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Intermittency in the presence of noise

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Abstract. We analyse iterations of maps on an interval with an added noise term, in the neighbourhood of an intermittency threshold. We rigorously derive a universal scaling function for the laminar time expressed as a function of the distance from the threshold and the variance of the noise.

1. Introduction

Discrete dynamical systems, i.e. iterations of the form $x_{n+1} = f(x_n)$, are a subject of intense study, by mathematicians and physicists alike. The reasons for interest in these systems include the following.

(i) They model various continuous evolution equations of physics, e.g. Hamilton's equations, equations of hydrodynamics. (A connection between discrete dynamical systems and continuous evolution equations such as the Navier–Stokes equations can, at least partially, be established through the use of Poincaré sections (cf Abraham and Marsden 1978).)

(ii) They can exhibit apparent chaotic behaviour, which bears on questions of instability (e.g. hydrodynamic instabilities and (weak) turbulence in fluids).

(iii) They are amenable to numerical study.

Since some of the equations of physics, such as Navier–Stokes, are of a phenomenological nature, neglecting e.g. quantum mechanical effects and the size of molecules, doubts have been raised as to whether the beautiful structures such as strange attractors survive the addition of noise terms to the deterministic equations. Surprisingly, the answer is yes for Axiom A attractors (Kifer 1974). These attractors have a strong hyperbolic structure, i.e. nearby points evolve in exponentially diverging or converging orbits.

In this paper, we consider a general class of one-dimensional discrete dynamical systems, which have deterministic part depending on a parameter ε, and which are perturbed by a noise of mean zero and standard deviation proportional to a parameter σ, i.e.

$$x_{n+1} = f_\varepsilon(x_n) + \sigma\xi_n, \tag{1.1}$$

with the $\{\xi_n\}$ independent identically distributed random variables. In particular, we analyse the situation in which a stable and an unstable fixed point of the deterministic part collide as ε varies. The situation differs from the one considered by Kifer; here, the linearisation of the deterministic part has eigenvalue 1 so that the evolution of

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neighbouring points is governed by higher-order corrections. In this situation, we show that the noise does influence the behaviour of the system, and we give quantitative results (scaling law in ε, σ) for the invariant measure of the process.

To clarify the situation envisaged, consider the family of maps $f_\varepsilon : [-1, 1] \to [-1, 1]$ defined by

$$f_\varepsilon(x) = \begin{cases} g_\varepsilon(x) & \text{if } |g_\varepsilon(x)| \leqslant 1 \\ g_\varepsilon(x) - 2 & \text{if } |g_\varepsilon(x)| > 1 \end{cases} \tag{1.2}$$

and

$$g_\varepsilon(x) = x + \tfrac{7}{4}x^2 + \varepsilon.$$

For $\varepsilon = 0$, the graph of f_ε is illustrated below.

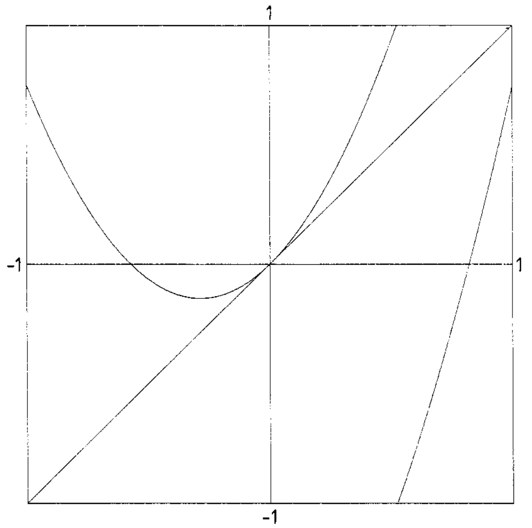

Figure 1. Graph of $f_\varepsilon(x)$, with $\varepsilon = 0$.

For $\varepsilon = 0$, $x = 0$ is a (one-sided) stable fixed point, and almost every initial point x_0 will be attracted to 0, i.e. $\lim_{n\to\infty} f_\varepsilon^n(x_0) = 0$ where $f_\varepsilon^n(x) = f_\varepsilon(f_\varepsilon^{n-1}(x))$, $n > 1$. When $\varepsilon < 0$ the curve in figure 1 moves downward and f_ε has a stable and an unstable fixed point close to zero. When $\varepsilon > 0$ an interesting situation arises for a typical initial point x_0, as can be seen from figure 2.

Near $x = 0$ the motion is calm, *laminar*, but away from it there are *bursts* of *turbulent behaviour.* This observation was systematically analysed by Pomeau and Manneville (1980), who also showed that similar behaviour can be traced to the same causes for the Lorenz system (1963), showing thus the relevance of the problem for more realistic dynamical equations. In addition they argued that the mean duration of the laminar phase is $O(\varepsilon^{-1/2})$, while the duration of the bursts is of mean length which is essentially independent of ε for ε near zero. In this paper, we study and solve the associated problem in which the deterministic evolution is perturbed by a noise term (cf equation (1.1)).

We fix a small interval $[-a, a]$ around $x = 0$ and call the motion in $[-a, a]$ laminar. The principal object of interest is the expected duration of the laminar phase,

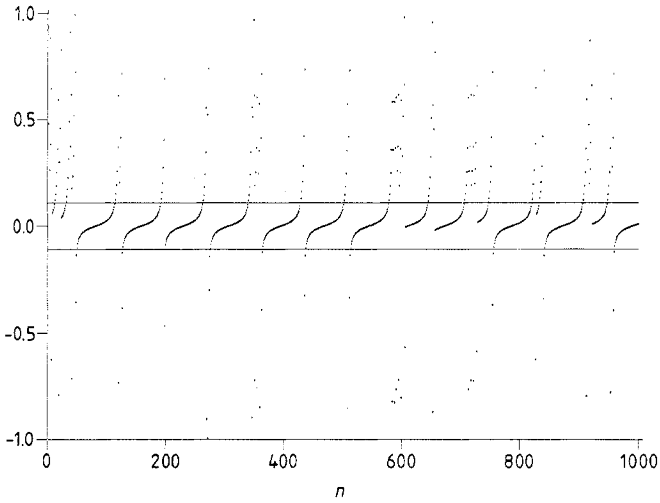

Figure 2. Successive laminar and burst motion for $\varepsilon \gtrsim 0$. The horizontal axis represents the number of iterations, the vertical axis represents the interval $[-1, 1]$. The interval $[-7/64, 7/64]$ is the laminar region.

$T(\varepsilon, \sigma, \kappa)$, for a 'noisy' map of the form $f_\varepsilon(x) = x + \kappa x^2 + \varepsilon + \mathrm{O}(x^3) + \sigma\xi$, $\kappa > 0$, in a neighbourhood of this interval. Our main result, holding for a typical family of maps, is the following (we discuss typicality below and in § 4).

Theorem 1.1. There are universal functions $\mathbb{T}_\pm$ such that for all typical families of maps as above and a sufficiently small, the laminar time satisfies

$$\lim_{\substack{\varepsilon,\sigma\to 0\\ \sigma/|\varepsilon|^{3/4}\text{ fixed}}} |\varepsilon|^{1/2} T(\varepsilon, \sigma, \kappa) = \text{constant } \mathbb{T}_\pm(\sigma'), \tag{1.3}$$

with $\sigma' = (\sigma/|\varepsilon|^{3/4}) \exp(\xi^2)^{1/2} \kappa^{1/4}$ ($\pm$ corresponding to sign ε). The functions $\mathbb{T}_\pm$ are given by

$$\mathbb{T}_\pm(\sigma') = \int_0^\infty \mathrm{d}s\, s^{-1/2} \exp\left[-\left(\frac{\sigma'^4 s^3}{48} \pm s\right)\right]. \tag{1.4}$$

We now briefly describe the main line of reasoning. (Since κ is held fixed, we omit the κ dependence in our formulae.) In § 2 we define $T(\varepsilon, \sigma)$ in terms of an integral, with integrand $\tau_a(x, \varepsilon, \sigma)\rho(x)$ where $\tau_a(x, \varepsilon, \sigma)$ is the expected escape time (number of iterations) from $[-a, a]$, assuming that the particle starts at $x \in [-a, a]$ at time zero, and where $\rho(x)\,\mathrm{d}x$, the reinjection density, is the probability to enter $[-a, a]$ in $[x, x+\mathrm{d}x]$ from the outside burst region. Since we condition the state space $[-1, 1]$ by identifying all points in the burst region, the resulting process is no longer Markovian; we find it convenient to regard the resulting process as a semi-Markov process, the relevant facts of which are outlined in the next section.

In order to obtain an explicit approximate expression for $\tau_a(x, \varepsilon, \sigma)$, it is at first sight natural to approximate the discrete time process equation (1.1) by a continuous time diffusion governed by the Langevin equation

$$\mathrm{d}x = (\kappa x^2 + \varepsilon)\,\mathrm{d}t + \tilde{\sigma}\,\mathrm{d}w. \tag{1.5}$$

Here, w is the standard Wiener process with $\tilde{\sigma}$ chosen appropriately so that the solution $x(t, x_0)$ to this equation, evaluated at integer times $t = 0, 1, 2, \ldots$, approximates the solution to equation (1.1). One could hope that $\tau_a(x, \varepsilon, \sigma)$ is given approximately by an analogously defined expected escape time $\tau^0(x, \varepsilon, \sigma)$ for equation (1.5). The time $\tau^0(x, \varepsilon, \sigma)$ itself is the solution to a second-order ordinary differential equation in x which can be solved by quadrature. Note, moreover, that under the transformation $x' = \kappa^{1/2}|\varepsilon|^{-1/2}x$, $t' = \kappa^{1/2}|\varepsilon|^{1/2}t$, $w' = \kappa^{1/4}|\varepsilon|^{1/4}w$ (recall $(\mathrm{d}w)^2 = \mathrm{d}t$), equation (1.5) becomes

$$\mathrm{d}x' = (x'^2 \pm 1)\,\mathrm{d}t' + \sigma'\,\mathrm{d}w',$$

with σ' as in equation (1.3); this observation provides an intuitive explanation for the $\sigma|\varepsilon|^{-3/4}$ scaling we find for $T(\varepsilon, \sigma)$.

The difficulty with the above reasoning would be to justify rigorously the replacement of equation (1.1) by (1.5), and so we do not proceed in this manner. Rather, in § 3, we obtain an integral equation for $\tau_a(x, \varepsilon, \sigma)$ and then show directly that τ^0 defined above, plus corrections of higher order in $\varepsilon \sim \sigma^{4/3}$, satisfies this integral equation; τ^0 therefore contains the leading singular behaviour of τ_a.

It should be noted that the critical behaviour of $\tau_a(x, \varepsilon, \sigma)$, $\varepsilon, \sigma \to 0$, accounts entirely for the critical behaviour of $T(\varepsilon, \sigma)$ since $\rho(x)$, the reinjection density, which also depends on ε, σ, is assumed to be a non-atomic measure density, continuous in ε, σ at $\varepsilon, \sigma = 0$. It is this assumption that defines 'typical'. Section 4 contains a discussion of this assumption.

A final section (§ 5) is devoted to the description of a numerical test of our theory for the particular family of maps f_ε defined in equation (1.2), with the ξ_n uniformly dsitributed in $[-\frac{1}{2}, \frac{1}{2}]$.

2. Invariant measure for the dynamical system

As described in the Introduction, the state space for the process we consider consists of the laminar region, i.e. an interval $[-a, a]$ around the contact point, and the burst region, i.e. the complement of $[-a, a]$ in which all points are identified. (The points of the burst region are identified in order to achieve some universality of our statements—we are not interested in the detailed behaviour of the process in this region.) In the following, x denotes a point in $[-a, a]$, b the burst region. The resulting conditioned dynamics are modelled as a semi-Markov process (Feller 1964), one in which the sojourn time in a state depends on the next state to which the particle jumps.

Let $F_{xb}(t)$ be the probability that a sojourn in the laminar region ends with a jump to the burst region before $t+s$, given that the laminar region was entered at x at time s, s arbitrary. Similarly let $F_{bx}(t)\,\mathrm{d}x$ be the probability of a burst ending before $t+s$ with a jump to $[x, x+\mathrm{d}x]$, assuming that the burst started at time s, s arbitrary. See figure 3 below. (In the next section we derive models for F_{xb}, F_{bx}.)
The principal task of this section is to derive an integral equation for the invariant measure ν of this process. In fact this has been done already (see Feller 1964), but for the convenience of the reader we include the derivation here.

To simplify notation, we consider a problem with discrete states, $i, j, \ldots$ instead of the continuum of states $\{x\}$ and b; at the end of the section we write out the integral equation for the case at hand. Let $F_{ij}(t)$, $i \neq j$, be the probability that a sojourn at state i ends by a jump to j before time $t+s$, assuming i was entered at time s, s arbitrary. Let

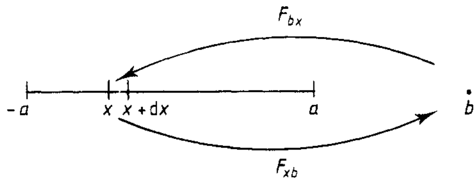

Figure 3. The semi-Markov process.

$P_{ij}(t)$ be the probability of being at j at time $t+s$, assuming that i was entered at time s, s arbitrary. The *backward* equation for $P_{ij}(t)$ is given by

$$P_{ij}(t)=\delta_{ij}\Big(1-\sum_{\substack{k\\k\neq i}} F_{ik}(t)\Big)+\sum_{\substack{k\\k\neq i}}\int_0^t \mathrm{d}F_{ik}(s)P_{kj}(t-s). \tag{2.1}$$

(Note that discrete time processes as well as continuous time processes are included in this analysis.)

An integral equation for the invariant measure ν can be extracted from the corresponding *forward equation.* It is, however, not immediate to obtain the forward equation directly from equation (2.1); rather, we proceed through Laplace transforms. Set

$$\pi_{ij}(\lambda)=\int_0^\infty \mathrm{e}^{-\lambda t}P_{ij}(t)\,\mathrm{d}t, \tag{2.2}$$

$$\Phi_{ij}(\lambda)=\int_0^\infty \mathrm{e}^{-\lambda t}\,\mathrm{d}F_{ij}(t), \tag{2.3}$$

$$\Gamma_{ij}(\lambda)=\int_0^\infty \mathrm{e}^{-\lambda t}\delta_{ij}\Big(1-\sum_{\substack{k\\k\neq i}} F_{ik}(t)\Big)\,\mathrm{d}t=\frac{1}{\lambda}\delta_{ij}\Big(1-\sum_{\substack{k\\k\neq i}}\Phi_{ik}(\lambda)\Big). \tag{2.4}$$

Then equation (2.1) becomes, using matrix notation,

$$\pi(\lambda)=\Gamma(\lambda)+\Phi(\lambda)\pi(\lambda), \tag{2.5}$$

from which it follows that

$$\pi(\lambda)=\Gamma(\lambda)+\pi(\lambda)(1/\Gamma(\lambda))\Phi(\lambda)\Gamma(\lambda). \tag{2.6}$$

This latter equation is the Laplace transform of the forward equation.

The invariant measure ν is given by the Abelian limit,

$$\nu_j=\lim_{t\to\infty}P_{ij}(t)=\lim_{\lambda\to 0}\lambda\int_0^\infty \mathrm{e}^{-\lambda t}P_{ij}(t)\,\mathrm{d}t=\lim_{\lambda\to 0}\lambda\pi_{ij}(\lambda). \tag{2.7}$$

Applying this relation to equation (2.6), we obtain the desired integral equation for ν,

$$\nu_j=\sum_{k\neq j}\nu_k\frac{1}{\tau_k}\rho_{kj}\tau_j, \tag{2.8}$$

where

$$\tau_i=\Gamma_{ii}(0)=\int_0^\infty\Big(1-\sum_{k\neq i}F_{ik}(t)\Big)\,\mathrm{d}t=\int_0^\infty t\sum_{k\neq i}\mathrm{d}F_{ik}(t) \tag{2.9}$$

is the expected sojourn time at i, and

$$\rho_{ij} = \Phi_{ij}(0) = \int_0^\infty \mathrm{d}F_{ij}(t) = F_{ij}(\infty) \tag{2.10}$$

is the probability that a sojourn at i terminates by a jump to j.

Returning to the problem at hand and suppressing the ε, σ dependence, we have that $\tau_a(x)$, the expected time to escape from the laminar region, is given by

$$\tau_a(x) = \int_0^\infty t\,\mathrm{d}F_{xb}(t), \tag{2.11}$$

whereas the expected sojourn time τ_b in the burst region is

$$\tau_b = \int_{-a}^{a} \mathrm{d}x \int_0^\infty t\,\mathrm{d}F_{bx}(t). \tag{2.12}$$

The re-entry density $\rho(x)\,\mathrm{d}x$, i.e. the probability density for the burst ending with a jump to $[x, x+\mathrm{d}x]$, is

$$\rho(x) = \int_0^\infty \mathrm{d}F_{bx}(t) = F_{bx}(\infty). \tag{2.13}$$

Combining these quantities with the continuous analogue of equation (2.8), we obtain the equation for the invariant measure,

$$\nu(x) = \frac{\tau_a(x)\rho(x)}{\tau_b}\nu_b, \qquad \nu_b = \tau_b \int_{-a}^{a} \frac{\nu(x)}{\tau_a(x)}\,\mathrm{d}x. \tag{2.14}$$

In particular, the ratio R of the laminar time to the burst time is

$$R = \frac{\nu_{[-a,a]}}{\nu_b} = \tau_b^{-1} \int_{-a}^{a} \tau_a(x)\rho(x)\,\mathrm{d}x. \tag{2.15}$$

It is this quantity which we study numerically (§ 5). The behaviour of the integral in equation (2.15), $\varepsilon \sim \sigma^{4/3} \to 0$, is precisely the content of theorem 1.1.

3. Expected laminar time

Consider the discrete process

$$x_{n+1} = x_n + \kappa x_n^2 + \varepsilon + \sigma\xi_n, \tag{3.1}$$

where the $\{\xi_n\}$ are mean zero, bounded, independent, identically distributed random variables with common density $\mathrm{d}\mu$. Denote the solution to this equation starting at x_0 by $x_n(x_0, \omega)$ where ω is a sample path. Let $\eta_c(x_0, \omega)$ be the escape time, starting at x_0, defined by

$$\eta_c(x_0, \omega) = \min\{n | x_n(x_0, \omega) > c\}. \tag{3.2}$$

The expected value of η_c, which we write as $\tau_c(x) \equiv \tau_c(x, \varepsilon, \sigma)$, is given by

$$\tau_c(x) = \sum_{n=1}^{\infty} nP(\eta_c(x) = n) = \sum_{n=0}^{\infty} P(\eta_c(x) > n). \tag{3.3}$$

But

$$P(\eta_c(x)>n)=S_c^n 1(x), \tag{3.4}$$

where 1 is the function which is identically 1 and S_c is the integral operator, acting on continuous functions defined on $(-\infty, c]$, defined by

$$S_c f(x)=\int_{x+\kappa x^2+\varepsilon+\sigma\xi\leq c} f(x+\kappa x^2+\varepsilon+\sigma\xi)\,\mathrm{d}\mu(\xi). \tag{3.5}$$

Combining equations (3.3) and (3.4), we obtain the integral equation

$$(\mathbb{1}-S_c)\tau_c(x)=1, \qquad x\leq c, \tag{3.6}$$

for $\tau_c(x)$. Equation (3.4) can be used further to show for example that

$$\operatorname{var}\eta_c(x)=(\mathbb{1}+S_c)(\mathbb{1}-S_c)^{-1}\tau_c(x)-(\tau_c(x))^2. \tag{3.7}$$

The *expected laminar time* starting at $x\in[-a, a]$ is $\tau_a(x)$. (Because of the deterministic part of equation (3.1), escapes through the left boundary of the laminar interval are impossible for ε, σ sufficiently small.) The principal result of this section is to give an approximation to $\tau_a(x)$, i.e. an approximate solution to equation (3.6) with $c=a$.

Theorem 3.1. Let $\sigma/|\varepsilon|^{3/4}$ be held constant. Then

$$\tau_a(x,\varepsilon,\sigma)=\tau^0(x,\varepsilon,\sigma)+\mathrm{O}(|\varepsilon|^{-2/5}) \tag{3.8}$$

where

$$\tau^0(x,\varepsilon,\sigma)=\frac{2}{\tilde{\sigma}^2}\int_x^\infty \mathrm{d}t\,\mathrm{e}^{-h(t)}\int_{-\infty}^t \mathrm{d}s\,\mathrm{e}^{h(s)}, \tag{3.9}$$

with $\tilde{\sigma}^2=\sigma^2\exp(\xi^2)$ and

$$h(x)=h_{\varepsilon,\tilde{\sigma}}(x)=(2/\tilde{\sigma}^2)(\kappa x^3/3+\varepsilon x). \tag{3.10}$$

The quantity $\tau^0(x,\varepsilon,\sigma)$ is of order $|\varepsilon|^{-1/2}$ and hence gives the leading-order singular behaviour of $\tau_a(x,\varepsilon,\sigma)$ for $\varepsilon\to 0$.

Remarks. As discussed in the Introduction, τ^0 is the expected time to escape (actually to $+\infty$) for a particle diffusing according to the continuous time Langevin equation (1.5). If $\eta^0(x, w)$ denotes the escape time, starting at x, the probability that $\eta^0(x,\cdot)$ exceeds t is given by

$$P(\eta^0(x,\cdot)>t)=\mathrm{e}^{Gt}1(x), \tag{3.11}$$

where G is the differential operator

$$G=\tfrac{1}{2}\tilde{\sigma}^2\,\mathrm{d}^2/\mathrm{d}x^2+(\kappa x^2+\varepsilon)\,\mathrm{d}/\mathrm{d}x \tag{3.12}$$

with the Dirichlet boundary condition imposed at $+\infty$. Using equation (3.11) one finds that $\tau^0(x)=\exp(\eta^0(x,\cdot))$ satisfies

$$G\tau^0(x)=-1; \tag{3.13}$$

cf equations (3.3) and (3.6). Equation (3.13) is easily integrated to yield (3.9) (cf Feller 1966).

Proof. Set $\delta=\frac{1}{10}$ and define $c(\varepsilon)=|\varepsilon|^{\delta}$. For $x\in[-a,-c(\varepsilon)]$, the expected time to reach $-c(\varepsilon)$ starting at x can be estimated from the deterministic part of equation (3.1), since for x in this interval the $\sigma\xi$ part is negligible with respect to the deterministic part. This time is certainly bounded by $x+c(\varepsilon)$ divided by the minimal step which is $O(|\varepsilon|^{2\delta})$, from which one finds the estimate $0<\tau_a(x)-\tau_a[-c(\varepsilon)]=O(|\varepsilon|^{-2\delta})$. Similarly, for $x\in[c(\varepsilon),a]$, one finds $0\leqslant\tau_a(x)\leqslant\tau_a[c(\varepsilon)]=O(|\varepsilon|^{-2\delta})$. As can be seen from detailed estimates on τ^0 which we give below, the above inequalities hold with τ_a replaced by τ^0. Thus for $|x|>c(\varepsilon)$,

$$\begin{aligned}|\tau_a(x)-\tau^0(x)|&\\ &\leqslant|\tau_a[\pm c(\varepsilon)]-\tau^0[\pm c(\varepsilon)]|+|\tau_a(x)-\tau_a[\pm c(\varepsilon)]|+|\tau^0(x)-\tau^0[\pm c(\varepsilon)]|\\ &=|\tau_a[\pm c(\varepsilon)]-\tau^0[\pm c(\varepsilon)]|+O(|\varepsilon|^{-2\delta});\end{aligned}$$

hence it suffices to prove equation (3.8) for $|x|\leqslant c(\varepsilon)$. Moreover $\tau_a(x)=\tau_{c(\varepsilon)}(x)+\tau_a[c(\varepsilon)]+O(1)=\tau_{c(\varepsilon)}(x)+O(|\varepsilon|^{-2\delta})$. Thus, it suffices to prove $|\tau_{c(\varepsilon)}(x)-\tau^0(x)|=O(|\varepsilon|^{-2/5})$ for $|x|\leqslant c(\varepsilon)$, where, again, $\tau_{c(\varepsilon)}$ is the solution to equation (3.6) with $c=c(\varepsilon)$.

Recall $c(\varepsilon)=|\varepsilon|^{1/10}$. Define $I_0(\varepsilon)=\{x\,|\,|x|<2|\varepsilon|^{2/5}\}$, $I_{\rm R}(\varepsilon)=\{x\,|\,|\varepsilon|^{2/5}<x\leqslant c(\varepsilon)\}$, $I_{\rm L}(\varepsilon)=\{x\,|-c(\varepsilon)\leqslant x<-|\varepsilon|^{2/5}\}$. The proof proceeds by considering $\tau_{c(\varepsilon)}(x)$, $\tau^0(x)$ for x in each of these intervals.

Lemma 3.2. For all $x\in\mathbb{R}$, $\tau^0(x)$ satisfies the estimates

$$d^n(\tau^0(x))/dx^n=O(|\varepsilon|^{-(n+1)/2}),\qquad n=0,1,2,3.\tag{3.14}$$

Proof. By means of the substitution $x'=|\varepsilon|^{-1/2}x$ one obtains from equations (3.9) and (3.10) that $\tau^0(x,\varepsilon,\sigma)=|\varepsilon|^{-1/2}\tau^0(|\varepsilon|^{-1/2}x,\pm1,\sigma|\varepsilon|^{-3/4})$; hence it suffices to show that the first three derivatives of $\tau^0(x',\pm1,\sigma|\varepsilon|^{-3/4})$ are uniformly bounded in x', $-\infty<x'<\infty$. We do not give the proof in its entirety but rather illustrate the technique involved for the particular case of the first derivative of

$$\tau^0(x')=\frac{2|\varepsilon|^{3/2}}{\tilde\sigma^2}\int_{x'}^{\infty}dt\,e^{-h(t)}\int_{-\infty}^{t}ds\,e^{h(s)},\tag{3.15}$$

with

$$h(t)=h_{\pm1,\tilde\sigma|\varepsilon|^{-3/4}}(t)=(2|\varepsilon|^{3/2}/\tilde\sigma^2)(\kappa t^3/3\pm t)\tag{3.16}$$

and $x'\to+\infty$. We have that

$$\begin{aligned}\frac{d\tau^0(x')}{dx'}&=-\frac{2|\varepsilon|^{3/2}}{\tilde\sigma^2}\exp[-h(x')]\int_{-\infty}^{x'}dt\,e^{h(t)}\\ &=-\frac{2|\varepsilon|^{3/2}}{\tilde\sigma^2}\exp[-h(x')]\left(\int_{-\infty}^{x_0'}dt\,e^{h(t)}+\int_{x_0'}^{x'-1}e^{h(t)}\,dt+\int_{x'-1}^{x'}dt\,e^{h(t)}\right),\end{aligned}\tag{3.17}$$

with $x_0'>0$ fixed so that $h'(t)\geqslant1$, for $t\geqslant x_0'$.

The second integral on the RHS of this equation can be estimated by

$$\int_{x_0'}^{x'-1}dt\,e^{h(t)}\leqslant\int_{x_0'}^{x'-1}dt\,h'(t)\,e^{h(t)}=\exp[h(x'-1)]-\exp[h(x_0')].\tag{3.18}$$

The third integral on the RHS of equation (3.17) can be estimated using integration by parts,

$$\int_{x'-1}^{x'} \mathrm{d}t \frac{h'(t)}{h'(t)} \mathrm{e}^{h(t)} = \frac{\mathrm{e}^{h(t)}}{h'(t)}\bigg|_{x'-1}^{x'} + \int_{x'-1}^{x'} \mathrm{d}t\, h''(t) h'^{-2}(t)\, \mathrm{e}^{h(t)}$$

$$< \frac{\mathrm{e}^{h(t)}}{h'(t)}\bigg|_{x'-1}^{x'} - \exp[h(x')] \frac{1}{h'(t)}\bigg|_{x'-1}^{x'}. \tag{3.19}$$

Note that $\exp[h(t)] = \exp[\mathrm{O}(t^3)]$. Therefore inequalities (3.18), (3.19), combined with (3.17) show not only that $\mathrm{d}\tau^0(x')/\mathrm{d}x'$ is bounded as $x' \to \infty$, but that it behaves like $h'(x')^{-1}$. The other cases are handled by the same methods.

Lemma 3.3. For $x \in I_{\mathrm{R}}(\varepsilon)$, and $n \geq 2$ one has

$$\tau^0(x) = \frac{1}{\kappa x} + |\varepsilon|^{-1/2}\left(\alpha_2 \frac{|\varepsilon|}{x^2} + \ldots + \alpha_n \frac{|\varepsilon|^{n/2}}{x^n}\right) + \mathrm{O}(|\varepsilon|^{(n+1)/10-1/2}), \tag{3.20}$$

while for $x \in I_{\mathrm{L}}(\varepsilon)$,

$$\tau^0(x) = \tau^0(-\infty) + \frac{1}{\kappa x} + |\varepsilon|^{-1/2}\left(\alpha_2 \frac{|\varepsilon|}{x^2} + \ldots + \alpha_n \frac{|\varepsilon|^{n/2}}{x^n}\right) + \mathrm{O}(|\varepsilon|^{(n+1)/10-1/2}), \tag{3.21}$$

where the α_i are independent of ε.

Proof. Again one uses the scaling $\tau^0(x, \varepsilon, \sigma) = |\varepsilon|^{-1/2}\tau^0(x', \pm 1, \sigma|\varepsilon|^{-3/4})$, $x' = |\varepsilon|^{-1/2}x$. For $x' \to -\infty$ and using integration by parts, we have

$$\frac{\mathrm{d}}{\mathrm{d}x'}\tau^0(x', \pm 1, \sigma|\varepsilon|^{-3/4})$$

$$= -\frac{2|\varepsilon|^{3/2}}{\tilde{\sigma}^2} \exp[-h(x')] \int_{-\infty}^{x'} \mathrm{d}t\, h'(t)^{-1}(h'(t)\, \mathrm{e}^{h(t)})$$

$$= -\frac{2|\varepsilon|^{3/2}}{\tilde{\sigma}^2} \exp[-h(x')]\left(\frac{\exp[h(x')]}{h'(x')} + \int_{-\infty}^{x'} \mathrm{d}t \frac{h''(t)}{(h'(t))^2}(h'(t)\, \mathrm{e}^{h(t)})\right).$$

Integrating by parts repeatedly and expanding inverse powers of $h'(x)$ appropriately generates equation (3.21). The case $x' \to +\infty$ can be treated similarly after breaking up the integrals as in equation (3.17).

The two preceding lemmas are the main bounds required for our estimate of $(\mathbb{1} - S_{c(\varepsilon)})\tau^0$ away from the right endpoint $c(\varepsilon)$.

Lemma 3.4. For $x \in [-c(\varepsilon), c(\varepsilon) - 2|\varepsilon|^{2\delta}]$, one has

$$(\mathbb{1} - S_{c(\varepsilon)})\tau^0(x, \varepsilon, \sigma) = 1 + \mathrm{O}(|\varepsilon|^{1/10}). \tag{3.22}$$

Proof. When $x \in I_0(\varepsilon)$, the ξ integration in the definition of $S_{c(\varepsilon)}$ is unrestricted. Thus

$$(\mathbb{1} - S_{c(\varepsilon)})\tau^0(x)$$

$$= \int \mathrm{d}\mu\ (\xi)[\tau^0(x) - \tau^0(x + \kappa x^2 + \varepsilon + \sigma\xi)]$$

$$= -\int \mathrm{d}\mu\ (\xi)\Big((\kappa x^2+\varepsilon+\sigma\xi)\frac{\mathrm{d}\tau^0}{\mathrm{d}x}(x)+\tfrac{1}{2}(\kappa x^2+\varepsilon+\sigma\xi)^2\frac{\mathrm{d}^2\tau^0}{\mathrm{d}x^2}(x)$$

$$+\mathrm{O}(\sigma^3)\mathrm{O}(|\varepsilon|^{-2})\Big)$$

$$= 1+\mathrm{O}(|\varepsilon|^{1/10}) \tag{3.23}$$

by the properties of μ, equation (3.12), and the bound $\kappa x^2+\varepsilon+\sigma\xi=\mathrm{O}(\sigma)$, valid for $x\in I_0(\varepsilon)$.

When $x\in I_\mathrm{R}(\varepsilon)\backslash[c(\varepsilon)-2|\varepsilon|^{2\delta}, c(\varepsilon)]$ or $x\in I_\mathrm{L}(\varepsilon)$, then we consider $(\mathbb{1}-S_{c(\varepsilon)})$ applied to each term in the asymptotic expansion (3.20) or (3.21). For example,

$$(\mathbb{1}-S_{c(\varepsilon)})\frac{1}{\kappa x}(x)=\int \mathrm{d}\mu\ (\xi)\Big(\frac{1}{\kappa x}-\frac{1}{\kappa(x+\kappa x^2+\varepsilon+\sigma\xi)}\Big)$$

$$=\frac{1}{\kappa x}\int \mathrm{d}\mu\ (\xi)\Big[\frac{\kappa x^2+\varepsilon+\sigma\xi}{x}-\sum_{p=2}^{\infty}\Big(-\frac{\kappa x^2+\varepsilon+\sigma\xi}{x}\Big)^p\Big]$$

$$=1+\mathrm{O}(|\varepsilon|^{1/10}) \tag{3.24}$$

(use $\int \mathrm{d}\mu\ (\xi)\xi=0$).

Similarly one can show that for $m\geqslant 2$

$$|\varepsilon|^{-1/2}(\mathbb{1}-S_{c(\varepsilon)})\Big(\frac{|\varepsilon|^{1/2}}{x}\Big)^m(x)=\mathrm{O}(|\varepsilon|^{(m-1)/10}). \tag{3.25}$$

Taking n suitably large ($n\geqslant 5$) so that $\mathbb{1}-S_{c(\varepsilon)}$ applied to the remainders of (3.20) and (3.21) is $\mathrm{O}(|\varepsilon|^{1/10})$ or smaller, and applying the estimates (3.24) and (3.25), we obtain the result.

At the right endpoint $c(\varepsilon)$, the operator $S_{c(\varepsilon)}$ introduces Dirichlet boundary conditions, because of the restriction in the integration, but G has Dirichlet boundary conditions at $x=+\infty$. Therefore we cannot expect a bound as good as lemma 3.4, near $c(\varepsilon)$. However we shall make use later of the fact that this larger error k_ε has small support and contributes an error $\mathrm{O}(|\varepsilon|^{-\delta})$ to our estimate of $\tau_{c(\varepsilon)}(x)$.

Let $k_\varepsilon(x)$ be the function defined by

$$k_\varepsilon(x)=\begin{cases}(\mathbb{1}-S_{c(\varepsilon)})\tau^0(x)-1, & x\in[c(\varepsilon)-2|\varepsilon|^{2\delta}, c(\varepsilon)],\\ 0 & \text{otherwise.}\end{cases} \tag{3.26}$$

Lemma 3.5. The function $k_\varepsilon(x)$ is $\mathrm{O}(|\varepsilon|^{-\delta})$.

Proof. Since τ^0 is $\mathrm{O}(\varepsilon^{-\delta})$ in the interval $[c(\varepsilon)-2|\varepsilon|^{2\delta}, c(\varepsilon)]$ by lemma 3.3, and $S_{c(\varepsilon)}$ is of norm 1, the assertion follows.

Lemmas (3.4) and (3.5) can be summarised as follows.

Corollary 3.6. For $x\in[-c(\varepsilon), c(\varepsilon)]$, $\delta=\frac{1}{10}$, $c(\varepsilon)=|\varepsilon|^{1/10}$, one has

$$(\mathbb{1}-S_{c(\varepsilon)})\tau^0(x)=1+\mathrm{O}(|\varepsilon|^{1/10})+k_\varepsilon(x), \tag{3.27}$$

with $k_\varepsilon(x)=\mathrm{O}(|\varepsilon|^{-\delta})$ and supported in $[c(\varepsilon)-2|\varepsilon|^{2\delta}, c(\varepsilon)]$.

To conclude the proof of theorem 3.1, we need the following additional lemma.

Lemma 3.7. Let $u_\varepsilon(x)$ be the solution to the equation

$$(\mathbb{1}-S_{c(\varepsilon)})u_\varepsilon(x)=v_\varepsilon(x), \tag{3.28}$$

with $v_\varepsilon(x)=\mathrm{O}(\varepsilon^\gamma)$ (γ arbitrary of either sign) and supported in $[c(\varepsilon)-2|\varepsilon|^{2\delta}, c(\varepsilon)]$. Then $u_\varepsilon(x)=\mathrm{O}(\varepsilon^\gamma)$ as well.

Proof. Assume first $0\leqslant v_\varepsilon(x)\leqslant 1$. Then

$$\tau_{c(\varepsilon)-2|\varepsilon|^{2\delta}}(x)=\sum_n S^n_{c(\varepsilon)-2|\varepsilon|^{2\delta}}1(x)=\sum_n S^n_{c(\varepsilon)-2|\varepsilon|^{2\delta}}(1-v_\varepsilon)(x)$$

$$\leqslant\sum_n S^n_{c(\varepsilon)}(1-v_\varepsilon)(x)\leqslant\sum_n S^n_{c(\varepsilon)}1(x)=\tau_{c(\varepsilon)}(x),$$

which implies that

$$0\leqslant u_\varepsilon(x)=\sum_n S^n_{c(\varepsilon)}v_\varepsilon(x)\leqslant\tau_{c(\varepsilon)}(x)-\tau_{c(\varepsilon)-2|\varepsilon|^{2\delta}}(x)=\mathrm{O}(1)$$

since, again, the time required to cross from $c(\varepsilon)-2|\varepsilon|^{2\delta}$ to $c(\varepsilon)$ can be estimated deterministically. The assertion of the lemma follows from the linearity of this inequality.

We can now complete the proof of theorem 3.1. Corollary 3.6 implies that

$$(\mathbb{1}-S_{c(\varepsilon)})(\tau^0-\tau_{c(\varepsilon)})(x)=\mathrm{O}(|\varepsilon|^{1/10})+k_\varepsilon(x), \tag{3.29}$$

i.e.

$$\tau^0(x)-\tau_{c(\varepsilon)}(x)=(\mathbb{1}-S_{c(\varepsilon)})^{-1}\mathrm{O}(|\varepsilon|^{1/10})+(\mathbb{1}-S_{c(\varepsilon)})^{-1}k_\varepsilon(x). \tag{3.30}$$

But

$$|(\mathbb{1}-S_{c(\varepsilon)})^{-1}\mathrm{O}(|\varepsilon|^{1/10})|=\left|\sum_n S^n_{c(\varepsilon)}\mathrm{O}(|\varepsilon|^{1/10})\right|$$

$$\leqslant\mathrm{O}(|\varepsilon|^{1/10})\sum_n S^n_{c(\varepsilon)}1(x)$$

$$=\mathrm{O}(|\varepsilon|^{1/10})\tau_{c(\varepsilon)}(x) \tag{3.31}$$

and

$$(\mathbb{1}-S_{c(\varepsilon)})^{-1}k_\varepsilon(x)=\mathrm{O}(|\varepsilon|^{-\delta}) \tag{3.32}$$

by corollary 3.6 and lemma 3.7. Equations (3.30), (3.31) and (3.32) and the fact that τ^0 itself is $\mathrm{O}(|\varepsilon|^{-1/2})$ imply that $\tau_{c(\varepsilon)}-\tau^0$ is $\mathrm{O}(|\varepsilon|^{-1/2+1/10})=\mathrm{O}(|\varepsilon|^{-2/5})$. This concludes the proof of the theorem.

Remark. Theorem 3.1 also holds if $\mathrm{O}(x^3)$ terms are included in the stochastic difference equation, equation (3.1) and the definition of S_c, equation (3.5). For simplicity of the exposition, we have omitted these terms, although this strengthened form of the theorem is implicitly assumed in the proof of theorem 1.1 below.

We can now complete the proof of the theorem 1.1 in the Introduction. Keeping $\sigma|\varepsilon|^{-3/4}$ fixed and assuming ρ has no atom at the origin, we have with $\tau_a(x)=\tau_a(x, \varepsilon, \sigma)$ that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}|\varepsilon|^{1/2}T(\varepsilon,\sigma)\\
&\quad=\lim_{\varepsilon\to 0}|\varepsilon|^{1/2}\int_{-a}^{a}\mathrm{d}x\,\rho(x)\tau_a(x)\\
&\quad=\lim_{\varepsilon\to 0}|\varepsilon|^{1/2}\tau_a(-a,\varepsilon,\sigma)\int_{-a}^{0}\mathrm{d}x\,\rho(x)\\
&\quad\quad+\lim_{\varepsilon\to 0}|\varepsilon|^{1/2}\biggl(\int_{-a}^{-c(\varepsilon)}\mathrm{d}x\,\rho(x)(\tau_a(x)-\tau_a(-a))+\int_{-c(\varepsilon)}^{0}\mathrm{d}x\,\rho(x)(\tau_a(x)-\tau_a(-a))\\
&\quad\quad+\int_{0}^{c(\varepsilon)}\mathrm{d}x\,\rho(x)\tau_a(x)+\int_{c(\varepsilon)}^{a}\mathrm{d}x\,\rho(x)\tau_a(x)\biggr),
\end{aligned}\tag{3.33}$$

where $c(\varepsilon)=|\varepsilon|^{1/10}$.

By theorem 3.1, $\tau_a(x)=\tau^0(x)+\mathrm{O}(\varepsilon^{-2/5})$ so that τ_a may be replaced by τ^0 in the above expression. Making this replacement, and using the scaling $\tau^0(x,\varepsilon,\sigma)=|\varepsilon|^{-1/2}\tau^0(|\varepsilon|^{-1/2}x, \pm 1, \sigma|\varepsilon|^{-3/4})$, we have that the first limit on the RHS of (3.33) is simply

$$\tau^0(-\infty,\pm 1,\sigma|\varepsilon|^{-3/4})\int_{-a}^{0}\mathrm{d}x\,\rho(x).\tag{3.34}$$

For $x\leqslant -c(\varepsilon)$ we have that $\tau^0(x)-\tau^0(-a)=\mathrm{O}(|\varepsilon|^{-1/5})$, and for $x\geqslant c(\varepsilon)$, $\tau^0(x)=\mathrm{O}(|\varepsilon|^{-1/5})$; from this and the replacement of τ_a by τ^0, it follows that

$$\lim_{\varepsilon\to 0}|\varepsilon|^{1/2}\biggl(\int_{-a}^{-c(\varepsilon)}\mathrm{d}x\,\rho(x)(\tau_a(x)-\tau_a(-a))+\int_{c(\varepsilon)}^{a}\mathrm{d}x\,\rho(x)\tau_a(x)\biggr)=0.\tag{3.35}$$

Finally, again replacing τ_a by τ^0, we have

$$\lim_{\varepsilon\to 0}|\varepsilon|^{1/2}\biggl(\int_{-c(\varepsilon)}^{0}\mathrm{d}x\,\rho(x)[\tau_a(x)-\tau_a(-a)]+\int_{0}^{c(\varepsilon)}\mathrm{d}x\,\rho(x)\tau_a(x)\biggr)=0\tag{3.36}$$

by the bounded convergence theorem. Combining these results, we obtain the theorem

$$\lim_{\varepsilon\to 0}|\varepsilon|^{1/2}T(\varepsilon,\sigma)=\biggl(\int_{-\infty}^{0}\mathrm{d}x\,\rho(x)\biggr)\tau^0(-\infty,\pm 1,\sigma|\varepsilon|^{-3/4}).\tag{3.37}$$

The double integral for $\tau^0(x)$ can be reduced to a single integral when $x=-\infty$. Let $u=s+t$, $v=t-s$. Then

$$\begin{aligned}
&\tau^0(-\infty,\pm 1,\sigma|\varepsilon|^{-3/4})\\
&\quad=\frac{2}{\sigma'^2}\kappa^{-1/2}\int_{-\infty}^{\infty}\mathrm{d}t\,\exp\biggl[-\frac{2}{\sigma'^2}\Bigl(\frac{t^3}{3}\pm t\Bigr)\biggr]\int_{-\infty}^{t}\mathrm{d}s\,\exp\biggl[\frac{2}{\sigma'^2}\Bigl(\frac{s^3}{3}\pm s\Bigr)\biggr]\\
&\quad=\sigma'^{-2}\kappa^{-1/2}\iint_{v\geqslant 0}\mathrm{d}u\,\mathrm{d}v\,\exp\biggl[\frac{-2}{\sigma'^2}\Bigl(\frac{v^3}{12}+\frac{u^2v}{4}\pm v\Bigr)\biggr]\\
&\quad=\pi^{1/2}\kappa^{-1/2}\int_{0}^{\infty}\mathrm{d}v\,v^{-1/2}\exp\biggl[-\Bigl(\frac{\sigma'^4v^3}{48}\pm v\Bigr)\biggr].
\end{aligned}\tag{3.38}$$

This observation accounts for the form of the integral, equation (1.4).

4. The reinjection

When the orbit leaves the laminar region, i.e. when $x_n \in [-a, a]$ but $x_{n+1} \notin [-a, a]$, then a burst starts, and the point will wander (as a function of n) in the complement of $[-a, a]$, until it re-enters the laminar region again. It is impossible to describe in full generality the motion of the burst, and we are forced to make some plausibility assumptions about that motion. Otherwise, no prediction about the reinjection density could be made. As we have said before, the only thing we need to know to obtain scaling is the non-atomicity of $\rho(x)$ at the origin. But even this we cannot prove.

Consider the function $f = f_\varepsilon$ in a neighbourhood of the contact point. We assume for simplicity that it is locally of the form shown in figure 4,

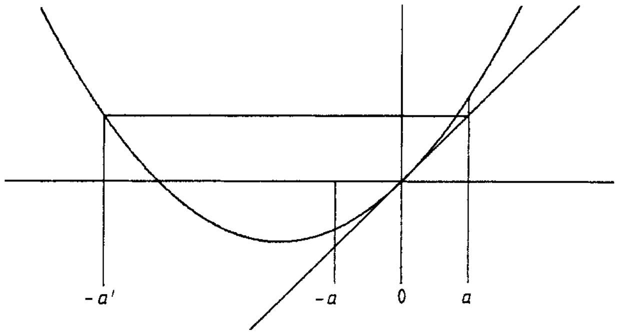

Figure 4. Assumed local picture near the contact point.

i.e. it is essentially quadratic down to a point $-a'$ such that $f(-a') = a$. The map has then a critical point in $[-a', a]$ and we assume it has none outside. Any orbit which enters $[-a', a]$ can only leave this interval at a. In particular, after the critical point has been encountered, a laminar period starts.

We now discuss the motion in the complement of $[-a', a]$. Since f has no other critical point in the complement of $[-a', a]$ and if f has negative Schwarzian derivative (Collet and Eckmann 1980, II.4), f is expanding in the complement of $[-a', a]$ in the sense that for every $x \notin [-a', a]$ there is an n such that $|df^n(x)/dx| > 1$ provided $f^k(x) \notin [-a', a]$ for $k \leq n$ (Misiurewicz 1980). In particular it is not necessary to have $|f'|_{[-1,1]\setminus[-a',a]} > 1$ for this result to hold (otherwise we can choose $n = 1$, trivially). It is now well known that expansiveness is the main ingredient to prove the existence of absolutely continuous invariant measures, and our claim about the non-atomic and uniform injection to $[-a', a]$ is thus based on the following assumptions.

(a) The expansion is strong and sufficiently uniform.

(b) The start of a burst is at a random position.

(c) $\sigma > 0$ helps.

We do not believe that counterexamples to these conjectures are impossible, by conspiracies of effects of the boundary of $[-1, 1]$, direct reinjections through accidental form of f, or perhaps even conspiracies with the noise terms. But typically these pathologies do not occur.

If a is sufficiently small, a point injected in $[-a', a]$ will be 'laminar' for at least one iteration and will thus contribute to $\rho(\cdot)$. We study ρ under the assumption that the reinjection is *uniform into the interval* $[-a', a]$ (*not into* $[-a, a]$), as we have argued above. It is easy to see that $\rho(x)$ satisfies (for $\sigma = 0$)

$$\rho(x) = \sum_{n \geqslant 0} \sum_{\{y \mid f^n(y)=x, f^k(y) \in [-a', a], k=0,1,\ldots,n-1\}} \frac{1}{|df^n(y)/dy|} \frac{1}{a+a'}. \tag{4.1}$$

Similarly if the injection to $[-a', a]$ has a non-atomic density, then the same is true for ρ. In addition, if the reinjection to $[-a', a]$ is uniform, and f is known, then ρ is calculable. It usually looks as in figure 5, where q is given by the size of the last step, i.e. essentially $f(-a) + a$, and the singularity is of square root nature. This compares very well with the numerical evidence, cf figure 6.

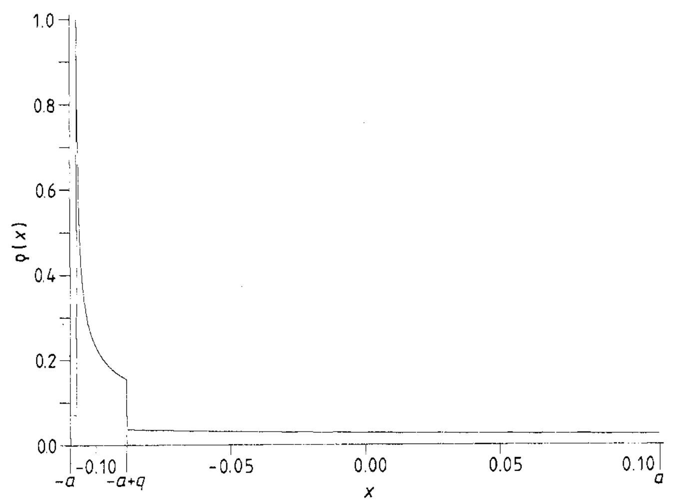

Figure 5. Reinjection density $\rho(x)$ calculated from uniform density in $[-a', a]$.

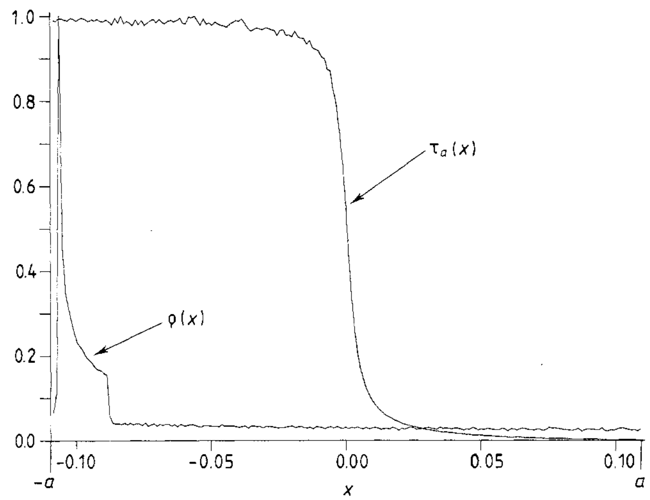

Figure 6. A typical result for $\tau_a(x)$, $\rho(x)$ ($\varepsilon = 1.5 \times 10^{-5}$, $\sigma_1 = 1$).

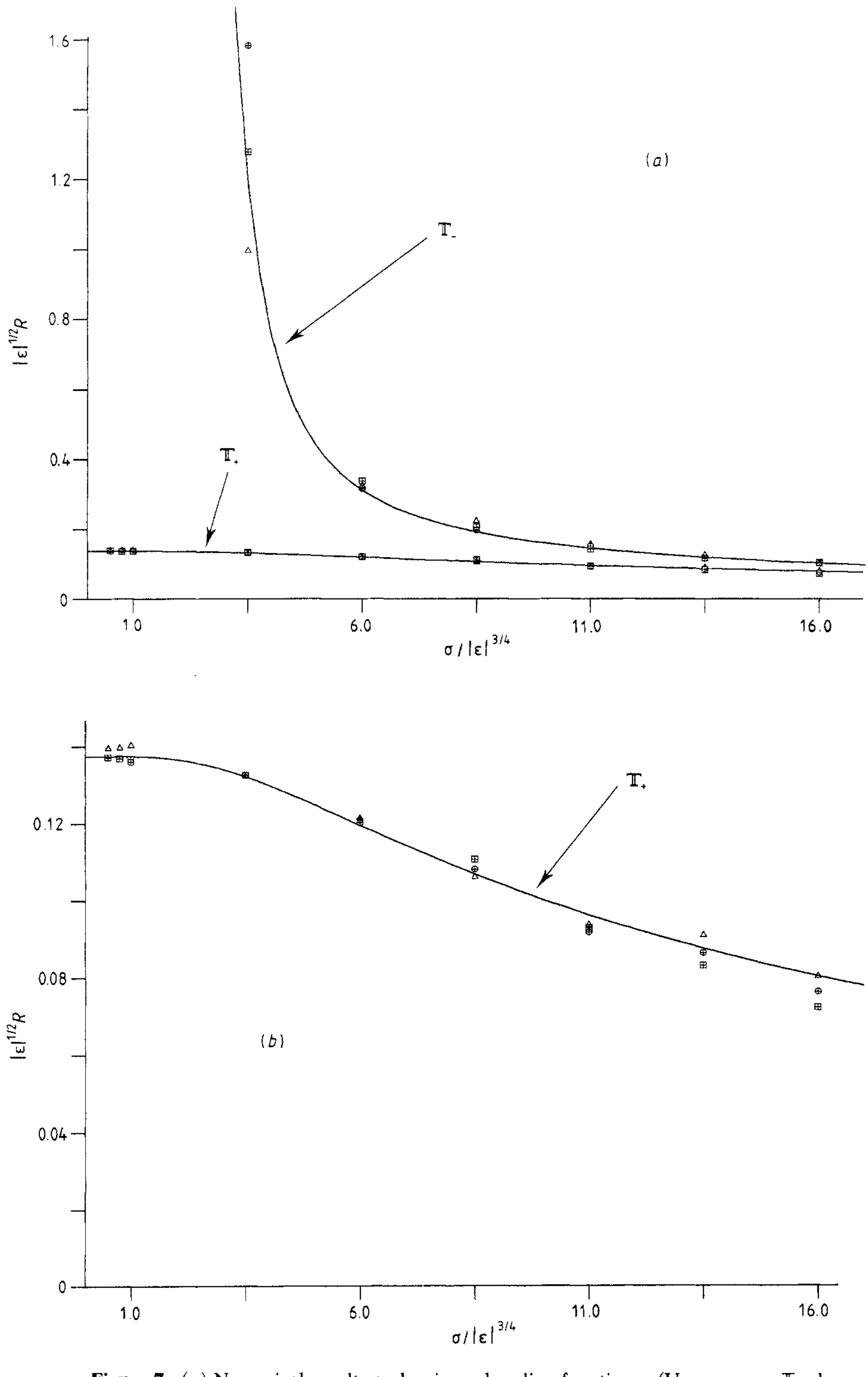

Figure 7. (*a*) Numerical results and universal scaling functions. (Upper curve $\mathbb{T}_-$, lower curve $\mathbb{T}_+$, $\Delta\varepsilon = \pm 1.5\times 10^{-5}$, ⊞ $\varepsilon = \pm 8.0\times 10^{-5}$, ⊕ $\varepsilon = \pm 12\times 10^{-5}$.) (*b*) Numerical results and universal scaling function $\mathbb{T}_+$ ($\varepsilon > 0$ enlarged).

5. Numerical tests

We have performed extensive numerical tests of our scaling predictions, using the maps $f_\varepsilon(x)$ defined by equation (1.2). In figure 1, we have shown the graph of $f_{\varepsilon=0}$. Before we present the results of the calculations, we briefly indicate some of the considerations which have dictated our choice of parameters. As we have shown, the theory applies to all values of $\sigma_1=\sigma/|\varepsilon|^{3/4}$, as soon as $|\varepsilon|$ is sufficiently small. But there are obvious numerical problems of which we list the three most important ones.

(1) The numerical precision is finite. In our case we used a word length of 32 significant bits. If we require that (a) the region of quadratic contact is traversed by steps whose numerical size is close to the ideal quadratic case and (b) that the reinjection density ρ is numerically a smooth function, we see that $1+\varepsilon^2$ should be representable on the computer.
This leads to $|\varepsilon|\gtrsim 1.5\times10^{-5}$. We have chosen $|\varepsilon|=1.5\times10^{-5}$, 8×10^{-5}, 1.2×10^{-4}.

(2) The computing time should not be excessive. This also leads to a bound $|\varepsilon|\gtrsim 1.5\times10^{-5}$ when $\sigma_1\sim1$.

(3) In order to be able to test the scaling even for the moderately large values of $|\varepsilon|$ and σ_1 we have taken, we require a set-up for which ρ is almost constant in the region where $|\tau_a'|$ is large. This motivates our choice of f_ε (almost all points have the same number of pre-images), the size of $[-a, a]$, and the upper bound on $|\varepsilon|$.

Our final parameters are as follows: $\varepsilon=\pm1.5\times10^{-5}$, $\pm8\times10^{-5}$, 1.2×10^{-4}, $\sigma/|\varepsilon|^{3/4}=0.5, 0.75, 1, 3.5, 6, 8.5, 11, 14.5, 16$, $a=\frac{7}{64}$ (see figure 7). We use a random number generator with uniform distribution in $[-\frac{1}{2},\frac{1}{2}]$. We divide the interval $[-a, a]$ into 200 bins, and count the number of reinjections into each bin, fixing a total of 130 000 ($\varepsilon>0$) and 40 000–130 000 ($\varepsilon<0$) reinjections per run. For each injection, we note the time to leave $[-a, a]$ on the right. This permits calculation of $\tau(x)$, and var $\tau(x)$. A typical result is shown in figure 6.

Finally, for each of the above choices, we compute $|\varepsilon|^{1/2}R=|\varepsilon|^{1/2}T(\varepsilon,\sigma)/\tau_b=|\varepsilon|^{1/2}\nu[-a,a]/\nu_b$ where $T(\varepsilon,\sigma)$ is the total time spent in the laminar region and τ_b is, again, the expected total time spent in the burst region (cf § 2). Assuming that the expectation τ_b is independent of ε and σ (actually we are assuming τ_b is continuous in ε, σ at $\varepsilon=\sigma=0$), we see that the quantity $|\varepsilon|^{1/2}R$ should be proportional to

$$\tau^0(-\infty, \pm1; \sigma_1=\sigma|\varepsilon|^{-3/4}) \qquad (\pm \text{ being the sign of } \varepsilon)$$

(cf equations (3.9) and (3.15), $\kappa=1.75$).

This function, along with the numerical results is shown in the figures 7(a) and 7(b) with good agreement.

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