Generalized Einstein-Aether theories and the Solar System

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It has been shown that generalized Einstein-Aether theories may lead to significant modifications to the non-relativistic limit of the Einstein equations. In this paper we study the effect of a general class of such theories on the Solar System. We consider corrections to the gravitational potential in negative *and* positive powers of distance from the source. Using measurements of the perihelion shift of Mercury and time delay of radar signals to Cassini, we place constraints on these corrections. We find that a subclass of generalized Einstein-Aether theories are compatible with these constraints.

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I. INTRODUCTION

The theory of general relativity proposed by Einstein explains a wealth of phenomena over a wide range of scales. At one extreme, one obtains equations of motion for artificial satellites in orbit around the Earth in particular and for Solar-System bodies more generally that considerably surpass Newtonian gravity in the accuracy of their agreement with observations. At the other extreme we find a comprehensive description of the evolution of the Universe and of the growth of structure from a spectrum of primordial perturbations which is backed up by a handful of precise astronomical observations. This apparent success over such a wide range of scales leads us to accept general relativity as the correct theory of gravitation valid on all scales.

It is well known that there are a number of imperfections in our understanding of the Universe, if we adopt Einstein's theory as the theory of gravity and the Standard Model of particle physics as the theory of matter. For example, there is more gravity in galaxies and clusters of galaxies than can be accounted for by the matter (stars, gas, *etc.*) that we detect directly through absorption or emission of electromagnetic radiation [1]. This is explained by invoking the presence of dark matter, for which there are several reasonable candidates within plausible extensions of the Standard Model of particle physics. Furthermore, the Universe seems to be expanding in a way that general relativity can only accomodate by including an exotic form of energy that possesses sufficient negative pressure to be gravitationally repulsive and that currently is the most significant form of energy density on cosmological scales. This is given the name dark energy if it varies in time or space, and the cosmological constant otherwise [2]. Finally, in order to understand the current homogeneity and isotropy of the Universe and to provide an origin for the primordial fluctuations that grew into cosmic structures, we are led to infer the existence of another such form of energy density with negative pressure that dominated the evolution of the Universe at early times and were the source of the quantum fluctuations which were the progenitors of all structure. This period of early dark-energy domination is called inflation. There are no particularly compelling candidates either for dark energy or for inflation within extensions to the Standard Model not devised expressly for that purpose [3].

As remarked, these imperfections can be explained by invoking the presence of dark matter and dark energy. It is also conceivable that we do not yet have the correct theory of gravity and that all or some of these effects are due to this fact. This possibility has been the subject of intermittent attention for a considerable time [4–6], but perhaps especially in the last few years [7, 8]. In particular, there has been progress recently in devising covariant theories that can, apparently, accommodate all these effects [9–13]. At some level these are, consequently, promising alternative theories of gravity. As with general relativity, they must therefore satisfy the constraints one infers from precise observations of the Solar System.

In this paper we undertake the task of comparing one class of modified gravity theories – generalized Einstein-Aether theories – to Solar-System observations. In these theories, one assumes the existence of a vector field with a non-standard kinetic term, and with a time-like vacuum expectation value, at least in the cosmological background. Such theories lead to differences from general relativistic phenomenology which may be substantial, for instance producing modifications to Poisson's equation and the Friedmann equation of precisely the form posited as alternatives to dark matter [4] and dark energy [14] respectively. They should therefore be constrained by observations such as the perihelion precession of Mercury and the time delay of radio pulses around the Solar System.

The structure of the paper is as follows. We first review the Einstein-Aether theory in Section II, presenting the complete action and the field equations. We then focus on the spherically symmetric static metric in Section III and consider a systematic expansion around the Newtonian solution including terms that decrease more

quickly with distance as well as terms that grow with distance. By considering such a wide range of solutions we find constraints on the parameters that define the action presented in Section II. In Section IV we discuss our results. We have organized the text in such a way that the main thrust of the calculations are presented in the main body of the paper while the details are given in a set of appendices at the end.

II.   FIELD EQUATIONS

A general action for a vector field A^α coupled to gravity has the following form:

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi G_N} + \mathcal{L}(g, A) \right] + S_M \qquad (1)$$

where g is the metric, R the Ricci scalar and S_M the matter action. The action of the vector field $\mathcal{L}(g, A)$ is constructed to be both covariant and local. We use units with $c = \hbar = 1$ and the metric signature is $(-, +, +, +)$. Furthermore we demand that the vector field is time-like with a fixed length $A^\alpha A_\alpha = -1$.

In this paper we will consider a Lagrangian that depends only on covariant derivatives of A and the time-like constraint. It can be written in the form:

$$\mathcal{L}(A, g) = \frac{M^2}{16\pi G_N} [\mathcal{F}(\mathcal{K}) + \lambda(A^\alpha A_\alpha + 1)] \qquad (2)$$

$$\mathcal{K} = M^{-2} \mathcal{K}^{\alpha\beta}_{\gamma\sigma} \nabla_\alpha A^\gamma \nabla_\beta A^\sigma$$

$$\mathcal{K}^{\alpha\beta}_{\gamma\sigma} = c_1 g^{\alpha\beta} g_{\gamma\sigma} + c_2 \delta^\alpha_\gamma \delta^\beta_\sigma + c_3 \delta^\alpha_\sigma \delta^\beta_\gamma$$

where c_i are dimensionless constants and M has the dimensions of mass. λ is a non-dynamical Lagrange-multiplier.

Note that in the particular case $c_1 = -c_3$ and $c_2 = 0$ we recover the canonical form $\mathcal{K} \propto F_{\alpha\beta} F^{\alpha\beta}$ where **F** is the field-strength of the four-vector **A**.

The gravitational field equations for this theory are

$$G_{\alpha\beta} = \tilde{T}_{\alpha\beta} + 8\pi G T^{matter}_{\alpha\beta} + M^2 \lambda A_\alpha A_\beta \qquad (3)$$

where the stress-energy tensor for the vector field is given by

$$\tilde{T}_{\alpha\beta} \equiv \frac{1}{2} \nabla_\sigma (\mathcal{F}'(J_{(\alpha}{}^\sigma A_{\beta)} - J^\sigma{}_{(\alpha} A_{\beta)} - J_{(\alpha\beta)} A^\sigma))$$
$$- \mathcal{F}' Y_{\alpha\beta} + \frac{1}{2} g_{\alpha\beta} M^2 \mathcal{F}, \qquad (4)$$

where

$$\mathcal{F}' \equiv \frac{d\mathcal{F}}{d\mathcal{K}}$$
$$J^\alpha{}_\sigma \equiv 2\mathcal{K}^{\alpha\beta}{}_{\sigma\gamma} \nabla_\beta \mathcal{A}^\gamma \qquad (5)$$
$$Y_{\alpha\beta} \equiv c_1 [(\nabla_\alpha A_\sigma)(\nabla_\beta A^\sigma) - (\nabla_\sigma A_\alpha)(\nabla^\sigma A_\beta)].$$

Brackets around indices denote symmetrization.

The equations of motion for the vector field are

$$\nabla_\alpha(\mathcal{F}' J^\alpha{}_\beta) = 2M^2 \lambda A_\beta. \qquad (6)$$

Variation of the action with respect to λ impose on the vector field the crucial constraint

$$A^\alpha A_\alpha = -1. \qquad (7)$$

III.   SPHERICALLY SYMMETRIC STATIC METRIC

In this paper we restrict ourselves to the particular case of a spherically symmetric static metric. In isotropic coordinates (t, r, θ, ϕ), it takes the form:

$$ds^2 = -e^\nu dt^2 + e^\xi (dr^2 + r^2 d\Omega^2), \qquad (8)$$

where $\nu(r)$ and $\xi(r)$ are functions of r only. The vector field has only two non-zero components:

$$A^\alpha = (A^t(r), A^r(r), 0, 0). \qquad (9)$$

With these restrictions, there are 5 unknown fields, ν, ξ, A^t, A^r and the Lagrange multiplier λ, all of which are functions of r only. We need 5 equations to determine them. We choose the tt and rr components of the Einstein equations (3), the t and r components of the vector field equation (6) and the constraint equation (7). We can then combine these in order to eliminate the Lagrange multiplier field λ, leaving us with 3 dynamical equations and the constraint.

The equations of motion depend on the function $\mathcal{F}(\mathcal{K})$ and its first and second derivatives with respect to $\mathcal{K}$. In [12], it was shown that **in order to get 'MONDian' corrections on galactic scales one has to choose the mass parameter M to be of the order $M \sim H_0 \sim 10^{-42}$ GeV.** Therefore in the Solar System, where the gravitational field is strong with respect to M, the function $\mathcal{K}$ is much larger than one, and $\mathcal{F}'(\mathcal{K})$ can be expanded in inverse powers of $\mathcal{K}^{1/2}$

$$\mathcal{F}'(\mathcal{K}) = \sum_{i=1}^\infty \frac{\alpha_i}{\mathcal{K}^{i/2}} , \qquad (10)$$

where this specific form is suggested by comparison to particular modifications of Poisson's equation compatible with galaxy phenomenology [12]. We therefore write the dynamical equations for a generic term $\mathcal{F}'(\mathcal{K}) = \frac{\alpha_{2n}}{\mathcal{K}^n}$. Note that the leading-order term corresponds to $n = 1/2$. We find

$$-\left(\xi'' + 2\frac{\xi'}{r} + \frac{\xi'^2}{4} \right) \frac{\mathcal{K}^{n+1}}{\alpha_{2n}} = -\frac{M^2 e^\xi}{2} \cdot N(\mathcal{K})\mathcal{K} \qquad (11)$$
$$+ \left(f_1 \mathcal{K} + n f_2 \mathcal{K}' \right)(A^r)^2 + \left(f_3 \mathcal{K} - n c_2 e^\xi \mathcal{K}' \right) A^r A^{r\prime}$$
$$+ c_2 e^\xi \mathcal{K} \left((A^{r\prime})^2 + A^r A^{r\prime\prime} \right) + \left(f_4 \mathcal{K} + n f_5 \mathcal{K}' \right)(A^t)^2$$
$$+ \left(f_6 \mathcal{K} - n c_3 e^\nu \mathcal{K}' \right) A^t A^{t\prime} + c_3 e^\nu \mathcal{K} \left((A^{t\prime})^2 + A^t A^{t\prime\prime} \right),$$

$$\left(\frac{\xi' + \nu'}{r} + \frac{\xi'^2}{4} + \frac{\xi'\nu'}{2}\right)\frac{\mathcal{K}^n}{\alpha_{2n}} = \frac{M^2 e^\xi}{2}\cdot N(\mathcal{K}) \quad (12)$$
$$+ g_1(A^r)^2 + g_2 A^r A^{r\prime} - (c_1 + c_2 + c_3)e^\xi(A^{r\prime})^2$$
$$+ \frac{c_1 - c_3}{2}(\nu')^2 e^\nu (A^t)^2 + (c_1 - c_3)\nu' e^\nu A^t A^{t\prime}$$
$$+ c_1 e^\nu (A^{t\prime})^2,$$

and

$$0 = \left(h_1\mathcal{K} + nh_2\mathcal{K}'\right)A^r A^t + \left(h_3\mathcal{K} + 2nc_1\mathcal{K}'\right)A^{t\prime}A^r$$
$$+ \left(h_4\mathcal{K} - 2n(c_1 + c_2 + c_3)\mathcal{K}'\right)A^{r\prime}A^t \quad (13)$$
$$+ 2(c_1 + c_2 + c_3)\mathcal{K}A^{r\prime\prime}A^t - 2c_1\mathcal{K}A^{t\prime\prime}A^r ,$$

where

$$N(\mathcal{K}) = \begin{cases} \frac{\mathcal{K}}{1-n} & \text{if } n \neq 1 \\ \ln(\mathcal{K})\mathcal{K} & \text{if } n = 1 \end{cases} \quad (14)$$

and f_i, g_i and h_i are functions of c_1, c_2, c_3, the metric fields and their first and second derivatives with respect to r. The specific expressions for these functions and for $\mathcal{K}$ in the metric (8) are given in appendix A.

Note that equation (12) contains only first derivatives of the metric and vector fields. It is therefore a constraint. For the leading order term at large $\mathcal{K}$, $\mathcal{F}'(\mathcal{K}) = \frac{\alpha_1}{\sqrt{\mathcal{K}}}$, the right-hand side vanishes and therefore this equation becomes

$$\left(\frac{\xi' + \nu'}{r} + \frac{\xi'^2}{4} + \frac{\xi'\nu'}{2}\right) = 0 . \quad (15)$$

A. Weak field approximation

In the post-Newtonian parametrization, the metric fields are expanded in power of $\frac{r_s}{r}$, where r_s is the Schwarzschild radius of the Sun. This expansion is a generalization of the asymptotic behaviour of the Schwarzschild metric in isotropic coordinates far from the source. Observations of the precession of Mercury and the time delay of a signal emitted from the Earth and reflected by a satellite or a planet allow us to constrain the first coefficients of the expansion. Hence we try the post-Newtonian ansatz for the four fields

$$e^\nu = 1 + a_1\frac{r_s}{r} + a_2\left(\frac{r_s}{r}\right)^2 + ...$$
$$e^\xi = 1 + b_1\frac{r_s}{r} + b_2\left(\frac{r_s}{r}\right)^2 + ... \quad (16)$$
$$A^r = d_1\frac{r_s}{r} + d_2\left(\frac{r_s}{r}\right)^2 + ...$$
$$A^t = 1 + e_1\frac{r_s}{r} + e_2\left(\frac{r_s}{r}\right)^2 + ... ,$$

where a_i, b_i, d_i and e_i are free coefficients which are determined by the equations of motions.

We shall take A^r to have no constant component. In fact, there is a preferred cosmological frame in which $A^r = 0$, and the motion of the Sun with respect to that frame will undoubtedly induce an A^r [15]. Because the time-like vacuum expectation value of the aether breaks Lorentz invariance, the effect of the solar motion with respect to this frame cannot necessarily be accounted for by a boost, however it suggests that the contribution to A^r will be $\mathcal{O}(\gamma_{sun}\beta_{sun}) \sim 10^{-3}$. We defer further consideration of velocity-induced effects to future work.

In order to recover Newton's theory we need $a_1 < 0$; the choice $a_1 = -1$ defines Newton's constant. Post-Newtonian corrections are associated with the parameters a_2 and b_1. We insert the trial solutions (16) in the equations (7), (11), (12) and (13), and we match the coefficients order by order in $\frac{r_s}{r}$.

¿From equation (11) at lowest order, we find:

$$nc_1 = 0. \quad (17)$$

The case $n = 0$ corresponds to the standard aether theory with $\mathcal{F}(\mathcal{K}) = \mathcal{K}$, and has already been studied [15, 16]. Here we are interested in $n \neq 0$ which therefore demands $c_1 = 0$.

It can be shown that $c_1 = 0$ leads to no modification to Poisson's equation at all [12]. Furthermore, it implies the existence of modes which propagate superluminally. There is some debate as to whether this can be phenomenologically acceptable [17]. We therefore try here to see whether the inclusion of additional terms in the expansion of the metric and vector field can lead to a consistent weak field approximation for Lagrangians with $c_1 \neq 0$.

B. Additional terms

In a modified theory of gravity, we expect the metric to have additional terms to the ones given in (16). At the scale of a galaxy, the next-to-leading term of the metric has to be a positive power or logarithm of r if the modified theory is to successfully mimic dark matter on the scales of galaxies. More generally, unless there are additional terms that at least decay more slowly than $1/r$, the modified theory will have only minor phenomenological consequences on large scales. Since these additional terms have to be completely subdominant in the Solar System, their contribution to the equations of motion have traditionally been neglected. Nevertheless the form of the equations (11) and (12) suggests that they could play a role even in the Solar System. Indeed, in equation (11) the left-hand side is proportional to $\mathcal{K}^{n+1}$, whereas the right-hand side is proportional to $\mathcal{K}$. Since $\mathcal{K}$ is expected to be much larger than one in the Solar System [12], the parenthesis on the left-hand side must be substantially suppressed. Therefore even if the additional terms are small in the Solar System, especially because they appear on the left-hand side of equation (11) or (12) they may measurably alter the dynamic of the fields in the Solar System. Indeed, [18, 19] show that, in modified

theories of gravity that seek to explain the observed accelerated expansion of the Universe, corrections to the effective Newtonian potential of a massive body that grow with radius are generic.

Let us take the Minkowski metric as the background and add a spherically symmetric perturbation due to the Sun,

$$e^\nu = 1 + \phi(r), \qquad (18)$$
$$e^\xi = 1 + \psi(r).$$

We now expand the perturbations on a complete set, including both positive and negative powers of r:

$$\phi(r) = \sum_{i=1}^\infty a_i \left(\frac{r_s}{r}\right)^i + \sum_{i=0}^\infty A_i \left(\frac{r}{r_{gm}}\right)^i \qquad (19)$$
$$\psi(r) = \sum_{i=1}^\infty b_i \left(\frac{r_s}{r}\right)^i + \sum_{i=0}^\infty B_i \left(\frac{r}{r_{gm}}\right)^i .$$

Here $r_{gm} \equiv (r_s/M)^{1/2}$. One can show that this is the scale at which modifications of gravity occur in theories that attempt to modify the Einstein equations to get MOND [13]. For the Sun, $r_{gm} \simeq 10^{11}$ km. Since inside the Solar System $r \lesssim 10^9$ km, we have $\frac{r}{r_{gm}} \lesssim 10^{-2}$.

By a rescaling of the coordinates t and r, we can eliminate A_0 and B_0 such that the constant term is equal to 1. The coefficients in front of the additional terms, A_i and B_i can be constrained by observations in the Solar System. Indeed, observations of the perihelion shift of Mercury allow to constrain the parameter a_2 to be 0.5 with an accuracy of $\delta a_2 \simeq 10^{-3}$, while time-delay observations allow us to constrain b_1 to be 1 with an accuracy of $\delta b_1 \simeq 10^{-5}$.

We derive limits on the coefficients A_i and B_i by requiring $|A_i| \left(\frac{r_M}{r_{gm}}\right)^i \lesssim \delta a_2 \left(\frac{r_s}{r_M}\right)^2$, where $r_M \simeq 6 \times 10^7$ km is the distance between Mercury and the Sun. The same argument applies for the other metric field. We find $|B_i| \left(\frac{r}{r_{gm}}\right)^i \lesssim \delta b_1 \frac{r_s}{r}$. The best constraint on b_1 comes from the Cassini satellite when it was at $\sim 10^9$ km from the Sun [20].

These limits would be exact if the only non-zero term were the one under consideration. Otherwise, the true limits may be either weaker or stronger (as the sum of a series can easily be smaller than some of its terms), or even have the opposite sense ($>$ instead of $<$ or vice versa). For example, a correction of the form $\exp(-Cr/r_{gm})$ with $C > 0$ requires a lower limit on C rather than the upper limits that would be derived from a term-by-term analysis. Thus the limits we shall proceed to derive from a term-by-term approach will be sufficient, but not necessary.

Indeed, we show that a set of consistency conditions between the coefficients of the powers of r/r_{gm} can be satisfied simultaneously. Thus an acceptable solution exists which has an expansion that includes both negative

and positive powers of r/r_{gm}. Future detailed study of the Schwarzschild metric in Einstein-Aether theories may yet identify an expansion basis that is better adapted to the physics than polynomials in r.

One might also ask if these extra-terms in the metric are physically acceptable if considered in isolation. After all, they diverge when r goes to infinity. However, the expansion (10) is itself appropriate only for $r \ll r_{gm}$. As mentioned though, to obtain the correct modification to Poisson's equation for $r \gg r_{gm}$ one expects terms in the metric growing as rapidly as r^v where $v \geq 1$. We might expect the spacetime to be asymptotically flat far from the source; although, we should be prepared that the modified gravity theory may only allow for cosmological solutions – Minkowski space need not be a global solution of the theory. Nevertheless, the physical quantity related to the flatness is the curvature $Q \equiv R^{abcd}R_{abcd}$, which behaves like $g_{rr}^{-2}r^{-4}$ as r approaches infinity for a metric of the form (19). We see that Q tends to zero at large r, even if g_{rr} diverges. Therefore positive powers of r are compatible **with asymptotic flatness**.

Through the equations of motion, the perturbations of the metric created by the Sun lead to perturbations of the vector field:

$$A^r(r) = \alpha(r) \qquad (20)$$
$$A^t(r) = 1 + \beta(r).$$

We can again expand these perturbations on the complete set:

$$\alpha(r) = \sum_{i=1}^\infty d_i \left(\frac{r_s}{r}\right)^i + \sum_{i=0}^\infty D_i \left(\frac{r}{r_{gm}}\right)^i \qquad (21)$$
$$\beta(r) = \sum_{i=1}^\infty e_i \left(\frac{r_s}{r}\right)^i + \sum_{i=0}^\infty E_i \left(\frac{r}{r_{gm}}\right)^i .$$

In the following we restrict ourselves to the leading order term $n = 1/2$ in the expansion (10) for $\mathcal{F}'$. In appendix B we show that $\alpha(r)$ and $\beta(r)$ are of the same order-of-magnitude as the perturbations of the metric $\phi(r)$ and $\psi(r)$. Since the leading term of the metric in the Solar System is $\frac{r_s}{r}$, we will take this value at the edge of the Solar System as a limit for the peculiar terms. This means that

$$|D_i| \left(\frac{r}{r_{gm}}\right)^i \lesssim \frac{r_s}{r} \quad \text{and} \quad |E_i| \left(\frac{r}{r_{gm}}\right)^i \lesssim \frac{r_s}{r} . \qquad (22)$$

In table I, we summarize the resulting constraints on the coefficients A_i, B_i, D_i and E_i with $i \leq 2$.

Next we wish to use our expansion of the metric (19) and the vector field (21) to solve equations (11), (12), (13) and the constraint equation (7) order-by-order in r. We first rewrite the positive powers of r in the form

$$A_n \left(\frac{r}{r_{gm}}\right)^n = A_n \left(\frac{r_s}{r_{gm}}\right)^n \left(\frac{r}{r_s}\right)^n \equiv \tilde{A}_n \left(\frac{r}{r_s}\right)^n , \qquad (23)$$

e^ν	e^ξ	A^t, A^r
$A_0 = 0$	$B_0 = 0$	$\|D_0\|, \|E_0\| \lesssim 3 \cdot 10^{-9}$
$\|A_1\| \lesssim 10^{-14}$	$\|B_1\| \lesssim 10^{-11}$	$\|D_1\|, \|E_1\| \lesssim 10^{-6}$
$\|A_2\| \lesssim 10^{-10}$	$\|B_2\| \lesssim 5 \cdot 10^{-9}$	$\|D_2\|, \|E_2\| \lesssim 5 \cdot 10^{-4}$

TABLE I: Constraints from observations on the coefficients of positive powers of $\frac{r}{r_{gm}}$.

where $\tilde{A}_n \equiv A_n \left(\frac{r_s}{r_{gm}} \right)^n$. We do the same for the coefficients of the other fields: B_n, E_n and D_n. Since $\frac{r_s}{r_{gm}} \simeq 10^{-11}$, the constraints on the A_n in table I become much more stringent constraints on the $\tilde{A}_n$ (see table II in appendix C 2).

We use a perturbative expansion in the coefficients. Moreover we assume the hierarchy $1 \gg |\tilde{A}_1| \gg |\tilde{A}_2| \gg ...$ for each of the four fields in order to truncate the number of terms present at each order. The detailed calculation is presented in appendix C. Here we summarize the results.

Equation (13) leads to two possibilities. The first is that $d_s = 0$ and $\tilde{D}_s = 0$ for all s, i.e. $A^r(r) = 0$. This means that the only degree of freedom of the vector field is $A^t(r)$ **which is then completely determined by the metric through the unit time-like constraint. We do not study in this specific case in this paper, but rather concentrate on the other possibility $A^r(r) \neq 0$. Nevertheless the former solution could be relevant in some particular cases. For example in [21], it has been shown that in the usual Einstein-Aether theory, corresponding to $n = 0$, there are regular perfect fluid stars with a static (i.e. $A^r(r) = 0$) aether exterior.**

The choice $A^r(r) \neq 0$ implies that the parameters c_1, c_2 and c_3 satisfy the constraint

$$(c_1+c_2+c_3)s^2 - 3(c_1+c_2+c_3)s - 2(c_1-c_2+c_3) = 0 , \quad (24)$$

for some positive integer s.

We have explicitly calculated the case $s = 1$, which corresponds to the constraint $c_3 = -c_1$. From the positive orders in $\frac{r_s}{r}$ we find that a_1 is unconstrained. If we set $a_1 = -1$ to recover Newton's theory, we find

$$b_1 = 1 + O(10^{-9}) \quad \text{and} \quad a_2 = \frac{1}{2} + O(10^{-9}) , \quad (25)$$

which is in complete agreement with the observations which measure

$$b_1^{obs} = 1 \pm 10^{-3} \quad \text{and} \quad a_2^{obs} = \frac{1}{2} \pm 10^{-5} . \quad (26)$$

Moreover, we find that c_1 no longer has to be equal to zero. We have four solutions for c_1 and c_2 which are negative and therefore causal [12]. They are given in appendix C 4. In each solution, c_1 and c_2 are of the order of $\left(\frac{\tilde{B}_2}{Mr_s} \right)^2 \frac{1}{\alpha_1^2}$. Since $\left(\frac{\tilde{B}_2}{Mr_s} \right)^2 \sim 10^{-18}$, either α_1 must be very small, or c_1 and c_2 are very small.
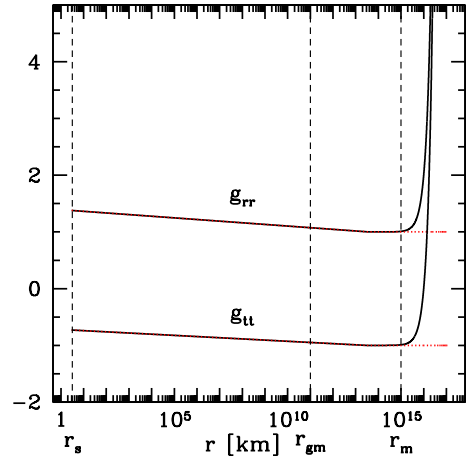

FIG. 1: The metric fields g_{tt} and g_{rr} as functions of r, in the aether theory (solid line) and in general relativity (dotted line). We see that at the scale $r_m \simeq 10^{15}$ km the two models start to diverge. $r_{gm} \simeq 10^{11}$ **km is the scale at which all growing corrections are expected to start to dominate.**

In figure 1 we plot $g_{tt}(r) = -e^{\nu(r)}$ and $g_{rr}(r) = e^{\xi(r)}$ (solid line) for the values of a_1 to a_4, b_1 to b_4, A_1, A_2 and B_1 calculated in appendix C 4 b. We find that B_2 is unconstrained by the equation of motion, except that $|B_2| < 5 \cdot 10^{-9}$ (see table I). We choose to saturate this maximum value for the plot. We also plot g_{tt} and g_{rr} from General Relativity, i.e. the Schwarzschild metric up to the order $\left(\frac{r_s}{r} \right)^4$ (dotted line). We see that inside the Solar System ($r \leq 10^9$ km) the two models are equivalent, but at a scale $r_m \simeq 10^{15}$ km they start to diverge. The Schwarzschild metric tends to $g_{tt} = -1$ and $g_{rr} = 1$, whereas the aether metric starts to grow. **The scale r_m is the one at which the first two growing corrections (r and r^2) become dominant. The subsequent growing terms could start to grow earlier. Nevertheless the limits set by hand on the coefficients A_i and B_i (see table I) ensure that they cannot become important inside the Solar System. Moreover $r_{gm} = 10^{11}$ km is the scale at which modifications of gravity should occur. Hence we expect the growing terms to remain small for $r \lesssim r_{gm}$. Nevertheless, only the full calculation of all the coefficients could confirm this limit.** Note that $g_{tt}(r)$ passes through zero at $r \simeq 10^{16}$ km and is therefore singular. However the series (19) cannot be taken seriously for $r > r_{gm}$, where they may not converge.

Finally we have determined the structure of the equa-

tions at each order. For negative powers in r ($s > 1$),

$$S\begin{pmatrix} a_s \\ b_s \\ d_s \\ e_s \end{pmatrix} = \begin{pmatrix} H_1^{(s)} \\ H_2^{(s)} \\ H_3^{(s)} \\ H_4^{(s)} \end{pmatrix}, \qquad \text{where} \qquad (27)$$

$$S \equiv \begin{pmatrix} -1 & 0 & 0 & -2 \\ 0 & s(s-1) & 0 & 0 \\ -s & -s & 0 & 0 \\ \frac{d_1 s(s-1)}{2} & 0 & -\frac{c_1(s-1)(s-2)}{4} & \frac{d_1 s(s-1)}{2} \end{pmatrix} \quad (28)$$

and $H_i^{(s)}$ are functions of the previous coefficients – $a_1, ... a_{s-1}, b_1, ..., b_{s-1}, d_1, .., d_{s-1}, e_1, .., e_{s-1}$. Therefore at order s a unique solution (a_s, b_s, d_s, e_s) exists if

$$\det S = -\frac{c_1}{2} s^2 (s-1)^2 (s-2) \neq 0 . \qquad (29)$$

Since $c_1 \neq 0$, this condition is satisfied for $s > 2$. The order $s = 2$, for which the determinant vanishes, has been solved explicitly. The solutions are not unique and are given in appendix C 4. For each order $s > 2$, we can find the unique solution as a function of the lower orders.

The same structure repeats for positive powers of r ($s > 2$):

$$\hat{S}\begin{pmatrix} \tilde{A}_s \\ \tilde{B}_s \\ \tilde{D}_s \\ \tilde{E}_s \end{pmatrix} = \begin{pmatrix} \hat{H}_1^{(s)} + \hat{Q}_1^{(s)} \\ \hat{H}_2^{(s)} + \hat{Q}_2^{(s)} \\ \hat{H}_3^{(s)} + \hat{Q}_3^{(s)} \\ \hat{H}_4^{(s)} + \hat{Q}_4^{(s)} \end{pmatrix}, \qquad \text{where} \qquad (30)$$

$$\hat{S} \equiv \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -s(s+1) & 0 & 0 \\ s & s & 0 & 0 \\ \frac{c_1 s(s+1)d_1}{2} & 0 & [c_2(14-s)d_1^2 - \frac{c_1(s-1)(s-2)}{4}] & \frac{c_1 s(s+1)d_1}{2} \end{pmatrix} \quad (31)$$

and $\hat{H}_i^{(s)}$ are functions of a_1, b_1, d_1 and e_1, and of the previous coefficients – $\tilde{A}_1, ..., \tilde{A}_{s-1}$, $\tilde{B}_1, ..., \tilde{B}_{s-1}$, $\tilde{D}_1, ..., \tilde{D}_{s-1}, \tilde{E}_1, ..., \tilde{E}_{s-1}$. In appendix C 2 we argue that these functions are at most of the same order of magnitude as the coefficient of order s. The $\hat{Q}_i^{(s)}$ are functions of the coefficients $\tilde{A}_{s+1}, \tilde{A}_{s+2}, ..., \tilde{B}_{s+1}, ..., \tilde{D}_{s+1}, ..., \tilde{E}_{s+1}, ...$ Generically the $\hat{H}_i^{(s)}$ are much larger than the $\hat{Q}_i^{(s)}$ since the constraints on the coefficients $\tilde{A}_s$ are much more stringent for larger s. We can, in general, therefore neglect the $\hat{Q}_i^{(s)}$. (See appendix C 2 for a more detailed discussion).

A unique solution $(\tilde{A}_s, \tilde{B}_s, \tilde{D}_s, \tilde{E}_s)$ exists then at order s if

$$\det \hat{S} = -2s^2(s+1)\left(c_2(14-s)d_1^2 - \frac{c_1}{4}(s-1)(s-2)\right) \neq 0 . \qquad (32)$$

The order $s = 0, 1$ and 2 have been calculated in appendix C 4. For each of the solutions (C42) to (C45) we find that $\det \hat{S} \neq 0 \; \forall \; s$.

The structure of equations (27) and (30) shows that there exist solutions to the equations of motion that can be expressed as the expansions (19) for the metric and (21) for the vector field. Indeed, since at each order the determinants $\det S$ and $\det \hat{S}$ are non zero, we are ensured that a solution exists. We conclude that it is not sufficient to calculate the first two orders, even if we are only interested in the values of the Post-Newtonian parameters. In order to be sure that we have a *bona fide* solution of the equations of motion and that we will not encounter an inconsistency at any given order, we need to calculate the determinant of the system at every order.

IV. CONCLUSION

In this paper, we have studied the constraints we imposed from Solar System observations on generalized Einstein-Aether theory. We have considered an expansion of the metric around the Newtonian solution including the usual Post-Newtonian terms, which are negative powers of the distance r from the Sun, but also terms increasing with the distance (positive powers of r). The aim of this complete expansion is to take into account in the Solar System the effect of modification of gravity at large (Galactic and cosmological) scales. These effects are usually neglected in the Solar System and one considers only the decreasing terms in the metric expansion [15, 16, 22]. Nevertheless, as long as the increasing terms are sufficiently small to be consistent with observations, nothing forces them to be completely absent in the Solar System. Moreover in the case of generalized Einstein-Aether theory, we found that the increasing terms play a crucial role, since without them the theory suffers from acausality. Indeed, if we neglect them, the equations of motion imply that one parameter of the theory $c_1 = 0$, leading to superluminal propagation of the aether field perturbations. If we consider the full expansion, the pathology disappears.

Moreover, we found that the Post-Newtonian parameters related to the precession of the perihelion of Mercury and the time delay of radio pulses are in agreement with observations. We found also constraints on the increasing terms coming from these observations, and constraints on the parameters c_1, c_2, c_3 and α_1 from the equations of motion. **TomTom Comments Follow**: Indeed, whereas the α_i represent behaviour in a particular regime of the theory (i.e. quasistatic configurations where the gravitational field is typically much larger than the mass scale M), the c_i can be expected to appear in calculations throughout. It may be shown [12, 23] that simultaneous consideration of very weak gravitational fields, the requirement for a realistic background cosmology, and suitable growth of large scale structure will favour the c_i

to be O(1). We may immediately see then from equations (C42)-(C45) that this will tend to correspond to an upper limit on the number α_1 of 10^{-9} to 10^{-8}. It is interesting to compare this bound to that obtained by rather considering the influence of such a term on the Poisson equation then deducing the perihelion precession due to the extra force provided by the modified potential in the context of Newtonian gravity. This was done in [4] where it was found that the resulting perihelion shift $\delta\phi_\alpha$ of Mercury's orbit per revolution, to lowest order in eccentricity, was given by:

$$\delta\phi_\alpha \sim 10\pi M\alpha_1 \frac{r^2}{r_s} \qquad (33)$$

where r is the semi-major axis length of Mercury's orbit. Taking $M = 1.2 \times 10^{-8} cm/s^2$, this yields a precession per revolution which is approximately $\alpha_1/5$ of the prediction due to general relativity. Therefore, as observation agrees with the prediction of general relativity to within $\sim 10^{-3}$, it was argued, α_1 must be less than around 5×10^{-3}, therefore several orders of magnitude greater than our analysis will allow. Clearly then, a benefit of recovering the modified Poisson equation from a set of generally covariant field equations has been to allow a broadening of the scope of analysis of the theory's implications in the solar system and in this case, in concert with additional constraints placed on the theory's parameters in other regimes, this has allowed for a significantly more severe restriction on the permissable size of such a modification. **Comments Ended**

Furthermore, we have developed a general method to test that the metric expansion is a solution of the equations of motion. The constraints that we obtain must be interpreted, in part, as consistency conditions on the theory. Indeed, it is not sufficient to calculate the first coefficients in the expansion to confront the theory with Solar System observations, but one has to study carefully the structure of the equations at each order to be sure that no inconsistency will invalidate the results. Our results show that, even though we have an infinite hierarchy of equations, these are solvable in terms of the fundamental parameters of the theory. Hence, generalized Einstein-Aether theories are viable theories of gravity within the Solar System.

APPENDIX A: SPECIFIC EXPRESSION OF THE EQUATIONS OF MOTION

The explicit expression for $\mathcal{K}$ in the spherically symmetric and static metric is

$$
\begin{aligned}
M^2\mathcal{K} =\ & (A^r)^2\Bigg[(c_1+c_3)\Big(\frac{2}{r^2}+\frac{2\xi'}{r}+\frac{\nu'^2}{4}+\frac{3\xi'^2}{4}\Big) \\
& + c_2\Big(\frac{4}{r^2}+\frac{2\nu'}{r}+\frac{6\xi'}{r}+\frac{3\nu'\xi'}{2}+\frac{\nu'^2}{4}+\frac{9\xi'^2}{4}\Big)\Bigg] \\
& + \Big[(c_1+c_3)\xi'+c_2\Big(\frac{4}{r}+\nu'+3\xi'\Big)\Big]A^r A^{r'} \quad (A1) \\
& + e^{\nu-\xi}(c_3-c_1)\Big[\frac{\nu'^2}{2}(A^t)^2+\nu' A^t A^{t'}\Big] \\
& + (c_1+c_2+c_3)(A^{r'})^2 - c_1 e^{\nu-\xi}(A^{t'})^2\ .
\end{aligned}
$$

The functions appearing in the equations of motions (11), (12) and (13) are given by:

$$
\begin{aligned}
f_1 =\ & \frac{e^\xi}{2}\Big[(c_1+c_3)\Big(\frac{\nu'^2}{2}+\frac{3\nu'\xi'}{2}+\frac{2\nu'}{r}+\nu''\Big) , \quad (A2) \\
& +c_2\Big(\frac{4}{r^2}+\frac{4(\nu'+3\xi')}{r}+\frac{(\nu'+3\xi')^2}{2}+\nu''+3\xi''\Big)\Big] , \\
f_2 =\ & -(c_1+c_3)\nu'-c_2\Big(\nu'+3\xi'+\frac{4}{r}\Big) , \\
f_3 =\ & e^\xi\Big[(c_1+c_3)\nu'+\frac{c_2}{2}\Big(3\nu'+9\xi'+\frac{12}{r}\Big)\Big] , \\
f_4 =\ & \frac{e^\nu}{2}(c_3-c_1)\Big(\frac{3\nu'^2}{2}+\frac{\nu'\xi'}{2}+\frac{2\nu'}{r}+\nu''\Big) , \\
f_5 =\ & -\frac{e^\nu}{2}(c_3-c_1)\nu' , \\
f_6 =\ & e^\nu\Big[\frac{c_3}{2}\Big(\frac{4}{r}+\xi'+5\nu'\Big)-c_1\nu'\Big] , \\
g_1 =\ & -\frac{e^\xi}{2}\Big[(c_1+c_3)\Big(\frac{\nu'^2}{2}+\frac{3\xi'^2}{2}+\frac{4\xi'}{r}+\frac{4}{r^2}\Big) , \\
& +c_2\Big(\frac{8}{r^2}+\frac{\nu'^2}{2}+\frac{9\xi'^2}{2}+\frac{4(\nu'+3\xi')}{r}+3\nu'\xi'\Big)\Big] , \\
g_2 =\ & -e^\xi\Big[(c_1+c_3)\xi'+c_2\Big(\nu'+3\xi'+\frac{4}{r}\Big)\Big] , \\
h_1 =\ & -c_1\Big(\nu'^2+\frac{4}{r^2}+\frac{2(\nu'+\xi')}{r}+\nu''-\xi''\Big) , \\
& +c_3\Big(\nu'\xi'-\frac{4}{r^2}+\frac{2(\nu'-\xi')}{r}+\nu''+\xi''\Big) , \\
& +c_2\Big(\nu''+3\xi''-\frac{4}{r^2}\Big) , \\
h_2 =\ & c_1(\nu'-\xi')-c_2\Big(\nu'+3\xi'+\frac{4}{r}\Big)-c_3(\nu'+\xi') , \\
h_3 =\ & -c_1\Big(3\nu'+\xi'+\frac{4}{r}\Big) \quad \text{and}
\end{aligned}
$$

$$h_4 \;=\; (c_1 + c_2 + c_3)\Big(\nu' + 3\xi' + \frac{4}{r}\Big)$$

APPENDIX B: ORDER OF MAGNITUDE OF THE VECTOR FIELD PERTURBATIONS

In this appendix, we compare orders of magnitude of the different field perturbations. We know that in the Solar System the metric perturbations are of the order $\phi(r) \simeq \psi(r) \simeq \frac{r_s}{r}$. We want to calculate the order of magnitude of the vector field perturbations $\alpha(r)$ and $\beta(r)$.

At first order in the perturbations:

$$\nu' \;=\; \frac{(e^\nu)'}{e^\nu} \simeq \phi' \;, \qquad \xi' \simeq \psi' \;, \tag{B1}$$

$$A^{t'} \;=\; \beta' \;\text{ and }\; A^{r'} = \alpha' \;.$$

The constraint (7) gives

$$-(1+\phi)(1+\beta)^2 + (1+\psi)\alpha^2 = -1 \;. \tag{B2}$$

At the lowest order in each perturbation we have

$$-\phi - 2\beta + \alpha^2 = 0 \;. \tag{B3}$$

We have four possibilities:

1. The three terms are or the same order of magnitude $\beta \sim \alpha^2 \sim \phi$ which implies $\alpha^2 = \phi + 2\beta$;

2. $\alpha^2 \ll \phi \sim \beta$ which implies $\phi \simeq -2\beta$;

3. $\beta \ll \phi \sim \alpha^2$ which implies $\alpha^2 \simeq \phi$;

4. $\phi \ll \beta \sim \alpha^2$ which implies $\alpha^2 \simeq 2\beta$.

We see directly that case 3 is not possible, since $\phi = -r_s/r < 0$ at first order in the Solar system. We use equation (13) to exclude the possibilities 1 and 4. Indeed, in these two cases we have $\alpha \gg \beta, \phi$. Using these constraints we find

$$\mathcal{K} = \frac{1}{M^2}\Big[\frac{2\alpha(c_1 + 2c_2 + c_3)}{r^2} \;+\; \frac{4\alpha'\alpha c_2}{r} \tag{B4}$$
$$+\; (c_1 + c_2 + c_3)\alpha'^2\Big] \;,$$

and equation (13) becomes

$$0 \;=\; 2(c_1 + c_2 + c_3)\Big[-2\frac{\alpha}{r^2} + 2\frac{\alpha'}{r} + \alpha''\Big]\mathcal{K} \\ -\Big[c_2\frac{\alpha}{r} + (c_1 + c_2 + c_3)\alpha'\Big]\mathcal{K}' = 0 \;. \tag{B5}$$

The only solution of this equation is $\alpha(r) = 0$, which implies that equation (13) doesn't allow the spatial component A^r to be much larger than the other perturbations. So the only possibility is that the dominant contribution to the time component of the vector field is $\beta = -\frac{r_s}{2r}$, and that the dominant contribution to the space component satisfies $\alpha^2 \ll \frac{r_s}{r}$. So it is legitimate to assume that also $\alpha \sim \frac{r_s}{r}$ at first order.

APPENDIX C: RESOLUTION

In this appendix we present the detailed derivation of the solutions. The aim is to take the expansion (19) for the metric and (21) for the vector field, to insert it in the equations of motions and to solve order by order in power of r, using a perturbative expansion for the positive powers of r. The orders r^{-1} and r^{-2} give the values of the post-Newtonian parameters b_1 and a_2. The orders r^{-3}, r^{-4} and r^0, r^1 and r^2 allow us to constrain the values of the c_i which are compatible with the data. Nevertheless it is not sufficient to calculate only these orders. Indeed, each equation generates an infinite numbers of positive and negative orders, and the expansions (19) and (21) are solutions of the equations of motion only if each of these equations has a solution. Therefore, it is crucial to understand the structure of the equations at each order and to verify that one introduces a sufficient number of new coefficients at each order so that the equations do not lead to constraints between coefficients which have already been determined, leading to a possible inconsistency.

In C 1, we consider only the terms with negative powers of r, that means the terms which appear in the usual Post-Newtonian parametrization . In C 2 we will consider the positive powers. Finally, in C 3 and C 4 we apply our result to equations (11), (12), (13) and (7).

1. Negative powers of r

We have four fields A^r, A^t, e^ξ and e^ν, satisfying four equations. Three fields, A^t, e^ξ and e^ν have an expansion of the form

$$e^\nu = 1 + \sum_{n=1}^{\infty} a_n x^n \;, \tag{C1}$$

where $x = \frac{r_s}{r}$. The other field has no constant term in the expansion

$$A^r = \sum_{n=1}^{\infty} d_n x^n \;. \tag{C2}$$

In the following we restrict ourselves to the two fields: A^r and e^ν and two equations. The generalization for the four fields will then be straightforward.

We assume that the two equations governing the two fields have the form

$$\sum_{i=1}^{m} f_i r^{\sigma_i} (e^\nu)^{\alpha_i} (e^{\nu'})^{\beta_i} (e^{\nu''})^{\gamma_i} (A^r)^{\mu_i} (A^{r'})^{\nu_i} (A^{r''})^{\rho_i} = 0 \;, \tag{C3}$$

where a′ stands for $\frac{d}{dr}$ and the exponents are all positive integers or null, except σ_i which is negative. Actually α_i will in principle be negative, since $\nu' = (e^\nu)'(e^\nu)^{-1}$ and $\nu'' = (e^\nu)''(e^\nu)^{-1} - (e^\nu)^2(e^\nu)^{-2}$, but we can always

multiply by $(e^\nu)^j$ so that α_i is a positive integer or null for all i.

The form (C3) is not exactly the one which appears in equations (11) and (12), because of the term K^n, where n is an integer or a half integer. Nevertheless, we will see later that we can rewrite the problem in such a way that the equation is the form (C3).

Since all the terms in the sum must have the same dimension, we have that $-\sigma_i + \beta_i + 2\gamma_i + \nu_i + 2\rho_i = c$ is the same for all i. Therefore, we can solve for σ_i, and then multiply every term by r^c. We insert then the expansion (C1) and (C2), and their derivatives to obtain

$$
\sum_{i=1}^{m} f_i(-1)^{\beta_i+\nu_i} x^{-(\beta_i+2\gamma_i+\nu_i+2\rho_i)} \left(1 + \sum_{n=1}^{\infty} a_n x^n\right)^{\alpha_i}
$$
$$
\cdot \left(\sum_{n=1}^{\infty} n a_n x^{n+1}\right)^{\beta_i} \left(\sum_{n=1}^{\infty} n(n+1) a_n x^{n+2}\right)^{\gamma_i}
$$
$$
\cdot \left(\sum_{n=1}^{\infty} d_n x^n\right)^{\mu_i} \left(\sum_{n=1}^{\infty} n d_n x^{n+1}\right)^{\nu_i}
$$
$$
\cdot \left(\sum_{n=1}^{\infty} n(n+1) d_n x^{n+2}\right)^{\rho_i} = 0 . \tag{C4}
$$

Each term of the product can be expanded in power of x using the binomial theorem

$$
(e^\nu)^{\alpha_i} = \left(1 + \sum_{n=1}^{\infty} a_n x^n\right)^{\alpha_i} = 1 + \alpha_i a_1 x + ... \tag{C5}
$$

$$
(A^r)^{\mu_i} = \left(\sum_{n=1}^{\infty} d_n x^n\right)^{\mu_i} \tag{C6}
$$
$$
= d_1^{\mu_i} x^{\mu_i} + \mu_i d_1^{\mu_i-1} d_2 x^{\mu_i+1} + ...
$$
$$
+ \left\{\mu_i d_1^{\mu_i-1} d_s + g_s(d_1, \cdots, d_{s-1})\right\} x^{\mu_i+s-1} + ...
$$

where $g_s(d_1, \cdots, d_{s-1})$ is a function of the coefficients of order 1 to $s-1$.

$$
(e^{\nu'})^{\beta_i} = \left(\sum_{n=1}^{\infty} n a_n x^{n+1}\right)^{\beta_i}
$$
$$
= d_1^{\beta_i} x^{2\beta_i} + \cdots \tag{C7}
$$

and the same for $A^{r''}$.

We can insert these developments in equation (C4) and solve order by order in x. The lowest order is

$$
\sum_{i \in O_1} f_i(-1)^{\beta_i+\nu_i} 2^{\gamma_i+\rho_i} a_1^{\beta_i+\gamma_i} d_1^{\mu_i+\nu_i+\rho_i} x^{\beta_i+\gamma_i+\mu_i+\nu_i+\rho_i} = 0 ,
$$
$$
\tag{C8}
$$
where $O_1 = \Big\{i \in \{1, ..., m\}$ such that $\beta_i + \gamma_i + \mu_i + \nu_i + \rho_i = \delta$ is the smallest exponent of the sum$\Big\}$.

The next order in x contains two different contributions. The first contribution comes from the terms in O_1 where we take the next order in the expansion for one term of the product, and the lowest order for the others. This contribution contains two new coefficients a_2 and d_2 which appear linearly, and also the previous coefficients a_1 and d_1. The second contribution comes from other terms in the sum, $i \in O_2$, where $O_2 = \Big\{i \in \{1, ..., m\}$ such that $\beta_i + \gamma_i + \mu_i + \nu_i + \rho_i = \delta + 1\Big\}$. This contribution contains only a_1 and d_1. The next order $x^{\delta+1}$ is therefore

$$
\sum_{i \in O_1} f_i(-1)^{\beta_i+\nu_i} 2^{\gamma_i+\rho_i} \Big\{ (2\beta_i + 3\gamma_i) a_1^{\beta_i+\gamma_i-1} d_1^{\mu_i+\nu_i+\rho_i} a_2
$$
$$
+ (\mu_i + 2\nu_i + 3\rho_i) a_1^{\beta_i+\gamma_i} d_1^{\mu_i+\nu_i+\rho_i-1} d_2 \Big\} x^{\delta+1}
$$
$$
+ H_1^{(1)}(a_1, d_1) x^{\delta+1} = 0 . \tag{C9}
$$

More generally, at the order $x^{\delta+s-1}$ ($s > 1$) the two equations have the form

$$
F_1^{(s)}(a_1, d_1) \cdot a_s + G_1^{(s)}(a_1, d_1) \cdot d_s
$$
$$
= H_1^{(s)}(a_1, \cdots, a_{s-1}, d_1, \cdots, d_{s-1}) ,
$$
$$
F_2^{(s)}(a_1, d_1) \cdot a_s + G_2^{(s)}(a_1, d_1) \cdot d_s
$$
$$
= H_2^{(s)}(a_1, \cdots, a_{s-1}, d_1, \cdots, d_{s-1}) , \tag{C10}
$$

where

$$
F_1^{(s)}(a_1, d_1) = \sum_{i \in O_1} f_i(-1)^{\beta_i+\nu_i} 2^{\gamma_i+\rho_i-1} \cdot
$$
$$
\times \left(2s\beta_i + s(s+1)\gamma_i\right) a_1^{\beta_i+\gamma_i-1} d_1^{\mu_i+\nu_i+\rho_i}
$$
$$
G_1^{(s)}(a_1, d_1) = \sum_{i \in O_1} f_i(-1)^{\beta_i+\nu_i} 2^{\gamma_i+\rho_i-1}
$$
$$
\times \left(2\mu_i + 2s\nu_i + s(s+1)\rho_i\right) a_1^{\beta_i+\gamma_i} d_1^{\mu_i+\nu_i+\rho_i-1} ,
$$
$$
\tag{C11}
$$

and $H_1^{(s)}(a_1, \cdots, a_{s-1}, d_1, \cdots, d_{s-1})$ is a function of the coefficients of order less than s. $F_2^{(s)}(a_1, d_1)$ and $G_2^{(s)}(a_1, d_1)$ have the same form but with different f_i and O_1.

A unique solution (a_s, d_s) exists at order s if

$$
\det \begin{pmatrix} F_1^{(s)}(a_1, d_1) & G_1^{(s)}(a_1, d_1) \\ F_2^{(s)}(a_1, d_1) & G_2^{(s)}(a_1, d_1)) \end{pmatrix} \neq 0 . \tag{C12}
$$

Since the functions $F_j^{(s)}$ and $G_j^{(s)}$ contain only a_1, d_1 and the parameters f_i for $i \in O_1$, it is easy to calculate them at each order, without solving the entire system. We only have to solve explicitly the first order. Then if the determinant is non zero for all $s > 1$ we know that a unique solution exists which can be determined as a function of the previous coefficients. On the other hand if the determinant vanishes at some order s, an

inconsistency may occur. Indeed, equations (C10) can become a constraint between the previous coefficients and lead to a contradiction. Therefore, if one wants to test the validity of a theory in the PPN parametrization, it is not sufficient to calculate the first coefficients in the expansion and to compare them with the observations. One has also to calculate the determinant (C12) at each order s, to insure that no additional constraints on the first coefficients will occur from higher orders.

The generalization of this method to four equations with four fields is straightforward. Each field with no constant component generates a function of the form $F_j^{(s)}$, whereas fields with a constant component generate a function of the form $G_j^{(s)}$.

2. Positive powers of r

In this section, we add positive powers of r to the expansions (C1) and (C2):

$$
\begin{aligned}
e^\nu &= 1 + \sum_{n=1}^{\infty} A_n \left(\frac{r}{r_{gm}}\right)^n + \sum_{n=1}^{\infty} a_n x^n \\
&= 1 + \sum_{n=1}^{\infty} A_n \epsilon^n \frac{1}{x^n} + \sum_{n=1}^{\infty} a_n x^n ,
\end{aligned}
$$
(C13)

where $\epsilon = \frac{r_s}{r_{gm}} \simeq 10^{-11}$. Equivalently for A^r

$$
A^r = \sum_{n=0}^{\infty} D_n \epsilon^n \frac{1}{x^n} + \sum_{n=1}^{\infty} d_n x^n .
$$
(C14)

These new terms add an infinite number of contributions to each previous order $x^{\delta+s}$. Furthermore they generate new lower orders $x^{\delta-s}$. To simplify the problem we must take into account the fact that the coefficients in front of the negative orders $A_n \epsilon^n$ (respectively $D_n \epsilon^n$) have to be small in order not to be observed in the Solar System. Indeed the constraints of table I are equivalent to $|A_n| \epsilon^n \lesssim \delta a_2 \left(\frac{r_s}{r_M}\right)^n \ll 1$. Therefore, we use a perturbative expansion for $A_n \epsilon^n$ and $D_n \epsilon^n$. Furthermore since $\frac{r_s}{r_M} \simeq 5 \cdot 10^{-8} \ll 1$ we have the following hierarchy: $1 \gg |A_1| \epsilon \gg |A_2| \epsilon^2 \dots$ (And similarly for the $D_n \epsilon^n$) So we start studying the effect of D_0 (remember that A_0 has been put to zero by a rescaling of t and r), then $\frac{A_1 \epsilon}{x}$ and $\frac{D_1 \epsilon}{x}$ and so on. And we neglect all these new terms with respect to 1. In other words, we take them into account only when they introduce a new order to the equation $x^{\delta-s}$.

Let us define new coefficients to simplify the formulae.

$$
\tilde{A}_n = \epsilon^n A_n \qquad \tilde{D}_n = \epsilon^n D_n .
$$
(C15)

In terms of these coefficients, the constraints coming from observations are as follows:

e^ν	e^ξ	A^t, A^r
$\tilde{A}_0 = 0$	$\tilde{B}_0 = 0$	$|\tilde{D}_0|, |\tilde{E}_0| \lesssim 3 \cdot 10^{-9}$
$|\tilde{A}_1| \lesssim 10^{-25}$	$|\tilde{B}_1| \lesssim 10^{-22}$	$|\tilde{D}_1|, |\tilde{E}_1| \lesssim 10^{-17}$
$|\tilde{A}_2| \lesssim 10^{-32}$	$|\tilde{B}_2| \lesssim 5 \cdot 10^{-31}$	$|\tilde{D}_2|, |\tilde{E}_2| \lesssim \cdot 10^{-26}$

TABLE II: Constraints from observations on the new coefficients .

a. Effect of $\tilde{D}_0$

$$
\left(\tilde{D}_0 + \sum_{n=1}^{\infty} d_n x^n\right)^{\mu_i} = \tilde{D}_0 d_1^{\mu_i-1} x^{\mu_i-1} + d_1^{\mu_i} x^{\mu_i} + \cdots O(\tilde{D}_0^2)
$$
(C16)

The term $\tilde{D}_0$ introduces a new order $x^{\delta-1}$

$$
\sum_{i \in O_1} f_i (-1)^{\beta_i+\nu_i} 2^{\gamma_i+\rho_i} a_1^{\beta_i+\gamma_i} d_1^{\mu_i+\nu_i+\rho_i-1} \tilde{D}_0 \cdot x^{\delta-1} \quad (C17)
$$

b. Effect of $\tilde{A}_1 x^{-1}$ and $\tilde{D}_1 x^{-1}$

At first order in $\tilde{A}_1$ and $\tilde{D}_1$ we have

$$
\begin{aligned}
(e^\nu)^{\alpha_i} &= \left(1 + \frac{\tilde{A}_1}{x} + \sum_{n=1}^{\infty} a_n x^n\right)^{\alpha_i} \\
&= \alpha_i \frac{\tilde{A}_1}{x} + 1 + \alpha_i a_1 x + \cdots
\end{aligned}
$$
(C18)

$$
\begin{aligned}
(A^r)^{\mu_i} &= \left(\frac{\tilde{D}_1}{x} + \sum_{n=1}^{\infty} d_n x^n\right)^{\mu_i} = \mu_i \tilde{D}_1 d_1^{\mu_i-1} x^{\mu_i-2} \\
&+ \mu_i(\mu_i-1)\tilde{D}_1 d_1^{\mu_i-2} d_2 x^{\mu_i-1} + d_1^{\mu_i} x^{\mu_i} + \cdots
\end{aligned}
$$
(C19)

$$
\begin{aligned}
(e^{\nu\prime})^{\beta_i} &= \left(\tilde{A}_1 - \sum_{n=1}^{\infty} n a_n x^{n+1}\right)^{\beta_i} = \\
&(-1)^{\beta_i-1} \frac{\beta_i(\beta_i-1)}{2} \tilde{A}_1 a_1^{\beta_i-1} x^{2\beta_i-2} \\
&+ (-1)^{\beta_i-1} \beta_i(\beta_i-1)^2 \tilde{A}_1 a_1^{\beta_i-2} a_2 x^{2\beta_i-1} \\
&+ (-1)^{\beta_i} \beta_i a_1^{\beta_i} x^{2\beta_i} + \cdots ,
\end{aligned}
$$
(C20)

and equivalently for $A^{r\prime}$. $e^{\nu\prime\prime}$ and $A^{r\prime\prime}$ remain the same as previously. Therefore, we see that $\tilde{A}_1 x^{-1}$ and $\tilde{D}_1 x^{-1}$ introduce a new order $x^{\delta-2}$ and also a contribution to the order $x^{\delta-1}$.

c. General: $\tilde{A}_s x^{-s}$ and $\tilde{D}_s x^{-s}$

More generally, the term $\tilde{A}_s x^{-s}$ introduces a lowest order $x^{\delta-s-1}$ linear in $\tilde{A}_s$. The following terms

$\tilde{A}_{s+1}, \tilde{A}_{s+2}...$ also contribute to the order $x^{\delta-s-1}$, but there are negligible with respect to $\tilde{A}_s$. Furthermore, the previous terms $\tilde{A}_{s-1}, \tilde{A}_{s-2}...$ also appear at order $x^{\delta-s-1}$, but non-linearly. We can show from the observational constraints that we have on the coefficients, that these non-linear contributions can be at most of the same order of magnitude as $\tilde{A}_s$, but not larger, since they contain a term of the order ϵ^s. Of course the same kind of terms are introduced by $\tilde{D}_s x^{-s}$.

Therefore, the equations at order $x^{\delta-s-1}$ $(s \geq 0)$ are

$$\hat{F}_1^{(s)}(a_1, d_1) \cdot \tilde{A}_s + \hat{G}_1^{(s)}(a_1, d_1) \cdot \tilde{D}_s$$
$$= \hat{H}_1^{(s)}(\tilde{A}_1, ..., \tilde{A}_{s-1}, \tilde{D}_1, ..., \tilde{D}_{s-1}, a_1, d_1)$$
$$+ \hat{Q}_1^{(s)}(\tilde{A}_{s+1}, ..., \tilde{D}_{s+1}, ...)$$
$$\hat{F}_2^{(s)}(a_1, d_1) \cdot \tilde{A}_s + \hat{G}_2^{(s)}(a_1, d_1) \cdot \tilde{D}_s$$
$$= \hat{H}_2^{(s)}(\tilde{A}_1, ..., \tilde{A}_{s-1}, \tilde{D}_1, ..., \tilde{D}_{s-1}, a_1, d_1)$$
$$+ \hat{Q}_2^{(s)}(\tilde{A}_{s+1}, ..., \tilde{D}_{s+1}, ...)$$
$$\text{(C21)}$$

where

$$\hat{F}_1^{(s)}(a_1, d_1) = \sum_{i \in O_1} f_i (-1)^{\beta_i + \nu_i - 1} 2^{\gamma_i + \rho_i - 1}$$
$$\times \left(2s\beta_i - s(s-1)\gamma_i \right) a_1^{\beta_i + \gamma_i - 1} d_1^{\mu_i + \nu_i + \rho_i} ,$$
$$\hat{G}_1^{(s)}(a_1, d_1) = \sum_{i \in O_1} f_i (-1)^{\beta_i + \nu_i - 1} 2^{\gamma_i + \rho_i - 1}$$
$$\times \left(-2\mu_i + 2s\nu_i - s(s-1)\rho_i \right) a_1^{\beta_i + \gamma_i} d_1^{\mu_i + \nu_i + \rho_i - 1} .$$
$$\text{(C22)}$$

$\hat{H}_j^{(s)}$ can be at most of the order of magnitude of $\tilde{A}_s$ and $\tilde{D}_s$ whereas $\hat{Q}_j^{(s)}$ are much smaller since they are of the order of magnitude of $\tilde{A}_{s+1}$ and $\tilde{D}_{s+1}$.

A unique solution $(\tilde{A}_s, \tilde{D}_s)$ exists at order s if

$$\det \begin{pmatrix} \hat{F}_1^{(s)}(a_1, d_1) & \hat{G}_1^{(s)}(a_1, d_1) \\ \hat{F}_2^{(s)}(a_1, d_1) & \hat{G}_2^{(s)}(a_1, d_1)) \end{pmatrix} \neq 0 . \quad \text{(C23)}$$

In this case, we can determine the solution at order s as a function of the previous coefficients and the following coefficients. Generically the contribution $\hat{H}_i^{(s)}$ from the previous coefficients is much larger than the one from the following coefficients $\hat{Q}_i^{(s)}$, since the constraints on the previous coefficients is less stringent (see table II). We can therefore neglect the following coefficients and at each order determine the solution as function of the previous coefficients. Nevertheless for some specific case the contribution from the previous coefficients can become very small and even vanish. In this case, we have to take into account the contribution of the following coefficients. This means that the solution $(\tilde{A}_s, \tilde{D}_s)$ will be of the order of magnitude of $(\tilde{A}_{s+1}, \tilde{D}_{s+1})$. At order $s+1$ we can then neglect $\tilde{A}_s$ and $\tilde{D}_s$ in $\hat{H}_i^{(s+1)}$ which appear non-linearly

and are therefore much smaller than the order $s+1$ linearly. This means that we can determine $(\tilde{A}_{s+1}, \tilde{D}_{s+1})$ as a function of the previous coefficients, but neglecting the order s. If the determinant (C23) is non zero for all $s \geq 0$, we can find a solution order by order. On the other hand if the determinant vanishes for some s, the equations can lead to inconsistencies.

One remark has to be made about the order of magnitude of the coefficients. In the general case with four fields, we have found constraints on the coefficients $\tilde{A}_n, \tilde{B}_n, \tilde{D}_n$ and $\tilde{E}_n$ from the observations. Since the different fields are not related to the same observation, the constraints are different for each field. We have therefore found that $\tilde{D}_n$ and $\tilde{E}_n$ can be much larger than $\tilde{A}_n$ and $\tilde{B}_n$ for small n, see table (II). For large n the situation is reversed. Nevertheless, from the previous analysis of the equations, we see that this situation is not satisfying. Indeed, if two coefficients are much larger than the two others at order s, it means that we can neglect the two small coefficients. Therefore, we introduce four new equations at order s but only two new coefficients and the determinant (C23) vanishes. Hence we can not insure that a solution exists. Therefore, even if the observations allow less stringent constraints on the order of magnitude of some coefficients, the equations of motion generically imply that all the coefficients can have at most the order of magnitude of the smallest one at each order.

3. Application to our problem

We need to transform the four equations of motion to apply the method described above. First, we multiply each equation by the correct power of e^ν and e^ξ such that each power in the equation (C3) is positive or null.

After this modification, equation (7) and (13) have the correct form and the method can be applied directly. For $n = 1/2$, which is the case we consider in detail, equation (12) reduces to

$$\left(\frac{\xi' + \nu'}{r} + \frac{\xi'^2}{4} + \frac{\xi'\nu'}{2} \right) \sqrt{\mathcal{K}} = 0 . \quad \text{(C24)}$$

We can expand the two terms $f = \left(\frac{\xi' + \nu'}{r} + \frac{\xi'^2}{4} + \frac{\xi'\nu'}{2} \right)$ and $g = \sqrt{\mathcal{K}}$ in power of x, using the method described above. Let's call ρ the lowest order of the development of f, containing only the coefficients a_1, b_1, d_1 and e_1 and σ the lowest order of g.

At lowest order the equation becomes

$$f^{(\rho)} g^{(\sigma)} = 0 . \quad \text{(C25)}$$

We have three possibilities:

- $f^{(\rho)} = 0$ and $g^{(\sigma)} \neq 0$. The following order is then: $f^{(\rho+1)} g^{(\sigma)} = 0$. Since $g^{(\sigma)} \neq 0$, it implies $f^{(\rho+1)} = 0$. The same development can be made at each order, and therefore we find $f \equiv 0$.

- $f^{(\rho)} \neq 0$ and $g^{(\sigma)} = 0$. As previously this implies $g \equiv 0$. This case is the trivial case $\mathcal{K} = 0$ and is therefore not interesting.

- $f^{(\rho)} = g^{(\sigma)} = 0$.

  In this case, the following order is $f^{(\rho+1)}g^{(\sigma+1)} = 0$, which has exactly the same form as equation (C25) and implies therefore either $f \equiv 0$ or $g \equiv 0$.

Since we want $\mathcal{K} \neq 0$, equation (12) becomes

$$f = \left(\frac{\xi' + \nu'}{r} + \frac{\xi'^2}{4} + \frac{\xi'\nu'}{2} \right) = 0 \qquad \text{(C26)}$$

at each order. We recover one of the Schwarzschild equation for which the method can be applied easily. Note that for the three equations (7), (13) and (C26), the relations between the usual coefficients of Post-Newtonian parametrization are not modified by the additional coefficients up to an order 10^{-9} which is the order of magnitude of the largest additional coefficient.

Equation (11) is slightly different. Indeed, the left-hand side is proportional to $\mathcal{K}^{n+1}$, that means proportional to $\left(\frac{1}{Mr_s} \right)^{2(n+1)}$, whereas the right-hand side is proportional to $\left(\frac{1}{Mr_s} \right)^2$. So the left-hand side is $\left(\frac{1}{Mr_s} \right)^{2n} \simeq (10^{23})^{2n}$ times larger than the right-hand side. Therefore, this equation can modify the relation between the usual coefficients. Indeed, we will now mix the usual coefficients coming from the right-hand side, with the additional coefficients of the left-hand side which are multiplied by $\left(\frac{1}{Mr_s} \right)^{2n}$.

We consider in the following the case $n = 1/2$. The lowest order of the left-hand side is 8 whereas the lowest order of the right-hand side is 5. At order 5 equation (11) will then have the following form

$$\frac{1}{Mr_s\alpha} F(\tilde{A}_1, \tilde{B}_1, \tilde{D}_1, \tilde{E}_1, a_1, b_1, d_1, e_1) = G(a_1, b_1, c_1, d_1) \,, \tag{C27}$$

where F is proportional to $\tilde{A}_1, \tilde{B}_1, \tilde{D}_1$ and $\tilde{E}_1$. This equation gives a relation between the usual and the additional coefficients.

Orders 6 and 7 have the same form, except that they contain also the coefficients $a_2, b_2...$ and $\tilde{D}_0$. Order 8 is different. Indeed, at this order the left-hand side contains also a_1, b_1, d_1 and e_1. Since they are multiplied by 10^{23}, we can neglect the terms coming from the right-hand side. The same occurs for all the following orders. Therefore, for orders 8 and larger, equation (11) becomes

$$\left(\xi'' + 2\frac{\xi'}{r} + \frac{\xi'^2}{4} \right) \mathcal{K}^{n+1} = 0 \,. \tag{C28}$$

The same argument as in Eq. (C24) implies that

$$\left(\xi'' + 2\frac{\xi'}{r} + \frac{\xi'^2}{4} \right) = 0 \,. \tag{C29}$$

This is the second Schwarzschild equation, but it is valid only for orders larger than 7 in the development in power of x. For smaller orders we have to take into account the right-hand side.

The same situation occurs for powers smaller than 5. The left-hand side can be neglected with respect to the right-hand side, and therefore we can apply the method to equation (C29).

To summarize, the method can be directly applied to equations (7), (13) and (C26) which replace (12). Then we have to solve orders x^5, x^6 and x^7 of equation (11). Our method can then be applied to equation (C29) for orders larger than x^7 and smaller than x^5.

4. Solutions

Let us divide each equation by the correct power of x such that the lowest order , containing only a_1, b_1, d_1 and e_1 is x for each equation. Concerning equation (11), this means that we have to divide by x^6.

At order x for the equations (7), (13) and (C26) we find:

$$\begin{aligned} b_1 &= -a_1 \,, \\ e_1 &= -\frac{a_1}{2} \,, \\ 0 &= (c_1 + c_3)d_1 \,. \tag{C30} \end{aligned}$$

Order x of equation (11) will be treated separately since it contains also $\tilde{A}_1, \tilde{B}_1, \tilde{D}_1$ and $\tilde{E}_1$.

We impose $a_1 = -1$ to recover Newton's theory and therefore we find

$$\begin{aligned} b_1 &= 1 \,, \\ e_1 &= 1/2 \,. \tag{C31} \end{aligned}$$

¿From the last equality of equation (C30) we have two possibilities: either $d_1 = 0$ or $c_3 = -c_1$.

$$a. \quad d_1 = 0$$

At order x^2 equation (13) implies

$$-8(c_1 + c_3)d_2 = 0 \,. \tag{C32}$$

Hence, either $c_3 = -c_1$ or $d_2 = 0$. At order x^3 we have

$$\left[c_2 - (c_1 + c_3) \right] d_3 = 0. \tag{C33}$$

Again we have two possibilities: either $c_2 = c_1 + c_3$, or $d_3 = 0$. The situation is the same at each order. Indeed if we assume that we have chosen $d_1 = d_2 = ... = d_{s-1} = 0$ from the order x to x^{s-1}, order x^s gives

$$\left[(c_1+c_2+c_3)s^2-3(c_1+c_2+c_3)s-2(c_1-c_2+c_3)\right]d_s = 0 .$$
(C34)

This means that the only possibility to have at least one of the $d_s \neq 0$, is to satisfy one of the relation

$$(c_1+c_2+c_3)s^2-3(c_1+c_2+c_3)s-2(c_1-c_2+c_3) = 0 ,$$
(C35)

for some positive integer s. Therefore, we have to consider two situations: either $d_s = 0 + O(\tilde{D}_0, \tilde{E}_0)$ for all s, or one of the relation (C35) is satisfied.

In the first case, the usual parameters d_s are equal to zero up to the order of magnitude of the additional coefficients which of course modify equation (C34). We can calculate the positive order $x, x^2...$ including the additional coefficients. And we have also the negative order $x^0, x^{-1}, ...$ We can show that these sets of equations imply either

$$d_s \sim \tilde{D}_0 , \ \forall s \ \text{ and } \ \tilde{D}_0 \sim \tilde{D}_1 \sim \tilde{D}_2... \sim \tilde{D}_\infty \quad \text{(C36)}$$

or

$$d_1 \sim \tilde{E}_0 d_2 \sim \tilde{E}_0^2 d_3 \sim ... \sim \tilde{E}_0^\infty d_\infty$$
$$d_s \sim \tilde{E}_0 d_{s+1} \sim \tilde{E}_0^2 d_{s+2} \sim ... \sim \tilde{E}_0^\infty d_\infty$$
$$\tilde{D}_0 \sim \tilde{E}_0 d_1 \sim \tilde{E}_0^\infty d_\infty$$
$$\tilde{D}_s \sim \tilde{E}_0 \tilde{D}_{s-1} \sim ... \sim \tilde{E}_0^\infty d_\infty \quad \text{(C37)}$$

Since $\tilde{D}_\infty \to 0$ in order that the expansion (21) converges in the Solar System, the first case implies $A^r(r) = 0$.

In the second case, since $\tilde{E}_0^\infty \to 0$ and d_∞ is finite, we also have $A^r(r) = 0$.

Hence $A^t(r)$ becomes the only degree of freedom of the vector field, which is then completely determined by the constraint (7).

In the following we will study in details the second situation where one of the d_s at least is different from zero. We consider the simplest case where $d_1 \neq 0$ and therefore $c_3 = -c_1$.

b. $c_3 = -c_1$

¿From order x^2 of the four equations (7), (13), (C26) and (C29), and using eq. (C31) we find

$$a_2 = \frac{1}{2} ,$$

$$b_2 = \frac{3}{8} ,$$
$$e_2 = \frac{1}{8} + \frac{d_1^2}{2} ,$$
$$0 = 2c_2 d_1^2 - 4c_2 d_1 + c_1 - \frac{1}{2}c_2 . \quad \text{(C38)}$$

We see that $b_1 = 1$ and $a_2 = 1/2$ are in complete agreement with the observations. The additional coefficients imply only a contribution of order 10^{-9}, which we have safely neglected since they are well beyond the precision of the measurements. The last equation allows to calculate d_1 as a function of c_1 and c_2.

We have to determine the equations for orders x^3 and x^4 which come from the mixed terms of equation (11). We use the solutions for the previous coefficients.

¿From order x^3, we then find

$$a_3 = -\frac{3}{16} ,$$
$$b_3 = \frac{1}{16} ,$$
$$e_3 = \frac{1}{32} + \frac{3}{4}d_1^2 + d_1 d_2 ,$$
$$d_3 = \frac{1}{8c_2}\Big(-8d_2 c_2 - 4c_1 d_1 - 24c_1 d_1^3 + 52d_1^3 c_2$$
$$+4c_1 d_2 + c_2 d_1 + 32d_2 c_2 d_1^2\Big) . \quad \text{(C39)}$$

¿From the order x^4 we obtain

$$a_4 = \frac{1}{16} ,$$
$$b_4 = \frac{1}{256} ,$$
$$e_4 = \frac{1}{128c_2}\Big(c_2 + 80c_2 d_1^2 + 64d_1 d_2 c_2 + 64d_1 c_1 d_2$$
$$+816c_2 d_1^4 - 64c_1 d_1^2 - 384c_1 d_1^4 + 512d_2 c_2 d_1^3$$
$$+64d_2^2 c_2\Big)$$
$$d_4 = \frac{1}{192c_2^2}\Big(-104c_1 d_2 c_2 + 70c_1 d_1 c_2 - 208c_1 d_1^3 c_2$$
$$+16c_1^2 d_2 - 16c_1^2 d_1 - 32c_1 d_2 c_2 d_1^2 - 96c_1^2 d_1^3$$
$$-3264c_1 d_1^5 c_2 + 132d_2 c_2^2 - 652d_1^3 c_2^2 - 7d_1 c_2^2$$
$$-384d_2 c_2^2 d_1^2 + 3840c_2^2 d_1^4 d_2 + 7104c_2^2 d_1^5$$
$$+384c_2^2 d_1 d_2^2\Big) . \quad \text{(C40)}$$

We also need to solve for the negative powers. Using the hierarchy between the additional coefficients $|\tilde{A}_1| \gg |\tilde{A}_2|...$ we find from equations (7), (13) and (C26) at order x^0, x^{-1} and x^{-2} that

$$\tilde{D}_0 = -\frac{3\tilde{B}_2}{16d_1 c_2}\Big(24c_2 d_1^4 - 8c_1 d_1^2 + 14c_2 d_1^2 - 11c_2 - 2c_1\Big)$$

$$\tilde{A}_1 = \tilde{B}_2, \quad \tilde{B}_1 = \frac{5\tilde{B}_3}{4}, \quad \tilde{E}_1 = -\frac{\tilde{B}_2}{2}, \quad \tilde{D}_1 = \frac{3\tilde{B}_2}{4d_1},$$

$$\tilde{A}_2 = -\tilde{B}_2, \quad \tilde{E}_2 = \frac{\tilde{B}_2}{2}, \quad \tilde{D}_2 = \frac{\tilde{B}_2 c_1}{8c_2 d_1}. \tag{C41}$$

We remark that $\tilde{B}_1$ is of the order of $\tilde{B}_3$, therefore it can be neglected with respect to $\tilde{A}_1, \tilde{D}_1$ and $\tilde{E}_1$, but also with respect to the order 2. $\tilde{B}_2$ remains undetermined. $\tilde{B}_3$ will be determined by the lower order x^{-3}, but we are not interested in its value here.

We now consider equation (11) at the order x, x^0 and x^{-1}, which contains both the coefficients of negative and positive powers. We solve this system of three equations plus the equation (C30) with the help of maple. It contains the five variables d_1, d_2, c_1, c_2 and $\tilde{B}_2$ and the two parameters α_1 and M. We find a set of solutions, from which we only consider those with c_1 and c_2 negative in order to ensure a positive-definite Hamiltonian for perturbations and non superluminal propagation of spin-0 degrees of freedom in the approximately-Minkowski regime of the theory [24]

$$c_1 = \frac{-33.11}{\alpha_1^2}\Big(\frac{\tilde{B}_2}{Mr_s}\Big)^2, \qquad c_2 = \frac{-30.75}{\alpha_1^2}\Big(\frac{\tilde{B}_2}{Mr_s}\Big)^2,$$
$$d_1 = 0.16, \qquad\qquad d_2 = -0.62, \tag{C42}$$

$$c_1 = \frac{-0.48}{\alpha_1^2}\Big(\frac{\tilde{B}_2}{Mr_s}\Big)^2, \qquad c_2 = \frac{-6.36}{\alpha_1^2}\Big(\frac{\tilde{B}_2}{Mr_s}\Big)^2,$$
$$d_1 = -0.10, \qquad\qquad d_2 = 0.28, \tag{C43}$$

$$c_1 = \frac{-24.11}{\alpha_1^2}\Big(\frac{\tilde{B}_2}{Mr_s}\Big)^2, \qquad c_2 = \frac{-143.48}{\alpha_1^2}\Big(\frac{\tilde{B}_2}{Mr_s}\Big)^2,$$
$$d_1 = -0.80, \qquad\qquad d_2 = 0.72, \tag{C44}$$

$$c_1 = \frac{-140.29}{\alpha_1^2}\Big(\frac{\tilde{B}_2}{Mr_s}\Big)^2, \qquad c_2 = \frac{-61.93}{\alpha_1^2}\Big(\frac{\tilde{B}_2}{Mr_s}\Big)^2,$$
$$d_1 = 0.66, \qquad\qquad d_2 = -7.89. \tag{C45}$$

Each solution implies a strong constraint on the parameters c_1, c_2 and α_1. Indeed $\tilde{B}_2 = B_2\left(\frac{r_s}{r_{gm}}\right)^2$ can be at most 10^{-32} in order not the be detectable in the Solar System, and therefore $\left(\frac{\tilde{B}_2}{Mr_s}\right)^2 \simeq 10^{-18}$. This means that either c_1 and c_2 or α_1 has to be very small if $\alpha_1 \neq 0$.

Finally, we have to calculate the determinant of the system to determine if a solution exists at each order. Using the method described above we find from the positive powers in x (which correspond to negative powers in r)($s > 1$)

$$S\begin{pmatrix} a_s \\ b_s \\ d_s \\ e_s \end{pmatrix} = \begin{pmatrix} H_1^{(s)} \\ H_2^{(s)} \\ H_3^{(s)} \\ H_4^{(s)} \end{pmatrix} \qquad \text{where} \tag{C46}$$

$$S = \begin{pmatrix} -1 & 0 & 0 & -2 \\ 0 & s(s-1) & 0 & 0 \\ -s & -s & 0 & 0 \\ \frac{d_1 s(s-1)}{2} & 0 & -\frac{c_1(s-1)(s-2)}{4} & \frac{d_1 s(s-1)}{2} \end{pmatrix} \tag{C47}$$

and $H_i^{(s)}$ are functions of the previous coefficients $a_1, ...a_{s-1}, b_1, ..., b_{s-1}, d_1, .., d_{s-1}, e_1, .., e_{s-1}$. Therefore, at order s a unique solution (a_s, b_s, d_s, e_s) exists if

$$\det S = -\frac{c_1}{2} s^2(s-1)^2(s-2) \neq 0. \tag{C48}$$

Since $c_1 \neq 0$, the condition is satisfied for $s > 2$. The order $s = 2$ for which the determinant vanishes has been solved explicitly above and leads to no inconsistency. The solution is not unique, since we have found four different values for d_2. For each order $(s > 2)$ the determinant is non zero and therefore we can determine the unique solution as a function of the previous orders.

The same structure repeats for positive powers of r $(s > 2)$

$$\hat{S}\begin{pmatrix} \tilde{A}_s \\ \tilde{B}_s \\ \tilde{D}_s \\ \tilde{E}_s \end{pmatrix} = \begin{pmatrix} \hat{H}_1^{(s)} + \hat{Q}_1^{(s)} \\ \hat{H}_2^{(s)} + \hat{Q}_2^{(s)} \\ \hat{H}_3^{(s)} + \hat{Q}_3^{(s)} \\ \hat{H}_4^{(s)} + \hat{Q}_4^{(s)} \end{pmatrix} \qquad \text{where} \tag{C49}$$

$$\hat{S} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -s(s+1) & 0 & 0 \\ s & s & 0 & 0 \\ \frac{c_1 s(s+1)d_1}{2} & 0 & \begin{matrix} c_2(14-s)d_1^2- \\ \frac{c_1(s-1)(s-2)}{4} \end{matrix} & \frac{c_1 s(s+1)d_1}{2} \end{pmatrix} \tag{C50}$$

and $\hat{H}_i^{(s)}$ are functions of a_1, b_1, d_1 and e_1, and of the previous coefficients $\tilde{A}_1, ..., \tilde{A}_{s-1}$, $\tilde{B}_1, ..., \tilde{B}_{s-1}$, $\tilde{D}_1, ..., \tilde{D}_{s-1}, \tilde{E}_1, ..., \tilde{E}_{s-1}$, and $\hat{Q}_i^{(s)} \ll \hat{H}_i^{(s)}$. A unique solution exists at order s if

$$\det \hat{S} = -2s^2(s+1)\Big(c_2(14-s)d_1^2 - \frac{c_1}{4}(s-1)(s-2)\Big) \neq 0. \tag{C51}$$

The orders $s = 0, 1$ and 2 have been calculated above. For each of the solutions (C42) to (C45) we find that $\det \hat{S} \neq 0 \,\forall\, s$.

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