Kalb-Ramond axion production in anisotropic string cosmologies

Ruth Durrer[1] and Mairi Sakellariadou[2]

[1] *Département de Physique Théorique, Université de Genève, 24 quai E. Ansermet, CH-1211 Genève 4, Switzerland.*

[2] *DARC, Observatoire de Paris, UPR 176 CNRS, 92195 Meudon Cedex, France.*

(March 2000)

We compute the energy spectra for massless Kalb-Ramond axions in four-dimensional anisotropic string cosmological models. We show that, when integrated over directions, the four-dimensional anisotropic model leads to infra-red divergent spectra similar to the one found in the isotropic case.

PACS number(s): 98.80.Cq

I. INTRODUCTION

The pre-big-bang (PBB) model of cosmology [1] inspired by the duality properties of string theory, is faced, on the phenomenological side, with the question whether or not it can reproduce the amplitude and slope of the observed temperature anisotropy spectrum [2] and of large-scale density perturbations.

Within the PBB model, the inflationary expansion due to the dilaton field in the low-energy effective action of string theory, leads to an amplification of metric fluctuations as well as of quantum fluctuations of all the fields present in PBB cosmology. Such fields, which are not part of the homogeneous background whose perturbations we study, are for example the gauge fields and the pseudo-scalar partner of the dilaton field in the string theory effective action.

At first, it was thought that the PBB scenario could not lead to the observed scale-invariant Harrison-Zel'dovich spectrum of perturbations at large-scales. First-order scalar and tensor metric perturbations were found to lead to primordial spectra that grow with frequency [3], with a normalization imposed by the string cut-off at the shortest amplified scales. These blue spectra have too little power at scales relevant for the observed anisotropies in the cosmic microwave background (CMB). In contrast, the axion energy spectra were found to be diverging at large scales, red spectra, leading to very large CMB anisotropies, in conflict with observations.

These results already rule out four-dimensional isotropic PPB cosmology. However, if one allows for internal contracting dimensions in addition to the three expanding ones, the situation is different. The axion field can lead to a flat Harrison-Zel'dovich spectrum of fluctuations for an appropriate relative evolution of the external and the compactified internal dimensions [4,5]. Thus, it is possible that the amplification of quantum fluctuations of fields which are present in the PBB scenario, can generate via the seed mechanism [6] the observed anisotropy of the CMB radiation.

Considering an isotropic PBB model with extra dimensions, the amplification of electromagnetic vacuum fluctuations and of Kalb-Ramond axion vacuum fluctuations lead to interesting observational consequences within the context of primordial magnetic fields [7] and large-scale temperature anisotropies [8]. In particular, massless axions as well as very light axions can exhibit a flat or slightly tilted blue spectrum which may reasonably fit the observational data [8,9]. (Even though an acoustic peak at $\ell \sim 350$ is excluded by experiments published after Ref. [9] was completed, it is possible to shift this peak to $\ell \sim 220$ by closing the universe with a cosmological constant. More details about this model can be found in Ref. [10].)

Recently it has been suggested that four-dimensional string cosmology models which expand anisotropically can also lead to blue or flat energy spectra for axionic perturbations [11]. According to Ref. [11], one can instead of assuming internal extra dimensions [8], consider an anisotropic four-dimensional background. This has become especially interesting in view of new results which show that the pre-big-bang phase may generically be homogeneous but anisotropic [12].

In Ref. [11], the axion spectrum is only computed for the part of phase space where the longitudinal component of the wave vector is sufficiently large. In this work we correct the result of Ref. [11] and complete the computation to contain all directions in phase space. We then integrate the obtained spectrum over directions and compare it with the result for the isotropic PBB. We find that the anisotropic spectrum, when averaged over directions agrees roughly with the isotropic one. Therefore, anisotropic expansion during the pre-big-bang phase cannot solve the axion problem of four-dimensional string cosmology.

II. AXION PRODUCTION IN THE PRE-BIG-BANG COSMOLOGICAL MODEL

Let us consider a four-dimensional spatially flat anisotropic PBB cosmological model, with metric

$$(g_{\mu\nu}) = \mathrm{diag}[1, -a^2(t), -b^2(t), -b^2(t)] \ ; \qquad (1)$$

the internal compactified radii (if present) are assumed to be frozen. For simplicity, we assume two directions to expand with the same scale factor b. Varying the low-energy string theory effective action (in the string frame)

$$S = -\frac{1}{2\lambda_s^2}\int d^4x\sqrt{-g}\; e^{-\phi} \times \left[R + g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \frac{1}{12}H_{\mu\nu\alpha}H^{\mu\nu\alpha}\right] \qquad (2)$$

($H^{\mu\nu\alpha}$ denotes the antisymmetric tensor field), with respect to the metric and the dilaton field ϕ, we obtain the dilaton driven vacuum solutions of the tree-level evolution equations. As derived in Ref. [1], these solutions read

$$a(\eta) = \left[-\frac{\eta}{\eta_1}\right]^{\frac{\alpha}{1-\alpha}} \quad , \quad b(\eta) = \left[-\frac{\eta}{\eta_1}\right]^{\frac{\beta}{1-\alpha}} \quad , \qquad (3)$$

and

$$\phi(\eta) = \left(\frac{\alpha + 2\beta - 1}{1-\alpha}\right)\log\left[-\frac{\eta}{\eta_1}\right] \ , \qquad (4)$$

with α and β satisfying the Kasner condition

$$\alpha^2 + 2\beta^2 = 1 \ . \qquad (5)$$

Here η denotes conformal time with respect to the scale factor a. It is negative during the pre-big-bang era and $\eta = -\eta_1$ stands for the transition time from the dilaton driven pre-big-bang era to the radiation dominated post-big-bang era. To obtain the axion evolution equation, we vary the effective action, Eq. (2), with respect to the Kalb-Ramond axion field σ, given by

$$H^{\mu\nu\alpha} = e^{\phi}\frac{\epsilon^{\mu\nu\alpha\rho}}{\sqrt{-g}}\partial_\rho\sigma \ . \qquad (6)$$

The evolution equation of the canonical field $\psi = e^{\phi/2}b\sigma$, in Fourier space, reads [11]

$$\psi_k'' + \left[k_L^2 + k_T^2\frac{a^2}{b^2} - \frac{\mathcal{P}''}{\mathcal{P}}\right]\psi_k = 0 \quad , \quad \mathcal{P} = e^{\phi/2}b \ , \qquad (7)$$

where k_L denotes the modulus of the comoving longitudinal momentum and $k_T = \sqrt{k_y^2 + k_z^2}$ is the modulus of the transverse momentum. Equation (7) describes the generation of axionic modes, where the anisotropy of the spacetime has been translated into an asymmetry between the longitudinal and transverse momenta.

The choice $\alpha = \beta = -1/\sqrt{3}$ corresponds to the isotropic case, for which $a = b$ and the evolution of axionic fluctuations is given by

$$\psi_k'' + \left[k^2 - \frac{\mathcal{P}''}{\mathcal{P}}\right]\psi_k = 0 \ , \qquad (8)$$

$$\text{with} \ \ \psi_k = \mathcal{P}\sigma_k \ \ , \ \ \mathcal{P} = e^{\phi/2}a \propto (-\eta)^p \ , \qquad (9)$$

$$\text{so that} \ \frac{\mathcal{P}''}{\mathcal{P}} = \frac{(\mu^2 - 1/4)}{\eta^2}, \ \text{with} \ \ \mu^2 = (p - \frac{1}{2})^2 \ . \qquad (10)$$

The solution of Eq. (8), normalized to an initial vacuum fluctuation spectrum, can be written as

$$\psi_k = \eta^{1/2}H_\mu^{(2)}(|k\eta|) \ , \ \mu = \left|p - \frac{1}{2}\right| \ , \ \eta \le -\eta_1 \ , \qquad (11)$$

with $\mu = \sqrt{3}$. $H_\mu^{(2)}$ denotes the Hankel function of second kind (we adopt the conventions of Ref. [13]).

Assuming that the dilaton driven era is followed by a radiation dominated era, the density parameter of produced Kalb-Ramond axions per logarithmic frequency interval is [4]

$$\Omega_\sigma(\omega, \eta) = \frac{\rho(\omega)}{\rho_c} = \frac{1}{\rho_c}\frac{d\rho_\sigma}{d\log\omega} \simeq g_1^2\Omega_\gamma(\eta)\left[\frac{\omega}{\omega_1}\right]^{3-2\mu} \ , \qquad (12)$$

where $\rho(\omega)$ denotes their spectral energy density and $\rho_c = 3M_p^2H^2/(8\pi)$ stands for the critical energy density. Note that $\omega_1 = k_1/a_1 = 1/(a_1|\eta_1|)$ represents the maximal amplified frequency, $g_1 = H_1/M_p$ is the transition scale in units of the Planck mass, $H_1 \simeq \omega_1$ denotes the Hubble scale at which the universe becomes radiation dominated. Hence $\Omega_\gamma(\eta) = (H_1/H)^2(a_1/a)^4$ is the radiation density parameter at a given time η.

Clearly a flat spectrum corresponds to $\mu = 3/2$ and the value $\mu = \sqrt{3}$ obtained in a four-dimensional isotropic pre-big-bang model implies a red spectrum, leading to an unacceptable divergence at low frequencies.

Let us now go back to the case of a four-dimensional anisotropic background. We first study the evolution of axionic fluctuations and we then calculate the spectral energy density of the axionic inhomogeneities ($d\rho_\sigma/d\log\omega$), as they re-enter the horizon during the isotropic radiation dominated era, after being amplified during the anisotropic dilaton driven era. Inserting Eqs. (3) and (4) into Eq. (7), we obtain [11]

$$\psi_k'' + \left(k_L^2 + k_T^2\left[-\frac{\eta}{\eta_1}\right]^\gamma - \frac{\mu^2 - 1/4}{\eta^2}\right)\psi_k = 0 \ , \qquad (13)$$

where

$$\gamma = \tfrac{2(\alpha-\beta)}{1-\alpha} \ , \ 2\mu = |2p - 1| \ , \ \ \text{where} \qquad (14)$$

$$p = \tfrac{\alpha+4\beta-1}{2(1-\alpha)} \ , 2\mu = 2 - \frac{4\beta}{1-\alpha} \ . \qquad (15)$$

If $\gamma < 0$, the k_T-term as well as the η^{-2}-term go to zero for $\eta \to -\infty$; and initially the parentheses in Eq. (13) is dominated by k_L^2 (except if $k_L \equiv 0$). If k_T is not very large, namely if

$$k_T < k_L (k_1/k_L)^{-\gamma/2} \ , \tag{16}$$

the scale k_L becomes super-horizon, *i.e.* the parentheses in Eq. (13) is dominated by the $1/\eta^2$-term, before the k_T-term takes over. In this case, we may entirely neglect the k_T-term in Eq. (13), which then reduces to Eq. (8) with k replaced by k_L. Therefore, the spectrum for these modes is flat for $\mu = 3/2$ which corresponds to

$$\alpha = -7/9 \ , \quad \beta = -4/9 \quad \text{and} \quad \gamma = -3/8 \ . \tag{17}$$

We also require the solution to expand, *i.e.* $\alpha, \beta < 0$. We first concentrate mainly on these values of the Kasner exponents since they lead to a scale invariant spectrum of fluctuations for directions with a sufficiently large k_L-component, but we express our results in terms of α and β so that they can then also be applied also to other values of the Kasner indices. In the part of phase-space defined by the inequality given in Eq. (16), the energy density of the produced axions has already been determined in Ref. [11]. Here we correct the result of Ref. [11] and generalize it to the entire phase space.

To solve Eq. (13), we distinguish among the following two cases:
(I) The modulus of the longitudinal momentum, k_L, always dominates until $\eta^2 < 1/k_L^2$ at which point the $1/\eta^2$ term comes to dominate. This is equivalent to the condition given in Eq. (16).
(II) At some conformal time $\eta = \eta_T < -\eta_1$, the modulus of the transverse momentum, k_T, comes to dominate over k_L, but the mode is still well within the horizon, *i.e.* $\sqrt{k_L^2 + k_T^2(-\eta_T/\eta_1)^\gamma} > \eta_T^{-2}$. Equation (13) implies

$$\eta_T = -\eta_1 \left(\frac{k_L}{k_T}\right)^{2/\gamma} \ . \tag{18}$$

Case (I) : Let us first discuss this case which is also the one studied in Ref. [11]. Here, the inequality given in Eq. (16) holds. For low frequency modes, $\omega \ll \omega_1$ this is the case outside a very thin slice around the plane $k_L = 0$ if $\gamma < 0$. In this situation we may entirely neglect the second term inside the parentheses of Eq. (13) which yields a Bessel differential equation. Its solution during the pre-big-bang era , is simply

$$\psi_k^{\rm PBB}(k,\eta) = \sqrt{\frac{|k_L\eta|}{k_L}} \ H_\mu^{(2)}(k_L\eta) \ , \qquad \text{for} \quad \eta \le -\eta_1 \ , \tag{19}$$

After the transition to the radiation dominated era (RD), we assume the dilaton to be frozen and the expansion to have become isotropic. This implies $\mathcal{P}'' = 0$, $a/b = 1$ and Eq. (7) reduces to a simple harmonic equation with general solution

$$\psi_k^{\rm RD}(k,\eta) = \frac{1}{\sqrt{k}} \ \left[c_+ e^{-ik(\eta+\eta_1)} + c_- e^{ik(\eta+\eta_1)}\right] , \tag{20}$$

$$\text{for} \quad \eta \ge -\eta_1 \ . \tag{21}$$

By matching the in-coming solution $\psi_k^{\rm PBB}$ to the outgoing one $\psi_k^{\rm RD}$, and by also matching their first derivatives, at the transition time $\eta = -\eta_1$, we obtain the frequency mixing coefficient $c_-(k)$:

$$c_- = \frac{-1}{\sqrt{2\pi}} \sqrt{\frac{1}{(k\eta_1)(k_L\eta_1)^{2\mu}}} \ . \tag{22}$$

The coefficient c_- determines the occupation numbers of produced axions. The spectral energy density of the produced axions reads

$$\rho_L(\omega, s) = \frac{d\rho_\sigma}{d\log\omega} \approx \frac{\omega^4}{\pi^2} |c_-(\omega)|^2 \ . \tag{23}$$

From Eqs. (22), (23) we obtain with $\omega = k/a$ for $\mu = 3/2$

$$\rho_L(\omega, s) \approx \frac{1}{2\pi^3} \omega_1^4 / s^3 \ , \tag{24}$$

where $s = k_L/k$.

Thus, if the longitudinal momentum k_L dominates, the spectrum of produced Kalb-Ramond axions is flat, *i.e.* independent of ω, but anisotropic. This result generically agrees with the finding of Ref. [11] (up to a factor $1/s^2$, which we think is missing in Ref. [11]).

Case (II) : We now assume that the k_T-term comes to dominate before the perturbation becomes superhorizon. As long as the perturbation is sub-horizon, we may approximate Eq. (13) by

$$\psi_k'' + \left(k_L^2 + k_T^2 \left[-\frac{\eta}{\eta_1}\right]^\gamma\right) \psi_k = 0 \ , \tag{25}$$

An approximate solution to this equation is

$$\psi \simeq \frac{\exp\left(\eta\sqrt{k_L^2 + q^2(-\eta/\eta_1)^\gamma k_T^2}\right)}{\sqrt{\pi/2}\,[k_L^2 + (-\eta/\eta_1)^\gamma k_T^2]^{1/4}} \ , \tag{26}$$

with $q = 1/(1+\gamma/2) = (1-\alpha)/(1-\beta)$.

In the regime considered, $\eta\sqrt{k_L^2 + q^2(-\eta/\eta_1)^\gamma k_T^2} \gg 1$, this solution becomes exact, if either k_L or k_T vanishes and it is a good approximation if one of the two terms dominates. If the k_L-term and the k_T-term are of the same order, the relative error is about $|\gamma/2| = 3/16$. It is also clear that this represents the correctly normalized incoming vacuum solution.

At conformal time $\eta = \eta_T$, the transverse momentum k_T comes to dominate over the k_L- term in Eq. (13). At even later times, the η^{-2}-term will eventually dominate. After η_T Eq. (13) can be approximated by

$$\psi_k'' + \left(k_T^2 \left[-\frac{\eta}{\eta_1}\right]^\gamma - \frac{\mu^2 - 1/4}{\eta^2}\right) \psi_k = 0 \ , \tag{27}$$

with general solution [13]

$$\psi_k(k_T,\eta) = c_T^{(1)}\sqrt{|k_T\eta|}H_{\mu q}^{(1)}\left(|k_T\eta|q\left[\frac{-\eta}{\eta_1}\right]^{\gamma/2}\right) -ic_T^{(2)}\sqrt{|k_T\eta|}H_{\mu q}^{(2)}\left(|k_T\eta|q\left[\frac{-\eta}{\eta_1}\right]^{\gamma/2}\right) \ , \quad (28)$$

where q is as above, and $H_{\mu q}^{(1)}, H_{\mu q}^{(2)}$ are Hankel functions of the 1st and 2nd kind of order μq. For large $k_T|\eta|$ the second term just corresponds to the solution (26) in the limit where k_L can be neglected. Therefore, by matching the solutions we find

$$c_T^{(1)} = 0 \qquad c_T^{(2)} = \frac{i}{\sqrt{k_T}} \ , \quad (29)$$

up to an irrelevant phase.

Next, we have to match the solution for the field σ of the pre-big-bang era to the solution for σ during the radiation era at the transition time $\eta = -\eta_1$.

As we go from the pre- to the post-big-bang era, we assume the universe to become isotropic and the dilaton field ϕ to become frozen. Thus, here the matching of the in-coming to the out-going solution for σ, is not equivalent to matching ψ. The relation between the the axion field σ and the canonical field ψ at conformal time η is

$$\sigma^{\rm RD}(\eta) = \left[\frac{\eta}{\eta_1}\right]^{-1}\psi^{\rm RD}(\eta) \ , \quad (30)$$

$$\sigma^{\rm PBB}(\eta) = \left[-\frac{\eta}{\eta_1}\right]^{-\lambda}\psi^{\rm PBB}(\eta) \ . \quad (31)$$

The canonical field in Fourier space during RD is given in Eq. (20). Matching the solutions and their first derivatives for σ, as we pass from PBB to RD at time $\eta = -\eta_1$, we obtain for $|k_T\eta_1| \ll 1$, the Bogoliubov coefficient c_- given by

$$|c_-|^2 = \left[\frac{\Gamma^2(\mu q)}{4\pi^2}2^{2\mu q}\left(\frac{3}{2}-\mu q\right)^2\right]\left(\frac{k_T}{k_L}\right)^{-2\mu q} \times s^{-2\mu q}\left(\frac{\omega}{\omega_1}\right)^{-1-2\mu q} \ . \quad (32)$$

With Eq. (23) we then obtain that the energy density of the produced Kalb-Ramond axions, in the case where the transverse momentum k_T comes to dominate, *i.e* the inequality given in Eq. (16) is violated:

$$\rho_T(\omega,s) = \left[\frac{\Gamma^2(\mu q)}{4\pi^2}2^{2\mu q}\left(\frac{3}{2}-\mu q\right)^2\right]\frac{1}{\pi^2} \times \left(\frac{k}{k_T}\right)^{2\mu q}\omega_1^{1+2\mu q}\omega^{3-2\mu q} \ . \quad (33)$$

Inserting the values $\mu = 3/2$, $\alpha = -7/9$ $\beta = -4/9$ which lead to a flat spectrum in case I one finds a somewhat blue spectrum in case II,

$$\rho_T(\omega,s) \propto \omega^{-9/13} \ . \quad (34)$$

Of course this case also gives a finite answer on the plane $k_L = 0$ for which the result obtained under case I diverges.

III. RESULTS AND CONCLUSION

In total we can summarize the calculated spectrum by

$$\rho(\omega,s) \simeq \frac{\omega_1^4}{2\pi^3}\begin{cases} s^{-2\mu}\left(\frac{\omega}{\omega_1}\right)^{3-2\mu} & \text{if } k_T < k_L(k_1/k_L)^{-\gamma/2} \\ (1-s^2)^{-\mu q}\left(\frac{\omega}{\omega_1}\right)^{3-2\mu q} & \text{else,}\end{cases} \quad (35)$$

where $s = k_L/k$, $\mu = 1-2\beta/(1-\alpha)$ and $q = (1-\alpha)/(1-\beta)$. For our prefered values, $\alpha = -7/9$ and $\beta = -4/9$ which imply $\mu = 3/2$ and $q = 16/13$, the above result reduces to

$$\Omega_\sigma(\omega,s,\eta) \simeq g_1^2\Omega_\gamma(\eta)\begin{cases} s^{-3} & \text{if} \quad k_T < k_L(k_1/k_L)^{-\gamma/2} \\ (1-s^2)^{-\frac{24}{13}}\left(\frac{\omega}{\omega_1}\right)^{-9/13} & \text{else.}\end{cases} \quad (36)$$

In the regime of phase space where the longitudinal mode of the momentum is very small *i.e.* when the condition given in Eq. (16) is violated, the spectrum of the produced Kalb-Ramond axions is not flat. For a given value of ω, this is the case if s is smaller than the critical value s_c which is well approximated by

$$s_c(\omega) \simeq \left(\frac{\omega}{\omega_1}\right)^{3/13} \ , \text{ if } \quad \omega \lesssim 0.1\omega_1 \ , \quad (37)$$

a very small value for cosmologically interesting frequencies.

In Figure 1 the the energy density $\rho(\omega)$ is shown as a function of s for different values of ω.

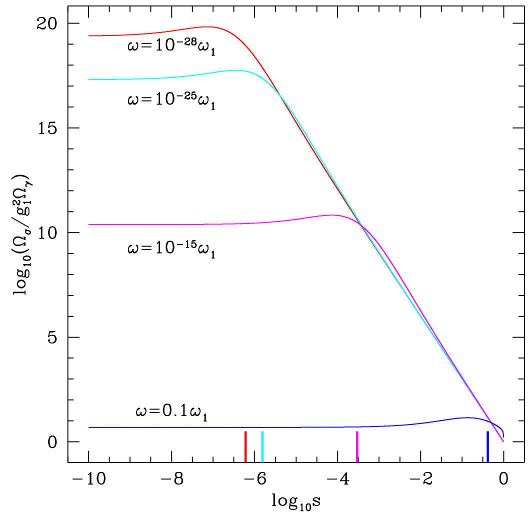

FIG. 1. The density parameter of the produced Kalb-Ramond axions is shown in units of $g_1^2\Omega_\gamma$ as a function of s for different values of ω. The value $\omega_1 \sim (0.01-0.1)M_{Pl}$ is the string scale. For this curve the results obtained in Eq. (36) have been interpolated logarithmically. The little bars on the $\log(s)$-axis indicate the values of s_c at which the condition (16) becomes an equality. For values of s smaller than $s_c(\omega)$, the spectral density is approximated by $\rho_T(\omega)$.

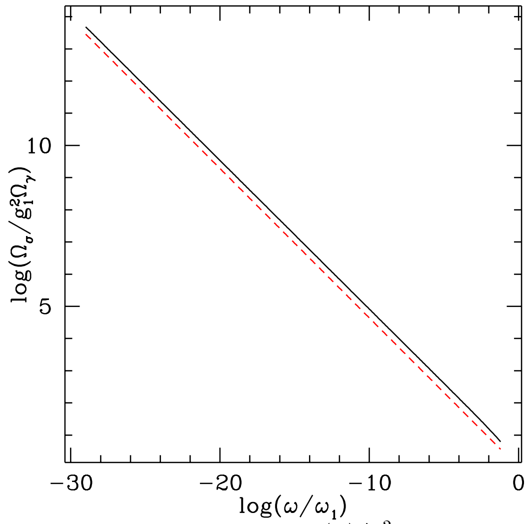

FIG. 2. The energy density $\Omega_\sigma(\omega)/g_1^2\Omega_\gamma$, integrated over directions s is shown as a function of ω (solid line). Comparing it with the isotropic result (dashed line) we conclude that, the two spectral indices are the same and, within our accuracy, also the amplitudes are comparable.

For s fixed, if the modulus of the longitudinal momentum dominates in Eq. (13), more precisely if it satisfies the condition $k_L > k_T(k_T/k_1)^{-\gamma/2/(1+\gamma/2)}$, the spectrum of the produced Kalb-Ramond axions is flat.

To estimate the total energy density per logarithmic frequency interval we integrate the axion density $\Omega_\sigma(\omega, s)$ over s. For this we use

$$d^3k = 2\pi k_T dk_L \wedge dk_T = 4\pi k^2 ds \wedge dk \ , \quad (38)$$

where we have used $dk_L = kds + sdk$ and

$$dk_T = \frac{-s}{\sqrt{1-s^2}}kds + \sqrt{1-s^2}dk \ .$$

Hence, we have

$$\Omega_\sigma(\omega,\eta) = \int \Omega_\sigma(\omega, s, \eta)ds$$
$$\simeq \frac{1}{\rho_c}\left[\int_0^{s_c(\omega)} \rho_T(\omega,s)ds + \int_{s_c(\omega)}^1 \rho_L(\omega,s)ds\right] \quad (39)$$
$$\simeq g_1^2\Omega_\gamma(\eta)\left[s_c\left(\frac{\omega_1}{\omega}\right)^{9/13} + \frac{0.5}{s_c^2}\right]. \quad (40)$$

This spectrum is shown in Fig. 2.

Using $s_c \simeq (\omega/\omega_1)^{3/13}$, which is a good approximation as long as $\omega \le 0.1\omega_1$ it can also be seen directly from Eq. (40) that the isotropic energy spectrum is nearly reproduced. The isotropic spectral index, $3-2\sqrt{3} \sim -0.464$ is actually replaced by $-6/13 \sim -0.463$. Inserting reasonable values for the string scale, $0.01 \le g_1 < 1$, we see that also in the anisotropic case axions are overproduced in unacceptable amounts. Even if the spectrum of the axions from wave vectors directed sufficiently far from the plane $k_L = 0$, is scale-invariant, the enhancement of the spectrum in the vicinity of the plane $k_L = 0$ leads to a total contribution which agrees with the one obtained in the isotropic case. Therefore, the model is excluded (see Ref. [4]).

So far we have mainly considered the case $\alpha = -7/9$ and $\beta = -4/9$, but our results apply quite generically, as long as $\gamma < 0$ and thus the k_L-term dominates at sufficiently early times. But also if $\gamma > 0$, Eq. (26) is an approximate solution on sub-horizon scales. In this situation, however the k_T-term dominates at sufficiently early times and continues to do so until the perturbation becomes super-horizon if the inequality given in Eq. (16) is violated. For $\gamma > 0$ this is the case outside a narrow cylinder around the $k_T = 0$ axis. Therefore, the generic formula given in Eq. (39) always applies, but $s_c \ll 1$, if $\gamma < 0$ and $s_c \simeq 1$, if $\gamma > 0$.

For general values of α and β we obtain

$$\Omega_\sigma(\omega,\eta) = \int \Omega_\sigma(\omega, s, \eta)ds$$
$$\simeq g_1^2\Omega_\gamma(\eta)\left[\left(\frac{\omega}{\omega_1}\right)^{3-2\mu q}\int_0^{s_c(\omega)}(1-s^2)^{-\mu q}ds\right.$$

$$+ \left(\frac{\omega}{\omega_1}\right)^{3-2\mu} \int_{s_c(\omega)}^{1} s^{-2\mu} ds \Bigg] . \qquad (41)$$

The transition value of s is given by

$$\sqrt{1-s_c^2} = s_c^{1+\gamma/2} \left(\frac{\omega}{\omega_1}\right)^{\gamma/2} . \qquad (42)$$

If $\gamma < 0$ (i.e. $\alpha < \beta$), the factor $\left(\frac{\omega}{\omega_1}\right)^{\gamma/2}$ is very large in most of phase space and hence $s_c \ll 1$. On the other hand, if $\gamma > 0$ (i.e. $\alpha > \beta$), the above factor is very small for the relevant frequencies, $\omega \ll \omega_1$ and $s_c \simeq 1$. A reasonable approximation is

$$s_c \simeq \left(\frac{\omega}{\omega_1}\right)^{q-1} \quad \text{if} \quad \gamma < 0 \qquad (43)$$

$$1 - s_c^2 \simeq \left(\frac{\omega}{\omega_1}\right)^{2/q-2} \quad \text{if} \quad \gamma > 0 , \qquad (44)$$

where we have used the relation $q = 1/(1+\gamma/2)$. Inserting these results in Eq. (41), the integrals can be approximated by

$$\Omega_\sigma(\omega,\eta) \sim g_1^2 \Omega_\gamma(\eta) \left(\frac{\omega}{\omega_1}\right)^n , \quad \text{where} \qquad (45)$$

$$n = 2 + q - 2\mu q = \frac{1+\alpha+2\beta}{1-\beta} \quad \text{if } \alpha < \beta , \qquad (46)$$

$$n = 1 + 2/q - 2\mu = \frac{1+\alpha+2\beta}{1-\alpha} \quad \text{if } \alpha > \beta . \qquad (47)$$

Clearly, since $\alpha^2 + 2\beta^2 = 1$ and $\alpha, \beta \le 0$ it is $\alpha + 2\beta \le -1$. This shows that the spectrum is never blue and becomes scale invariant only in the degenerate case with two static dimensions, $\beta = 0$. This is also shown in Fig. 3, where the above approximation for the spectral index plotted as a function of α: the spectrum always remains red with a spectral index relatively close to the isotropic value, $n_{\rm iso} = 3 - 2\sqrt{3} \sim -0.46$, except in the extremal case, when two dimensions are frozen and $\alpha = -1$.

If one relaxes the condition that both a and b be expanding and just asks for volume expansion, $\alpha + 2\beta < 0$, there is another pair of values for the Kasner indices leading to a flat spectrum, namely $\alpha = 1/3$ and $\beta = -2/3$. However, if we want expansion in all three dimensions the spectrum is always red.

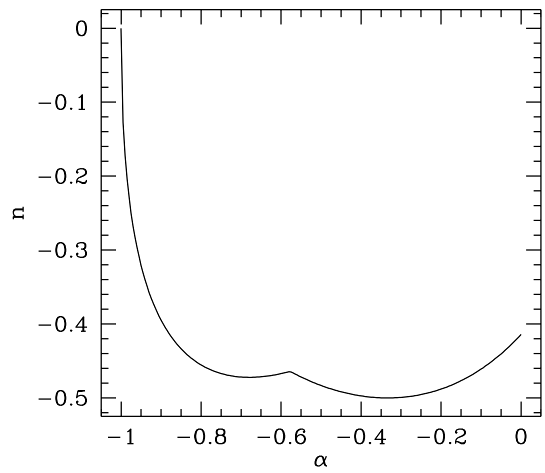

FIG. 3. The spectral index n is shown as a function of the exponent α of the expansion law. For $\alpha = -1/\sqrt{3}$, the isotropic case, our approximation is not very good since there $\alpha = \beta$. This is reflected in the unphysical kink at this value of α. Clearly, the resulting spectrum is always red ($n < 0$), with $-0.4 > n > -0.5$ except close to the degenerate case $\alpha \to -1$.

To summarize, we find that anisotropic expansion has very little influence on the overall axion production and cannot cure the axion problem of four-dimensional pre-big-bang models. Only by allowing for extra dimensions one can escape this conclusion and obtain a scale invariant spectrum of axions as described in Refs. [8,9]. A 'realistic' string cosmology with a Kalb-Ramond axion can therefore be realized only in models with extra dimensions.

ACKNOWLEDGMENTS

It is a pleasure to thank A. Buonanno, T. Damour, K. Kunze, G. Veneziano and A. Vilenkin for useful discussions. This work is supported by the Swiss National Science Foundation.

[1] G. Veneziano, Phys. Lett. B **265**, 287 (1991); M. Gasperini and G. Veneziano, Astropart. Phys. **1**, 317 (1993); Phys. Rev. D **50**, 2519 (1994).

[2] G. F. Smoot et al. Astrophys. J. **396**, L1 (1992); C. L. Bennet et al. Astrophys. J. **430**, 423 (1994).

[3] M. Gasperini and M. Giovannini, Phys. Lett. B **282**, 36 (1991); Phys. Rev. D **47**, 1519 (1993); R. Brustein, M. Gasperini, M. Giovannini, V. F. Mukhanov and G. Veneziano, Phys. Rev. D **51**, 6744 (1995).

[4] E. J. Copeland, R. Easther and D. Wands, Phys. Rev. D **56**, 874 (1997); E. J. Copeland, J. E. Lidsey and D. Wands, Nucl. Phys. **B56**, 506 (1997).
[5] A. Buonanno, K. A. Meissner, C. Ungarelli and G. Veneziano, J. High Energy Phys. **01** 004 (1998);
[6] R. Durrer, Phys. Rev. D **42**, 2533 (1990).
[7] M. Gasperini, M. Giovannini and G. Veneziano, Phys. Rev. Lett. **75**, 3796 (1995).
[8] R. Durrer, M. Gasperini, M. Sakellariadou and G. Veneziano, Phys. Rev. D **59**, 043511 (1999); Phys. Lett. B **436**, 66 (1998).
[9] A. Melchiorri, F. Vernizzi, R. Durrer and G. Veneziano, Phys. Rev. Lett. **83**, 4464 (1999); archived under `astro-ph/9905327`.
[10] F. Vernizzi, A. Melchiorri, R. Durrer and G. Veneziano, in preparation (2000).
[11] M. Giovannini, Phys. Rev. D **59**, 063503 (1999).
[12] K. Kunze and R. Durrer, preprint archived under `gr-qc/9912081` (1999).
[13] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972).