Anisotropies in the Cosmic Microwave Background: Theoretical Foundations

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Abstract. The analysis of anisotropies in the cosmic microwave background (CMB) has become an extremely valuable tool for cosmology. We even have hopes that planned CMB anisotropy experiments may revolutionize cosmology. Together with determinations of the CMB spectrum, they represent the first cosmological precision measurements. This is illustrated in the talk by Anthony Lasenby. The value of CMB anisotropies lies to a big part in the simplicity of the theoretical analysis. Fluctuations in the CMB can be determined almost fully within linear cosmological perturbations theory and are not severely influenced by complicated nonlinear physics.

In this contribution the different physical processes causing or influencing anisotropies in the CMB are discussed. The geometry perturbations at and after last scattering, the acoustic oscillations in the baryon–photon–plasma prior to recombination, and the diffusion damping during the process of recombination.

The perturbations due to the fluctuating gravitational field, the so called Sachs–Wolfe contribution, is described in a very general form using the Weyl tensor of the perturbed geometry.

1 Introduction

The formation of cosmological structure in the universe, inhomogeneities in the matter distribution like quasars at redshifts up to $z \sim 5$, galaxies, clusters, super clusters, voids and walls, is an outstanding basically unsolved problem within the standard model of cosmology. We assume, that the observed inhomogeneities formed from small initial fluctuations by gravitational clustering.

At first sight it seems obvious that small density enhancements can grow sufficiently rapidly by gravitational instability. But global expansion of the universe and radiation pressure counteract gravity, so that, e.g., in the case of a radiation dominated, expanding universe no density inhomogeneities can grow significantly. Even in a universe dominated by pressure-less matter, cosmic dust, growth of density perturbations is strongly reduced by the expansion of the universe.

Furthermore, we know that the universe was extremely homogeneous and isotropic at early times. This follows from the isotropy of the 3K Cosmic Microwave Background (CMB), which represents a relic of the plasma of baryons, electrons and radiation at times before protons and electrons combined to neutral hydrogen. After a long series of upper bounds, measurements with the DMR instrument aboard the COsmic Background Explorer satellite (COBE) have finally established anisotropies in this radiation [1] at the level of

$$\left\langle \frac{(T(\boldsymbol{n}) - T(\boldsymbol{n}'))^2}{T^2} \right\rangle_{(\mathbf{n}\cdot\mathbf{n}' = cos\theta)} \sim 10^{-10} \quad \text{on angular scales } 7^o \le \theta \le 90^o \ .$$

Such an angle independent spectrum of fluctuations on large angular scales is called Harrison Zel'dovich spectrum [2]. It is defined by yielding constant mass fluctuations on horizon scales at all time, i.e., if $l_H(t)$ denotes the expansion scale at time t,

$$\langle (\Delta M/M)^2(\lambda = l_H) \rangle = \text{ const. } , \quad \text{independent of time.}$$

The COBE result, the observed spectrum and amplitude of fluctuations, strongly support the gravitational instability picture.

Presently, there exist two main classes of models which predict a Harrison–Zel'dovich spectrum of primordial fluctuations: In the first class, quantum fluctuations expand to super Hubble scales during a period of inflationary expansion in the very early universe and 'freeze in' as classical fluctuations in energy density and geometry [3] (*see also the contribution by V. Mukhanov*). In the second class, a phase transition in the early universe, at a temperature of about 10^{16}GeV leads to topological defects which induce perturbations in the geometry and in the matter content of the universe [4]. Both classes of models are in basic agreement with the COBE findings, but differ in their prediction of anisotropies on smaller angular scales.

On smaller angular scales the observational situation is at present somewhat confusing and contradictory [5, 6], but many anisotropies have been measured with a maximum of about $\Delta T/T \approx (3 \pm 2) \times 10^{-5}$ at angular scale $\theta \approx (1 \pm 0.5)^o$. There is justified hope, that the experiments planned and under way will improve this situation within the next few years (see contribution by A. Lasenby) In Fig. 1, the experimental situation as of spring '96 is presented.

In this paper we outline a formal derivation of general formulas which can be used to calculate the CMB anisotropies in a given cosmological model. Since we have the chance

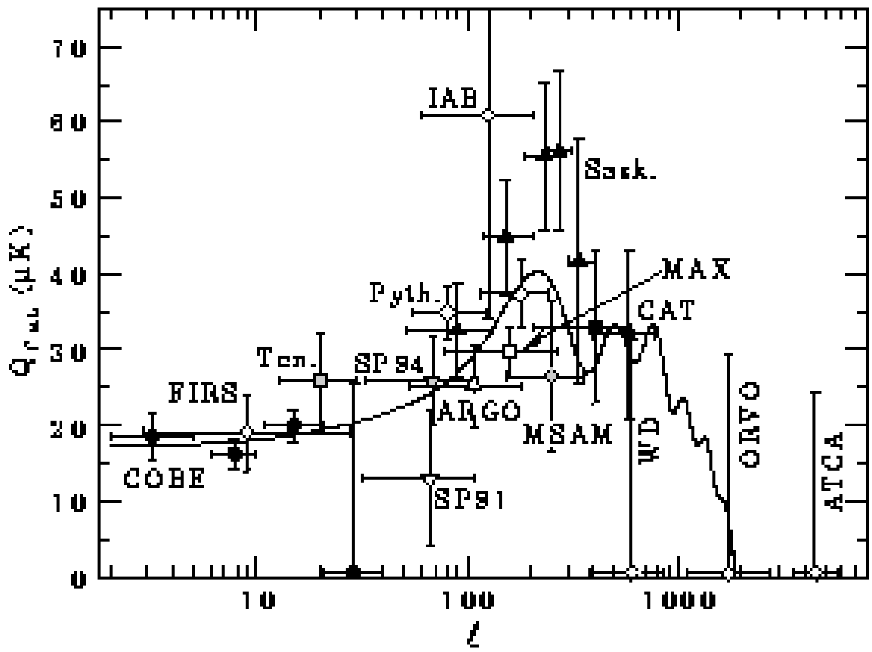

Figure 1: The corresponding quadrupole amplitude Q_{flat} is shown versus the corresponding spherical harmonic index ℓ. The amplitude $Q_{flat}(\ell)$ corresponds roughly to the temperature fluctuation on the angular scale $\theta \sim \pi/\ell$. The solid line indicates the predictions from a standard cold dark matter model. (Figure taken from ref. [5]).

to address a community of relativists, we make full use of the relativistic formulation of the problem. In Section 2 we derive Liouville's equation for massless particles in a perturbed Friedmann universe. In Section 3 we discuss the effects of non-relativistic Compton scattering prior to decoupling. This fixes the initial conditions for the solution to the Liouville equation and leads to a simple approximation of the effect of collisional damping. In the next Section we illustrate our results with a few simple examples. Finally, we summarize our conclusions.

Notation: We denote conformal time by t. Greek indices run from 0 to 3, Latin indices run from 1 to 3. The metric signature is chosen $(- + ++)$. The Friedmann metric is thus given by $ds^2 = a^2(t)(-dt^2 + \gamma_{ij}dx^i dx^j)$, where γ denotes the metric of a 3–space with constant curvature K. Three dimensional vectors are denoted by bold face symbols.
We set $\hbar = c = k_{Boltzmann} = 1$ throughout.

2 The Liouville equation for massless particles

2.1 Generalities

Collision-less particles are described by their one particle distribution function which lives on the seven dimensional phase space

$$\mathcal{P}_m = \{(x,p) \in T\mathcal{M} | g(x)(p,p) = -m^2\} \ .$$

Here $\mathcal{M}$ denotes the spacetime manifold and $T\mathcal{M}$ its tangent space. The fact that collision-less particles move on geodesics translates to the Liouville equation for the one particle distribution function, f. The Liouville equation reads [7]

$$X_g(f) = 0 \ . \tag{2.1}$$

In a tetrad basis $(e_\mu)_{\mu=0}^3$ of $\mathcal{M}$, the vector field X_g on $\mathcal{P}_m$ is given by (*see*, e.g., [7])

$$X_g = (p^\mu e_\mu - \omega^i_{\ \mu}(p) p^\mu \frac{\partial}{\partial p^i}) \ , \tag{2.2}$$

where $\omega^\nu_{\ \mu}$ are the connection 1–forms of $(\mathcal{M}, g)$ in the basis e^μ, and we have chosen the basis

$$(e_\mu)_{\mu=0}^3 \quad \text{and} \quad (\frac{\partial}{\partial p^i})_{i=1}^3 \quad \text{on} \quad T\mathcal{P}_m \ , \qquad p = p^\mu e_\mu \ .$$

We now show that for massless particles and conformally related metrics,

$$g_{\mu\nu} = a^2 \tilde{g}_{\mu\nu} \ ,$$

$$(X_g f)(x,p) = 0 \quad \text{is equivalent to} \quad (X_{\tilde{g}} f)(x, ap) = 0 \ . \tag{2.3}$$

This is easily seen if we write X_g in a coordinate basis:

$$X_g = b^\mu \partial_\mu - \Gamma^i_{\alpha\beta} b^\alpha b^\beta \frac{\partial}{\partial b^i} \ ,$$

with

$$\Gamma^i_{\alpha\beta} = \frac{1}{2} g^{i\mu} (g_{\alpha\mu},_\beta + g_{\beta\mu},_\alpha - g_{\alpha\beta},_\mu) \ .$$

The variables b^μ are the components of the momentum p with respect to the *coordinate* basis:

$$p = p^\mu e_\mu = b^\mu \partial_\mu \ .$$

If (e_μ) is a tetrad with respect to g, then $\tilde{e}_\mu = a e_\mu$ is a tetrad basis for $\tilde{g}$. Therefore, the coordinates of of $ap = ap^\mu \tilde{e}_\mu = a^2 p^\mu e_\mu = a^2 b^\mu \partial_\mu$, with respect to the basis ∂_μ on $(\mathcal{M}, \tilde{g})$ are given by $a^2 b^\mu$. In the coordinate basis thus our statement Eq. (2.3) follows, if we can show that

$$(X_{\tilde{g}} f)(x^\mu, a^2 b^i) = 0 \quad \text{iff} \quad (X_g f)(x^\mu, b^i) = 0 \tag{2.4}$$

Setting $v = ap = v^\mu \tilde{e}_\mu = w^\mu \partial_\mu$, we have $v^\mu = ap^\mu$ and $w^\mu = a^2 b^\mu$. Using $p^2 = 0$, we obtain the following relation for the Christoffel symbols of g and $\tilde{g}$:

$$\Gamma^i_{\alpha\beta} b^\alpha b^\beta = \tilde{\Gamma}^i_{\alpha\beta} b^\alpha b^\beta + \frac{2a,_\alpha}{a} b^\alpha b^i \ .$$

For this step it is crucial that the particles are massless! For massive particles the statement is of course not true. Inserting this result into the Liouville equation we find

$$a^2 X_g f = w^\mu (\partial_\mu f|_b - 2\frac{a,_\mu}{a} b^i \frac{\partial f}{\partial b^i}) - \tilde{\Gamma}^i_{\alpha\beta} w^\alpha w^\beta \frac{\partial f}{\partial w^i} \ , \tag{2.5}$$

where $\partial_\mu f|_b$ denotes the derivative of f w.r.t. x^μ at constant (b^i). Using

$$\partial_\mu f|_b = \partial_\mu f|_w + 2\frac{a,_\mu}{a} b^i \frac{\partial f}{\partial b^i} \ ,$$

we see, that the braces in Eq. (2.5) just correspond to $\partial_\mu f|_w$. Therefore,

$$a^2 X_g f(x,p) = w^\mu \partial_\mu f|_w - \tilde{\Gamma}^i_{\alpha\beta} w^\alpha w^\beta \frac{\partial f}{\partial w^i} = X_{\tilde{g}} f(x, ap) \ ,$$

which proves our claim. This statement is just a precise way of expressing conformal invariance of massless particles.

2.2 Free, massless particles in a perturbed Friedmann universe

We now apply this general framework to the case of a perturbed Friedmann universe. For simplicity, we restrict our analysis to the case $K =$, i.e., $\Omega = 1$. The metric of a perturbed Friedmann universe with density parameter $\Omega = 1$ is given by $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ with

$$g_{\mu\nu} = a^2(\eta_{\mu\nu} + h_{\mu\nu}) = a^2 \tilde{g}_{\mu\nu} \ , \tag{2.6}$$

where $(\eta_{\mu\nu}) = diag(-,+,+,+)$ is the flat Minkowski metric and $(h_{\mu\nu})$ is a small perturbation, $|h_{\mu\nu}| \ll 1$.

From Eq. (2.3), we conclude that the Liouville equation in a perturbed Friedmann universe is equivalent to the Liouville equation in perturbed Minkowski space,

$$(X_{\tilde{g}} f)(x, v) = 0 \ , \tag{2.7}$$

with $v = v^\mu \tilde{e}_\mu = ap^\mu \tilde{e}_\mu$.[1]

We now want to derive a linear perturbation equation for Eq. (2.7). If $\bar{e}^\mu$ is a tetrad in Minkowski space, $\tilde{e}_\mu = \bar{e}_\mu + \frac{1}{2} h^\nu_\mu \bar{e}_\nu$ is a tetrad w.r.t the perturbed geometry $\tilde{g}$. For

[1] Note that also Friedmann universes with non vanishing spatial curvature, $K \neq 0$, are conformally flat and thus this procedure can also be applied for $K \neq 0$. Of course, in this case the conformal factor a^2 is no longer just the scale factor but depends on position. A coordinate transformation which transforms the metric of $K \neq 0$ Friedmann universes into a conformally flat form can be found, e.g., in [8].

$(x, v^\mu \bar{e}_\mu) \in \bar{P}_0$, thus, $(x, v^\mu \tilde{e}_\mu) \in \tilde{P}_0$. Here $\bar{P}_0$ denotes the zero mass one particle phase space in Minkowski space and $\tilde{P}_0$ is the phase space with respect to $\tilde{g}$, perturbed Minkowski space. We define the perturbation, F, of the distribution function by

$$f(x, v^\mu \tilde{e}_\mu) = \bar{f}(x, v^\mu \bar{e}_\mu) + F(x, v^\mu \bar{e}_\mu) \ . \tag{2.8}$$

Liouville's equation for f then leads to a perturbation equation for F. We choose the natural tetrad

$$\tilde{e}_\mu = \partial_\mu - \frac{1}{2} h^\nu_\mu \partial_\nu$$

with the corresponding basis of 1–forms

$$\tilde{\theta}^\mu = dx^\mu + \frac{1}{2} h^\mu_\nu dx^\nu \ .$$

Inserting this into the first structure equation, $d\tilde{\theta}^\mu = -\omega^\mu{}_\nu \wedge dx^\nu$, one finds

$$\omega_{\mu\nu} = -\frac{1}{2}(h_{\mu\lambda},_\nu - h_{\nu\lambda},_\mu)\theta^\lambda \ .$$

Using the background Liouville equation, namely that $\bar{f}$ is only a function of $v = ap$, we obtain the perturbation equation

$$(\partial_t + n^i \partial_i) F = -\frac{v}{2}[(\dot{h}_{i0} - h_{00},_i)n^i + (\dot{h}_{ij} - h_{0j},_i)n^i n^j]\frac{d\bar{f}}{dv} \ ,$$

where we have set $v_i = v n_i$, with $v^2 = \sum_{i=1}^3 (v_i)^2$, i.e., $\mathbf{n}$ gives the momentum direction of the particle. Let us parameterize the perturbations of the metric by

$$(h_{\mu\nu}) = \left(\begin{array}{ll} -2A & B_i \\ B_i & 2H_L \delta_{ij} + 2H_{ij} \end{array} \right), \tag{2.9}$$

with $H^i_i = 0$. Inserting this above we obtain

$$(\partial_t + n^i \partial_i) F = -[\dot{H}_L + (A,_i + \frac{1}{2}\dot{B}_i)n^i + (\dot{H}_{ij} - \frac{1}{2}B_{i,j})n^i n^j] v \frac{d\bar{f}}{dv} \ . \tag{2.10}$$

From Eq. (2.10) we see that the perturbation in the distribution function in each spectral band is proportional to $v\frac{d\bar{f}}{dv}$. This shows once more that gravity is achromatic. We thus do not loose any information if we integrate this equation over photon energies. We define

$$m = \frac{\pi}{\rho_r a^4} \int F v^3 dv \ .$$

$4m$ is the fractional perturbation of the brightness ι,

$$\iota = a^{-4} \int f v^3 dv \ .$$

Setting $\iota(\mathbf{n}, \mathbf{x}) = \bar{\iota}(T(\mathbf{n}, \mathbf{x}))$, one obtains that $\iota = (\pi/60) T^4(\mathbf{n}, \mathbf{x})$. Hence, m corresponds to the fractional perturbation in the temperature,

$$T(\mathbf{n}, \mathbf{x}) = \bar{T}(1 + m(\mathbf{n}, \mathbf{x})) \ . \tag{2.11}$$

Another derivation of Eq. (2.11) is given in [10]. According to Eq. (2.10), the v dependence of F is of the form $v\frac{d\bar{f}}{dv}$. Using now

$$4\pi\int\frac{d\bar{f}}{dv}v^4dv = -4\int\bar{f}v^3dvd\Omega = -4\rho_r a^4 \ , \tag{2.12}$$

we find

$$F(x^\mu,n^i,v) = -m(x^\mu,n^i)v\frac{d\bar{f}}{dv} \ .$$

This shows that m is indeed the quantity which is measured in a CMB anisotropy experiment, where the spectral information is used to verify that the spectrum of perturbations is the derivative of a blackbody spectrum. Of course, in a real experiment located at a fixed position in the Universe, the monopole and dipole contributions to m cannot be measured. They cannot be distinguished from a background component and from a dipole due to our peculiar motion w.r.t. the CMB radiation.

Multiplying Eq. (2.10) with v^3 and integrating over v, we obtain the equation of motion for m

$$\partial_t m + n^i\partial_i m = \dot{H}_L + (A,_i + \frac{1}{2}\dot{B}_i)n^i + (\dot{H}_{ij} - \frac{1}{2}B_{i},_j)n^in^j \ . \tag{2.13}$$

It is well known that the equation of motion for photons only couples to the Weyl part of the curvature (null geodesics are conformally invariant). However, the r.h.s. of Eq. (2.13) is given by first derivatives of the metric only which could at best represent integrals of the Weyl tensor. To obtain a local, non integral equation, we thus rewrite Eq. (2.13) in terms of $\nabla^2 m$. It turns out, that the most suitable variable is however not $\nabla^2 m$ but χ, which is defined by

$$\chi \equiv \nabla^2 m - (\nabla^2 H_L - \frac{1}{2}H,_{ij}^{ij}) - \frac{1}{2}(\nabla^2 B_i - 3\partial^j\sigma_{ij})n^i \ ,$$

$$\text{where} \quad \sigma_{ij} \equiv -\frac{1}{2}(B_i,_j + B_j,_i) + \frac{1}{3}\delta_{ij}B_l^{;l} + \dot{H}_{ij}.$$

Note that χ and $\nabla^2 m$ only differ by the monopole contribution, $\nabla^2 H_L - (1/2)H^{ij},_{ij}$, and the dipole term, $(1/2)(\nabla^2 B_i - 3\partial^j\sigma_{ij})n^i$. The higher multipoles of χ and $\nabla^2 m$ agree. An observer at fixed position and time cannot distinguish a monopole contribution from an isotropic background and a dipole contribution from a peculiar motion. Only the higher multipoles, $l \geq 2$ contain information about temperature anisotropies. For a fixed observer therefore, we can identify $\nabla^{-2}\chi$ with $\delta T/T$.

In terms of metric perturbations, the electric and magnetic part of the Weyl tensor are given by (*see*, e.g. [11, 10])

$$\begin{aligned} \mathcal{E}_{ij} &= \frac{1}{2}[\triangle_{ij}(A - H_L) - \dot{\sigma}_{ij} - \nabla^2 H_{ij} - \frac{2}{3}H_{lm}^{;lm}\delta_{ij} + H_{il}^{;l},_j + H_{jl}^{;l},_i] && (2.14)\\ \mathcal{B}_{ij} &= -\frac{1}{2}(\epsilon_{ilm}\sigma_{jm},_l + \epsilon_{jlm}\sigma_{im},_l) \ , && (2.15)\end{aligned}$$

$$\text{with} \quad \triangle_{ij} = \partial_i\partial_j - (1/3)\delta_{ij}\nabla^2 \ .$$

Explicitly working out $(\partial_t + n^i\partial_i)\chi$ using Eq. (2.13), yields after some algebra the equation of motion for χ:

$$(\partial_t + n^i\partial_i)\chi = 3n^i\partial^j\mathcal{E}_{ij} + n^k n^j \epsilon_{kli}\partial_l\mathcal{B}_{ij} \equiv \mathcal{S}(t, \boldsymbol{x}, \boldsymbol{n}) \ , \tag{2.16}$$

where ϵ_{kli} is the totally antisymmetric tensor in three dimensions with $\epsilon_{123} = 1$. The spatial indices in this equation are raised and lowered with δ_{ij} and thus index positions are irrelevant. Double indices are summed over irrespective of their positions.

Eq. (2.16) is the main result of this paper. We now discuss it, rewrite it in integral form and specify initial conditions for adiabatic scalar perturbations with or without seeds.

In Eq. (2.16) the contribution from the electric part of the Weyl tensor is a divergence, and therefore does not contain tensor perturbations. On the other hand, scalar perturbations do not induce a magnetic gravitational field. The second contribution to the source term in Eq. (2.16) thus represents a combination of vector and tensor perturbations. If vector perturbations are negligible (like, e.g., in models where initial fluctuations are generated during an epoch of inflation), the two terms on the r.h.s of Eq. (2.16) yield thus a split into scalar and tensor perturbations which is local.

Since the Weyl tensor of Friedmann Lemaître universes vanishes, the r.h.s. of Eq. (2.16) is manifestly gauge invariant (this is the so called Stewart–Walker lemma [12]). Hence also the variable χ is gauge invariant. Another proof of the gauge invariance of χ, discussing the behavior of F under infinitesimal coordinate transformations is presented in [10].

The general solution of Eq. (2.16) is given by

$$\chi(t, \boldsymbol{x}, \boldsymbol{n}) = \int_{t_i}^{t} \mathcal{S}(t', \boldsymbol{x} + (t'-t)\boldsymbol{n}, \boldsymbol{n})dt' \ + \ \chi(t_i, \boldsymbol{x} + (t_i - t)\boldsymbol{n}, \boldsymbol{n}) \ , \tag{2.17}$$

where $\mathcal{S}$ is the source term on the r.h.s. of Eq. (2.16).

In Appendix A we derive the relations between the geometric source term $\mathcal{S}$ and the energy momentum tensor in a perturbed Friedmann universe.

3 The collision term

In order for Eq. (2.17) to provide a useful solution, we need to determine the correct initial conditions, $\chi(t_{dec})$, at the moment of decoupling of matter and radiation. Before recombination, photons, electrons and baryons form a tightly coupled plasma, and thus χ can not develop higher moments in $\boldsymbol{n}$. The main collision process is non–relativistic Compton scattering of electrons and photons. The only non vanishing moments in the distribution function before decoupling are the zeroth, i.e., the energy density, and the first, the energy flow. We therefore set

$$\chi(t_{dec}) = \nabla^2\left(\frac{1}{4}D_g^{(r)}(t_{dec}) - \boldsymbol{n}\cdot\boldsymbol{V}^{(r)}(t_{dec})\right) \ , \tag{3.1}$$

where

$$
\begin{aligned}
D_g^{(r)}(t_{dec}) &= \nabla^{-2}\left(\frac{1}{\pi}\int\chi(t_{dec})d\Omega\right) && (3.2)\\
&= \frac{\delta\rho^{(r)}}{\rho} - 4H_L + 2\nabla^{-2}(H_{ij}^{|ij}) \quad \text{and} \\
V^{j(r)}(t_{dec}) &= -\nabla^{-2}\left(\frac{3}{4\pi}\int\chi(t_{dec})n^j d\Omega\right) && (3.3)\\
&= -T_0^{(r)j}/(\frac{4}{3}\rho^{(r)}) + B^i - \frac{3}{2}\nabla^{-2}(\partial^i\sigma_{ij}) \ .
\end{aligned}
$$

$D_g^{(r)}$ and $\boldsymbol{V}^{(r)}$ are gauge invariant density and velocity perturbation variables [9, 10].

In the tight coupling or fluid limit, the initial conditions can also be obtained from the collision term. Setting $\mathcal{M} \equiv \nabla^{-2}\chi$ one finds the following expression for the collision integral [10],

$$
C[\mathcal{M}] = a\sigma_T n_e[\frac{1}{4}D_g^{(r)} - \mathcal{M} + \boldsymbol{n}\cdot V^{(b)} + \frac{1}{2}n_i n_j M^{ij}] \ . \tag{3.4}
$$

The last term is due to the anisotropy of the cross section for non–relativistic Compton scattering, with

$$
M^{ij} = \frac{3}{8\pi}\int(n^i n^j - \frac{1}{3}\delta^{ij})\mathcal{M}d\Omega \ .
$$

$\mathcal{M}$ is a gauge invariant perturbation variable for the distribution function of photons. $\boldsymbol{V}^{(b)}$ denotes the baryon velocity field, σ_T and n_e are the Thomson cross section and the free electron density respectively. To make contact with other literature, we note that $\mathcal{M} = \Theta + \Phi$, where Θ is the perturbation variable describing the CMB anisotropies defined in [13] and Φ denotes a Bardeen potential (*see Section 4*). Since $\mathcal{M}$ and Θ differ only by a monopole term, they give rise to the same spectrum of temperature anisotropies for $\ell \geq 1$. $\mathcal{M}$ satisfies the Boltzmann equation

$$
(\partial_t + n^i\partial_i)\mathcal{M} = \nabla^{-2}\mathcal{S} + C[\mathcal{M}] \ , \tag{3.5}
$$

where $\mathcal{S}$ is the gravitational source term given in Eq. (2.16). In the tight coupling limit, $t_T \equiv (a\sigma_T n_e)^{-1} \ll t$, we may, to lowest order in (t_T/t), just set the square bracket on the right hand side of Eq. (3.4) equal to zero. Together with Eq. (3.3) this yields

$$
\boldsymbol{V}^{(b)} = \boldsymbol{V}^{(r)} \ .
$$

Neglecting gravitational effects, the right hand side of Boltzmann's equation then leads to

$$
\dot{D}_g^{(r)} = \frac{4}{3}\boldsymbol{\nabla}\cdot\boldsymbol{V}^{(b)} = \frac{4}{3}D_g^{(b)} \ , \tag{3.6}
$$

where the last equal sign is due to baryon number conservation. In other words, photons and baryons are adiabatically coupled. Expanding Eq. (3.5) one order higher in t_T, one obtains Silk damping [14], the damping of radiation perturbations due to imperfect coupling.

Let us estimate this damping by neglecting gravitational effects and the time dependence of the coefficients in the Boltzmann equation (3.5) since we are interested in time scales $t_T \ll t$. We can then look for solutions of the form

$$
V^{(b)} \propto \mathcal{M} \propto \exp(i(\boldsymbol{kx} - \omega t)) \ .
$$

We also neglect the angular dependence of the collision term. Solving Eq. (3.5) for $\mathcal{M}$, we then find

$$\mathcal{M} = \frac{(1/4)D_g^{(r)} + i\boldsymbol{k}\cdot\boldsymbol{n}V^{(b)}}{1 - it_T(\omega - \boldsymbol{k}\cdot\boldsymbol{n})} \ . \tag{3.7}$$

The collisions also induce a drag force in the equation of motion of the baryons which is given by

$$F_i = \frac{a\sigma_T n_e \rho_r}{\pi}\int C[\mathcal{M}]n_i d\Omega = \frac{4\rho_r}{3t_T}(\boldsymbol{V}^{(r)} - i\boldsymbol{k}V^{(b)}) \ .$$

With this force, the baryon equation of motion becomes

$$\boldsymbol{k}\omega V^{(b)} + i(\dot{a}/a)\boldsymbol{k}V^{(b)} = i\boldsymbol{k}\Psi - \boldsymbol{F}/\rho_b \ .$$

To lowest order in t_T/t and kt_T, this leads to the following correction to the adiabatic condition $\boldsymbol{V}^{(b)} = \boldsymbol{V}^{(r)}$:

$$t_T\omega\boldsymbol{k}V^{(b)} = \frac{4\rho_r}{3\rho_b}(i\boldsymbol{k}V^{(b)} - \boldsymbol{V}^{(r)}) \ , \tag{3.8}$$

From Eq. (3.6) we obtain the relation $\boldsymbol{k}\cdot\boldsymbol{V}^{(r)} = -(3/4)\omega D_g^{(r)}$ to lowest order. Using this approximation, we find, after multiplying Eq. (3.8) with $\boldsymbol{k}$,

$$V^{(b)} = \frac{(3/4)\omega}{t_T k^2\omega R - ik^2}D_g^{(r)} \ , \tag{3.9}$$

with $R = 3\rho_b/\rho_r$. The densities ρ_b and ρ_r denote the baryon and radiation densities respectively. Inserting this result in Eq. (3.7) leads to

$$\mathcal{M} = \frac{1 + \frac{3\mu\omega/k}{1 - it_T\omega R}}{1 - it_T(\omega - k\mu)}D_g^{(r)}/4 \ , \tag{3.10}$$

where we have set $\mu = \boldsymbol{k}\cdot\boldsymbol{n}/k$. From this result, which is valid on time scales shorter than the expansion time (length scales smaller than the horizon), we can derive a dispersion relation $\omega(k)$. In lowest order ωt_T we obtain

$$\omega = \omega_0 - i\gamma \quad \text{with} \tag{3.11}$$

$$\omega_0 = \frac{k}{\sqrt{3(1+R)}} \quad \text{and} \quad \gamma = k^2 t_T \frac{R^2 + \frac{4}{5}(R+1)}{6(R+1)^2} \ . \tag{3.12}$$

At recombination $R \sim 0.1$ so that $\gamma \sim 2k^2 t_T/15$.

We have thus found that, due to diffusion damping, the photon perturbations thus undergo an exponential decay which can be approximated by

$$|\mathcal{M}| \propto \exp(-2k^2 t_T t/15) \ , \ \text{on scales} \ \ t \gg 1/k \gg t_T \ . \tag{3.13}$$

In general, the temporal evolution of radiation perturbations can be split into three regimes: Before recombination, $t \ll t_{dec}$ the evolution of photons can be determined in the fluid limit. After recombination, the free Liouville equation is valid. Only during recombination the full Boltzmann equation has to be considered, but also there collisional damping can be reasonable well approximated by an exponential damping envelope [15], which is a somewhat sophisticated version of (3.13).

4 Example: Adiabatic scalar perturbations

We now want to discuss Eq. (2.16) with initial conditions given by Eq. (3.1) in some examples.

Perturbations are called 'scalar' if all 3 dimensional tensors (tensors w.r.t their spatial components on hyper-surfaces of constant time) can be obtained as derivatives of scalar potentials.

Scalar perturbations of the geometry can be described by two gauge invariant variables, the Bardeen potentials [16] Φ and Ψ. The variable Ψ is the relativistic analog of the Newtonian potential. In the Newtonian limit, $-\Phi = \Psi =$ the Newtonian gravitational potential. In the relativistic situation, Φ is better interpreted as the perturbation in the scalar curvature on the hyper-surfaces of constant time [17]. In terms of the Bardeen potentials, the electric and magnetic components of the Weyl tensor are given by [11]

$$\mathcal{E}_{ij} = \frac{1}{2}\triangle_{ij}(\Phi - \Psi) \quad , \quad \mathcal{B}_{ij} = 0 \ , \tag{4.1}$$

where $\triangle_{ij}$ denotes the traceless part of the second derivative, $\triangle_{ij} = \partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2$. The Liouville equation, (2.16) then reduces to

$$(\partial_t + n^i\partial_i)\mathcal{M} = n^i\partial_i(\Phi - \Psi) \ . \tag{4.2}$$

With the initial conditions given in Eq. (3.1) we find the solution

$$\frac{\delta T}{T}(t_0, \boldsymbol{x}_0, \boldsymbol{n}) = \mathcal{M}(t_0, \boldsymbol{x}_0, \boldsymbol{n}) = [\frac{1}{4}D_g^{(r)} + n^i\partial_i V^{(b)} + \Psi - \Phi](t_{dec}, \boldsymbol{x}_{dec}) - \int_{t_{dec}}^{t_0} (\dot{\Phi} - \dot{\Psi})(t, \boldsymbol{x}(t))dt \ , \tag{4.3}$$

where $\boldsymbol{x}_{dec} = \boldsymbol{x}_0 - (t_0 - t_{dec})\boldsymbol{n}$ and correspondingly $\boldsymbol{x}(t) = \boldsymbol{x}_0 - (t_0 - t)\boldsymbol{n}$.

We now want to replace the fluid variables, $D_g^{(r)}$ and $V^{(b)}$, wherever possible, by perturbations in the geometry. To this goal, let us first consider the general situation, when one part of the geometry perturbation is due to perturbations in the cosmic matter components and another part is due to some type of seeds, which do not contribute to the background energy and pressure. The Bardeen potentials can then be split into contributions from matter and seeds:

$$\Phi = \Phi_m + \Phi_s \quad , \Psi = \Psi_m + \Psi_s \ . \tag{4.4}$$

To proceed further, we must assume a relation between the perturbations in the total energy density and energy flow, D_g and V, and the corresponding perturbations in the photon component. The most natural assumption here is that perturbations are adiabatic, i.e., that

$$D_g^{(r)}/(1 + w_r) = D_g/(1 + w) \quad \text{and} \qquad V^{(b)} = V^{(r)} = V \ ,$$

where $w \equiv p/\rho$ denotes the enthalpy, i.e. $w_r = 1/3$. For $w_r \neq w$ this condition can only be maintained on super–horizon scales or for tightly coupled fluids. For decoupled fluid components, the different equations of state lead to a violation of this initial condition on sub–horizon scales.

In order to use the perturbed Einstein equations to replace D_g and V by geometric perturbations we define yet another density perturbation variable,

$$\begin{aligned} D &\equiv D_g + 3(1+w)\frac{\dot{a}}{a}V - 3(1+w)\Phi \quad \text{and} \\ D^{(r)} &\equiv D_g^{(r)} + 4\frac{\dot{a}}{a}V^{(r)} - 4\Phi \ . \end{aligned}$$

The matter perturbations D and V determine the matter part of the Bardeen potentials via the perturbed Einstein equations (*see*, e.g. [10]). The following relation between Φ_m and D can also be obtained using Eqs. (4.1) and (A16) in the absence of seeds.

$$D = -\frac{2}{3}\left(\frac{\dot{a}}{a}\right)^{-2}\nabla^2\Phi_m \sim (kt)^2\Phi_m \quad \text{and}$$

$$\frac{\dot{a}}{a}\Psi_m - \dot{\Phi}_m = \frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2(1+w)V \ .$$

The term D rsp. $D^{(r)}$, is much smaller than the Bardeen potentials on super–horizon scales and it starts to dominate on sub–horizon scales, $kt \gg 1$. For this term therefore, the adiabatic relation is not useful and we should not replace $D^{(r)}$ by $\frac{4}{3(1+w)}D$. The same holds for $\partial_i V^{(b)}$ which is of the order of $kt\Phi_m$. However, $(\dot{a}/a)V^{(r)}$ is of the same order of magnitude as the Bardeen potentials and thus mainly relevant on super horizon scales. There the adiabatic condition makes sense and we may replace $(\dot{a}/a)V$ by its expression in terms geometric perturbations. Keeping only $D^{(r)}$ and $\partial_i V^{(b)}$ in terms of photon fluid variables, Eq. (4.3) becomes

$$\begin{aligned} \frac{\delta T}{T}(\boldsymbol{x}_0, t_0, \boldsymbol{n}) &= [\Psi_s + \frac{1+3w}{3+3w}\Psi_m + \frac{2}{3(1+w)}\left(\frac{\dot{a}}{a}\right)^{-1}\dot{\Phi}_m + \frac{1}{4}D^{(r)} + n^i\partial_i V^{(b)}](\boldsymbol{x}_{dec}, t_{dec}) \\ &\quad - \int_{t_{dec}}^{t_0}(\dot{\Phi} - \dot{\Psi})(\boldsymbol{x}(t), t) \ . \end{aligned} \tag{4.5}$$

This is the most general result for adiabatic scalar perturbations in the photon temperature. It contains geometric perturbations, acoustic oscillations prior to recombination and the Doppler term. Silk damping, which is relevant on very small angular scales (see the contribution by [6]) is neglected, i.e., we assume 'instantaneous recombination'. Eq. (4.5) is valid for all types of matter models, with or without cosmological constant and/or spatial curvature (we just assumed that the latter is negligible at the last scattering surface, which is clearly required by observational constraints). The first two terms in the square bracket are usually called the ordinary Sachs–Wolfe contribution. The integral is the 'integrated Sachs–Wolfe effect'. The third and fourth term in the square bracket describe the acoustic Doppler oscillations respectively. On super horizon scales, $kt \ll 1$, they can be neglected.

To make contact with the formula usually found in textbooks, we finally constrain ourselves to a universe dominated by cold dark matter (CDM), i.e., $w = 0$ without any seed perturbations. In this case $\Psi_s = \Phi_s = 0$ and it is easy to show that $\Psi = -\Phi$ and that

$\dot{\Phi} = \dot{\Psi} = 0$ (see, e.g., [10]). Our results then simplifies on super–horizon scales, $kt \ll 1$, to the well–known relation of Sachs and Wolfe [18]

$$\left(\frac{\delta T}{T}\right)_{SW} = \frac{1}{3}\Psi(\boldsymbol{x}_0 - t_0\boldsymbol{n}, t_{dec}) \ . \tag{4.6}$$

5 Conclusions

We have derived all the basic ingredients to determine the temperature fluctuations in the CMB. Since the fluctuations are so small, they can be calculated fully within linear cosmological perturbation theory. Note however that density perturbations along the line of sight to the last scattering surface might be large, and thus the Bardeen potentials inside the Sachs Wolfe integral might have to be calculated within non–linear Newtonian gravity. But the Bardeen potentials themselves remain small (as long as the photons never come close to black holes) such that Eq. (4.5) remains valid. In this way, even a CDM model can lead to an integrated Sachs Wolfe effect which then is known under the name 'Rees Sciama effect'. Furthermore, do to ultra violet radiation of the first objects formed by gravitational collapse, the universe might become reionized and electrons and radiation become coupled again. If this reionization happens early enough ($z > 30$) the subsequent collisions lead to additional damping of anisotropies on angular scales up to about 5^o. However, present CMB anisotropy measurements do not support early reionization and the Rees Sciama effect is probably very small. Apart from these effects due to non–linearities in the matter distribution, which depend on the details of the structure formation process, CMB anisotropies can be determined within linear perturbation theory.

This is one of the main reason, why observations of CMB anisotropies may provide detailed information about the cosmological parameters (see contribution by A. Lasenby): The main physics is linear and well known and the anisotropies can thus be calculated within an accuracy of 1% or so. The detailed results do depend in several ways on the parameters of the cosmological model which can thus be determined by comparing calculations with observations.

There is however one caveat: If the perturbations are induced by seeds (e.g. topological defects), the evolution of the seeds themselves is in general non–linear and complicated. Therefore, much less accurate predictions have been made so far for models where perturbations are induced by seeds (see, e.g., [19, 20, 21]). In this case, the observation of CMB anisotropies might not help very much to constrain cosmological parameters, but it might contain very interesting information about the seeds, which according to present understanding originate from very high temperatures, $T \sim 10^{16}$GeV. The CMB anisotropies might thus bury some 'fossils' of the very early universe, of the physics at an energy scale which we can never probe directly by accelerator experiments.

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A An equation of motion for the Weyl tensor

The Weyl tensor of a spacetime $(\mathcal{M}, g)$ is defined by

$$C^{\mu\nu}{}_{\sigma\rho} = R^{\mu\nu}{}_{\sigma\rho} - 2g^{[\mu}{}_{[\sigma}R^{\nu]}{}_{\rho]} + \frac{1}{3}Rg^{[\mu}{}_{[\sigma}g^{\nu]}{}_{\rho]} \ , \tag{A1}$$

where $[\mu...\nu]$ denotes anti-symmetrization in the indices μ and ν. The Weyl curvature has the same symmetries as the Riemann curvature and it is traceless. In addition the Weyl tensor is invariant under conformal transformations:

$$C^{\mu}{}_{\nu\sigma\rho}(g) = C^{\mu}{}_{\nu\sigma\rho}(a^2 g)$$

(Careful: This equation only holds for the given index position.) In four dimensional spacetime, the Bianchi identities together with Einstein's equations yield equations of motion for the Weyl curvature. In four dimensions, the Bianchi identities,

$$R_{\mu\nu[\sigma\rho;\lambda]} = 0$$

are equivalent to [8]

$$C^{\alpha\beta\gamma\delta}{}_{;\delta} = R^{\gamma[\alpha;\beta]} - \frac{1}{6}g^{\gamma[\alpha}R^{;\beta]} \ . \tag{A2}$$

This together with Einstein's equations yields

$$C^{\alpha\beta\gamma\delta}{}_{;\delta} = 8\pi G(T^{\gamma[\alpha;\beta]} - \frac{1}{3}g^{\gamma[\alpha}T^{;\beta]}) \ , \tag{A3}$$

where $T_{\mu\nu}$ is the energy momentum tensor, $T = T^{\lambda}_{\lambda}$.

Let us now choose some time-like unit vector field u, $u^2 = -1$. We then can decompose any tensor field into longitudinal and transverse components with respect to u. We define

$$h^{\mu}{}_{\nu} \equiv g^{\mu}{}_{\nu} + u^{\mu}u_{\nu} \ ,$$

the projection onto the subspace of tangent space normal to u. The decomposition of the Weyl tensor yields its electric and magnetic contributions:

$$\mathcal{E}_{\mu\nu} = C_{\mu\lambda\nu\sigma}u^{\lambda}u^{\sigma} \tag{A4}$$

$$\mathcal{B}_{\mu\nu} = \frac{1}{2}C_{\mu\lambda\gamma\delta}u^{\lambda}\ \eta^{\gamma\delta}_{\nu\sigma}u^{\sigma} \ ; \tag{A5}$$

where $\eta^{\alpha\beta\gamma\delta}$ denotes the totally antisymmetric 4 tensor with $\eta_{0123} = \sqrt{-g}$. Due to symmetry properties and the tracelessness of the Weyl curvature, $\mathcal{E}$ and $\mathcal{B}$ are symmetric and traceless, and they fully determine the Weyl curvature. One easily checks that $\mathcal{E}_{\mu\nu}$ and $\mathcal{B}_{\mu\nu}$ are also conformally invariant. We now want to perform the corresponding decomposition for the energy momentum tensor of some arbitrary type of seed, $T^{S}_{\mu\nu}$. We define

$$\rho_S \equiv T^{(S)}_{\mu\nu}u^{\mu}u^{\nu} \tag{A6}$$

$$p_S \equiv \frac{1}{3}T^{(S)}_{\mu\nu}h^{\mu\nu} \tag{A7}$$

$$q_{\mu} \equiv -h_{\mu}{}^{\nu}T^{(S)}_{\nu\alpha}u^{\alpha} \qquad q_i = -\frac{1}{a}T^{(S)}_{0i} \tag{A8}$$

$$\tau_{\mu\nu} \equiv h_{\mu}{}^{\alpha}h_{\nu}{}^{\beta}T^{(S)}_{\alpha\beta} - h_{\mu\nu}p_S \ . \tag{A9}$$

We then can write

$$T^{(S)}_{\mu\nu} = \rho_S u_\mu u_\nu + p_S h_{\mu\nu} + q_\mu u_\nu + u_\mu q_\nu + \tau_{\mu\nu} \ . \qquad \text{(A10)}$$

This is the most general decomposition of a symmetric second rank tensor. It is usually interpreted as the energy momentum tensor of an imperfect fluid. In the frame of an observer moving with four velocity u, ρ_S is the energy density, p_S is the isotropic pressure, q is the energy flux, $u \cdot q = 0$, and τ is the tensor of anisotropic stresses, $\tau_{\mu\nu}h^{\mu\nu} = \tau_{\mu\nu}u^\mu = 0$.

We now want to focus on a perturbed Friedmann universe. We therefore consider a four velocity field u which deviates only in first order from the Hubble flow: $u = (1/a)\partial_0+$ first order. Friedmann universes are conformally flat, and we require the seed to represent a small perturbation on a universe dominated by radiation and cold dark matter (CDM). The seed energy momentum tensor and the Weyl tensor are of thus of first order, and (up to first order) their decomposition does not depend on the choice of the first order contribution to u, they are gauge–invariant. But the decomposition of the dark matter depends on this choice. Cold dark matter is a pressure-less perfect fluid We can thus choose u to denote the energy flux of the dark matter, $T^\mu_\nu u^\nu = -\rho_C u^\mu$. Then the energy momentum tensor of the dark matter has the simple decomposition

$$T^{(C)}_{\mu\nu} = \rho_C u_\mu u_\nu \ . \qquad \text{(A11)}$$

With this choice, the Einstein equations Eq. (A3) linearized about an $\Omega = 1$ Friedmann background yield the following 'Maxwell equations' for E and B [22]:
i) Constraint equations

$$\begin{aligned} \partial^i \mathcal{B}_{ij} &= 4\pi G \eta_{j\beta\mu\nu} u^\beta q^{[\mu;\nu]} && \text{(A12)} \\ \partial^i \mathcal{E}_{ij} &= 8\pi G(\frac{1}{3}a^2\rho_C D,_j + \frac{1}{3}a^2\rho_S,_j - \frac{1}{2}\partial^i\tau_{ij} - \frac{\dot a}{a^2}q_j) \ . && \text{(A13)} \end{aligned}$$

ii) Evolution equations

$$\begin{aligned} a\dot{\mathcal{B}}_{ij} + \dot a\mathcal{B}_{ij} - a^2 h_{(i}{}^\alpha \eta_{j)\beta\gamma\delta}u^\beta \mathcal{E}_\alpha{}^{\gamma;\delta} &= -4\pi G a^2 h_{\alpha(i}\eta_{j)\beta\mu\nu}u^\beta \tau^{\alpha\mu;\nu} && \text{(A14)} \\ \dot{\mathcal{E}}_{ij} + \frac{\dot a}{a}\mathcal{E}_{ij} + a h_{(i}{}^\alpha \eta_{j)\beta\gamma\delta}u^\beta \mathcal{B}_\alpha{}^{\gamma;\delta} &= -4\pi G(a q_{ij} - \frac{\dot a}{a}\tau_{ij} + \dot\tau_{ij} + a\rho_C u_{ij}), && \text{(A15)} \end{aligned}$$

where $(i...j)$ denotes symmetrization in the indices i and j. The symmetric traceless tensor fields $q_{\mu\nu}$ and $u_{\mu\nu}$ are defined by

$$\begin{aligned} q_{\mu\nu} &= q_{(\mu;\nu)} - \frac{1}{3}h_{\mu\nu}q^\lambda{}_{;\lambda} \\ u_{\mu\nu} &= u_{(\mu;\nu)} - \frac{1}{3}h_{\mu\nu}u^\lambda{}_{;\lambda} \ . \end{aligned}$$

In Eqs. (A14) and (A15) we have also used that for the dark matter perturbations only scalar perturbations are relevant, vector perturbations decay quickly. Therefore u is a gradient field, $u_i = U_{;i}$ for some suitably chosen function U. Hence the vorticity of the vector field u vanishes, $u_{[\mu;\nu]} = 0$. With

$$\eta_{0ijk} = a^4\epsilon_{ijk} \ , \quad \rho_S = a^{-2}T^S_{00} \quad \text{and} \quad q_i = -a^{-1}T^S_{0i} \ ,$$

we obtain from Eq. (A13)

$$\partial^i \mathcal{E}_{ij} = 8\pi G(\frac{1}{3}\rho_C a^2 D_{,j} + \frac{1}{3}T^S_{00}{}_{,j} - \frac{1}{2}\partial_i \tau_{ij} + \frac{\dot{a}}{a}T^S_{0j}) \ . \qquad \text{(A16)}$$

In Eq. (A16) and the following equations summation over double indices is understood, irrespective of their position.

To obtain the equation of motion for the magnetic part of the Weyl curvature we take the time derivative of Eq. (A14), using $u = (1/a)\partial_0 + 1.$order and $\eta_{0ijk} = a^4\epsilon_{ijk}$. This leads to

$$(a\mathcal{B}_{ij})^{\cdot\cdot} = -a(\epsilon_{lm(i}[\dot{\mathcal{E}}_{j)l} + \frac{\dot{a}}{a}\mathcal{E}_{j)l}]_{,m} - 4\pi G\epsilon_{lm(i}[\dot{\tau}_{j)l,m} + \frac{\dot{a}}{a}\tau_{j)l,m}]) \ , \qquad \text{(A17)}$$

where we have again used that u is a gradient field and thus terms like $\epsilon_{ijk}u_{lj,k}$ vanish. We now insert Eq. (A15) into the first square bracket above and replace product expressions of the form $\epsilon_{ijk}\epsilon_{ilm}$ and $\epsilon_{ijk}\epsilon_{lmn}$ with double and triple Kronecker deltas. Finally we replace divergences of B with the help of Eq. (A12). After some algebra, one obtains

$$\epsilon_{lm(i}[\dot{\mathcal{E}}_{j)l} + \frac{\dot{a}}{a}\mathcal{E}_{j)l}]_{,m} = -\nabla^2\mathcal{B}_{ij} - 4\pi G\epsilon_{lm(i}[2aq_{l,mj)} + \dot{\tau}_{j)l,m} - \frac{\dot{a}}{a^2}\tau_{j)l,m}] \ .$$

Inserting this into Eq. (A17) and using energy momentum conservation of the seed, we finally find the equation of motion for $\mathcal{B}$:

$$a^{-1}(a\mathcal{B})^{\cdot\cdot}_{ij} - \nabla^2\mathcal{B}_{ij} = 8\pi G\mathcal{S}^{(B)}_{ij} \ , \qquad \text{(A18)}$$

with

$$\mathcal{S}^{(B)}_{ij} = \epsilon_{lm(i}[-T^S_{0l}{}_{,j)m} + \dot{\tau}_{j)l,m}] \ . \qquad \text{(A19)}$$

Eq. (A18) is the linearized wave equation for the magnetic part of the Weyl tensor in an expanding universe. A similar equation can also be derived for $\mathcal{E}$.

Since dark matter just induces scalar perturbations and $\mathcal{B}_{ij}$ is sourced by vector and tensor perturbations only, it is independent of the dark matter fluctuations. Equations Eqs. (A16) and (A18) connect the source terms in the Liouville equation of section 2, $\partial^i\mathcal{E}_{ij}$ and $\mathcal{B}_{ij}$ to the perturbations of the energy momentum tensor.