilon > 0$ is given. We shall find a t_0 such that

$$|u(x,t) - u(x,\infty)| \le \varepsilon , \tag{8.16}$$

for $t > t_0$. First, since $u_0' \in L^1$ there is a y_0 such that $\int_{|y|>y_0} dy \, |u_0'(x-y)| < \varepsilon/4$. Thus, the contribution from $|y| > y_0$ in (8.15) to the difference $u(x,t) - u(x,\infty)$ is bounded by $\varepsilon/2$. Furthermore, there is a t_0 such that

$$\sup_{|y|\le y_0} \left| \mathrm{erf}(y/\sqrt{4t}) - \tfrac{1}{2} \right| \le \varepsilon/(2\|u_0'\|_1) , \tag{8.17}$$

for all $t \geq t_0$. Thus, the contribution from $|y| < y_0$ is also bounded by $\varepsilon/2$ and (8.16) follows.

Remark. The crucial step in the proof is the *pointwise* estimate of the fundamental solution in Eq.(8.17). Such a bound is *not provided* by the Harnack inequality, Eq.(8.10). On the other hand, the situation is very good with respect to *local convergence*, even when the coefficients a are not constant in time. One has the following result:

Theorem 8.4. [Moser, 1971]. *Consider the equation*

$$\partial_t u \,=\, \partial_x a(x,t) \partial_x u \;,$$

with measurable a satisfying $0 < \lambda < a(x,t) < \Lambda$, and bounded initial conditions. If the solution exists, then its "oscillation," i.e.,

$$\operatorname*{osc}_{[a,b]\times[T_1+T,T_2+T]} u \,\equiv\, \sup_{x\in[a,b]} \sup_{t\in[T_1+T,T_2+T]} u(x,t) - \inf_{x\in[a,b]} \inf_{t\in[T_1+T,T_2+T]} u(x,t) \;,$$

tends to zero as $T \to \infty$, for any finite interval $[a,b]$ and any finite times $T_1 < T_2$.

Thus, the solution reaches a "local" equilibrium over an arbitrarily large region, but a readaptation over very large times cannot be excluded. Thus, we see another variant of the result obtained for the GL-equation in Theorem 7.5.

Remark. If the solution converges in one point, it converges everywhere. Indeed, since the oscillation converges, we find convergence for all x.

We can apply the above discussion to the GL-equation:

Theorem 8.5. *For every initial datum satisfying the Main Assumption with sufficiently small $\varepsilon > 0$, and every compact interval I, there are functions $R_I(t)$, $Q_I(t)$, $\Phi_I(t)$ so that*

$$\lim_{t\to\infty} \sup_{x\in I} |u(x,t) - R_I(t) e^{iQ_I(t)x + i\Phi_I(t)}| \,=\, 0 \;. \tag{8.18}$$

Furthermore, one can choose the solutions to look "locally stationary," i.e., $R_I^2(t) + Q_I^2(t) = 1$.

The proof follows by integrating (7.23) in space. In other words, the system tries to adapt locally to a stationary solution, but this stationary solution will vary slowly in time due to global effects as shown in Theorem 7.4. The speed with which these effects take place goes to zero, but their cumulative effect will be visible forever. If the initial data are close to a "scaling solution," then convergence is guaranteed for the derivative, as we show in Subsection 8.4.4.

8.4.2. It is necessary that $u_0' \in L^1$

We show here that solutions of the heat equation on $\mathbf{R}$ do not necessarily tend to a limit, if the initial data have derivatives which are not in L^1 at infinity. This shows, in a sense, that Lemma 8.3 is sharp. This result might seem somewhat counterintuitive since the solutions *do* tend "locally in space-time" to constants as we have seen in Theorem 8.4. We now provide 2 examples.

Lemma 8.6. *There are initial data for the heat equation for which the solution at $x = 0$ oscillates forever as a function of time, with the amplitude of the oscillation of order 1. These initial data have derivatives in $L^{1+\varepsilon}$ for any $\varepsilon > 0$.*

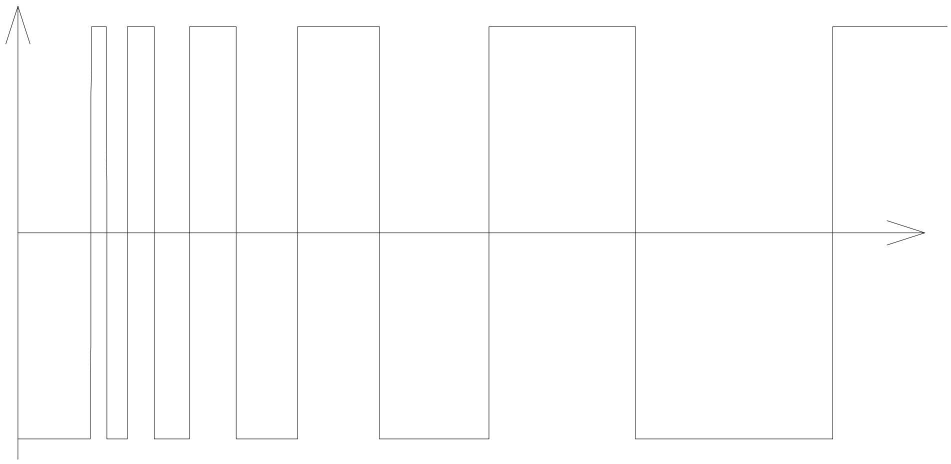

Fig. 13: The function f used in the proof of Lemma 8.6. The horizontal scale is squeezed. (The function is infinitely differentiable, but the corners look sharp at the scale used.) The solution of the diffusion equation with such initial data, when measured at $x = 0$, will oscillate forever in time, with an amplitude of almost 1.

Proof. The proof is by construction, see Fig. 13. To keep the argument simple, we just construct a particular example. Let

$$L_n = n!\,, \quad \ell_n = 2^n\,.$$

We define an even $\mathcal{C}^\infty$ function f by requiring

$$f(x) = (-1)^n \ \text{ for } \ x \in [L_n + \ell_n, L_{n+1} - \ell_{n+1}]\,,$$

when $n \geq 5$ and for other values of x we require $|f| \leq 1$. In the sequel, we assume throughout $n \geq 5$. If we take f as initial data, then the solution of the heat equation $\dot{u} = u''$ at $x = 0$ is

$$u(0,t) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} dy\, \frac{e^{-y^2/4t}}{\sqrt{t}} f(y)\,. \tag{8.19}$$

Let now $T_n = L_n L_{n+1} = (n+1)n!^2$. We shall show that

$$u(0,T_n) \approx (-1)^n \;, \tag{8.20}$$

when n is large, and this proves the lemma. Define

$$\alpha_n \;=\; \frac{L_n+\ell_n}{T_n^{1/2}}\;, \quad \beta_n \;=\; \frac{L_{n+1}-\ell_{n+1}}{T_n^{1/2}}\;.$$

By construction, we have

$$\alpha_n \;=\; \mathcal{O}(n^{-1/2})\;, \quad \beta_n \;=\; \mathcal{O}(n^{+1/2})\;. \tag{8.21}$$

We next set

$$I_n^0 \;=\; [0,\alpha_n T_n^{1/2}]\;, \quad I_n \;=\; [\alpha_n T_n^{1/2},\beta_n T_n^{1/2}]\;, \quad I_n^+ \;=\; [\beta_n T_n^{1/2},\infty]\;.$$

Note now that if $y\in I_n$, then, by construction, we have $f(y)=(-1)^n$. Thus, the integral (8.19) (evaluated for $t=T_n$), can be divided into 3 pieces, and the piece coming from $|y|\in I_n$ leads to

$$\begin{aligned}(-1)^n \int_{|y|\in I_n} dy\, \frac{1}{\sqrt{4\pi}}\frac{e^{-y^2/4T_n}}{\sqrt{T_n}} f(y) \;&=\; \int_{|y|\in I_n} dy\, \frac{1}{\sqrt{4\pi}}\frac{e^{-y^2/4T_n}}{\sqrt{T_n}}\\ &=\; 1-\mathcal{O}(\alpha_n T_n^{1/2}T_n^{-1/2}) - \mathcal{O}(e^{-\beta_n^2/3})\;.\end{aligned}$$

Similarly, since $|f|<1$, we find that the integrals over the sets $\{|y|\in I_n^0\}$ and $\{|y|\in I_n^+\}$ contribute $\mathcal{O}(\alpha_n)+\mathcal{O}(e^{-\beta_n^2/3})$. By (8.21), we see that the remainders are irrelevant as n gets large, and hence we have shown (8.20). Since the distances between $L_n-\ell_n$ and $L_n+\ell_n$ (which are the regions where f must oscillate between $(-1)^{n-1}$ and $(-1)^n$) grow like 2^n as $n\to\infty$, we see that in each of these gaps we can choose f in such a way that $|\partial_x f|\le \mathrm{const.}\, 2^{-n}$. In the complement of these sets we have $\partial_x f=0$, and therefore

$$\int |\partial_x f|^p \;\le\; \mathrm{const.}\; \sum_{n=1}^{\infty} 2^{-np}2^n\;,$$

from which we see that $f\in L^p$, when $p>1$. The proof of Lemma 8.6 is complete.

Another example. The following example "explains" the appearance of the factor $e^{i\vartheta^*\sqrt{t}}$ in Eq.(7.21). We thank J. Bricmont for illuminating discussions on this point. Consider the diffusion equation $\dot u = u''$ with initial data

$$u(x,0) \;=\; \begin{cases} v_-x, & \text{if } x\le 0,\\ v_+x, & \text{if } x\ge 0.\end{cases} \tag{8.22}$$

The derivative of this function is not in L^1. The methods of Section 8.1 can be applied to the derivative $v=u'$ and one obtains

$$v(x,t) \;=\; v_- + (v_+ - v_-)\,\mathrm{erf}(x/(2\sqrt{t}))\;.$$

Integrating, and adapting the integration constant (which is a function of t!) to solve the diffusion equation, we get

$$u(x,t) \;=\; \sqrt{t}\,\frac{v_+ - v_-}{\sqrt{\pi}} + v_-x + (v+-v-)\int_0^x dy\;\mathrm{erf}\bigl(\frac{y}{2\sqrt{t}}\bigr)\;. \tag{8.23}$$

Note that this function remains tangent to $u(x,0)$ at $x=\pm\infty$ for all times.

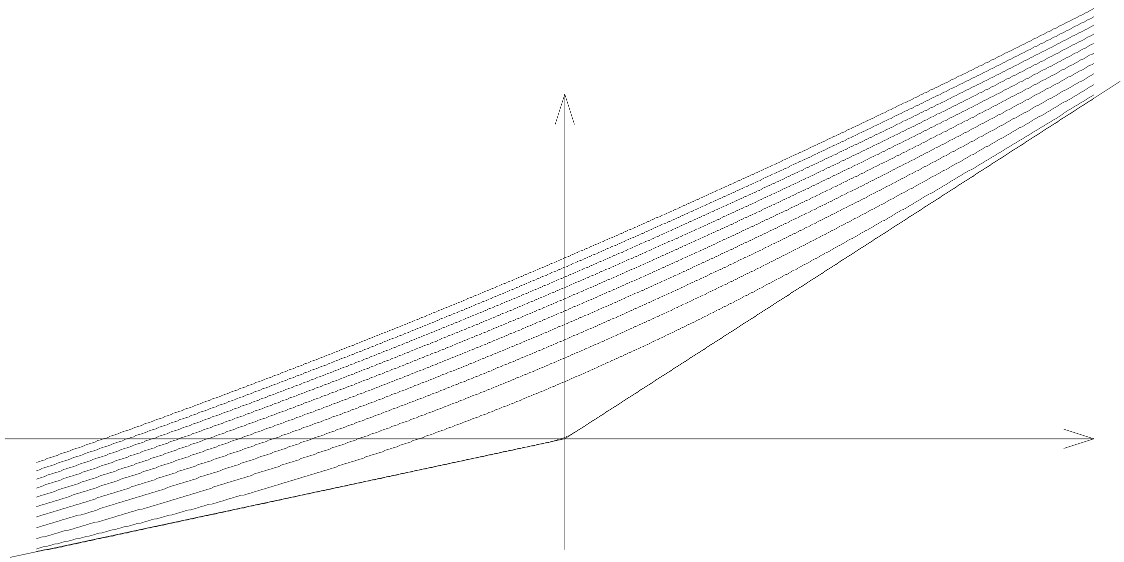

Fig. 14: Solution of the diffusion equation for initial data as in Eq.(8.22) with $v_+ = 3$, $v_- = 1$. Note the upward drift of the solution.The speed $\dot{u}(0,t)$ at $x = 0$ slows down like $1/\sqrt{t}$ but $u(0,t)$ will *diverge* as $t \to \infty$. This phenomenon also happens in the GL-equation, cf. Theorem 7.4.

8.4.3. The coefficients depend on space, but not on time

We consider here the problem

$$\dot{u} \;=\; \partial_x a(x) \partial_x u \;, \tag{8.24}$$

when $\infty > \Lambda > a(x) > \lambda > 0$ and $a(x)$ has limits as $x \to \infty$, and $u_0' \in L^1$, and $u_0(\pm\infty) = u_\pm$. Define $A_\pm^2 = \lim_{y\to\pm\infty} a(y)$ and define $A(x)$ as in Eq.(8.2). Let now $g(x,t) = g^*(x/(t+t_0)^{1/2})$, for some $t_0 \ge 0$, with g^* defined by Eq.(8.8), and with $g(\pm\infty, 0) = u_0(\pm\infty)$. (The constant t_0 is arbitrary and it makes g smooth. To shorten notation, we set $t_0 = 0$, and leave the generalization to arbitrary $t_0 > 0$ to the reader.)

Proposition 8.7. *Assume $a - A \in L^1$ and assume $u_0 - g(\cdot, 0) \in L^1$. Then, the solution of Eq.(8.24) converges pointwise as $t \to \infty$.*

Proof. We consider $v = u - g$. Then v must satisfy the equation

$$\dot{v} \;=\; \partial_x a(x) \partial_x v + \frac{1}{t^{1/2}} \partial_x \bigl(a(x) - A(x)\bigr) (g^*)'(x/t^{1/2}) \;,$$

with $v_0(\pm\infty) = v(\pm\infty, t = 0) = 0$. We already know from Theorem 8.4 that for every interval $I = [x_0 - \varepsilon^{-1}, x_0 + \varepsilon^{-1}]$, there is a $T = T(x_0, \varepsilon)$ such that for $t > T$ the spatial oscillation of v on this interval is less than ε. (It suffices to take $T_1 = 0$ and $T_2 = 1$ in Theorem 8.4.) Let now φ_0 be centered on this interval,

$$\varphi_0(x) \;=\; \begin{cases} \varepsilon - \varepsilon^2 |x - x_0|, & \text{when } |x - x_0| \le 1/\varepsilon \;, \\ 0, & \text{otherwise .} \end{cases}$$

We shall show that for $t > T$, one has

$$\int dx|\varphi_0 v_t| \,<\, 2(K+1)\varepsilon\,. \tag{8.25}$$

Since the oscillation of v is less than ε, and since $\int_{x\in I}\varphi_0 = 1$, this means that $\min_{x\in I}|v(x,t)| \le 2(K+1)\varepsilon$, and hence $\max_{x\in I}|v(x,t)| \le 3(K+1)\varepsilon$. Since ε is arbitrary, this means that $v \to 0$, pointwise.

To prove Eq.(8.25), we define $L = \partial_x a \partial_x$; L is self-adjoint. We have

$$v_t \,=\, e^{Lt}v_0 + \int_0^t ds\, e^{L(t-s)}\frac{1}{\sqrt{s}}\partial_x(a-A)(g^*)'(x/s^{1/2})\,,$$

and hence, for any test function φ_0, we have

$$\int dx\varphi_0 v_t \,=\, \int dx v_0 e^{Lt}\varphi_0 - \int dx \int_0^t ds\,\left(\partial_x e^{L(t-s)}\varphi_0\right)\frac{1}{\sqrt{s}}(a-A)(g^*)'(x/s^{1/2})\,. \tag{8.26}$$

Since $\varphi_0 \in L^1$, we get from Lemma 8.2 and Eq.(8.12) that

$$\int |\varphi_0 e^{Lt}v_0| \,=\, \int |\varphi_t v_0| \,\le\, \sup_x |e^{Lt}\varphi_0(x)|\cdot\int dx|v_0| \,\le\, \mathcal{O}(t^{-1/2})\|\varphi_0\|_1\|v_0\|_1\,.$$

By the assumption on $a - A$, the quantity $(a(x) - A(x))(g^*)'(x/s^{1/2})$ is bounded (uniformly in s!) in L^1. Therefore the second term in Eq.(8.26) can be bounded by

$$\text{const.}\int_0^t ds|\varphi'(x,t-s)|s^{-1/2}\,. \tag{8.27}$$

To bound this expression, we consider the problem

$$\dot\varphi \,=\, \partial_x a(x)\partial_x\varphi\,, \tag{8.28}$$

when $\varphi_0' \in L^1$.

Lemma 8.8. *There is a constant K such that*

$$\|\varphi_t'\|_\infty \,\le\, K\|\varphi_0'\|_1/\sqrt{t}\,. \tag{8.29}$$

Postponing the proof of Lemma 8.8, we continue the proof of Proposition 8.7. The term (8.27) is now bounded by

$$\text{const.}\,\|\varphi_0'\|_1\int_0^t ds\frac{1}{\sqrt{t-s}}\frac{1}{\sqrt{s}} \,\le\, \text{const.}\,\|\varphi_0'\|_1\,.$$

By the construction of φ_0, we have $\|\varphi_0'\|_1 = 2\varepsilon$. Hence, Eq.(8.25) follows, and the proof of Proposition 8.7 is complete, except for the proof of Lemma 8.8.

Proof of Lemma 8.8. Consider the equation for $\psi = \varphi'$: It is

$$\dot{\psi} = \partial_x^2 a\psi . \tag{8.30}$$

The following transformation brings this equation again to Sturm-Liouville form [Dunford-Schwartz, 1958, Chapter XIII]: Define

$$\alpha(x) = \int_0^x dy \, \frac{1}{a(y)} , \tag{8.31}$$

and set now

$$w(x,t) = a(\alpha^{-1}x)\psi(\alpha^{-1}x,t) , \tag{8.32}$$

where α^{-1} is the inverse function of α. We shall write $a \circ \alpha^{-1}$ for the composition $a(\alpha^{-1}(x))$ and $f \circ \alpha^{-1} = f(\alpha^{-1}x,t)$ for functions of 2 variables. Then w satisfies the equation

$$\dot{w} = \partial_x \frac{1}{a \circ \alpha^{-1}} \partial_x w . \tag{8.33}$$

Indeed, since $(\alpha^{-1})' = a \circ \alpha^{-1}$, we see that

$$\partial_x \frac{1}{a \circ \alpha^{-1}} \partial_x (f \circ \alpha^{-1}) = (a \cdot f'') \circ \alpha^{-1} .$$

Therefore, by Eq.(8.32) we get

$$a\dot{\psi} = (\dot{w}) \circ \alpha = \big(\partial_x \frac{1}{\alpha^{-1}} \partial_x w\big) \circ \alpha = a(a\psi)'' ,$$

which is Eq.(8.30), and hence (8.33) follows.

In view of the above discussion, it suffices to study the equation

$$\dot{w} = \partial_x b(x) \partial_x w , \tag{8.34}$$

with $b = 1/a \circ \alpha^{-1}$. Note that from the definition $w = (av) \circ \alpha^{-1}$ it follows that $\|w_t\|_1 = \|\psi_t\|_1$, where $w_t(x) \equiv w(x,t)$. So, since $\psi \in L^1$, so is w, and the Lemma 8.2 applies, with $u_0 = w_0$. It follows that $|\varphi_t'|_\infty \leq \mathcal{O}(t^{-1/2}) \|\varphi_0'\|_1$, as asserted. The proof of Lemma 8.8, and hence of Proposition 8.7 is complete.

8.4.4. Sketch of proof of Theorem 7.4

This subsection is based on the nice argument of [Bricmont, Kupiainen, Lin, 1992], [Bricmont, Kupiainen, 1992], and we work out what seems to us the salient feature of this argument. Instead of considering the GL-equation, we simplify the discussion by analyzing the analogue of the phase equation for η in (7.11). Therefore, consider

$$\partial_t u \;=\; \partial_x (1 + a(u, u'))\partial_x u \;, \tag{8.35}$$

with $a(\xi, \zeta)$ analytic at 0 in ξ, ζ, $a(0, 0) = 0$, and with initial data which satisfy $u(\pm\infty, 0) = u_\pm$. Below, we shall also specify a neighborhood in a function space in which $u_0 = u(\cdot, t = 0)$ should lie. The first observation is that a depends on x and t only through the solution u. Thus, e.g., if the solution is constant in space, then the same is true for $a(u, u')$. There are two main ideas:

i) View $u_\pm$ as small perturbations of zero boundary conditions.

ii) Consider the scaling solutions for *two* levels of approximation. Then the equations for the remainder will be seen to converge diffusively, with sufficiently strong bounds to guarantee the convergence of the type of Theorem 7.4, cf. the time-dependent integration constant in Eq.(8.23).

In view of diffusive nature of Eq.(8.35), and the example of Fig. 10 we expect u to be an approximate error function and we expect $u' \to 0$ as $t \to \infty$. Therefore, we start with an ansatz $u(x, t) = \varphi^*(x/t^{1/2}) + v(x, t)$, where $\varphi^*(\pm\infty) = u_\pm$. Substituting in Eq.(8.35) and dropping all terms involving v, one finds the differential equation

$$-\tfrac{1}{2}x\partial\varphi^* \;=\; \partial\big(1 + a(\varphi^*, 0)\big)\partial\varphi^* \;, \tag{8.36}$$

where ∂ denotes the derivative w.r.t. the argument of φ^*. This is a *non-linear* differential equation, and it is an ordinary differential equation because we are in 1 spatial dimension. Since the $|u_\pm|$ are supposed to be small, and a vanishes at the origin, a perturbation argument shows the existence and uniqueness of φ^*, starting from the problem with $a \equiv 0$. If φ^* solves (8.36), then v must satisfy

$$\partial_t v \;=\; \partial_x\big(1 + a(u, u')\big)\partial_x v + \partial_x\big(a(u, u') - a(u^*, 0)\big)\partial_x u^* \;, \tag{8.37}$$

where $u^*(x, t) = \varphi^*(x/t^{1/2})$. One writes $v(x, t) = t^{-1/2} f^*(x/t^{1/2}) + w(x, t)$. Dropping all terms involving w, one finds the following equation for f^*:

$$\partial \left[\partial(1 + a(\varphi^*, 0)) f^* + \tfrac{1}{2}x f^* + h\right] \;=\; 0 \;, \tag{8.38}$$

where

$$h \;=\; \frac{\partial a}{\partial u'}(\varphi^*, 0) \cdot (\partial\varphi^*)^2 \;. \tag{8.39}$$

Such equations are typical "transport equations" which are also found in WKB calculations. As already remarked, the *first* term (φ^*, Eq.(8.36)) satisfies a *non-linear* differential equation, all further terms (such as f^*) satisfy linear, inhomogeneous equations. These equations can be solved by quadratures. The function f^* is determined up to the addition of a multiple of $\partial\varphi^*$ (essentially a Gaussian). We define $v^*(x,t) = t^{-1/2} f^*(x/t^{1/2})$. The equation for w is finally of the form

$$\partial_t w \;=\; w'' + K \;, \tag{8.40}$$

where

$$\begin{aligned} K \;=\; &\partial\big(a(u,u') - a(u^*,0)\big)\partial v^* \\ &+ \partial a(u,u')\partial w \\ &+ \partial\big(a(u,u') - a(u^*,0) - \frac{\partial a}{\partial u'}(u^*,0)\cdot(\partial u^*)\big)\partial u^* \\ &- \partial v^* \partial a(u^*,0) \;. \end{aligned} \tag{8.41}$$

Here, $u = u^* + v^* + w$. One notes now that K contains no linear terms and no terms which are only powers of u^*. Assume now the initial data (at time $t = 1$) are of the form

$$u(x,1) \;=\; \varphi^*(x,1) + f^*(x,1) + w(x,1) \;,$$

where $\int dx w(x,1) = 0$, and f^* is one of the solutions of (8.38). One furthermore assumes that w is small in some suitable space of integrable functions (see [Bricmont, Kupiainen, Lin, 1992], [Bricmont, Kupiainen, 1992] for details): Essentially, w should decay like x^{-4} at ∞. Since the integral of w is 0, by assumption, this yields another small quantity at zero momentum. Therefore, all terms in K tend to zero as $x \to \pm\infty$. The asymmetry created by $u_\pm$ has been eliminated from the problem, by the subtraction of the *two* scaling terms, u^* and v^*. A careful power counting, as in the proof of Proposition 7.1 shows that $w \to 0$, diffusively, as $t \to \infty$.

Appendix A. Proof of Theorem 3.1

We give ourselves a weight function

$$h(x) = (1 + (x - x_0)^2)^{-1} \;, \tag{A.1}$$

with arbitrary x_0. It is easy to see that h satisfies

$$\|\partial_x^p h/h\|_\infty \le H \;, \text{ for } p = 1, \dots, 2k \;, \tag{A.2}$$

with H independent of x_0. The bounds are done in $L^2(h^2 dx)$ and we denote

$$(u,v)_h \;=\; \int dx\, h^2(x)\bar u(x) v(x) \;,$$

and

$$(u,v) \;=\; \int dx\, \bar u(x) v(x) \;.$$

The bounds can be derived from the energy estimates because the "error" coming from the cutoff function h gives rise to lower order operators: Let $Q(x,\xi)$ be of the form

$$Q(x,\xi) = b\xi^{2n} + S(x,\xi)\ , \tag{A.3}$$

where $b > 0$ and where S is a polynomial of degree $< 2n$ in ξ with sufficiently smooth coefficients which are bounded together with their derivatives. We denote by Q the operator $Q(x,-i\partial_x)$. Then one has the following

Lemma A.1. *There is a constant $G = G(Q,H)$ such that*

$$(u,Qu)_h \ge -G(u,u)_h\ .$$

Proof. We have

$$\begin{aligned} (u,Qu)_h &= (h^2u,Qu) \\ &= (uh,Quh) \\ &\quad + (uh,[h,Q]h^{-1}\cdot uh)\ , \end{aligned}$$

where $[\cdot,\cdot]$ denotes the commutator. Note now that $[h,\partial_x^p]h^{-1}$ is a sum of differential operators of degree less than p,

$$[h,\partial_x^p]h^{-1} = \sum_{\substack{j_1,\dots,j_s \\ 1\le\Sigma j_k\le p}} c_{j_1,\dots,j_s} \prod_{k=1}^{s} \left(\frac{\partial_x^{j_k}h}{h}\right) \partial_x^{p-\Sigma j_k}\ .$$

Therefore, we find

$$[h,Q]h^{-1} = \sum_{j=1}^{2n} S_j(x,-i\partial_x)\ ,$$

where the S_j are polynomials of degree at most $2n-j$ in the second argument, whose coefficients are bounded when h is of the form of (A.1). By our assumptions, we find that

$$Q_0(x,\xi) = Q(x,\xi) + \sum_{j=1}^{2n} S_j(x,\xi)$$

is a symbol of order $2n$ and it satisfies

$$Q_0(x,\xi) \ge -K\ ,$$

for some $K = K(Q,H)$. Therefore, we have

$$(uh^2,Qu) = (uh,Q_0(x,-i\partial_x)uh) \ge -G(uh,uh)\ ,$$

for some $G = G(Q,H)$, by [Lang, 1962, Theorem 18.1.14-15]. This completes the proof of Lemma A.1.

We consider next an initial condition $u_0 \in L^\infty$ and establish a bound on $(u_t, u_t)_h$.

Proposition A.2. *There is a constant* $C_1 = C_1(a, R, H, \|u_0\|_\infty)$ *such that*

$$\int dx\, h^2(x) u_t^2(x) \le C_1 \,.$$

Proof. Consider

$$\begin{aligned} \tfrac{1}{2}\partial_t (u_t, u_t)_h &= (u_t, \partial_t u_t)_h \\ &= -(u_t, P u_t)_h - (u_t, u_t f(u_t^2))_h \,. \end{aligned}$$

By the lemma, this quantity is bounded by

$$\tfrac{1}{2}\partial_t (u_t, u_t)_h \le G (u_t, u_t)_h - (u_t, u_t f(u_t^2))_h \,.$$

Define now U such that $f(U) \ge G + 1$. Then, using the assumptions on f, we obtain

$$\begin{aligned} \tfrac{1}{2}\partial_t (u, u)_h &\le G \int h^2 u^2 - \int_{u^2 \ge U} h^2 u^2 (G+1) \\ &\le G \int_{u^2 \le U} h^2 u^2 - \int_{u^2 \ge U} h^2 u^2 \\ &\le (G+1) \int_{u^2 \le U} h^2 u^2 - \int h^2 u^2 \,. \end{aligned}$$

Thus, $\frac{1}{2}\partial_t (u, u)_h \le C_2 (G+1) U - (u, u)_h$, where $C_2 = \int h^2$. From this differential inequality it follows at once that

$$(u_t, u_t)_h \le \max(C_2 (G+1) U \,,\ (u_0, u_0)_h) \,.$$

Assuming for simplicity that $P(x, \xi)$ in (3.4) only depends on ξ, we next bound

$$\begin{aligned} \tfrac{1}{2}\partial_t (u', u')_h = &- (u', P u')_h - (u', f(u^2) u')_h \\ &- 2(u u', f'(u^2) u u')_h \,. \end{aligned}$$

Note that the terms containing f are non-positive. Fix any $A \in \mathbf{R}$. Then the identity

$$(u', (P - A) u')_h = (u, Q u)_h \,, \tag{A.4}$$

holds with

$$Q(x, \xi) = (\xi^2 - 2i \frac{h'}{h}(x) \xi)(P(\xi) - A) \,.$$

Remark. When P is of the general form of (3.4), one can still find a Q such that (A.4) holds. The definition of Q is more complicated, but the terms of highest order remain the same in all cases.

We now apply Lemma A.1 with Q as above and get

$$(u, Qu)_h \geq -C(u,u)_h \ .$$

Therefore the identity (A.4) implies

$$(u', Pu')_h - A(u', u')_h = (u, Qu) \geq -C(u,u)_h \ .$$

It follows that

$$\tfrac{1}{2}\partial_t(u', u')_h \leq -(u', Pu')_h \leq C(u,u)_h - A(u', u')_h \ .$$

We have already seen that $(u_t, u_t)_h$ is bounded, and hence it follows that $(u_t', u_t')_h$ is bounded as well.

We now have shown that $(u_t, u_t)_h$ and $(u_t', u_t')_h$ are bounded in t. From this the proof of the theorem is completed as follows. We have

$$\begin{aligned} h^2(x)|u^2|(x) \;&=\; \int_{-\infty}^{x} dy\, (|u^2|h^2)'(y) \\ &= \int_{-\infty}^{x} dy\, |u^2|' h^2 + 2\int_{-\infty}^{x} dy\, |u|^2 \frac{h'}{h} h^2 \\ &\leq 2(u,u)_h^{1/2}(u', u')_h^{1/2} + 2(u,u)_h \|\frac{h'}{h}\|_\infty < \infty \ . \end{aligned}$$

Thus, we find

$$|u(x)| \leq K h(x)^{-1} \;=\; K(1 + (x - x_0)^2) \ ,$$

where K is independent of x_0. Setting $x_0 = x$, we find

$$|u(x)| \leq K \ .$$

Acknowledgements. We thank the IHES for its kind hospitality. This work was partially supported by the Fonds National Suisse.

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