# Biasing in Gaussian random fields and galaxy correlations

Andrea Gabrielli[1,2], Francesco Sylos Labini [2,3] and Ruth Durrer[3]
[1] *Laboratoire de Physique de la Matière Condensée, École Polytechnique, 91128 - Palaiseau Cedex, France*
[2] *INFM Sezione Roma1, Dip. di Fisica, Universitá "La Sapienza", P.le A. Moro, 2, I-00185 Roma, Italy.*
[3] *Département de Physique Théorique, Université de Genève, 24 quai Ernest Ansermet, CH-1211 Genève 4, Switzerland*

In this letter we show that the peaks of a Gaussian random field, with density-density correlations on small scales, are not more 'strongly correlated' than the field itself: they are more sparse. The peaks are (almost) connected regions identified by a certain threshold in the density field, and their spatial extension is of the order of the correlated patches of the original Gaussian field. The amplification of the correlation function of the peaks selected by a certain threshold, usually referred to as 'biasing', has nothing to do with how 'strongly clustered' the peaks are but is due to their sparseness. This clarifies an old-standing misconception in the literature. We also argue that this effect does not explain the observed increase of the amplitude of the correlation function $\xi(r)$ when galaxies of brighter luminosity or galaxy clusters of increasing richness are considered.

PACS Numbers : 98.65.Dx, 02.50.-r, 05.40.+j

We first explain, in mathematical terms, the notion of biasing for a Gaussian random field. Here we follow the ideas of Kaiser [1] which have been developed further in [2]. We then calculate biasing for some examples and we clarify the physical meaning of bias in the context of Ref. [1]. Finally we comment on the significance of our findings for the correlations of galaxies and clusters.

We consider a homogeneous, isotropic and correlated Gaussian random field, $\delta(\mathbf{x})$, with mean zero and variance $\sigma^2 = \langle\delta(\mathbf{x})^2\rangle$ in a volume $V$. We assume $V$ to be finite for definiteness, but it can go to infinity at the end. The marginal one-point probability density function of $\delta$ is

$$P(\delta) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\delta^2}{2\sigma^2}} \ .$$

Using $P$ we calculate the fraction of the volume $V$ with $\delta(\mathbf{x}) \geq \nu\sigma$, given by $P_1(\nu) = \int_{\nu\sigma}^{\infty} P(\delta) d\delta$.

The correlation function between two values of $\delta(\mathbf{x})$ in two points separated by a distance $r$ is given by $\xi(r) = \langle\delta(\mathbf{x})\delta(\mathbf{x}+r\mathbf{n})\rangle$. By definition, $\xi(0) = \sigma^2$. In this context, homogeneity means that the variance $\sigma^2$ and the correlation function, $\xi(r)$ do not depend on $\mathbf{x}$. Isotropy means that $\xi(r)$ does not depend on the direction $\mathbf{n}$[1]. An important application we have in mind are cosmological density fluctuations, $\delta(\mathbf{x}) = (\rho(\mathbf{x}) - \rho_0)/\rho_0$, where $\rho_0 = \langle\rho\rangle$ is the mean density; but the following arguments are completely general.[2] Here and in what follows we assume that the average density $\rho_0$ is a well defined quantity. This is not so if the distribution is fractal [4]. Then the correlation length is infinite, and the mean density does not exist.

The goal is, to determine the correlation function of maxima from the correlation function of the underlying density field. Like [1] we simplfy the problem by computing the correlations of the *region* above a certain threshold $\neq \sigma$ instead of the correlations of *maxima*; but these quantities are closely related for values of $\nu$ significanlty larger than 1. We define the threshold density, $\theta_\nu(\mathbf{x})$ by

$$\theta_\nu(\mathbf{x}) \equiv \theta(\delta(\mathbf{x}) - \nu\sigma) = \begin{cases} 1 & \text{if} \quad \delta(\mathbf{x}) \geq \nu\sigma \\ 0 & \text{else.} \end{cases} \tag{1}$$

Note the qualitative difference between $\delta$ which is a weighted density field and $\theta_\nu$ which just defines a set on $V$, each point having equal weight. We note the following simple facts concerning the threshold density, $\theta_\nu$, due only to its definition, independent of the correlation properties of $\delta(\mathbf{x})$:

$$\langle\theta_\nu\rangle \equiv P_1(\nu) \leq 1 \ , \quad (\theta_\nu(\mathbf{x}))^n = \theta_\nu(\mathbf{x}) \ , \tag{2}$$

$$\langle\theta_\nu(\mathbf{x})\theta_\nu(\mathbf{x}+r\mathbf{n})\rangle \leq P_1(\nu) \ ,$$

$$\frac{\langle\theta_\nu(\mathbf{x})\theta_\nu(\mathbf{x}+r\mathbf{n})\rangle}{P_1(\nu)^2} - 1 \equiv \xi_\nu(r) \leq \xi_\nu(0) = \frac{1}{P_1(\nu)} - 1 \ ,$$

$$\theta_{\nu'}(\mathbf{x}) \leq \theta_\nu(\mathbf{x}) \ , \quad P_1(\nu') \leq P_1(\nu) \quad \text{for} \quad \nu' > \nu \ ,$$

$$\xi_{\nu'}(0) \geq \xi_\nu(0) \quad \text{for} \quad \nu' > \nu \ . \tag{3}$$

The enhancement of $\xi_\nu(0)$ for higher thresholds has clearly nothing to do with how 'strongly clustered' the peaks are but is entirely due to the fact that the larger $\nu$ the lower the fraction of points above the threshold (*i.e.* $P_1(\nu') < P_1(\nu)$ for $\nu' > \nu$). In the case of white noise, $\xi(r) = 0$ for $r > 0$ the peaks are just spikes. When a threshold $\nu\sigma$ is considered the number of spikes decreases and hence $\xi_\nu(0)$ is amplified because they are much more sparse and not because they are 'more strongly clustered': This is the point where the misconception of bias, as usually referred to [1,2] comes from. In fact, in Refs. [1,2] $\xi_\nu(r)$ is used as measure for the 'clustering strength' of

[1]In other words, we assume $\delta(\mathbf{x})$ to be a so called 'stationary normal stochastic process' [3].

[2]Clearly, cosmological density fluctuations can never be perfectly Gaussian since $\rho(\mathbf{x}) \geq 0$ and thus $\delta(\mathbf{x}) \geq -1$, but for small fluctuations Gaussianity is a good approximation. Furthermore, our results remain at least qualitatively correct also in the non-Gaussian case.

the peaks, while the information contained in $\xi_\nu(r)$ is a non-trivial combination of the noise level related to sparseness of the peaks and the clustering properties (related to the correlation length) of the system. As we clarify below, also in correlated systems, like for white noise, the bias introduced in Ref. [1] leads to an increase of the average distance between the peaks which is a measure of the noise level and not an increase of the correlation length ('strength').

In the context of cosmological density fluctuations, if the average density is well-defined [4], the amplitude of $\xi(r)$ is very important, since its integral over a given radius is proportional to the over density on this scale,

$$\sigma(R) = 3R^{-3}\int_0^R \xi(r) r^2 dr \ .$$

The scale $R_l$, where $\sigma(R_l) \sim 1$ separates large, non-linear fluctuations from small ones. In contrary, $\xi_\nu$ is not simply related to the fluctuation amplitude and, as will become clear from this work, it should by no means be used to quantify the amplitude of cosmological density fluctuations on any scale.

The joint two-point probability density $\mathcal{P}_2(\delta,\delta';r)$ depends on the distance $r$ between $\mathbf{x}$ and $\mathbf{x}'$, where $\delta = \delta(\mathbf{x})$ and $\delta' = \delta(\mathbf{x}')$. For Gaussian fields, $\mathcal{P}_2$, it is entirely determined by the 2-point correlation function $\xi(r)$ [5,3]:

$$\begin{aligned} &\mathcal{P}_2(\delta,\delta';r) = \\ &= \frac{1}{2\pi\sqrt{\sigma^4 - \xi(r)^2}} \exp\left(-\frac{\sigma^2(\delta^2+\delta'^2) - 2\xi(r)\delta\delta'}{2(\sigma^4 - \xi^2(r))}\right) \ . \end{aligned} \tag{4}$$

By definition

$$\xi(r) \equiv \langle \delta(\mathbf{x} + r\mathbf{n})\delta(\mathbf{x})\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\delta d\delta' \delta\delta' \mathcal{P}_2(\delta,\delta';r) \ . \tag{5}$$

The probability that both, $\delta$ and $\delta'$ are larger than $\nu\sigma$ is

$$P_2(\nu, r) = \int_{\nu\sigma}^{\infty}\int_{\nu\sigma}^{\infty} \mathcal{P}_2(\delta,\delta',r) d\delta d\delta' \equiv \langle \theta_\nu(\mathbf{x})\theta_\nu(\mathbf{x}+r\mathbf{n})\rangle \ . \tag{6}$$

The conditional probability that $\delta(\mathbf{y}) \geq \nu\sigma$, given $\delta(\mathbf{x}) \geq \nu\sigma$, where $|\mathbf{x} - \mathbf{y}| = r$, is then just $P_2(\nu,r)/P_1(\nu)$. The two-point correlation function for the stochastic variable $\theta_\nu(\mathbf{x})$, introduced above can be expressed in terms of $P_1$ and $P_2$ by

$$\xi_\nu(r) = \frac{P_2(\nu,r)}{P_1^2(\nu)} - 1 \tag{7}$$

If we set $\xi_c(r) = \xi(r)/\sigma^2$, we obtain

$$\begin{aligned} P_1(\nu)^2(\xi_\nu(r)+1) &= \frac{1}{2\pi\sqrt{1-\xi_c^2}}\int_\nu^\infty\int_\nu^\infty dx dx' \\ &\times \exp\left(-\frac{(x^2+x'^2) - 2\xi_c(r)xx'}{2(1-\xi_c^2(r))}\right) \end{aligned} \tag{8}$$

which implies for $\nu \gg 1$ and for sufficiently large $r$ such that $\xi_c(r) \ll 1$ [6]

$$\xi_\nu(r) \simeq \exp\left(\nu^2\xi_c(r)\right) - 1 \ , \tag{9}$$

in lowest non-vanishing order in $\xi_c(r)$. If, in addition, $\nu^2\xi_c(r) \ll 1$ we find [6]

$$\xi_\nu(r) \simeq \nu^2\xi_c(r) \ . \tag{10}$$

This is the relation derived by Kaiser [1]. He only states the condition $\xi_c(r) \ll 1$ and separately $\nu \gg 1$, which is significantly weaker than the required $\nu^2\xi_c(r) \simeq \xi_\nu(r) \ll 1$, especially around the correlation length where $\xi$ is not yet very small.

It is important to note that in the cosmologically relevant regime, $\xi_\nu \gtrsim 1$ the Kaiser relation (10) does not apply and $\xi_\nu$ is actually exponentially enhanced. If this mechanism would be the cause for the observed cluster correlation function one would thus expect an exponential enhancement on scales where $\xi_{cc} \gtrsim 1$, *i.e.* $R \lesssim 20h^{-1}$Mpc. This is in complete contradiction with observations [7]![3]

If, within a range of scales, $\xi(r)$ can be approximated by a power law, $\xi = (\frac{r}{r_0})^{-\gamma}$, and if the threshold $\nu$ is such that Eq. (10) holds, which implies $\xi_\nu \ll 1$, it is of the form $\xi_\nu = (\frac{r}{r_\nu})^{-\gamma}$ where the scales $r_\nu$ for different biases are related by $r_{\nu'} = r_\nu(\nu'/\nu)^{2/\gamma}$ . For that reason Kaiser, who first derived relation (10), interpreted it as an increase in the 'correlation length', i.e. an increase of the 'clustering strength'[4] $r_\nu$.

In general a correlation function is characterized by two quantities, an amplitude $A$ strictly related to the dilution of objects or the noise level and a correlation length $r_c$ (which may however be infinite). In the above example a meaningful choice would be $A = (r_{\min}/r_0)^{-\gamma}$, where $r_{\min}$ is the lower cutoff (or smoothing scale) which has to be introduced.

In order to clarify the meaning of the amplitude of the correlation function, $A$ and the correlation length $r_c$, we first study an example with finite correlation

[3] One might argue that non-linearities which are important when the fluctuations are large can 'rescue' the Kaiser relation (10) also into the regime $\xi_\nu > 1$. There are two objections against this: First of all, as we pointed out above, $\xi_\nu > 1$ does not imply large fluctuations. Actually most cosmologists would agree that on $R \sim 20h^{-1}$Mpc, where the cluster correlation function, $\xi_{cc} \sim 1$, fluctuations are linear. Secondly, it seems very unphysical that Newtonian clustering should act as to change the exponential relation (9) into a linear one (10).

[4] Clearly, in the sense of statistical mechanics, power law correlation functions indicate infinite correlation length. Therefore, $r_\nu$ is not a correlation length in the statistical mechanics definition of this term [3,4]. Since it is called so in the cosmology literature, we use this term here in quotation marks.

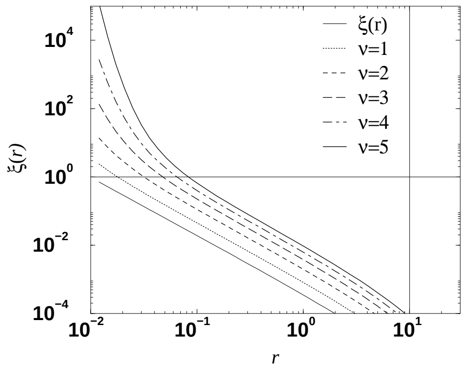


FIG. 1. Behavior of $\xi(r) \sim \sigma^2 (r/r_0)^{-\gamma} \exp(-r/r_c)$ (where $\gamma = -1.7$, $r_0 = 0.3$ and $r_c = 10$) and $\xi_\nu(r)$ are shown for different values of the threshold $\nu$ in a log-log plot.

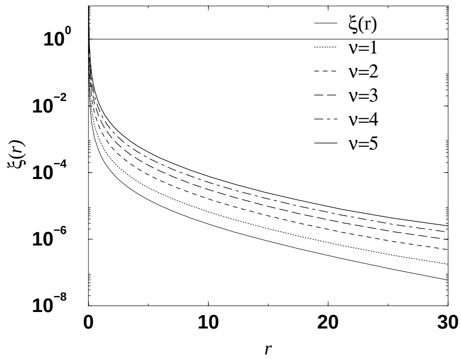


FIG. 2. As Fig. 1 but in a semi-log plot. The slope of $\xi_\nu(r)$ for $r \gtrsim 10$ is $-1/r_c$, independent of $\nu$ *i.e.* the correlation length of the system does not change for the peaks above the threshold.

length, which, however, is well approximated by a power law on a certain range of scales. We set $\xi(r) = (r/r_0)^{-\gamma} \exp(-r/r_c)$ with $r_0 \ll r_c$. In the region $r \ll r_c$ $\xi$ is well approximated by the power law $(r/r_0)^{-\gamma}$. The correlation length, $r_c$ is given by the slope of $\log \xi_\nu(r)$ at large $r$ which is clearly independent of bias (see Figs 1 and 2). In order to investigate whether $\xi_\nu(r) \sim (r/r_\nu)^{-\gamma_\nu}$, we plot $-d\log(\xi_\nu(r))/d\log(r) \sim \gamma_\nu$ in Fig 3. Only in the regime where $\xi_\nu(r) \ll 1$, $\gamma_\nu$ becomes constant and roughly independent of $\nu$. This behavior is very different from the one found in galaxy catalogs!

We also want to analyse briefly a more realistic example with a lower cutoff (smoothing scale) but infinite correlation length:

$$\xi(r) = \sigma^2/(1 + (r/r_{\min})^\gamma) \ . \tag{11}$$

On scales $r_{\min} < r < r_c$ this example should not differ very much from the above, but of course the correlation length is infinite here. The amplification of $\xi_\nu$ for this example is plotted in Fig. 4 and the scale dependence of the spectral index is shown in Fig. 5.

We have shown that bias does not influence the correlation length. It amplifies the correlation function by the fact that the mean density, $P_1(\nu)$, is reduced more

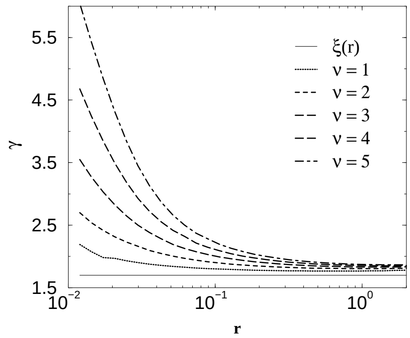


FIG. 3. The behavior of $\gamma_\nu(r)$ is shown for different values of the threshold $\nu$ for the correlation function shown in Figs. 1 and 2. Clearly $\gamma_\nu$ is strongly scale dependent on all scales where $\xi_\nu \gtrsim 1$, this is $r \lesssim 0.1$ in our units.

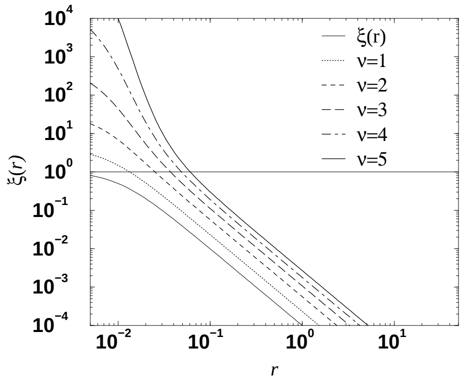


FIG. 4. Behavior of $\xi(r) \sim \sigma^2/(1 + (r/r_{\min})^\gamma)$ (with $\gamma = -1.7$, $r_{\min} = 0.01$) and $\xi_\nu(r)$ are shown for different values of the threshold $\nu$ in a log-log plot.

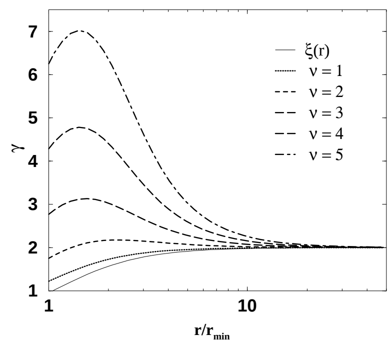


FIG. 5. The behavior of $\gamma_\nu(r)$ is shown for different values of the threshold $\nu$ for the correlation function shown in Figs. 3 and 4. Clearly $\gamma_\nu$ is strongly scale dependent on all scales where $\xi_\nu \gtrsim 1$, this is $r < 1$ in our units.

strongly than the conditional density, $P_2(\nu, r)/P_1(\nu)$. The fact that the amplitude of $\xi_\nu(r)$ increases with the threshold does not imply that the peaks of the Gaussian field are "more clustered" but that they are more sparse. According to Eq. (9), this amplification is very significant (more than exponential!) within correlated regions and becomes weak (linear) beyond the correlation length. If the correlation length is well defined the distribution is 'more strongly clustered' if the correlation length, and not the amplitude of $\xi(r)$, is larger.

This can also be understood when comparing the mean peak size, $R_p$ and the mean peak distance, $D_p$. For a Gaussian random field it is

$$R_p \simeq \sqrt{2\pi}\frac{R_0}{\nu} \quad \text{and} \quad D_p \simeq 2\pi R_0 \exp(\nu^2/2)$$
$$\text{so that} \quad D_p/R_p \simeq \nu \exp(\nu^2/2) \text{ for } \nu \gg 1 \; . \qquad (12)$$

Here $R_0$ is the ratio of the variances of the Gaussian random field and its derivative [8] which is of the order of the smoothing scale of the field, $R_0 \sim \sqrt{\xi(0)/\xi''(0)}$. It is mainly the inter peak distance, a measure of the peak sparseness, which increases.

Finally, we want to stress once more that the biasing mechanism introduced by Kaiser and discussed in this work cannot lead to a relation of the form $\xi_{\nu'}(r) = \alpha_{\nu'\nu}\xi_\nu(r)$ over a range of scales $r_1 < r < r_2$ such that $1 < \xi_\nu(r_1)$ and $\xi_\nu(r_2) < 1$. But exactly this behavior is found in galaxy and cluster catalogs. For example in [7] or [9], a constant biasing factor $\alpha_{\nu'\nu}$ over a range from about $1h^{-1}$Mpc to $20h^{-1}$Mpc is observed for correlation amplitudes varying from about 20 to 0.1. We therefore conclude that the explanation by Kaiser [1] cannot be the origin of the difference of the correlation functions observed in the distribution of galaxies with different intrinsic magnitude or in the distribution of clusters with different richness.

This result appears at first disappointing since it invalidates an explanation without proposing a new one. On the other hand, the search for an explanation of an observed phenomenon is only motivated if we are fully aware of the fact that we don't already have one.

Last but not least, we want to point out that fractal density fluctuations together with the fact that more luminous objects are seen out to larger distances do actually induce a increase in the amplitude of the correlation function $\xi(r)$ similar to the one observed in real galaxy catalogs [4]. In this explanation, the linear amplification found for the correlation function, has nothing to do with a correlation length but is a pure finite size effect, and the distribution of galaxies does not have any intrinsic characteristic scale.

It is a pleasure to thank L. Pietronero, A. Baldassarri, T. Buchert, P. Ferreira, R. Juszkiewicz, N. Kaiser, N. Turok, and F. Vernizzi for useful discussions, suggestions and comments. This work is partially supported by the EEC TMR Network "Fractal structures and self-organization" ERBFMRXCT980183 and by the Swiss NSF.

[1] N. Kaiser, Astrophys. J. Lett. **284**, L9 (1984).
[2] J. Bardeen, J.R. Bond, N. Kaiser and A. Szalay, Astrophys. J. **304**, 15 (1986).
[3] W. Feller, *An Introduction to Probability Theory and its Applications*, vol. 2, ed. J. Wiley& Sons, (New York, 1965).
[4] Pietronero Physica A **144**, 257 (1987), Coleman, P.H. & Pietronero, L., Phys.Rep. **231**, 311 (1992), Sylos Labini F., Montuori M., Pietronero L., Phys.Rep. **293**, 66 (1998).
[5] S.O. Rice in *Noise and Stochastic Processes*, ed. N. Wax, Dover Publications, (New York, 1954).
[6] H.D. Politzer & M.B. Wise Astrophys. J. Lett. **285**, L1 (1984).
[7] N. Bahcall and R. Soneira, Astron. Astrophys. **270**, 20 (1983).
[8] E. Vanmarcke, *Random Fields*, MIT press (Cambridge MA, 1983).
[9] C. Benoist *et al.*, Astrophys. J. **472**, 452 (1996).