

Primordial Magnetic Fields and Causality

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Abstract. We discuss the implications of causality on a primordial magnetic field. We show that the residual field on large scales is much more suppressed than usually assumed, and that a helical component is even more reduced. Due to this strong suppression, even maximal primordial fields generated at the electroweak phase transition can just marginally seed the fields in clusters but they cannot leave any detectable imprint on the cosmic microwave background.

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1. Introduction

The observed Universe is permeated with large scale coherent magnetic fields of the order of micro Gauss. It is still under debate whether these fields have been generated by charge separation processes in the late universe, or whether primordial seed fields are needed. The observational situation is described in Ref. [1]. More recent detections of magnetic fields in clusters are discussed in Ref. [2].

In this letter we want to clarify a point which is often missed when investigating cosmic magnetic fields (for a comprehensive review see [3]): namely that, because they are divergence-free, magnetic fields are suppressed stronger than white noise on large scales.

We assume that primordial magnetic fields are generated with a certain (comoving) coherence scale L , by a random process which is statistically homogeneous and isotropic. If the field generation occurs during a non-inflationary phase of the Universe, L must be smaller than the horizon scale. During inflation, L may diverge and the arguments presented below do not apply. We further assume that the field created by this causal process is a purely classical magnetic field satisfying Maxwell's equations, or that it might be considered to be so soon after its generation. Neglecting possible quantum mechanical fluctuations in the field, we can state that its amplitudes and directions must be uncorrelated for points which lay farther apart than L :

$$\langle B_i(\mathbf{x})B_j(\mathbf{y}) \rangle \equiv C_{ij}(\mathbf{x} - \mathbf{y}) = 0 \quad \forall \mathbf{x}, \mathbf{y} \text{ with } |\mathbf{x} - \mathbf{y}| > L . \quad (1)$$

The first equality comes from the fact that \mathbf{B} is statistically homogeneous and isotropic, so that the correlation tensor C_{ij} is only a function of $\mathbf{x} - \mathbf{y}$, and its trace $C = C_{ii}$ depends only on the distance $r = |\mathbf{x} - \mathbf{y}|$. For simplicity, we have set the correlation tensor to zero on separations larger than L , but an exponential decay would actually be sufficient for all our results.

In Ref. [4] Hogan has argued that the field averaged over a volume of size $\lambda^3 > L^3$ behaves like

$$B_\lambda \simeq B_0 \left(\frac{L}{\lambda} \right)^{3/2} , \quad (2)$$

where B_0 is the amplitude of the field averaged over a volume given by the correlation scale L , and B_λ is the field averaged over a volume V_λ of size λ^3 . Hogan's argument leading to the above result is very simple: V_λ contains $N = (\lambda/L)^3$ uncorrelated volumes. Within each of them the magnetic field has an average value of B_0 pointing in an arbitrary direction. The amplitude of the field averaged over a volume of size V_λ is therefore reduced by a factor \sqrt{N} , leading to the result (2).

In this paper we show that this result is not correct and has to be replaced by

$$B_\lambda \simeq B_0 \left(\frac{L}{\lambda} \right)^{5/2} . \quad (3)$$

In what follows we proof Eq. (3). We then apply it to some relevant cases and show that the difference is important. We also derive the corresponding scaling behaviour of an helical magnetic field component. We end with some conclusions.

We shall always use comoving length scales λ , L and wave numbers k . The scale factor today is normalised to unity, $a(\eta_0) = 1$, η denotes conformal time, and we assume a spatially flat universe with metric

$$ds^2 = a^2(\eta)(-d\eta^2 + \delta_{ij}dx^i dx^j) .$$

On scales larger than the coherence scale, we consider a field frozen into the plasma, and simply red-shifting with the expansion of the universe like $\mathbf{B}(\mathbf{x}, \eta) = \mathbf{B}(\mathbf{x}, \eta_0)a^{-2}(\eta) \equiv \mathbf{B}(\mathbf{x})a^{-2}(\eta)$. Here $\mathbf{B}(\mathbf{x})$ is the magnetic field scaled to its value today, the quantity which we will mainly use from now on. This behaviour is well justified on large scales; on smaller scales however, magnetic energy is converted into heat due to plasma viscosity. We account for this damping by introducing a cutoff in the spectrum of the magnetic field at the smallest scale at which the field is not affected by viscous processes. A suitable value for the cutoff scale is given in Eq. (1) of Ref. [5]. If the field has an helicity component, this model is no longer very appropriate since a process of inverse cascade may take place [6]. We will discuss that when analysing the helicity case.

2. Causal stochastic magnetic fields

The fact that magnetic fields are divergence-free implies that the integral over an arbitrary closed surface of the normal component of \mathbf{B} has to vanish. This shows that \mathbf{B} cannot take arbitrary mean values in all boxes of size L . To see what this implies, let us define the Fourier transform,

$$\hat{\mathbf{B}}(\mathbf{k}) = \int \exp(i\mathbf{k} \cdot \mathbf{x})\mathbf{B}(\mathbf{x})d^3x ,$$

so that

$$\mathbf{B}(\mathbf{x}) = \frac{1}{(2\pi)^3} \int \exp(-i\mathbf{k} \cdot \mathbf{x})\hat{\mathbf{B}}(\mathbf{k})d^3k .$$

Since $C_{ij}(\mathbf{x})$ is a function with compact support, its Fourier transform is analytic. If the magnetic field is truly stochastic, with no preferred direction, the only tensors which may enter into the Fourier transform of its correlation tensor are combinations of k_n , δ_{lm} and ϵ_{jlm} , where δ_{lm} denotes the Kronecker delta and ϵ_{jlm} is the completely antisymmetric tensor in three dimensions. The most general Ansatz for the magnetic field correlation tensor in Fourier space which respects stochastic homogeneity and isotropy is then

$$\langle \hat{B}_l(\mathbf{k})\hat{B}_m^*(\mathbf{k}') \rangle = \frac{(2\pi)^3}{2}\delta(\mathbf{k} - \mathbf{k}')[(\delta_{lm} - \hat{k}_l\hat{k}_m)S(k) + i\epsilon_{lmj}\hat{k}_j A(k)] . \quad (4)$$

With $\hat{\mathbf{k}}$ we denote the unit vector in direction of \mathbf{k} , $\hat{\mathbf{k}} = \mathbf{k}/k$ and $k = |\mathbf{k}|$. The square bracket in (4) is nothing else than the Fourier transform of C_{ij} and thus has to be analytic. As we shall see in the next section (see also Refs. [7, 8, 9]), the second term, which changes sign under the transition $\mathbf{k} \rightarrow -\mathbf{k}$ represents a non-vanishing helicity. We disregard it for this section.

Causality also implies that $S(k)$ cannot have any structure for values of the wave number smaller than L^{-1} , and hence can be approximated by a simple power law,

$$S(k) = S_0 k^n \left(1 + \mathcal{O}((kL)^2)\right). \quad (5)$$

Analyticity of $(\delta_{lm} - \hat{k}_l \hat{k}_m)S(k)$ then requires that

$$n \geq 2 \text{ is an even integer.} \quad (6)$$

Generically, if there are no additional constraints, we expect $n = 2$. Note that the ‘usual value’ $n = 0$ (white noise) is not allowed because of the non-analytic pre-factor $\hat{k}_l \hat{k}_m$ which is required to keep the magnetic field divergence-free, $\mathbf{k} \cdot \hat{\mathbf{B}} = 0$. In other words, the divergence-free condition forces a blue spectrum on the magnetic field energy density, $\langle |\hat{\mathbf{B}}(\mathbf{k})|^2 \rangle \propto k^2$.

Note that this can also be obtained by assuming that the vector potential $\mathbf{A}(\mathbf{k})$ has a white noise spectrum. With the Ansatz

$$\langle \mathbf{A}_i \mathbf{A}_j^* \rangle = \delta_{ij} V(k) + \epsilon_{ijl} \hat{\mathbf{k}}_l W(k), \quad (7)$$

one has that analyticity requires W to grow at least like k . Using $\mathbf{B} = -i\mathbf{k} \wedge \mathbf{A}$ one finds $S = k^2 V$ and $A = k^2 W$.

We want to estimate the average field on a given scale $\lambda \geq L$. At this aim, we perform a volume average of the field on a region of size λ^3 , following Ref. [10]. We convolve \mathbf{B} with a Gaussian window function,

$$\mathbf{B}_\lambda(\mathbf{x}) = \frac{1}{(\lambda\sqrt{2\pi})^3} \int d^3 y \mathbf{B}(\mathbf{y}) \exp\left(-\frac{(\mathbf{x} - \mathbf{y})^2}{2\lambda^2}\right). \quad (8)$$

A short computation shows that the magnetic energy density on scale λ , $B_\lambda^2 \equiv \langle \mathbf{B}_\lambda(\mathbf{x})^2 \rangle$, is given by

$$B_\lambda^2 = \frac{1}{(2\pi)^3} \int d^3 k S(k) \hat{f}_\lambda^2(k) = \frac{S_0}{(2\pi)^2} \frac{1}{\lambda^{n+3}} \Gamma\left(\frac{n+3}{2}\right), \quad (9)$$

where $\hat{f}_\lambda(k) = \exp(-\lambda^2 k^2/2)$ is the Fourier transform of our window function and Γ denotes the Gamma function [11].

For two different scales, λ_1 and λ_2 we therefore have

$$\left(\frac{B_{\lambda_1}}{B_{\lambda_2}}\right)^2 = \left(\frac{\lambda_2}{\lambda_1}\right)^{n+3},$$

(see also Ref. [10], where however $n = 0$ was concluded for causal fields), and especially, for the generically expected value $n = 2$

$$\frac{B_\lambda}{B_0} \simeq \left(\frac{L}{\lambda}\right)^{5/2}, \quad (10)$$

as claimed in Eq. (3). For $n > 2$, the suppression with scale is even stronger.

To demonstrate the importance of this additional L/λ factor with respect to Eq. (2), let us consider magnetic fields produced during the electroweak phase transition as it has been proposed by various authors, see *e.g.* [12, 13, 14, 15] (note that the

authors of [15] found a magnetic field spectrum $\propto k^2$. Following these references, at the scale $L_c \sim 10^5$ cm, a magnetic field with amplitude $B_{ew}(L_c) \sim 10^{-6}$ Gauss is produced (note that we have scaled the field value to today and we use conformal length scales normalising the scale factor to unity today, $a_0 = 1$). Now L_c is much smaller than the horizon scale, $\eta_{ew} \sim 10^{15}$ cm. But in Ref. [14], it is argued that a field is induced also on large scales, scaling like L^{-1} . As we have shown above, this scaling can only take place up to the horizon scale, and so we have at best a field of about $B_{ew}(\eta_{ew}) \sim B_{ew}(L_c)(L_c/\eta_{ew}) \sim 10^{-16}$ Gauss. As we have argued above, due to causality the field has to decay like $L^{-5/2}$ on super horizon scales. For $\lambda \sim 1$ Mpc $\sim 3 \times 10^{24}$ cm one can therefore have a field of only about $B_\lambda 10^{-39}$ Gauss, and not 10^{-20} Gauss as inferred in Ref. [14] and also in Ref. [10], where the authors have set $n = 0$ for ‘frozen-in’ magnetic fields.

3. Helicity

Let us now investigate limits due to causality on the helicity component in Eq. (4). This component can have been produced due to parity violating processes during the electroweak phase transition, as it has been proposed in [7, 8]. We rewrite the term proportional to $A(k)$ in Eq. (4) introducing the helicity basis,

$$\mathbf{e}^\pm(\mathbf{k}) = -\frac{i}{\sqrt{2}}(\mathbf{e}_1 \pm i\mathbf{e}_2), \quad (11)$$

where $(\mathbf{e}_1, \mathbf{e}_2, \hat{\mathbf{k}})$ form a right-handed orthonormal system with $\mathbf{e}_2 = \hat{\mathbf{k}} \times \mathbf{e}_1$. Setting $\hat{\mathbf{B}}(\mathbf{k}) = B^+\mathbf{e}^+ + B^-\mathbf{e}^-$ it is straightforward to see that

$$\langle B^+(\mathbf{k})B^+(-\mathbf{k}') - B^-(\mathbf{k})B^-(-\mathbf{k}') \rangle = (2\pi)^3 A(k)\delta(\mathbf{k} - \mathbf{k}'), \quad (12)$$

so that $A(k)$ determines the net circular polarisation of the Fourier mode $\hat{\mathbf{B}}(\mathbf{k})$. Again, causality requires that the function

$$\epsilon_{lmj}\hat{k}_j A(k)$$

must be analytic and featureless for $k < 1/L$, so that

$$A(k) = A_0 k^m \left(1 + \mathcal{O}((kL)^2)\right), \quad (13)$$

where m has to be a positive odd integer. But there is an additional constraint coming simply from the Schwarz inequality,

$$\lim_{\mathbf{k}' \rightarrow \mathbf{k}} |\langle (\hat{\mathbf{k}} \times \mathbf{B}(\mathbf{k})) \cdot \mathbf{B}(-\mathbf{k}') \rangle| \leq \lim_{\mathbf{k}' \rightarrow \mathbf{k}} \langle \mathbf{B}(\mathbf{k}) \cdot \mathbf{B}(-\mathbf{k}') \rangle$$

implying

$$|A(k)| \leq S(k) \quad (14)$$

(note that $S(k) \propto \langle |\hat{\mathbf{B}}|^2 \rangle$, and therefore $S(k) \geq 0$). For Eq. (14) to be valid for very small values of k we must require

$$m \geq n. \quad (15)$$

Together with the causality limit from above and (6) this implies

$$m \geq 3 \quad \text{is an odd integer.} \quad (16)$$

Again, generically we expect $m = 3$. Furthermore, applying Eq. (14) close to the correlation scale L , we have

$$|A_0| \leq S_0 L^{(m-n)}. \quad (17)$$

Vorticity is even more suppressed on large scales by causality, than a non vortical component of the magnetic field. To quantify this we define the amplitude of the vortical component on a scale λ by

$$\mathcal{B}_\lambda^2 = \frac{\lambda}{(2\pi)^3} \int d^3 k k |A(k)| \hat{f}_\lambda^2(k) = \frac{|A_0|}{(2\pi)^2} \frac{1}{\lambda^{m+3}} \Gamma\left(\frac{m+4}{2}\right). \quad (18)$$

With the generic value, $m = 3$ we therefore have

$$\frac{|\mathcal{B}_\lambda|}{|\mathcal{B}_0|} \simeq \left(\frac{L}{\lambda}\right)^3, \quad (19)$$

a factor $\sqrt{L/\lambda}$ more suppression than the non-vortical component, which for a coherence length of η_{ew} translate into an additional suppression of the order of 10^{-5} . However, for helical magnetic fields an inverse cascade effect takes place in the early universe, which causes a transfer of power from smaller to larger scales. This results in a larger coherence scale than the frozen in one, while the magnetic spectral index remains unchanged on larger scales [6, 5]. To account for this effect, we follow Ref. [8], in which the primordial helicity is given by $\mathcal{H} \sim L_c \mathcal{B}_0^2 = -n_b/\alpha$, where n_b is the baryon density of the universe today, $n_b(\eta_0) = 3 \times 10^{-7} \text{cm}^{-3}$ [16], and α is the fine structure constant. Just as the magnetic field strength, we have also scaled the helicity, which evolves like a^{-3} to today and is a conserved quantity like the baryon number. L_c is the comoving coherence length. According to analytical studies [17], the physical coherence scale aL_c evolves with cosmic time roughly like $t^{2/3}$ (note however that different scaling laws have been found in numerical simulations, see [6, 5]). In Ref. [8], the comoving coherence scale of a maximally helical component of the magnetic field is found to be $L_c = 0.1 \text{ pc}$. On this coherence scale, the amplitude of the magnetic field today becomes $\mathcal{B}_0 = \sqrt{n_b/\alpha/L_c} \sim 10^{-19} \text{ Gauss}$. Taking again a scale λ of 1 Mpc, we get a maximal helicity amplitude of $\mathcal{B}_\lambda = \mathcal{B}_0(L_c/1\text{Mpc})^3 \sim 10^{-40} \text{ Gauss}$.

4. Conclusions

In this paper we have shown that causally produced magnetic fields cannot have a white noise spectrum on large scales. They have a blue spectrum with index $n = 2$. A possible helical component of the field, having a spectral index $m = 3$, is even more suppressed on large scales. These spectral indexes are valid on scales larger than the coherence scale of the field. The helical component typically has a larger coherence scale, because of non-linear MHD processing which leads to an inverse cascade.

The estimates in the previous sections show that for a magnetic field causally generated at the electroweak phase transition, the amplitudes of the symmetric and helical components at $\lambda = 1$ Mpc are at most 10^{-39} and 10^{-40} Gauss: these amplitudes are too small to seed the magnetic fields observed in clusters today [3] (see however Ref. [18] which argues that 10^{-30} Gauss or even less might suffice).

To answer the question whether we might be able to see some effects of these fields in the cosmic microwave background (CMB), we have to estimate the field amplitudes on scales close to about 100 Mpc, which corresponds to an harmonic of about $\ell \sim 400$ (here we have used the angular diameter distance to the last scattering surface, $d_A \simeq 13700$ Mpc, from WMAP [16]). From the amplitudes for $\lambda = 1$ Mpc given above and using the scaling behaviour derived in this paper, we obtain residual fields of at best $B_{100\text{Mpc}} \simeq 10^{-42}$ Gauss and helicity of $\mathcal{B}_{100\text{Mpc}} \sim 10^{-46}$ Gauss on 100 Mpc. The amplitude of the induced fluctuations in the CMB is typically of the order of $\delta T/T \sim (\rho_B|_{100\text{Mpc}})/\rho_\gamma \ll 10^{-5}$.

We therefore conclude, that a magnetic field which has evolved on large scales simply via flux conservation from its creation at the electroweak phase transition until today is not sufficient to have seeded the large scale magnetic fields observed in clusters, even if a dynamo mechanism could amplify it during the process of structure formation. The same conclusion can be made for an helicity component of the magnetic field, if accounting for MHD processing in the simple way as explained in the previous paragraph. Such a field also does not lead to observable traces in the anisotropies or the polarisation of the CMB. This latter conclusion has also been drawn in previous works [19, 20].

Possible ways out are either that the magnetic fields observed in clusters are due to very small scale seed fields, coherent on scales of the order of a parsec or less. Another possibility is that the seed fields have been generated by a ‘non-causal’ mechanism, *e.g.* during an inflationary phase, see [21, 22]. But also in this latter case, a very red spectrum $n < -2$ is needed for the magnetic fields to play at the same time the role of seeds for large scale magnetic fields and to lead to visible imprints on the CMB. Such red spectra are actually also required by the limits from small scale gravitational waves which are induced by magnetic fields [23].

Our results strongly disfavour large scale seeds induced from small scale coherent magnetic fields which might be produced in the early Universe.

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References

- [1] P. Kronberg, Rep. Prog. Phys. **57**, 57 (1994).
- [2] J. Eilek and F. Owen, Astrophys. J. **567**, 202 (2002).
- [3] D. Grasso and H.R. Rubinstein, Phys. Rept. **348**, 163 (2001).
- [4] C. Hogan, Phys. Rev. Lett. **51**, 1488 (1983).
- [5] R. Banerjee and K. Jedamzik, preprint `astro-ph/0306211`
- [6] M. Christensson, M. Hindmarsh and A. Brandenburg, Phys. Rev. **E64** 056405 (2001).
- [7] J. M. Cornwall, Phys. Rev. **D56**, 6146 (1997)
- [8] T. Vachaspati, Phys. Rev. Lett. **87** 251302 (2001).
- [9] L. Pogosian, T. Vachaspati, S. Winitzki, Phys. Rev. D **65**, 3264 (2002).
- [10] M. Hindmarsh and A. Everett, Phys. Rev. **D58**, 103505 (1998).
- [11] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972).
- [12] T. Vachaspati, Phys. Lett. **B265**, 258 (1991).
- [13] M. Joyce and M. Shaposhnikov, Phys. Rev. Lett. **79**, 1193 (1997).
- [14] J.T. Ahonen and K. Enqvist, Phys. Rev. **D57**, 664 (1998).
- [15] D. Boyanovsky, H. J. de Vega and M. Simionato, preprint `astro-ph/0305131`
- [16] D. Spergel et al., preprint `astro-ph/0302209`.
- [17] D.T. Son, Phys. Rev. **59**, 063008 (1999)
- [18] A. Davis, M. Lilley and O. Törnkvist, Phys. Rev. **D60** 021301 (1999).
- [19] R. Durrer, P.G. Ferreira, and T. Kahniashvili, Phys. Rev. D **61**, 043001 (2000).
- [20] C. Caprini, T. Kahniashvili and R. Durrer, Phys. Rev. D submitted (2003).
- [21] E. Turner and L. Widrow Phys. Rev. **D37** 2743 (1988).
- [22] M. Gasperini, M. Giovannini and G. Veneziano, Phys. Rev. Lett. **75** 3796 (1995).
- [23] C. Caprini and R. Durrer, Phys. Rev. **D65** 023517 (2002).