q-analysis of fractal sets

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Abstract

q-derivatives can be identified with the generators of fractal and multifractal sets with discrete dilatation symmetries. Besides providing a natural language in which to discuss homogeneous functions with oscillatory amplitudes, this also allows one to discuss cascade models with continuous scale changes.

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q-analysis, q-special functions and the quantum groups to which they are related is a very active and fast growing field of research, as testified by the enormous amount of literature that has recently appeared on the subject [1-4]. Yet the topic still has not received the attention it deserves from the statistical physics community. Our purpose here is to point out that q-derivatives are naturally suited for describing systems with discrete dilatation symmetries such as certain fractal and multifractal sets, the limit of $q \to 1$ corresponding to the case of continuous scale changes.

We will introduce a slightly different notation than the usual [1-4]. In order to indicate the variable with respect to which a q-derivative is to be taken we use a subscript and we use a superscript to indicate the dilatation factor q, in parenthesis. Thus we define the q-derivative as

$$\partial_x^{(q)} f(x;y;\ldots) \equiv \frac{f(qx;y;\ldots) - f(x;y;\ldots)}{(q-1)x} \quad . \tag{1}$$

In contrast to the usual derivative, which measures the rate of change of the function in terms of an incremental *translation* of its argument, the q-derivative measures its rate of change with respect to a *dilatation* of its argument by a factor of q. It follows from the definition that if f is differentiable, then $\partial_x^{(q)}$ converges to ∂_x as $q \to 1$. Nowhere differentiable functions which characterize fractal or multifractal sets turn out to be perfectly well behaved under the q-derivative. Even for some functions for which the ordinary derivative exists nowhere, one may choose a sequence of q's going to 1 in such a way that the limit is well defined, as we shall describe below.

Self-similar sets are characterized by homogeneous functions. A homogeneous function f of degree ψ satisfies

$$\partial_x^{(q)} f(x) = \frac{q^\psi - 1}{q-1} \frac{f(x)}{x} \quad . \tag{2}$$

Using the definition [3,4] of the q-basic number $[\psi]_q$,

$$[\psi]_q \equiv \frac{q^\psi - 1}{q-1} \tag{3}$$

where $\lim_{q\to 1}[\psi]_q = \psi$, Eq. (2) may be written neatly as,

$$x\,\partial_x^{(q)} f(x) = [\psi]_q f(x) \quad . \tag{4}$$

It is useful to note that the solution A_q of the equation

$$\partial_x^{(q)} A_q(x) = 0 \tag{5}$$

is either a constant or a function that is periodic in $\ln x$, with period $\ln q$, such that $A_q(qx) = A_q(x)$. With the modified product rule

$$\partial_x^{(q)} g(x)f(x) = g(qx)\partial_x^{(q)} f(x) + f(x)\partial_x^{(q)} g(x) \quad , \tag{6}$$

one has, besides the arbitrary *additive* constant of integration that appears in the case of ordinary integration, an arbitrary multiplicative *function* of integration satisfying (5), so that the general solution (taking the constant of integration to be zero) to (4) is

$$f(x) = A_q(x)x^{\psi} \quad . \tag{7}$$

The function $A_q(x)$ can be recognized as the oscillatory amplitude [5,6] that arises, say, in homogeneous functions describing thermodynamic systems with discrete dilatation symmetries, such as spin systems on hierarchical lattices [7], where e.g., the free energy has the form (7) and can be expressed [8] in the form of a q-integral. For $0 < q < 1$, the q-integral is defined as [3,4],

$$\int_0^x D_t^{(q)}\ g(t) \equiv (1-q)x \sum_{n=0}^{\infty} q^n g(q^n x) \tag{8}$$

$$\int_x^{\infty} D_t^{(q)}\ g(t) \equiv (1-q)x \sum_{n=1}^{\infty} q^{-n} g(q^{-n} x) \tag{9}$$

where we have used $D_t^{(q)}$ to denote the *q-differential.*

For any given $\psi > 0$ and $q \neq 1$, a function $A_q(x)$ with the desired property (5) can be written,

$$A_q(x) = x^{-\psi} \sum_{n=-\infty}^{\infty} \frac{g(q^n\, x)}{q^{n\psi}} \tag{10}$$

$$= \frac{1}{1-q}\int_0^x \frac{g(t)}{t^{1+\psi}} D_t^{(q)} + \frac{1}{1-q}\int_x^{\infty} \frac{g(t)}{t^{1+\psi}} D_t^{(q)}. \tag{11}$$

The function $g(x)$ is quite arbitrary; it is easy to show that for the sum in (10) to converge, it suffices for $g(x)$ to be a periodic function vanishing at $x = 0$ together with its first n_0 derivatives, where n_0 is the smallest integer $> \psi$. Since the infinite sum is symmetric under $q \to q^{-1}$, for $q > 1$ one simply has to replace q by its inverse in order to be able to use the definitions (8, 9). For $q = 1$, the amplitude can only be a constant.

As an example, consider the Weierstrass-Mandelbrot function [9],

$$f_b(x) = \sum_{n=-\infty}^{\infty} b^{-n\psi}(1 - \cos(b^n x)) \tag{12}$$

with $b > 1,\ 0 < \psi < 1$. This function is nowhere differentiable and has a fractal graph with the fractal dimension $2 - \psi$. It can be expressed in the form of $f_b(x) = x^{\psi} A_b(x)$, (Eq. (7)), with the oscillatory amplitude $A_b(x)$ explicitly given by (11), for $q = b$, $g(x) = 1 - \cos(x)$. Under q-differentiation with $q = b$ it satisfies the eigenvalue equation (4), with the eigenvalue $[\psi]_b$. In fact, $f_b(x)$ has a discrete spectrum of dilatation factors b^m (m integer), equally spaced on the log scale, under which it is self-similar. In the limit that $b \to 1$, this spectrum becomes arbitrarily finely spaced. For $b = 1 + \delta$, with $\delta \ll 1$, one has $b^m \approx 1 + m\delta$, approaching 1 uniformly as $\delta \to 0$. So, as one increases m, one may speak of a *continuous dilatation symmetry.* We would like to point out that in the limit of $b \to 1$, with $q = b$, the q derivative of this function remains well defined in the sense that the following limit exists,

$$\lim_{b \to 1} \frac{x\, \partial_x^{(b)} f_b(x)}{f_b(x)} = \psi \quad . \tag{13}$$

Note that $\lim_{b\to 1} x\, f_b'(x)/f_b(x)$ is undefined. (If the limit $b \to 1$ is taken before taking the lower and upper limits of the sums in $f_b(x)$ and $f_b'(x)$ to ∞, then a limit exists.)

Now consider a fractal set S which is self-similar under successive scale changes by b and let $\rho_b(\mathbf{x})$ be the characteristic function of this set:

$$\rho_b(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in S \\ 0, & \text{otherwise}\ . \end{cases} \tag{14}$$

Assume $\rho_b(b\mathbf{x}) = \rho_b(\mathbf{x})$ so that $\rho_b(\mathbf{x})$ satisfies (5) with $q = b$, with respect to any one of its arguments, $x_k,\ k = 1, \ldots, d$, where d is the dimension of the Euclidean space in which the set S is embedded. This function can be constructed as the $n \to \infty$ limit of a succession of $\rho_b^{(n)}$ such that

$$\rho_b^{(n)}(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in \cup_i \Omega_i(b^n), \quad i = 1, \ldots N_n \\ 0, & \text{otherwise} \ . \end{cases} \tag{15}$$

where $\Omega_i(b^n)$ is a covering of the set S, consisting of $N_n = N^n$ balls of linear size b^n. Here $N > 1$ is an arbitrary integer, and $b < 1$. The fractal (self- similarity) dimension [10,11] of this set is clearly $d_f = \ln(N)/\ln(b^{-1})$. To see this, define,

$$M(\ell_n, R) \equiv \int_{\Gamma(R)} \rho_b^{(n)}(\mathbf{x}) d^d x = \Phi_b(\ell_n, R) \ell_n^d (\ell_n/R)^{-d_f} \quad , \tag{16}$$

where $\ell_n = b^n \ell_0$, $\Gamma(R)$ is a region of linear size R of the embedding space of dimension d. The oscillatory amplitude $\Phi_b(\ell, R)$ has been called lacunarity [10,12], and is periodic in the logarithm of both ℓ and R, with period $\ln b$. As an example one may consider the one dimensional Cantor set [10,11] with $d = 1$, $\ell_0 = 1$, $b = 1/3$ and $N = 2$. From (16) it is obvious that $M(\ell_n, 1/3)/M(\ell_n, 1) = 1/2 = (1/3)^{d_f}$, yielding $d_f = \ln 2/\ln 3$.

Taking the limit of $n \to \infty$, we now define,

$$M(R) \equiv \lim_{n \to \infty} M(\ell_n, R) = \int_{\Gamma(R)} \rho_b(\mathbf{x}) d^d x \quad . \tag{17}$$

The usefulness of our definition is that $M(R)$ is an eigenfunction of the operator $R\, \partial_R^{(q)}$ with $q = b$ being the scale change which takes the fractal set to itself, such that

$$R\, \partial_R^{(b)} M(R) = [d_f]_b \, M(R) \quad . \tag{18}$$

while its derivative fails to exist at an infinite number of points. One may think of other quantities defined on this set which will also satisfy similar equations, such as the correlation function, $G(r) = \int \rho(\mathbf{x})\rho(\mathbf{x} + \mathbf{r}) d^d x$, etc. For a set which is self similar under arbitrarily small dilatations by $b = 1 + \delta$, with $\delta \to 0$, the eigenvalue goes over to $\lim_{b \to 1}[d_f]_b = d_f$. In this limit the oscillatory amplitude can only be a constant.

It is convenient to introduce the dilatation operator such that $T_x^{(q)} f(x) = f(qx)$. One finds, $T_x^{(q)} = [(q-1)x\partial_x^{(q)} + 1]$, or $T_x^{(q)} = [\partial_x^{(q)}, x]$, where the brackets indicate the ordinary commutator. In the case that S is generated by an exact recursion relation, consider a fine-graining of the covering by b; this leads to

$$T_\ell^{(b)} M(\ell_n, R) = M(\ell_{n+1}, R) \quad , \tag{19}$$

or,

$$\left(T_\ell^{(b)}\right)^k M(\ell, R) = M(b^k \ell, R) \quad . \tag{20}$$

For small $b-1$ one may in fact exponentiate and obtain

$$\begin{aligned} M(b^k\ell, R) &= \left(T_\ell^{(b)}\right)^k M(\ell, R) \\ &\simeq e^{k[T_\ell^{(b)}-1]} M(\ell, R) \quad . \end{aligned} \tag{21}$$

Similar considerations also hold for multifractal measures[13] which have discrete self-similarity properties, say under finegraining by a factor q. The generalized partition function will satisfy, as $\ell \to 0$, the scaling relation similar to (16),

$$\chi_q(p; \ell, R) = \sum_i^{N(\ell,R)} \mu_i^p = A_q(p; \ell, R)(\frac{\ell}{R})^{\tau(p)} \quad , \tag{22}$$

where μ_i is the measure contained in the ith box of linear size ℓ, $N(\ell, R)$ is the number of such boxes needed to cover the set within a region of size R, and $A_q(p; \ell, R)$ is a periodic function of $\ln \ell/R$. Using (4) we see that the generalised partition function (22) satisfies,

$$\frac{\ell\, \partial_\ell^{(q)} \chi_q(p; \ell, R)}{\chi_q(p; \ell, R)} = [\tau(p)]_q \quad , \tag{23}$$

and an analogous identity holds for q-differentiation with respect to R. Henceforth we will not display the R dependence unless necessary. Assuming that the measure in each ith box scales as $\mu_i \sim \ell^{\alpha_i}$, let us define the partial sums over boxes with singularity strength α, with $\mu \sim \ell^\alpha$, to be

$$\chi_q^{(\alpha)}(p; \ell) = A_q^{(\alpha)}(p; \ell) \ell^{-f(\alpha)+p\alpha} \tag{24}$$

such that

$$\chi_q(p;\ell) = \sum_\alpha \chi_q^{(\alpha)}(p;\ell) \quad . \tag{25}$$

Acting with $\ell\, \partial_\ell^{(q)}$ on both sides of (25) gives,

$$[\tau(p)]_q \sum_\alpha \chi_q^\alpha(p;\ell) = \sum_\alpha [-f(\alpha) + p\alpha]_q \, \chi_q^\alpha(p;\ell) \quad . \tag{26}$$

Thus we see that *i)* if, for each p and $\ell \to 0$ the $\chi_q^\alpha(p;\ell)$ are very strongly peaked at some $\alpha^*(p)$, then one gets, as usual [13], $\tau(p) = -f(\alpha^*(p)) + p\alpha^*(p)$. *ii)* If all terms in the sum (26) are of the same order of magnitude, (equivalently, there is no spread in the spectrum of α's), then we are forced to accept $-f(\alpha^*) + p\alpha^* = \tau(p)$, the set is a homogeneous fractal with fractal dimension $d_f = f(\alpha^*)$. For a multifractal measure with an exact recursive structure, obtained via a multiplicative process[13], one may identify the q-dilatation operator with the generator. Consider that under a scale change by q, $\ell' = q\ell$, each ith box is finegrained in the same way so that the measure in the jth subbox always gets multiplied by β_j, where $j = 1, \ldots, q^{-D}$, with $D \leq d$ and moreover that the total measure is always conserved, i.e., $\sum_j \beta_j = 1$. Then, at each finegraining step, μ_i^p gets replaced by $\mu_i^p \sum_j \beta_j^p$. Assuming a scaling form for the β_j, such that $q^{\gamma_j} \equiv \beta_j$, one has

$$W \equiv \frac{\chi_q(p;\ell')}{\chi_q(p;\ell)} = \sum_j q^{p\gamma_j}. \tag{27}$$

Combining this with Eq.(23) gives,

$$\tau(p) = \ln\left(\sum_j q^{p\gamma_j}\right) / \ln q = \ln W / \ln q \quad . \tag{28}$$

The derivative of $\lim_{\ell\to 0} \chi_q(p;\ell,R)$ with respect to R does not exist at an ∞ number of points; for $D = d$, it becomes a nowhere differentiable function of R. Note that (27) implies,

$$\left(T_\ell^{(q)}\right)^k \chi_q(p;\ell) = \left(\sum_j q^{p\gamma_j}\right)^k \chi_q(p;\ell) = \chi_q(p, q^k\ell) \quad , \tag{29}$$

which should be compared with (20). It follows that,

$$\lim_{\ell\to 0} \chi_q(p;\ell) = \lim_{n\to\infty} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (q-1)^k \ell^k \left[\partial_\ell^{(q)}\right]^k \chi_q(p;\ell) \quad . \tag{30}$$

Fractal and multifractal cascades have found wide application in modeling turbulent media, such as in the so called β- model [14], random β-model [15], etc. As an example where the existence or non-existence of the derivative and the q-derivative show up neatly, we would like to mention random cascade models with a continuum of scales, such as proposed recently by Zhen-Su She et al. [16]. Consider a scale transformation from $\ell \to \ell' = b\ell$, with $b < 1$. It can be subdivided into arbitrarily fine steps in the following way [16]: First partition the box of size ℓ into $N_1 \gg 1$ subboxes, $N_1 = (\ell/\ell_1)^d$. Then construct the sequence $N_1, N_2, \ldots, N_n$ such that $N_m = (\ell/\ell_m)^d$ with $\ell_{m+1} = q\ell_m$, taking $q \simeq 1-\delta$, $\delta \ll 1$, $q^n = b$ and finally, $N_n = (\ell/\ell_n)^d = N_1 q^{-nd} = (\ell'/\ell'_1)^d b^{-d}$, where $\ell'_1 = b\ell_1$. It is understood that $\delta \to 0$ as $n \to \infty$, with $n\delta \sim O(1)$. Now at each step assign to the subboxes a fraction of the measure βq^γ with probability ϕ and the fraction q^γ with the probability $1-\phi$. In this probabilistic scheme, $\tau(p)$ will be given [16,17] by $\tau(p) = \ln\langle W\rangle / \ln q$, compare Eq.(28), with $\langle W\rangle = q^{-d}[\phi\beta q^\gamma + (1-\phi)\beta q^\gamma]$. Taking $(1-\phi) = q^{d-D}$, one gets

$$\tau(p) = -D + p\gamma + \ln\left[1 + \beta^p(q^{-d+D} - 1)\right] / \ln q \quad . \tag{31}$$

In the limit of $q \to 1$, holding D and γ fixed, one has $\tau_1(p) \equiv p\gamma - \beta^p d + (\beta^p - 1)D$. Note that the scaling of the partition function in this limit is *intermittent*, with a nonlinear dependence of τ_1 on p. Even though the probability $\phi \to 0$ as $q \to 1$, as $n \to \infty$ at each step and $\ell \to 0$, one still has infinitely many "discrete events" distributed over a fractal set of dimension D, where a subbox has been assigned the relative measure βq^γ rather than q^γ, and therefore where the ordinary derivative with respect to R does not exist. On the other hand, taking the limit $q \to 1$ as in Eq.(23) one gets,

$$\lim_{q\to 1} \frac{R\, \partial_R^{(q)} \chi_q(p;R)}{\chi_q(p;R)} = -\tau_1(p) \quad , \tag{32}$$

where, by $\chi_q(p;R)$ we mean $\lim_{\ell\to 0} \chi_q(p;\ell,R)$.

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