GAUGE-INVARIANT COSMOLOGICAL PERTURBATION THEORY

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...“our mistake is not that we take our theories too seriously, but that we do not take them seriously enough. It is always hard to realize that these numbers and equations we play with at our desks have something to do with the real world.” ...

(Steven Weinberg, *The First Three Minutes*)

Abstract

The linear perturbation theory of Friedman universes is treated in a gauge-invariant manner (i.e., invariant under linearized coordinate transformations). After presenting some mathematical tools in Chapter 1, the perturbation equations are derived with the help of the 3+1 formalism of general relativity in Chapter 2. Bardeens perturbation theory [4] is thereby extended to the treatment of collisionless matter. In Chapter 3 we solve the perturbation equations for perfect fluids explicitely. The results are applied to the discussion of the Sachs-Wolfe effect and of the traditional baryonic scenario of galaxy formation. In Chapter 4 numerical solutions of the gauge invariant perturbation equations are presented for a universe consisting of collisionless particles of mass $m_X \neq 0$, massless neutrinos and radiation. In this scenarios the masses of the first objects to collapse are of the order of m_{pl}^3/m_X^2.

Within the gauge-invariant treatment, the perturbation amplitudes exhibit no growth on scales which are larger than the present horizon size.

Zusammenfassung

In dieser Dissertation wird lineare Störungstheorie von Friedman Universen auf eichinvariante Weise (das heisst, invariant unter linearisierten Koordinatentransformationen) behandelt. Nach der Bereitstellung einiger mathematischer Hilfsmittel in Kapitel 1, werden im zweiten Kapitel die Störungsgleichungen unter Benutzung des 3+1 Formalismus' der allgemeinen Relativitätstheorie hergeleitet. Die Störungstheorie von Bardeen [4] wird dabei zur Behandlung von stossfreier Materie erweitert. Im dritten Kapitel lösen wir die Störungsgleichungen explizit für ideale Fluida. Die Resultate wenden wir in der Diskussion des Sachs-Wolfe Effekts und des konventionellen, baryonischen Szenarios der Galaxienbildung an. In Kapitel 4 werden numerische Lösungen der eichinvarianten Störungsgleichungen für ein Universum, dessen Materie aus stossfreien Teilchen der Masse $m_X \neq 0$, masselosen Neutrini und elektromagnetischer Strahlung besteht, präsentiert. Die Massen der Objekte, welche als erste gravitativ instabil werden, sind in solchen Szenarien von der Grössenordnung m_{pl}^3/m_X^2.

Im Rahmen der eichinvarianten Störungstheorie können Störungen auf Skalen, welche grösser sind als die momentane Ausdehnung des Horizontes, nicht anwachsen.

Contents

Introduction

Most modern theories of the origin of structure in the universe assume that the structure we observe today — from the smallest galaxies to the largest superclusters — grew from gravitationally unstable small density inhomogeneities. This generally accepted idea has already been spelled out by Newton in a remarkably clear manner in a famous letter to R. Bentley in 1692 [56]. With time the universe gets lumpier as the gravitational pull of galaxy clusters attracts other galaxies from neighboring regions. Conversely, the early universe was very smooth. The strongest evidence for this is provided by the astonishing isotropy of the microwave background radiation (MBR) whose cosmological nature implies that initial deviations from homogeneity and isotropy must have been very small ($< 2 \cdot 10^{-5}$ [12,34,44,63]) at the time of recombination of protons and electrons into hydrogen. It is a great challenge to theory to explain how the enormous variety of structure that is observed arose from tiny initial fluctuations. In addition, theory should also explain the origin of these fluctuations, which so far remains a mystery, in spite of interesting suggestions.

The first mathematical theory of galaxy formation was proposed by Sir James Jeans in 1902 [25]. Jeans assumed a static, homogeneous and isotropic background universe and found that small density perturbations can grow exponentially, if their typical size is larger than the Jeans length, $\lambda_J = (2\pi/k_J) = (\frac{\pi c_s^2}{G\rho})^{1/2}$. This is just the length where the gravitational attraction becomes comparable to the push due to pressure. In 1957, Bonner [6] expanded Jeans's theory to the treatment of an expanding universe. He found that disturbances with typical sizes larger than λ_J grow not exponentially but like a power of t.

Even earlier, in 1946, the first fully relativistic treatment of cosmological perturbation theory was given by Lifshitz [32]. In his classic paper he considered the perturbations of $K = \pm 1$ universes and found that only density perturbations of pressureless matter can grow, and also these only by a rather small amount. Therefore, he abandoned the idea of galaxies forming by gravitational instability from infinitesimal perturbations.

In our applications, we will always set $K = 0$ for reasons mentioned later in this introduction. In Section 3.1 we shall obtain in a gauge-invariant manner that also for $K = 0$ only perturbations of pressureless matter can grow, but by a considerably larger amount than in a $K = \pm 1$ universe. Since the baryons in the post-recombination universe provide such a matter component, this looks fine. However, as will be seen by simple arguments (Section 3.3), there was not enough time that the observed galaxies, clusters and superclusters could develop from the tiny inhomogeneities present at recombination, if only the well-known stable particles: the photons, nucleons, electrons and massless neutrinos are taken into account. In addition, one can not explain the typical scales of galaxies, clusters and superclusters within a such a purely baryonic scenario.

There are several possible ways out of this first basic problem:

- Another fundamental force (e.g., primordial magnetic fields [39]) might be involved in the structure formation process.
- One may go as far as to question the generally accepted interpretation of the MBR, or, even more radically, refute the standard big bang model.
- The most conservative alternative is to postulate another unknown matter component which would have played a major role in the structure formation process.

In view of the great successes of standard cosmology, the second possibility is very unattractive and will be discarded.

For the last of these alternatives there are also observational hints: The rotation curves of spiral galaxies (the velocities of orbiting objects as a function of their distance from the center) show an unexpected flat behaviour far outside the visible region of the galaxies [7,17,45]. This indicates that galaxies have massive dark halos extending far beyond their visible radius.

Furthermore, one finds by several different techniques [29,55] that the mass-to-light ratio of galaxies is about $(M/L) \approx 10 M_\odot / L_\odot$. This ratio increases for clusters and superclusters up to a value of $700 M_\odot / L_\odot$ or more for the largest coherent structures which are observed. We thus "see" only the shining core of the galaxies which is surrounded by a huge dark halo. This seems to be even more pronounced for clusters and superclusters.

Density measurements and determinations of the deceleration parameter $q = -\frac{\ddot{a}a}{\dot{a}^2}$ limit the density parameter Ω to $0.2 \leq \Omega \leq 2$. On the other hand, theoretical considerations like the instability of the value $\Omega = 1$ under the Friedman evolution, the flatness problem and its solution by inflationary models [22,33,50] strongly suggest $\Omega = 1$. However, the sucessfull theory of big bang nucleosynthesis [23,36] restricts the baryonic contribution to the density parameter to $\Omega_B \leq 0.2$. Thus, the simplest hypothesis — that all the dark matter consists of invisible baryons, "Jupiters", brown dwarfs, black holes — is very unlikely. There are a lot of non-baryonic candidates for dark matter: Quark nuggets, nuclearites [64,3] or primordial black holes [10,9,38], collisionless elementary particles like a massive neutrino with mass $m_\nu \approx 30eV - 100eV$, the lightest supersymmetric particle which could be the photino the gravitino, the sneutrino or a higgsino, the axion (the Goldstone boson originating from the Peccei-Quinn symmetry breaking) and many others. (A complete list is presented in [55].)

This huge variety of candidates, in fact, means that we really do not know what the dark matter consists of, and there is also no direct observation of it in sight. The nature of the dark matter will remain an outstanding unsolved problem of cosmology and particle physics for the next decade or probably even longer.

In this dissertation we assume that a massive collisionless fermion, which we call X-particle, provides the dark matter. We neglect its chemical potential, μ_X. Furthermore, we assume that this particle was in thermal equilibrium with the other matter components in the universe until an early time $t_{\rm dec}$, and that it was still extremely relativistic at $t_{\rm dec}$. (Otherwise the corresponding cosmological damping scale would be negligible and would, most probably, have nothing to do with the large-scale structure we observe today [5].)

We have calculated the implications of such a relic for the evolution of the large-scale structure within linear perturbation theory. These calculations have already been carried out by [5] and partly by [43], but the authors of these papers always worked in a specific gauge (choice of coordinates), the synchronous gauge. This gauge dependence of the results causes difficulties for the interpretation of perturbations which are larger than the horizon size of the universe. Since every perturbation was once of super-horizon size, this affects the whole perturbation spectrum.

Furthermore, in the synchronous gauge the coefficient h_{00} of the metric perturbation, which is usually identified with the gravitational potential in the Newtonian limit, is gauged away and there is thus no straightforward Newtonian limit which one can use as input for the nonlinear simulations of galaxy formation. We therefore performed our calculations with Bardeen's gauge-invariant formalism [4,28].

In Chapter 2 we re-derive Bardeen's perturbation equations applying the 3+1 formalism of general relativity, which we briefly recall in Chapter 1 (see also [14]). One finds that the 3+1 formalism is really very well adapted to the problem. In addition, we expand Bardeen's formalism to a gauge-invariant treatment of collisionless matter. (I recently noticed that similar work has been done independently by a Japanese group [26].)

After discussing some simple cases in Chapter 3, we integrate the gauge-invariant perturbation equations for a universe consisting of photons, massless neutrinos and X-particles numerically. We distinguish between two types of dark matter. Particles which dominate the energy content of the universe when they are still hot ($T > 0.1m_X$) are called hot dark matter (HDM). Those which dominate only when they are already cold ($T < 0.01m_X$) are termed cold dark matter (CDM). (Of course, the range inbetween "hot" and "cold" is also possible and is called "warm".) The resulting spectra for HDM and CDM are presented in Figures 4 and 5 of Chapter 4.

In order to find out which of the two candidates (HDM or CDM) leads to a better agreement with observations, one has to perform nonlinear Newtonian N-body simulations using the resulting linear spectrum as initial condition (see, e.g., [19,20]). For HDM scenarios, one finds that most of the galaxies have been formed uncomfortably recently ($z \approx 1$). Within biased CDM scenarios, besides the somewhat unnatural biasing condition, it is difficult to obtain the observed filamentary structure with large voids. Furthermore, the recently observed large streaming motions remain unexplained. An attempt to compare the two theories by the distribution of "marks" led J. Silk [29] to 21 points for CDM and 20 points for HDM. It is thus fair to say that neither of these theories does solve the galaxy formation problem in a convincing manner, but they are also not ruled out by observations. We need further theoretical work to obtain more precise predictions, and observational data which yields more stringent limits. o compare the theoretical results with observations, it is also important to develop efficient methods like those presented in [54].

New developments including cosmic strings and dark matter [8] or other "hybrid models" [57] seem to provide promising candidates for structure formation.

Notations:

We use the units $\hbar = c = k_B = 1$

The signature of the Lorentz metric is given by $(-, +, +, +)$.

The Ricci tensor is defined by $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$.

Greek indices, $\mu, \nu, \lambda \cdots$, usually run over the four space-time coordinates and Latin ones, $i, j, k, l \cdots$, label the three spatial dimensions.

Three-dimensional tensorial quantities are denoted by boldface letters.

Chapter 1

Preliminaries

1.1 Nonrelativistic collisionless matter perturbations

As mentioned in the introduction, the goal of this dissertation is a numerical solution of the gauge-invariant linear perturbation equations for a universe dominated by collisionless particles.

In this section we present, as an introduction, the Newtonian problem in a static universe. The Newtonian perturbation equations for collisionless particles in an expanding universe are derived in [21].

Let $\tilde{f}(\boldsymbol{x},\boldsymbol{p},t)$ be the distribution function of the collisionless particles in phase space. The equation of motion is the Vlasov equation for $\tilde{f}$:

$$\partial_t \tilde{f} + m^{-1}\boldsymbol{p}\nabla_{\boldsymbol{x}}\tilde{f} + \boldsymbol{F}\nabla_{\boldsymbol{p}}\tilde{f} = 0 \,, \tag{1.1}$$

with

$$\boldsymbol{F} = -\boldsymbol{\nabla}\phi \,, \tag{1.2}$$

and

$$\Delta\phi = 4\pi G\rho m \,. \tag{1.3}$$

Let us split $\tilde{f}$ into a homogeneous background contribution and a small perturbation:

$$\tilde{f} = f(t,p) + \epsilon f_1(t,\boldsymbol{x},\boldsymbol{p}) \,. \tag{1.4}$$

We do not take into account the contribution of the background mass density to the gravitational field. One can absorb it into a cosmological constant. Thus, we have

$$\boldsymbol{\nabla}\cdot\boldsymbol{F} = -4\pi G\rho_1 m \;\;,\;\; \rho_1 = m\int d^3p f_1(t,\boldsymbol{x},\boldsymbol{p}) \,. \tag{1.5}$$

In zeroth order, Vlasov's equation therefore yields

$$\partial_t f = 0 .$$

In first order we obtain

$$\partial_t f_1 + m^{-1}\boldsymbol{p}\nabla_{\boldsymbol{x}} f_1 + \boldsymbol{F}\nabla_{\boldsymbol{p}} f = 0 . \tag{1.6}$$

Let us make a plane wave ansatz for f_1:

$$f_1 = \hat{f}_1(\boldsymbol{p})e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} .$$

Equation (1.6) leads then to

$$- i\omega\hat{f}_1 + im^{-1}\boldsymbol{k}\cdot\boldsymbol{p}\hat{f}_1 - \widehat{\boldsymbol{F}}\nabla_{\boldsymbol{p}} f = 0 , \tag{1.7}$$

with

$$\widehat{\boldsymbol{F}} = -i\boldsymbol{k}\hat{\phi} ,$$

and

$$-k^2\hat{\phi} = 4\pi G m^2 \int d^3p \hat{f}_1 .$$

Thus

$$\widehat{\boldsymbol{F}} = i\boldsymbol{k}k^{-2}4\pi G m^2 \int d^3p \hat{f}_1 . \tag{1.8}$$

Inserting (1.8) into (1.7) results in

$$\hat{f}_1 = (m^{-1}\boldsymbol{k}\cdot\boldsymbol{p} - \omega)^{-1}k^{-2}4\pi G m^2 \boldsymbol{k}\cdot\nabla_{\boldsymbol{p}} f \int d^3p \hat{f}_1 .$$

After integration of this equation over momentum space, we obtain a dispersion relation $\omega(\boldsymbol{k})$ for a given f:

$$1 = \frac{m^2 4\pi G}{k^2} \int d^3p (m^{-1}\boldsymbol{k}\cdot\boldsymbol{p} - \omega)^{-1}\boldsymbol{k}\cdot\nabla_{\boldsymbol{p}} f . \tag{1.9}$$

Of course (1.9) is determined only if we know how to handle the singularity at $\omega = m^{-1}\boldsymbol{k}\cdot\boldsymbol{p}$. Let us assume, as in the electromagnetic case of Landau damping (see [30]), that the perturbation is slowly turned off for $t \to -\infty$. We thus have to add an infinitesimal imaginary part, $+i\delta$ to ω. Furthermore, we denote the momentum component in direction of $\boldsymbol{k}$ by p_k and set

$$F(p_k) = \int d^2p_\perp f(p_\perp, p_k) ,$$

where $p_\perp$ denotes the momentum orthogonal to $\boldsymbol{k}$. By the isotropy of f, we then obtain

$$1 = -\frac{m^2 4\pi G}{k} \int_{-\infty}^{\infty} \frac{dF}{dp_k} \frac{dp_k}{(\omega + i\delta - m^{-1}p_k k)} , \tag{1.10}$$

$$1 = -\frac{m^2 4\pi G}{k} \fint_{-\infty}^{\infty} \frac{dF}{dp_k} \frac{dp_k}{(\omega - m^{-1} p_k k)} - i\pi \frac{dF}{dp_k}(p_k = \omega m k^{-1}) , \quad (1.11)$$

where we applied the Sochozki-Plemelj formula in (1.11). We are interested in the wavenumber, k_J, when the perturbation becomes unstable. (In analogy to the fluid case, the Jeans theory, we call k_J the Jeans number.) The instability sets in when the imaginary part of ω changes its sign. To obey (1.11) we must require $\omega(k_J) = 0$ (since f is an even function of $\boldsymbol{p}$, the second term vanishes for $p_k = 0$). We thus obtain

$$1 = -\frac{m^2 4\pi G}{k^2} \fint_{-\infty}^{\infty} \frac{dF}{dp_k} \frac{dp_k}{p_k} .$$

Since for an isotropic distribution function

$$p_k^{-1} \frac{\partial f}{\partial p_k} = p_k^{-1} \frac{p_k}{p} \frac{df}{dp} ,$$

we end up with

$$\begin{aligned} k_J^2 &= -(4\pi)^2 G m^3 \int_0^\infty p \frac{df}{dp} dp \\ &= (4\pi)^2 G m^3 \int_0^\infty f(p) dp \\ &= (4\pi)^2 (m^2/m_{pl})^2 \int_0^\infty f(v) dv . \end{aligned} \quad (1.12)$$

(We use $m_{pl}^2 = G^{-1}$ and set $v = \frac{1}{m} p$.)
From (1.12) we calculate the Jeans mass, M_J, the mass inside a ball of diameter $\lambda_J = 2\pi/k_J$:

$$M_J = \frac{4\pi}{3} \rho (\pi/k_J)^3 . \quad (1.13)$$

Later we will find that the distribution function for collisionless fermions in a Friedman universe has the form

$$f = \frac{N_X}{(2\pi)^3} \frac{1}{e^{p/T_X} + 1} , \quad (1.14)$$

where N_X denotes the spin degrees of freedom of the X-particles. For $N_X = 2$, we obtain

$$\rho = 4\pi m \int_0^\infty p^2 f dp = \frac{\zeta(3) 3}{2\pi^2} m T_X^3 .$$

ζ denotes the Riemannian Zeta-function. Inserting (1.12) with f given by (1.14) in (1.13) leads then to

$$M_J = \frac{\zeta(3)\pi^{7/2}}{4(\ln 2)^{3/2}} a^{-3/2}(m_{pl}^3/m^2) \approx 30a^{-3/2}(m_{pl}^3/m^2) \,, \qquad (1.15)$$

where we have introduced the proportionality factor $a = m/T_X$.

In Section 4.3, we will obtain equation (1.15) as the nonrelativistic limit of a fully relativistic treatment of this problem.

1.2 The general relativistic treatment of collisionless matter

To set up our notation and to be self-contained, we repeat briefly as far as we need it for further developments, the Hamiltonian and Lagrangian formulations of classical mechanical systems and apply them to the geodesic spray. A more explicit treatment of this subject is given, for example, in [1]. Then, we use these tools for the definition of the general relativistic distribution function and derive Liouville's equation. In Subsection 2.4, we shall follow partly the exposition of Stewart [51].

1.2.1 Classical Hamiltonian systems

In classical mechanics the configuration space is an n-dimensional manifold $\mathcal{M}$ with cotangent space $T^*\mathcal{M}$, which is also called the phase space. Let π be the canonical projection and $T\pi$ its tangent map:

$$\pi \, : \, T^*\mathcal{M} \to \mathcal{M} : (x,p) \mapsto x \,,$$

$$T\pi \, : \, T(T^*\mathcal{M}) \to T\mathcal{M} : (x,p;\dot{x},\dot{p}) \mapsto (x,\dot{x}) \,.$$

We define the 1-form

$$\theta \, : \, T(T^*\mathcal{M}) \to \mathbf{R}$$

by

$$\theta(x,p,\dot{x},\dot{p}) = p(\dot{x}) \,.$$

In local coordinates,

$$x = (x^\mu) \ , \quad p = p_\mu dx^\mu \ , \quad \dot{x} = \dot{x}^\mu \frac{\partial}{\partial x^\nu} \ , \quad \dot{p} = \dot{p}_\mu \frac{\partial}{\partial p_\nu} \ ,$$

we find

$$\theta = p_\mu dx^\mu \ .$$

The exterior derivative of θ , $\omega := -d\theta$ is a non-degenerate closed 2-form on $T^*\mathcal{M}$, i.e., a symplectic form. θ and ω are called the canonical forms on $T^*\mathcal{M}$.

$$\Omega = \frac{(-1)^{[n/2]}}{n!}\omega^n \tag{1.16}$$

defines a volume form on $T^*\mathcal{M}$ which is called the Liouville measure. This shows in particular that the cotangent bundle of any manifold is orientable.

The symplectic form ω defines an isomorphism between vector fields and 1-forms on $T^*\mathcal{M}$:

$$\flat \ : \ \mathcal{X}(T^*\mathcal{M}) \to \mathcal{X}^*(T^*\mathcal{M}) : X \mapsto i_X\omega \ .$$

Thus, for a function $H \in \mathcal{F}(T^*\mathcal{M})$, there exists a vector field X_H with

$$i_{X_H}\omega = dH \ .$$

In local coordinates we find

$$X_H = \frac{\partial H}{\partial p_\mu}\frac{\partial}{\partial x^\mu} - \frac{\partial H}{\partial x^\mu}\frac{\partial}{\partial p_\mu} \ .$$

The triple $(T^*\mathcal{M}, \omega, X_H)$ is called a Hamiltonian system.

The flow of the vector field X_H , ϕ_t^H, leaves ω invariant:

$$L_{X_H}\omega = di_{X_H}\omega = ddH = 0 \ ,$$

i.e., $\phi_t^H(\omega) = \omega$.
Therefore, we have Liouville's theorem:

$$L_{X_H}\Omega = 0 \ . \tag{1.17}$$

It is readily seen that the hypersurfaces of constant energy,

$$\Sigma_E = \{(x,p) \in T^*\mathcal{M} \mid H(x,p) = E\} \ ,$$

are invariant under the flow ϕ_t^H . Hence, X_H is tangent to Σ_E and the restriction of X_H to Σ_E which we denote again by X_H is well defined.

We are now going to construct a volume element, μ_E, on Σ_E which is invariant under X_H: For the Liouville measure Ω on $T^*\mathcal{M}$ there exists a $(2n-1)$-form σ such that

$$\Omega = dH \wedge \sigma \,. \tag{1.18}$$

For two forms σ and σ' satisfying (1.18) we have

$$\sigma - \sigma' = dH \wedge \rho \,.$$

Let now $i : \Sigma_E \to T^*\mathcal{M}$ be the inclusion. We define

$$\mu_E = i^*\sigma \,. \tag{1.19}$$

Since

$$i^*(dH) = d(i^*H) = 0 \,,$$

we have

$$i^*\sigma = i^*\sigma' + i^*(dH \wedge \rho) = i^*\sigma' \,.$$

Therefore, μ_E does not depend on the particular choice of σ.

1.2.2 Lagrangian systems

The velocity space $T\mathcal{M}$ yields an alternative description of mechanical systems. Let

$$\mathcal{L} \,:\, T\mathcal{M} \to \mathbf{R}$$

be given. The fiber derivative of $\mathcal{L}$, $\mathcal{FL}$, is defined as follows:

$$\mathcal{FL} \,:\, T\mathcal{M} \to T^*\mathcal{M} : (x, \dot{x}) \mapsto (x, D\mathcal{L}_x(x, \dot{x})) \,,$$

where $\mathcal{L}_x$ denotes the restriction of $\mathcal{L}$ to the fiber over x. In local coordinates:

$$\dot{x} = \dot{x}^\mu \frac{\partial}{\partial x^\mu} \,,$$

$$D\mathcal{L}_x = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} dx^\mu \,.$$

Thus,

$$\mathcal{FL} \,:\, T\mathcal{M} \to T^*\mathcal{M} : (x^\mu, \dot{x}^\mu) \mapsto (x^\mu, p_\mu) \,, \quad p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \,.$$

The Lagrangian 2-form on $T\mathcal{M}$ belonging to $\mathcal{L}$ is given by:

$$\omega_{\mathcal{L}} := (\mathcal{FL})^*(\omega) .$$

Clearly, $\omega_{\mathcal{L}}$ is closed. If and only if $\mathcal{FL}$ is a local diffeomorphism, $\omega_{\mathcal{L}}$ is also non degenerate and therefore a symplectic form. In coordinates we find

$$\omega_{\mathcal{L}} = \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^\mu \partial x^\nu} dx^\mu \wedge dx^\nu + \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^\mu \partial \dot{x}^\nu} dx^\mu \wedge d\dot{x}^\nu . \tag{1.20}$$

The action A is defined by

$$A \,:\, T\mathcal{M} \to \mathbf{R} : (x, \dot{x}) \mapsto \mathcal{FL}(x, \dot{x})\dot{x} .$$

Let us further call

$$E = A - \mathcal{L}$$

the energy of the system. In coordinates we obtain

$$E = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \dot{x}^\mu - \mathcal{L} .$$

If $\mathcal{FL}$ is a local diffeomorphism, there exists the Lagrangian vector field $X_E \in \mathcal{X}(T\mathcal{M})$ with

$$i_{X_E} \omega_{\mathcal{L}} = dE .$$

We then call $(T\mathcal{M}, \omega_{\mathcal{L}}, X_E)$ a Lagrangian system and the equations of motion,

$$\frac{d}{dt}(x, \dot{x}) = X_E ,$$

are the Euler-Lagrange equations. In local coordinates X_E is given by

$$X_E = \dot{x}^\mu \frac{\partial}{\partial x^\mu} + M^{\mu\nu}(\mathcal{L}_{,x^\nu} - \dot{x}^\lambda \mathcal{L}_{,x^\lambda \dot{x}^\nu}) \frac{\partial}{\partial \dot{x}^\mu} ,$$

$$\text{with} \quad M^{\mu\nu} = (\frac{\partial^2 \mathcal{L}}{\partial \dot{x}^\mu \partial \dot{x}^\nu})^{-1} \qquad \text{and} \quad \mathcal{L}_{,x} = \frac{\partial \mathcal{L}}{\partial x} .$$

1.2.3 The geodesic spray

We now want to apply this general formalism to freely gravitating test particles. Thus, let $(\mathcal{M}, g)$ be a $n+1$-dimensional pseudo-Riemannian manifold and set

$$\mathcal{L}(x, \dot{x}) = \frac{1}{2} g(x)(\dot{x}, \dot{x}) .$$

Then, $\mathcal{FL}$ is given by

$$\mathcal{FL} : T\mathcal{M} \to T^*\mathcal{M} : (x^\mu, \dot{x}^\mu) \mapsto (x^\mu, p_\mu) \ , \ p_\mu = g_{\mu\nu}\dot{x}^\nu \ .$$

Hence, $\mathcal{FL}$ is the canonical isomorphism between tangent and cotangent bundles of (pseudo-) Riemannian manifolds, the usual "flat map" associated with the metric:

$$\mathcal{FL}(x, \dot{x}) = (x, \dot{x})^\flat \ .$$

Therefore, we have: $\dot{x}^\mu(x,p) = p^\mu$. One easily computes action and energy in this example:

$$A = 2\mathcal{L} \ , \qquad E = A - \mathcal{L} = \mathcal{L} \ .$$

If $\mathcal{FL}$ is a diffeomorphism, as here, the Lagrangian system is equivalent to a Hamiltonian system with Hamilton function

$$H = E \circ (\mathcal{FL})^{-1} \ .$$

In our example we find

$$H(x,p) = \frac{1}{2} g^{\mu\nu}(x) p_\mu p_\nu \ .$$

The flow corresponding to $\mathcal{L}$ is the geodesic flow. In local coordinates we obtain

$$X_E \equiv X_g = \dot{x}^\mu \frac{\partial}{\partial x^\mu} - \Gamma^\mu_{\nu\lambda} \dot{x}^\nu \dot{x}^\lambda \frac{\partial}{\partial \dot{x}^\mu} \ . \qquad (1.21)$$

By conservation of energy, the mass bundle

$$P_m = \{(x,p) \in T\mathcal{M} |\, g(x)(p,p) = -m^2\}$$

is invariant under the flow of X_g. Therefore, the restriction of X_g to P_m , $(X_g)_{|P_m}$ is well defined. It is called the Liouville vector field. Let

$$i \, : \, P_m \to T\mathcal{M}$$

be the inclusion. If we choose the coordinates $(x^\mu, p^j, j = 1, n)$ on P_m, we find

$$i^* X_g = p^\mu \partial_\mu - \Gamma^j_{\mu\nu} p^\mu p^\mu \frac{\partial}{\partial p^j} \ , \qquad (1.22)$$

where p^0 is given by the condition $p^2 = -m^2$. Equation (1.20) leads to

$$\omega_{\mathcal{L}} = g_{\mu\nu} dx^\mu \wedge dp^\nu \ ,$$

so that

$$\Omega_{\mathcal{L}} = \frac{(-1)^{(n+1)}}{(n+1)!}\omega_{\mathcal{L}}^{(n+1)} = |g| dx^0 \wedge \cdots \wedge dx^n \wedge dp^0 \wedge \cdots \wedge dp^n \ ,$$

where $|g|$ denotes the determinant of the metric g. The invariant measure μ_m on P_m defined in equation (1.19) is constructed as follows :

$$d(p^2) = g_{\mu\nu,\lambda} dx^\lambda p^\mu p^\nu + p_0 dp^0 + p_j dp^j \ .$$

Thus,

$$\Omega_{\mathcal{L}} = d(p^2) \wedge \sigma \ ,$$

with

$$\sigma = \frac{|g|}{p_0} d^{n+1}x dp^1 dp^2 \cdots dp^n \ .$$

Therefore,

$$\mu_m = i^* \sigma = \frac{|g|}{p_0} d^{n+1}x d^n p \ , \tag{1.23}$$

where $p_0(x^\mu, p^i)$ is given by the condition $p^2 = -m^2$. μ_m is an invariant measure on P_m, i.e.,

$$L_{X_g} \mu_m = 0 \ .$$

We can write μ_m in the form

$$\mu_m = \eta \wedge \pi_m \ ,$$

with

$$\eta = |g|^{1/2} d^{n+1}x$$

and

$$\pi_m(x) = \frac{|g|^{1/2}}{p_0} d^n p \ . \tag{1.24}$$

The mass bundle P_m is the general relativistic one particle phase space.

1.2.4 The distribution function

Let us now specialize to the physically most relevant case $n = 3$. To count the number of world lines of particles (integral curves of X_g on P_m) crossing a hypersurface Σ of P_m, we need a volume element for hypersurfaces of P_m. We define it by

$$\omega_m = i_{X_g} \mu_m \ .$$

In local coordinates, we obtain by (1.22) and (1.24)

$$\omega_m = \frac{1}{3!}p^\mu \eta_{\mu\alpha\beta\gamma} dx^\alpha dx^\beta dx^\gamma \wedge \pi_m - \frac{1}{2p_0}\eta_{0jkl}\Gamma^j_{\alpha\beta}p^\alpha b^\beta dp^k \wedge dp^l \wedge \eta \,. \quad (1.25)$$

The measure ω_m is uniquely determined (up to a numerical factor) by the requirement that it assigns a zero volume to any hypersurface tangent to X_g (i.e., $i_{X_g}\omega_m = 0$). If we thus consider a tube T of integral curves of X_g (i.e., phase orbits of particles) intersected by a hypersurface Σ, then $\int_\Sigma \omega_m$ is non-zero only if Σ is not parallel to the boundary of the tube. By the invariance of μ_m we know

$$0 = L_{X_g}\mu_m = di_{X_g}\mu_m = d\omega_m \,.$$

Hence, ω_m is closed on P_m. This leads to

$$L_{X_g}\omega_m = i_{X_g}d\omega_m + di_{X_g}\omega_m = 0 \,.$$

Let us consider again a tube T of integral curves of X_g bounded by the cross sections Σ and Σ' and the mantle Λ (see Figure 1). We find

$$0 = \int_T d\omega_m = \int_\Sigma \omega_m - \int_{\Sigma'} \omega_m + \int_\Lambda \omega_m \,.$$

The last term vanishes because X_g is tangent to Λ. We thus obtain

$$\int_\Sigma \omega_m = \int_{\Sigma'} \omega_m \,, \quad (1.26)$$

which is just another way of expressing Liouville's theorem.

To define a distribution function on the seven-dimensional relativistic phase space, P_m, it is helpful first to look at the Newtonian theory. There, the relativistic phase space corresponds to a direct product of the time axis with the six-dimensional Newtonian phase space. A hypersurface $\Sigma \subset P_m$ for which the projection onto the Lorentz manifold, $\mathcal{M}$, $\pi(\Sigma) = G \subset \mathcal{M}$ is a spacelike hypersurface of $\mathcal{M}$ corresponds to a part of ordinary Newtonian phase space. On such a hypersurface, ω_m reduces to the first summand in (1.25) which we can write in the form

$$\omega_m|\Sigma = p^\mu \sigma_\mu \wedge \pi_m \,.$$

If we choose G contained in the local rest frame of an observer at x, we obtain in coordinates

$$\omega_m|\Sigma = p^0\sigma_0 \wedge \pi_m = p^0 d^3x \wedge \frac{d^3p}{p_0} = d^3x d^3p \,.$$

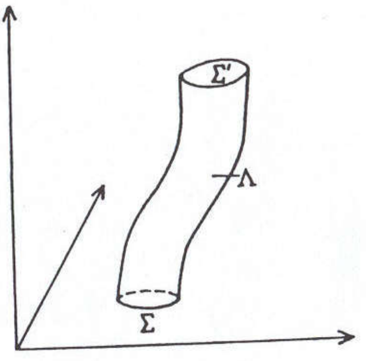


Figure 1.1 A tube of world lines tangent to X_g with cross sections Σ and Σ' in general relativistic phase space.

Now we want to define a distribution function on P_m. In the nonrelativistic phase space, the distribution function, $f(t, \boldsymbol{x}, \boldsymbol{p})$ determines the ensemble averaged number of particles at $(\boldsymbol{x}, \boldsymbol{p})$ at time t per volume:

$$dN(t, \boldsymbol{x}, \boldsymbol{p}) = f(t, \boldsymbol{x}, \boldsymbol{p}) d^3x d^3p \, .$$

If we want to generalize this notion to the general relativistic phase space, we first have to choose a hypersurface $\Sigma \subset P_m$. For this hypersurface we define, as in the Newtonian theory, a distribution function f_Σ such that the ensemble averaged number of particles in Σ, $N[\Sigma]$, is given by

$$N[\Sigma] = \int_\Sigma f_\Sigma \omega_m \, .$$

In [16] Ehlers shows the following theorem:

Theorem 1 *Let $N[\Sigma]$ be defined as above. If the following smoothness conditions are fulfilled:*
1) On any fixed hypersurface $\Sigma \subset P_m$ there exists a continous non-negative distribution function f_Σ such that for all hypersurfaces $\Sigma' \subset \Sigma$,

$$N[\Sigma'] = \int_{\Sigma'} f_\Sigma \omega_m,$$

and
2) for any sequence of regions, $\{D_i\} \subset P_m$, which shrinks to a point there

exists a positive number C such that

$$|N[\partial D_i]| \leq C|\int_{D_i} \mu_m| \,, \tag{1.27}$$

then there exists an invariant one particle distribution function f on P_m *such that*

$$N[\Sigma] = \int_\Sigma f\omega_m \qquad \textit{for any } \Sigma \subset P_m \,,$$

i.e., f is the same for all observers.

The condition (1.27) requires that the average number of collisions in a region $D \subset P_m$ is of order at most the μ_m measure of D. For collisionless matter we even have $N[\partial D] = 0$, thus (1.27) is trivially fulfilled.

The number of collisions (creations – annihilations) inside a region D of P_m is given by

$$N[\partial D] = \int_{\partial D} f\omega_m = \int_D d(f\omega_m) = \int_D df \wedge \omega_m \,.$$

But for any function h on P_m we have

$$L_{X_g}(h)\mu_m = L_{X_g}(h\mu_m) = d \circ i_{X_g}(h\mu_m) = dh \wedge i_{X_g}\mu_m \,.$$

Thus,

$$\int_{\partial D} h\omega_m = \int_D (L_{X_g}h)\mu_m \,. \tag{1.28}$$

Applying this identity to the distribution function, we find

$$\int_D (L_{X_g}f)\mu_m = N[\partial D]$$

for arbitrary regions $D \subset P_m$. For collsionless matter this leads to Liouvilles equation,

$$L_{X_g}f = 0 \,. \tag{1.29}$$

In local coordinates we obtain by (1.22)

$$(p^\mu\partial_\mu - \Gamma^i_{\mu\nu}p^\mu p^\nu \frac{\partial}{\partial p^i})f = 0 \,. \tag{1.30}$$

The energy-momentum tensor, $T(x)$, is given by the second moments of the distribution function integrated over the fiber $P_m(x)$:

$$T^{\mu\nu} = \int_{P_m(x)} fp^\mu p^\nu \pi_m \,. \tag{1.31}$$

It is now easy to see how the conservation equations follow from the matter equation (1.29): Let $V \subset \mathcal{M}$ be any region in spacetime, $D \subset P_m$

the corresponding region in phase space and v a vector field on $\mathcal{M}$ with $v_{\mu;\nu} = 0$ at a point $x_0 \in \mathcal{M}$. Then we have by Stoke's theorem and equation (1.28)

$$\begin{aligned} \textstyle\int_V (v_\mu T^{\mu\nu})_{;\nu}\eta &= \textstyle\int_{\partial V} v_\mu T^{\mu\nu}\sigma_\nu = \int_{\partial D} f v_\mu p^\mu (p^\nu \sigma_\nu \wedge \pi_m) = \\ \textstyle\int_{\partial D} f v_\mu p^\mu \omega_m &= \textstyle\int_D L_{X_g}(f p^\mu v_\mu)\mu_m \,. \end{aligned}$$

Thus,

$$(v_\mu T^{\mu\nu})_{;\nu}(x_0) = \int_{P_m(x_0)} L_{X_g}(f p^\mu v_\mu)\pi_m(x_0) \,.$$

At $x_0 \in \mathcal{M}$ we have $v_{\mu;\nu} = 0$. With this, one finds

$$L_{X_g}(v_\mu p^\mu f) = (L_{X_g} f) v_\mu p^\mu \text{ and } (v_\mu T^{\mu\nu})_{;\nu} = v_\mu T^{\mu\nu}_{;\nu} \,.$$

Because the choice of v and x_0 was arbitrary, we obtain

$$T^{\mu\nu}_{;\nu} = \int_{P_m} p^\mu (L_{X_g} f)\pi_m \,. \tag{1.32}$$

With the help of the matter equation (1.29), this leads to the conservation equations, $\nabla \cdot T = 0$, which are of course also a consequence of the gravitational field equations and the Bianchi identities, $\nabla \cdot G = 0$.

1.3 The 3+1 formalism of general relativity

Since we want to develop the gauge-invariant cosmological perturbation theory by means of the 3+1 formalism of general relativity, let us derive the nessecary tools in this section. A mathematically rigorous overview of the subject is given in [11,18], but the concrete calculations are not presented there. Therefore, we will be rather detailed. (See also [14].)

1.3.1 Generalities

We assume that spacetime $(\mathcal{M}, g)$ admits a slicing by slices Σ_t, i.e., there is a diffeomorphism $\phi : \mathcal{M} \to \Sigma \times I$, $I \subset \mathbf{R}$, such that the manifolds $\Sigma_t = \phi^{-1}(\Sigma \times \{t\})$ are spacelike and the curves $\phi^{-1}(\{x\} \times I)$ are timelike. These curves are what we call preferred timelike curves. They define a vector field ∂_t, which can be decomposed into normal and parallel components relative to the slicing (Figure 2):

$$\partial_t = \alpha n + \boldsymbol{\beta} \ . \tag{1.33}$$

Here n is a unit normal field and $\boldsymbol{\beta}$ is tangent to the slices Σ_t. α is the lapse function and $\boldsymbol{\beta}$ the shift vector field [11].

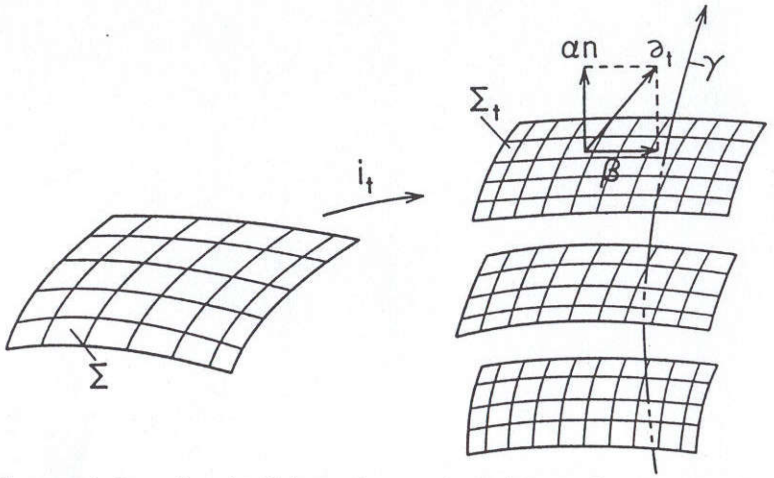


Figure 1.2 $3+1$ slicing of spacetime $\mathcal{M}$. i_t denotes the family of immersions of Σ into $\mathcal{M}$, i.e., $i_t(m) = \phi^{-1}(m,t)$.

A coordinate system $\{x^i\}$ on Σ induces natural coordinates on $\mathcal{M}$: $\phi^{-1}(m,t)$ has coordinates (t, x^i) if $m \in \Sigma$ has coordinates x^i. The preferred timelike curves have constant spatial coordinates. Let us set $\boldsymbol{\beta} = \beta^i \partial_i$ $(\partial_i = \frac{\partial}{\partial x^i})$. From $g(n, \partial_i) = 0$ and (1.33), we find

$$g(\partial_t, \partial_t) = -(\alpha^2 - \beta^i \beta_i) \ , \qquad g(\partial_t, \partial_i) = \beta_i \ .$$

Thus, in "comoving coordinates"

$$g = -(\alpha^2 - \beta^i \beta_i) dt^2 + 2\beta_i dx^i dt + g_{ij} dx^i dx^j \tag{1.34}$$

or

$$g = -\alpha^2 dt^2 + g_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \ . \tag{1.35}$$

This shows that the forms dt and $dx^i + \beta^i dt$ are orthogonal.

The tangent and cotangent spaces of $\mathcal{M}$ have two natural decompositions. One is defined by the slicing

$$T_p(\mathcal{M}) = H_p \oplus V_p \ , \tag{1.36}$$

where the "horizontal" space H_p consists of the vectors tangent to the slice through p and the "vertical" subspace is the one-dimensional space spanned by $(\partial_t)_p$ (preferred direction). The dual decomposition of (1.36) is

$$T^*_p(\mathcal{M}) = H^*_p \oplus V^*_p \ , \tag{1.37}$$

with

$$H^*_p = \{\omega \in T^*_p(\mathcal{M}) : \langle \omega, \partial_t \rangle = 0\}$$

and

$$V^*_p = \{\omega \in T^*_p(\mathcal{M}) : \langle \omega, H_p \rangle = 0\} \ ,$$

which is spanned by $(dt)_p$.

The metric defines—through the normal field n—yet another decomposition

$$T_p(\mathcal{M}) = H_p \oplus H_p^{\perp} \ , \tag{1.38}$$

where $H_p^{\perp}$ is spanned by n, and dually

$$T^*_p(\mathcal{M}) = (V^*_p)^{\perp} \oplus V^*_p \ . \tag{1.39}$$

Equation (1.33) reflects the fact that, in general, the two directions V_p and $H_p^{\perp}$ do not agree. Dually this implies that H^*_p and $(V^*_p)^{\perp}$ do not coincide. We have for $\omega^{\perp} \in (V^*_p)^{\perp}$ the following decomposition relative to (1.37):

$$\omega^{\perp} = \mathrm{hor}(\omega^{\perp}) + \langle \omega^{\perp}, \boldsymbol{\beta} \rangle dt \ . \tag{1.40}$$

The decompositions (1.36) to (1.39) induce two types of decompositions of arbitrary tensor fields on $\mathcal{M}$. We call a tensor field horizontal if it vanishes whenever at least one argument is ∂_t or dt. Relative to a comoving coordinate system such a tensor has the form

$$\boldsymbol{S} = \boldsymbol{S}^{i_1 \cdots i_r}_{j_1 \cdots j_s} \partial_{i_1} \otimes \cdots \otimes \partial_{i_r} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s} \ .$$

This shows that horizontal tensor fields can naturally be identified with families of tensor fields on Σ_t, or with time-dependent tensor fields on Σ ("absolute" space). We shall denote them with boldface letters (except ∂_i and dx^i).

As an often occuring example of a decomposition, we consider a horizontal p-form ω and its exterior derivative $d\omega$. We have

$$d\omega = \mathbf{d}\omega + dt \wedge \partial_t\omega \ ,$$

where $\mathbf{d}\omega$ is again horizontal. In comoving coordinates $\mathbf{d}$ involves only the dx^i ($\mathbf{d} = dx^i \wedge \partial_i$) and $\partial_t\omega$ is the partial time derivative. $\mathbf{d}\omega$ and $\partial_t\omega$ are horizontal and can be interpreted as t-dependent forms on Σ. In this interpretation $\mathbf{d}\omega$ is just the exterior derivative of ω. Similarly, other differential operators (covariant derivative, Lie derivative, etc.) can be decomposed. We use two types of bases of vector fields and 1-forms which are adapted to (1.36) and (1.37), respectively (1.38) and (1.39). Obviously, the dual pair $\{\partial_\mu\}$ and $\{dx^\mu\}$ for comoving coordinates $\{x^\mu\}$ are adapted to (1.36) and (1.37). On the other hand, equations (1.33) and (1.35) show that the dual pair

$$\{\partial_i, n\} \quad \text{and} \quad \{dx^i + \beta^i dt, \alpha dt\} \tag{1.41}$$

is adapted to (1.38) and (1.39).

Instead of $\{\partial_i\}$ we shall also use an orthonormal horizontal basis $\{\mathbf{e}_i\}$ ($g(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij}$), together with the dual basis $\{\boldsymbol{\vartheta}^i\}$ instead of $\{dx^i\}$. Then we have the following two dual pairs, which will be constantly used:

$$\{\mathbf{e}_i, \partial_t\} \ , \quad \{\boldsymbol{\vartheta}^i, dt\} \qquad \text{(adapted to slicing)}, \tag{1.42}$$

$$\{\mathbf{e}_i, e_0 = n\} \qquad \{\theta^\mu\} \qquad \text{(adapted to (1.38) and (1.39))} \ , \tag{1.43}$$

where the orthonormal tetrad $\{\theta^\mu\}$ is given by

$$\theta^0 = \alpha dt \ , \ \theta^i = \boldsymbol{\vartheta}^i + \beta^i dt \ , \tag{1.44}$$

with β^i defined by $\boldsymbol{\beta} = \beta^i \mathbf{e}_i$. We note also the relation

$$e_0 = n = \frac{1}{\alpha}(\partial_t - \beta^i \mathbf{e}_i) \ .$$

1.3.2 $3 + 1$ Split of the Liouville operator for a geodesic spray

In this section we derive a useful form of the Liouville operator for a geodesic spray for an arbitrary $3 + 1$ split. In the next chapter this will be worked out in a gauge-invariant manner in cosmological perturbation theory.

In terms of natural bundle coordinates the geodesic spray X_g is given by equation (1.21),

$$X_g = (p^\mu, -\Gamma^\mu{}_{\alpha\beta} p^\alpha p^\beta) , \qquad (1.45)$$

where $\Gamma^\mu{}_{\alpha\beta}$ are the Christoffel symbols for $(\mathcal{M}, g)$.

Let f be the distribution function on the one-particle phase space. If we consider the spatial components p^i, relative to an orthonormal tetrad $\{e^\mu\}$ as independent variables of f, then the Liouville operator L_{X_g} can be written as

$$L_{X_g} f = p^\mu e_\mu(f) - \omega^i{}_\alpha(p) p^\alpha \frac{\partial f}{\partial p^i} , \qquad (1.46)$$

where ω^μ_ν are the connection forms relative to the dual basis $\{\theta^\mu\}$.

We derive now a more explicit expression of (1.46) for an arbitrary $3+1$ slicing. In order to do this, we need the connection forms relative to the basis $\{\theta^\mu\}$ introduced in subsection 3.2. These are derived in detail in Appendix A. They can be expressed in terms of $\alpha, \boldsymbol{\beta}, \boldsymbol{\omega}^i{}_j, c^i{}_j$, where $\boldsymbol{\omega}^i{}_j$ are the connection forms of the slices Σ_t belonging to the induced metric $\mathbf{g}$ and $c^i{}_j$ is defined by

$$\partial_t \vartheta^i = c^i{}_j \vartheta^j . \qquad (1.47)$$

Using equations (A.5), (A.3), (A.6) and (A.2) we find ($\boldsymbol{p} = p^i \boldsymbol{e}_i$, $E = p^0 = \sqrt{\boldsymbol{p}^2 + m^2}$):

$$\begin{aligned}
\omega^i{}_\alpha(p) p^\alpha \tfrac{\partial}{\partial p^i} &= \omega^i{}_0(p) p^0 \tfrac{\partial}{\partial p^i} + \omega^i{}_j(p) p^j \tfrac{\partial}{\partial p^i} \\
&= [\omega^i{}_0(e^0) p^0 + \omega^i{}_0(\boldsymbol{p})] p^0 \tfrac{\partial}{\partial p^i} + [\omega^i{}_j(e^0) p^0 + \omega^i{}_j(\boldsymbol{p})] p^j \tfrac{\partial}{\partial p^i} \\
&= E^2 \alpha^{-1} \alpha^{|i} \tfrac{\partial}{\partial p^i} - K^i{}_j E p^j \tfrac{\partial}{\partial p^i} + \boldsymbol{\omega}^i{}_j(\boldsymbol{p}) p^j \tfrac{\partial}{\partial p^i} + \omega^i{}_j(e^0) E p^j \tfrac{\partial}{\partial p^i} \\
&= E^2 \alpha^{-1} \alpha^{|i} \tfrac{\partial}{\partial p^i} + \boldsymbol{\omega}^i{}_j(\boldsymbol{p} - \alpha^{-1} \boldsymbol{\beta} E) p^j \tfrac{\partial}{\partial p^i} - \tfrac{E}{\alpha}(\beta_j{}^{|i} - c^i_j) p^j \tfrac{\partial}{\partial p^i} .
\end{aligned}$$

Here $K^i{}_j$ are the components of the second fundamental form of Σ_t, for which we also use equation (A.7) of Appendix A. ($_|$ denotes the covariant derivative on $(\Sigma_t, \boldsymbol{g}_t)$.)

This leads to the following useful $3+1$ split of the Liouville operator:

$$L_{X_g} f = [\frac{E}{\alpha}\partial_t + L_{p-\frac{E}{\alpha}\beta}]f - [\omega^i{}_j(p - \frac{E}{\alpha}\beta)p^j + E^2(\ln\alpha)^{|i} - EH^i{}_j p^j]\frac{\partial f}{\partial p^i} \ , \tag{1.48}$$

where we have introduced the horizontal tensor field

$$H^i{}_j = \alpha^{-1}(\beta^i{}_j - c^i{}_j) \ . \tag{1.49}$$

1.3.3 3 + 1 Split of hydrodynamics

Calculations similar to those in the last section lead quite rapidly to a $3+1$ split of hydrodynamics. To this end, let us decompose the energy-momentum tensor into horizontal and vertical components:

$$T = \epsilon e_0 \otimes e_0 + e_0 \otimes \boldsymbol{S} + \boldsymbol{S} \otimes e_0 + \boldsymbol{T} \ . \tag{1.50}$$

For an perfect fluid with

$$T = (\rho + p)u \otimes u + pg^{\#} \tag{1.51}$$

we find, setting as in special relativity $u = \gamma(e_0 + \boldsymbol{v})$, $\gamma = (1 - \boldsymbol{v}^2)^{-1/2}$,

$$\epsilon = \gamma^2(\rho + p\boldsymbol{v}^2) \ , \tag{1.52}$$

$$\boldsymbol{S} = (\rho + p)\gamma^2 \boldsymbol{v} \ , \tag{1.53}$$

$$\boldsymbol{T} = (\rho + p)\gamma^2 \boldsymbol{v} \otimes \boldsymbol{v} + p\boldsymbol{g}^{\#} \ . \tag{1.54}$$

Now we compute $\nabla \cdot T$ for an arbitrary T. From

$$\nabla_{e_0}(\epsilon e_0 \otimes e_0) = L_{e_0}(\epsilon)e_0 \otimes e_0 + \epsilon\omega^i{}_0(e_0)e_i \otimes e_0 + \epsilon e_0 \otimes \omega^i{}_0(e_0)e_i$$

and

$$\nabla_{e_k}(\epsilon e_0 \otimes e_0) = L_{e_k}(\epsilon)e_0 \otimes e_0 + \epsilon\omega^i{}_0(e_k)e_i \otimes e_0 + \epsilon e_0 \otimes \omega^i{}_0(e_k)e_i$$

we obtain

$$\nabla \cdot (\epsilon e_0 \otimes e_0) = L_{e_0}(\epsilon)e_0 + \epsilon\omega^i{}_0(e_0)e_i + \epsilon\omega^i{}_0(e_i)e_0 \ .$$

In the same manner one finds the other contributions, with the result:

$$(\nabla \cdot T)^0 = L_{e_0}(\epsilon) + \epsilon\omega^i{}_0(e_i) + \omega^0_j(e_0)S^j + S^k{}_{|k} + \omega^0_j(e_0)S^j + \omega^0_j(e_i)T^{ij} \ .$$

Inserting the expressions for the connection forms given in Appendix A leads to the following form of the energy conservation:

$$\frac{1}{\alpha}(\partial_t - \boldsymbol{L}_{\boldsymbol{\beta}})\epsilon = -\boldsymbol{\nabla} \cdot \boldsymbol{S} - 2\,\mathbf{grad}(\ln\alpha) \cdot \mathrm{S} + \epsilon\,\mathrm{tr}(\boldsymbol{K}) + \mathrm{tr}(\boldsymbol{K} \cdot \boldsymbol{T}) \ . \quad (1.55)$$

Similarely one finds

$$\begin{aligned}(\nabla \cdot T)^i &= \omega^i{}_0(e_0)\epsilon + L_{e_0}(S^i) + [\omega^i{}_0(e_i) + \omega^i{}_j(e_0)]S^j + \omega^j{}_0(e_j)S^i \\ &\quad + \omega^0{}_j(e_0)T^{ji} + T^{ij}{}_{|j} \ ,\end{aligned}$$

and from this we obtain the momentum conservation equation

$$\frac{1}{\alpha}(\partial_t - \boldsymbol{L}_{\boldsymbol{\beta}})\boldsymbol{S} = -\mathbf{grad}(\ln\alpha)\epsilon + 2\boldsymbol{K} \cdot \boldsymbol{S} + \mathrm{tr}(\boldsymbol{K})\boldsymbol{S} - \alpha^{-1}\boldsymbol{\nabla}\cdot(\alpha\boldsymbol{T}) \ . \quad (1.56)$$

This form of the conservation laws will turn out to be very convenient in our treatment of cosmological perturbation theory.

1.3.4 3 + 1 Split of Einstein's field equations

For the sake of completeness and for later applications, we discuss also the often-used 3 + 1 split of the gravitational field equations. The calculation of the curvature forms relative to the basis (1.44) is presented in Appendix A. The reader will note that Cartan's calculus leads rather quickly to the required results.

We use the notation introduced in equation (1.50) for the various projections of the energy-momentum tensor T into normal and horizontal components:

$$T = \epsilon e_0 \otimes e_0 + e_0 \otimes \boldsymbol{S} + \boldsymbol{S} \otimes e_0 + \boldsymbol{T} \ . \quad (1.57)$$

From equations (A.14), (A.16) and (A.17) of Appendix A for the Einstein and Ricci tensors, Einsteins field equations can be written in the form (recall that boldface letters always refer to the slices Σ_t):

$$\boldsymbol{R} + (\mathrm{tr}\boldsymbol{K})^2 - \mathrm{tr}\boldsymbol{K}^2 = 16\pi G\epsilon \ , \quad (1.58)$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{K} - \boldsymbol{\nabla}\cdot\mathrm{tr}(\boldsymbol{K}) = 8\pi G\mathrm{S} \ , \quad (1.59)$$

$$\begin{aligned}\partial_t \boldsymbol{K} &= \boldsymbol{L}_{\boldsymbol{\beta}}\boldsymbol{K} - \mathbf{Hess}(\alpha) \\ &\quad + \alpha[\mathbf{Ric}(\boldsymbol{g}) - 2\boldsymbol{K} \cdot \boldsymbol{K} + (\mathrm{tr}\boldsymbol{K})\boldsymbol{K} - 8\pi G\boldsymbol{T} - 4\pi G\boldsymbol{g}(\epsilon - \mathrm{tr}(\boldsymbol{T}))] \quad (1.60)\end{aligned}$$

In addition to (1.58), (1.59) and (1.60), we have the following relation (Appendix A, equation (A.8)) between $\boldsymbol{g}$ and the second fundamental form $\boldsymbol{K}$:

$$\partial_t \boldsymbol{g} = -2\alpha \boldsymbol{K} + \boldsymbol{L}_{\boldsymbol{\beta}} \boldsymbol{g} . \tag{1.61}$$

Note that this decomposition into constraint equations, (1.58) and (1.59), and dynamical equations, (1.60) and (1.61), involves only horizontal quantities, and thus provides the 3+1 split of the gravitational field equations.

Later we shall also use the following consequence of (1.60) and (1.58):

$$\partial_t \mathrm{tr}(\boldsymbol{K}) = -\triangle\alpha + \boldsymbol{L}_{\boldsymbol{\beta}} \mathrm{tr}(\boldsymbol{K}) + \alpha[\mathrm{tr}(\boldsymbol{K}^2) + 4\pi G(\epsilon + \mathrm{tr}\boldsymbol{T})] . \tag{1.62}$$

Note that ∂_t and tr do not commute. With (1.61) one shows easily

$$\mathrm{tr}(\partial_t \mathrm{K} - \boldsymbol{L}_{\boldsymbol{\beta}} \boldsymbol{K}) = \partial_t \mathrm{tr}(\mathrm{K}) - \boldsymbol{L}_{\boldsymbol{\beta}} \mathrm{tr}(\boldsymbol{K}) - 2\alpha \mathrm{tr}(\boldsymbol{K}^2) . \tag{1.63}$$

Chapter 2

Cosmological Perturbation Theory

In this chapter, we study gauge-invariant linear perturbation theory on a Friedman universe.

We will thereby review the gauge-invariant formalism of Bardeen [4,28] with the help of the tools developed in the previous chapter. We will see that the 3+1 formalism is very well-suited to the problem and leads quickly to the required perturbation equations. Furthermore, we expand Bardeen's work to the gauge-invariant perturbation theory of collisionless particles.

2.1 The background

Let us split Friedman spacetime in the form

$$\mathcal{M} = \Sigma \times \boldsymbol{R}_+ ,$$

where (Σ, γ) is a space of constant curvature $K = \pm 1, 0$.

Choosing the conformal time as our time variable, we obtain for the lapse function, the shift vector field and the 3-metric on Σ

$$\alpha = a(t) \; , \; \beta = 0 \; , \; \boldsymbol{g} = a^2(t)\gamma \; , \tag{2.1}$$

where $a(t)$ denotes the scale factor. Using the notation of Section 1.3.3, the energy momentum tensor is given by equations (1.51) to (1.54) with $\boldsymbol{v} = 0$:

$$\epsilon = \rho \; , \quad \boldsymbol{T} = p\boldsymbol{g}^{\#} \; . \tag{2.2}$$

The unperturbed distribution function,

$$f : P_m \to \boldsymbol{R} \; ,$$

depends, for reasons of symmetry, only on the time t and the momentum $p = \sqrt{g(\boldsymbol{p},\boldsymbol{p})}$.

The background equations are easily derived with the help of sections 1.3.4 for the field equations, 1.3.3 for the conservation equations and 1.3.2 for Liouville's equation. The second fundamental form and the Ricci tensor of the slices are given in Appendices A and B. One obtains the following well known equations:

$$(\frac{\dot{a}}{a})^2 + K = \frac{8\pi G}{3}\rho a^2 , \tag{2.3}$$

$$2\frac{\ddot{a}}{a} - (\frac{\dot{a}}{a})^2 + K = -8\pi G p a^2 . \tag{2.4}$$

The energy conservation yields

$$\dot{\rho} = -3(1+w)\frac{\dot{a}}{a}\rho , \quad \text{where } w = \frac{p}{\rho} . \tag{2.5}$$

Liouville's equation leads to

$$\partial_t f - p\frac{\dot{a}}{a}\partial_p f = 0 , \tag{2.6}$$

where we used $\frac{\partial f}{\partial p^i} = \frac{p^i}{p}\frac{\partial f}{\partial p}$, and $\omega_{ij}p^i p^j = 0$. (2.6) implies that f is only a function of the variable

$$v = \frac{a}{m_X}p = \frac{p}{T_X} , \tag{2.7}$$

where we have set

$$T_X = \frac{m_X}{a} . \tag{2.8}$$

Since we are treating collisionless particles, they are not in thermodynamical equilibrium, and thus, T_X is not an equilibrium temperature. It is, however, related to the decoupling temperature of the X-particles as we will explain in Chapter 4.

2.2 Harmonic Analysis

The symmetry of the constant time slices $(\Sigma, a^2\gamma)$ of a Friedman universe leads one to perform a harmonic analysis of the perturbation. This corresponds to a decomposition of the tensor fields into parts which transform irreducibly under the isometry group of γ. One easily finds the following result:

Theorem 2 *Be (Σ, γ) an n-dimensional space of constant curvature $K = \pm 1$ or 0 with corresponding isometry group $G = SO(n+1)$, $SO(n,1)$ or $E(n)$. $E(n)$ denotes the n-dimensional Euclidean group, $E(n) = SO(n) \times \mathbf{R}^n$. We define $P_n(r, k^2)$, the space of totally symmetric, traceless, divergencefree tensor fields t of rank r on (Σ, γ) which are eigenvectors of the Laplace-Beltrami operator on (Σ, γ) with eigenvalue $-k^2$, i.e.,*

$$t^{i_1 \dots i_r}_{|i_1} = 0 ,$$

$$t^{i_1 \dots i_{r-1}}_{i_1} = 0 ,$$

$$(\Delta + k^2) t = 0 .$$

Here ' | ' denotes the covariant derivative with respect to γ. It is well known (see [59]), that k can take the values

$$k^2 \begin{cases} \geq (n-2)|K| & \text{for } K = -1 \\ \geq 0 & \text{for } K = 0 \\ = l(l+n-1)K \ , \ l \in \mathbf{N}_0 & \text{for } K = 1 \end{cases}$$

(K = curvature of Σ.) The spaces $P_n(r, k^2)$ provide a decomposition of the representation of the isometry group on the totally symmetric tensor fields on (Σ, γ).

In Appendix C we investigate in how far this decomposition is irreducible and how one obtains the decomposition of a generic, not totally symmetric tensor field.

It is clear that within the linear approximation the different components of a tensor field - modes - do not couple, and we can thus treat them independently. We therefore always consider one generic mode for scalar-, vector- and tensor-type perturbations (i.e., elements of $P_3(0, k^2)$, $P_3(1, k^2)$ and $P_3(2, k^2)$) and supress the index k.

2.2.1 Scalar perturbations

Since only scalar-type perturbations can lead to energy-density imhomogeneities, they must be responsible for the formation of galaxies. We will therefore handle them most explicitly. A scalar perturbation is an eigenfunction of the Laplace-Beltrami operator on (Σ, γ):

$$(\Delta + k^2)Y = 0 \tag{2.9}$$

"Scalar-type" contributions to vector and symmetric tensor fields can be expanded in terms of

$$Y_i = -k^{-1} Y_{|i} \tag{2.10}$$

$$Y_{ij} = k^{-2}(Y_{|ij} - \frac{1}{3}\gamma_{ij}\Delta Y) = k^{-2} Y_{|ij} + \frac{1}{3}\gamma_{ij} Y \tag{2.11}$$

$$\text{and} \qquad Y\gamma_{ij} \,.$$

Totally symmetric, traceless tensor contributions of rank $r > 2$ are defined inductively by

$$Y_{i_1 \cdots i_r} = \frac{-1}{rk} \sum_{l=1}^{r} Y_{\cdots \check{i}_l \cdots | i_l} + s(r, K) \sum_{l<m} \gamma_{i_l i_m} Y_{\cdots \check{i}_l \cdots \check{i}_m \cdots} \,. \tag{2.12}$$

Here, $s(r, K)$ is a complicated function of K and r which is determined by the trace-condition, $Y_{i_1 \cdots i_{r-1}}{}^{i_{r-1}} = 0$. (The check '$\check{}$', indicates which subscripts have to be left out.) We will not use the concrete form of $s(r, K)$ in the following. We will just make use of the result

$$\Delta Y_{i_1 \cdots i_r} = -[k^2 - K(r+1)r] Y_{i_1 \cdots i_r} \,. \tag{2.13}$$

Below, we prove this equation inductively from the general form (2.12) using the Riemann tensor $R^i{}_{jlm} = K(\delta^i_l \gamma_{jm} - \delta^i_m \gamma_{jl})$ to interchange covariant derivatives. One also easily finds the following identities by computing covariant derivatives:

$$
\begin{aligned}
\nabla_i Y^i &= kY\,, \\
\triangle Y_i &= -(k^2-2K)Y_i\,, \\
\nabla_j Y_i &= -k(Y_{ij} - 1/3\gamma_{ij}Y)\,, \\
\nabla^j Y_{ij} &= (2/3)k^{-1}(k^2-3K)Y_i\,, \\
\nabla_j \nabla^m Y_{im} &= (2/3)(3K-k^2)(Y_{ij}-(1/3)\gamma_{ij}Y)\,, \\
\triangle Y_{ij} &= -(k^2-6K)Y_{ij}\,, \\
\nabla_m Y_{ij} - \nabla_j Y_{im} &= (k/3)(1-(3K/k^2))(\gamma_{im}Y_j - \gamma_{ij}Y_m)\,.
\end{aligned}
\tag{2.14}
$$

Let us, as an illustration, derive equation (2.13) by induction. For $r=0$, (2.13) coincides with (2.9). We thus have only to show how (2.13) follows for r if it is true for $j<r$. By (2.12)

$$\triangle Y_{i_1\cdots i_r} = \frac{-1}{rk}\sum_{l=1}^{r} Y_{\cdots\check{i}_l\cdots|i_l}{}^{|j}{}_{|j} + s(r,K)\sum_{l<m}\gamma_{i_l i_m}\triangle Y_{\cdots\check{i}_l\cdots\check{i}_m\cdots}\,. \tag{2.15}$$

For a generic term of the first sum, we find with the help of the general rule for the interchanging of covariant derivatives

$$
\begin{aligned}
Y_{\cdots\check{i}_l\cdots|i_l}{}^{|j}{}_{|j} &= -(R^s{}_{i_1 i_l}{}^j Y_{s\cdots\check{i}_l\cdots} + \cdots + R^s{}_{i_r i_l}{}^j Y_{\cdots\check{i}_l\cdots s})_{|j} + Y_{\cdots\check{i}_l\cdots}{}^{|j}{}_{|i_l|j} \\
&= K(Y_{i_2\cdots i_r|i_1} + \cdots + Y_{i_1\cdots i_{r-1}|i_r} \\
&\quad -\gamma_{i_1 i_l}Y^j{}_{i_2\cdots\check{i}_l\cdots|j} - \cdots - \gamma_{i_r i_l}Y_{\cdots\check{i}_l\cdots i_{r-1}}{}^j{}_{|j}) + Y_{\cdots\check{i}_l\cdots}{}^{|j}{}_{|i_l|j} \\
&= 2K\sum_{s=1}^{r} Y_{\cdots\check{i}_s\cdots|i_s} - 2K\sum_{m\neq l}\gamma_{i_l i_m}Y_{\cdots\check{i}_l\cdots\check{i}_m\cdots}{}^j{}_{|j} + (\triangle Y_{\cdots\check{i}_l\cdots})_{|i_l}
\end{aligned}
\tag{2.16}
$$

(2.17)

On the other hand, one finds from the trace condition, $Y_{i_1\cdots i_m}{}^{|i_m}=0$,

$$\frac{-1}{(m+1)k}Y_{i_1\cdots i_m}{}^{|i_m} = -\frac{1}{2}s(m+1,K)[2(m-1)+3]Y_{i_1\cdots i_{m-1}}\,.$$

Inserting this in (2.16) and using (2.13) for $r-1$ yields

$$
\begin{aligned}
Y_{\cdots\check{i}_l\cdots|i_l}{}^{|j}{}_{|j} &= 2K\sum_{s=1}^{r} Y_{\cdots\check{i}_s\cdots|i_s} + (Kr(r-1)-k^2)Y_{\cdots\check{i}_l\cdots|i_l} \\
&\quad -\frac{K}{kr}s(r,K)(2r-1)\sum_{m\neq l}\gamma_{i_l i_m}Y_{\cdots\check{i}_l\cdots\check{i}_m\cdots}\,.
\end{aligned}
$$

With this result for the terms in the first sum of (2.15), and with (2.13) for $r-2$ for the terms in the second sum in (2.15) we find

$$\begin{aligned}\triangle Y_{i_1\cdots i_r} &= \frac{-[k^2+2rK+Kr(r-1)]}{kr}\sum_{l=1}^{r} Y_{\cdots i_l\cdots|i_l} + \\ &\quad s(r,K)[K(r-1)(r-2)-k^2+2K(2r-1)]\sum_{l<m}\gamma_{i_l i_m} Y_{\cdots \check{i}_l\cdots\check{i}_m\cdots}\,.\end{aligned}$$

One easily verifys that the two square brackets are equal, namely

$$[\cdots] = -k^2 + K(r+1)r\,.$$

Hence, we obtain finally (2.13).

After this somewhat technical diversion, let us define the scalar perturbations of the metric tensor. They are given by the perturbed quantities $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{g}$. Using the same notation as [28], we set

$$\begin{aligned}\tilde{\alpha} &= a(1+A(t)Y)\,, \\ \tilde{\beta} &= -B(t)Y^i\partial_i \quad \text{and} \\ \tilde{g} &= a^2[(1+2H_L(t)Y)\gamma_{ij}+2H_T(t)Y_{ij}]dx^i dx^j\,.\end{aligned} \tag{2.18}$$

(We indicate perturbed quantities by a twiddle.)

Next we consider the perturbed energy-momentum tensor $\widetilde{T}$. The 4-velocity $\tilde{u}$ is defined to be the normalized timelike eigenvector of $\widetilde{T}$,

$$\widetilde{T}\tilde{u} = -\tilde{\rho}\tilde{u}\,, \quad \tilde{g}(\tilde{u},\tilde{u}) = -1\,. \tag{2.19}$$

We decompose $\widetilde{T}$ as follows

$$\widetilde{T} = \tilde{\rho}\tilde{u}\otimes\tilde{u}^\flat + \tilde{\tau}\,, \tag{2.20}$$

where $\tilde{\tau}$ is orthogonal to $\tilde{u}$,

$$\tilde{\tau}^\mu_\nu\tilde{u}^\nu = 0\,. \tag{2.21}$$

For a scalar perturbation $\tilde{\rho}$, $\tilde{u}$ and $\tilde{\tau}$ are given by

$$\tilde{\rho} = \rho(1+\delta Y)\,, \tag{2.22}$$

$$\tilde{u} = a^{-1}((1-AY)\partial_t + vY^i\partial_i)\,. \tag{2.23}$$

The coefficient in the first term of $\tilde{u}$ is fixed by the normalization condition in (2.19). For the spatial components of the stress tensor $\tilde{\tau}$ we set

$$\tilde{\tau} = p[(1+\pi_L Y)\delta^i_{\ j} + \pi_T Y^i_{\ ;j}]\partial_i \otimes dx^j - (\rho+p)[vY^i\partial_i \otimes dt - (v-B)Y_{;i}\partial_t \otimes dx^i] \,. \tag{2.24}$$

The $\tilde{\tau}^\mu_0$ and $\tilde{\tau}^0_\mu$ are determined by the orthogonality condition (2.21).

Let us finally consider the perturbed distribution function $\tilde{f}$. For $p \in \widetilde{P}_m = \{(x,p) \in T\mathcal{M} |\ \tilde{g}(x)(p,p) = -m^2\}$, the one-particle phase space, we split f as follows into an unperturbed contribution f and a perturbation $f^{(1)}$:

$$\tilde{f}(p^\mu \tilde{e}_\mu) = f(p^\mu e_\mu) + f^{(1)}(p^\mu e_\mu) \,, \tag{2.25}$$

where $\{\tilde{e}_\mu\}$ is the adapted orthonormal basis (1.43) and $\{e_\mu\}$ is the corresponding basis for the unperturbed metric. For the harmonic expansion of $f^{(1)}$ into scalar modes, we assume that there are no other preferred directions of space than those given by the covariant derivatives of $\boldsymbol{Y}$. The most general ansatz for $f^{(1)}$ is then given by

$$f^{(1)} = \sum_{r=0}^{\infty} F_{(r)}(t,v)\gamma^{i_1}\cdots\gamma^{i_r} \boldsymbol{Y}_{i_1\cdots i_r} \,, \tag{2.26}$$

where v is defined in (2.7) and $\boldsymbol{\gamma}$ denotes the unit vector in direction of $\boldsymbol{p}$.

<u>**Remark:**</u> In the case $K = 0$, which will be relevant for all our applications, $\boldsymbol{Y} = \exp i\boldsymbol{k}\cdot\boldsymbol{x}$, and therefore

$$\boldsymbol{Y}_{i_1\cdots i_r}\gamma^{i_1}\cdots\gamma^{i_r} \propto \mu^r \boldsymbol{Y} \,,$$

where μ denotes the cosine of the angle between $\boldsymbol{\gamma}$ and $\boldsymbol{k}$, i.e.,

$$\mu = \frac{\boldsymbol{\gamma}\cdot\boldsymbol{k}}{k} \,. \tag{2.27}$$

In this case, we can thus simplify (2.26) to

$$f^{(1)} = F(t,v,\mu)\boldsymbol{Y} \,. \tag{2.28}$$

2.2.2 Vector perturbations

Vectorial perturbations are expanded in vector harmonics,

$$(\Delta + k^2)X^i = 0 \,, \quad X^i_{\ ;i} = 0 \,. \tag{2.29}$$

Vector-type contributions to symmetric 2-tensors are expanded in

$$\boldsymbol{X}_{ij} = \frac{-1}{2k}(\boldsymbol{X}_{i|j} + \boldsymbol{X}_{j|i}) . \tag{2.30}$$

One easily derives

$$\begin{aligned} \boldsymbol{X}^i{}_i &= 0 , \\ \boldsymbol{X}_{ij}^{\ |j} &= \tfrac{1}{2k}(k^2 - 2K)\boldsymbol{X}_i , \\ \boldsymbol{X}_{im}{}^{|m}{}_{|j} + \boldsymbol{X}_{jm}{}^{|m}{}_{|i} &= -(k^2 - 2K)X_{ij} . \end{aligned} \tag{2.31}$$

As in the scalar case, totally symmetric, traceless tensors of rank $r > 2$ are defined inductively by

$$\boldsymbol{X}_{i_1\cdots i_r} = \frac{-1}{rk}\sum_{l=1}^{r}\boldsymbol{X}_{\cdots\bar{i}_l\cdots|i_l} + v(r,K)\sum_{l<m}\gamma_{i_l i_m}\boldsymbol{X}_{\cdots\bar{i}_l\cdots\bar{i}_m\cdots} . \tag{2.32}$$

Again, $v(r,K)$ is a complicated function of K and r which is determined by the trace-condition, $\boldsymbol{X}_{i_1\cdots i_{r-1}}{}^{i_{r-1}} = 0$. We will not use the concrete form of $v(r,K)$ in the following. We will just make use of the result

$$\triangle\boldsymbol{X}_{i_1\cdots i_r} = -(k^2 - K[(r+1)r - 2])\boldsymbol{X}_{i_1\cdots i_r} , \tag{2.33}$$

which one derives as equation (2.13).

The vector perturbations of the metric are given by

$$\tilde{\beta} = BX^i\partial_i , \tag{2.34}$$

$$\tilde{g} = a^2[\gamma_{ij} + 2H_T\boldsymbol{X}_{ij}]dx^i \otimes dx^j . \tag{2.35}$$

Defining $\tilde{\rho}$, $\tilde{u}$ and $\tilde{\tau}$ as in Subsection 2.1, we obtain for their vectorial perturbations

$$\tilde{u} = a^{-1}\partial_t + a^{-1}vX^i\partial_i , \tag{2.36}$$

$$\tilde{\tau} = p[\delta^i_j + \pi_T\boldsymbol{X}^i_j]\partial_i \otimes dx^j - (p+\rho)v\boldsymbol{X}^i\partial_i \otimes dt + (p+\rho)(v-B)\boldsymbol{X}_i\partial_t \otimes dx^i . \tag{2.37}$$

Assuming again that there are no other preferred spatial directions than those of the $\boldsymbol{X}_{i_1\cdots i_r}$, the vector-type perturbations of the distribution function are given by

$$f^{(1)} = \sum_{r=1}^{\infty} F_{(r)}(t,v)\gamma^{i_1}\cdots\gamma^{i_r}\boldsymbol{X}_{i_1\cdots i_r} . \tag{2.38}$$

Like for scalar perturbations, we have to note that in the case $K = 0$, where

$$\boldsymbol{X}_j = \boldsymbol{w}_j \exp(i\boldsymbol{k}\cdot\boldsymbol{x}) , \quad \text{with} \quad \boldsymbol{k}\cdot\boldsymbol{w} = 0 , \quad \boldsymbol{w}^2 = 1 , \tag{2.39}$$

we can set

$$f^{(1)} = F(t, v, \mu)\gamma^l \boldsymbol{X}_l . \tag{2.40}$$

2.2.3 Tensor perturbations

They are expanded by tensor harmonics:

$$(\triangle + k^2)\boldsymbol{Z}_{ij} = 0 . \tag{2.41}$$

The totally symmetric, traceless higher rank tensors are defined inductively by

$$\boldsymbol{Z}_{i_1\cdots i_r} = \frac{-1}{rk}\sum_{l=1}^{r} \boldsymbol{Z}_{\cdots i_l \cdots | i_l} + t(r, K)\sum_{l<m} \gamma_{i_l i_m} \boldsymbol{Z}_{\cdots i_l \cdots i_m \cdots} , \tag{2.42}$$

where we do not have to specify $t(r, K)$ any further. One can show by induction

$$\triangle \boldsymbol{Z}_{i_1\cdots i_r} = -(k^2 - K[(r+1)r - 6])\boldsymbol{Z}_{i_1\cdots i_r} . \tag{2.43}$$

The perturbations are then given by

$$\tilde{g} = a^2[\gamma_{ij} + 2H_T \boldsymbol{Z}_{ij}]dx^i \otimes dx^j , \tag{2.44}$$

$$\tilde{\tau} = p[\delta^i_j + \pi_T \boldsymbol{X}^i_j]\partial_i \otimes dx^j , \tag{2.45}$$

$$f^{(1)} = \sum_{r=2}^{\infty} F_{(r)}(t, v)\gamma^{i_1}\cdots\gamma^{i_r}\boldsymbol{Z}_{i_1\cdots i_r} . \tag{2.46}$$

Also here, for the relevant case $K = 0$, $f^{(1)}$ simplifies: We can then set

$$f^{(1)} = F(t, v, \mu)\gamma^i\gamma^i Z_{ij} . \tag{2.47}$$

2.3 Gauge-invariant amplitudes

2.3.1 Gauge transformations

The separation into an isotropic background and perturbations which we performed in the preceding section is not unique. An infinitesimal change in this separation induces changes in the perturbations but leaves the background invariant. This is equivalent to a linearized coordinate transformation, which is called gauge transformation in this context. A gauge transformation is given by the infinitesimal flow ϕ_ϵ of a vector field ξ.

Transformations of the metric and the energy-momentum tensor

The perturbation of a tensor field $\tilde{t} = t + \epsilon T$ of arbitrary rank $r \geq 0$ transforms under ξ with the Lie-derivative of the corresponding background contribution:

$$\tilde{t} \rightarrow \tilde{t} + \epsilon L_\xi \tilde{t} \, . \tag{2.48}$$

Hence,

$$T \rightarrow T + L_\xi t \, . \tag{2.49}$$

Thus, T is gauge-invariant, if and only if, its background contribution t is constant on $\mathcal{M}$.

With this general rule, we can easily derive the gauge transformation properties of the metric and energy-momentum perturbations.

Scalar perturbations

Gauge transformations (2.49) affect scalar modes only for vector fields of the "scalar" type. These have the form

$$\xi = T(t) Y \partial_t + L(t) Y^i \partial_i \, . \tag{2.50}$$

Using

$$L_\xi dt = d(L_\xi t) = d(TY) = \dot{T} Y dt - kTY_{;i} dx^i \ , \ \text{etc.} \ , \tag{2.51}$$

one finds the transformation laws

$$
\begin{aligned}
A &\rightarrow A + \dot{T} + (\dot{a}/a)T \,, \\
B &\rightarrow B - \dot{L} - kT \,, \\
H_L &\rightarrow H_L + (k/3)L + (\dot{a}/a)T \,, \\
H_T &\rightarrow H_T - kL \,.
\end{aligned}
\tag{2.52}
$$

With the help of the unperturbed equation (2.5), and defining the velocity of sound as $c_s^2 = \dot{p}/\dot{\rho}$, we obtain the transformation laws for the energy-momentum perturbations:

$$
\begin{aligned}
\delta &\rightarrow \delta - 3(1+w)(\dot{a}/a)T \,, \\
v &\rightarrow v - \dot{L} \,, \\
\pi_L &\rightarrow \pi_L - 3(c_s^2/w)(1+w)(\dot{a}/a)T \,, \\
\pi_T &\rightarrow \pi_T \,.
\end{aligned}
\tag{2.53}
$$

Vector perturbations

Vector-type perturbations transform non-trivial only under vector-type gauge transformations:

$$\xi = L(t)\boldsymbol{X}^i\partial_i \,. \tag{2.54}$$

One easily derives the transformation laws

$$
\begin{aligned}
B &\rightarrow B - \dot{L} \,, \\
H_T &\rightarrow H_T - kL \,,
\end{aligned}
\tag{2.55}
$$

$$
\begin{aligned}
v &\rightarrow v - \dot{L} \,, \\
\pi_T &\rightarrow \pi_T \,.
\end{aligned}
\tag{2.56}
$$

Since there are no tensorial gauge transformations, tensor perturbations are always gauge-invariant.

Transformations of the distribution function

The discussion of the gauge transformations of the distribution function $\tilde{f}$ is slightly more complicated, since $\tilde{f}$ is defined on $\widetilde{P}_m$, which is itself not gauge-invariant. Let us, as before, consider a gauge transformation given by a vector field ξ . Then f transforms into $\bar{f} = f + L_{T\xi} f$, where $T\xi$ is the natural lift of ξ to $T\mathcal{M}$. Similar to (2.25), we set

$$\bar{f}(p^\mu \bar{e}_\mu) = f(p^\mu e_\mu) + \bar{f}^{(1)}(p^\mu e_\mu) \,. \tag{2.57}$$

The subtraction of these two decompositions shows that the change $\delta_\xi f^{(1)}$ of $f^{(1)}$ under a gauge transformation is given (ignoring higher order terms) by

$$\begin{aligned} \delta_\xi f^{(1)}(p^\mu e_\mu) &= \bar{f}(p^\mu \bar{e}_\mu) - \tilde{f}(p^\mu \tilde{e}_\mu) \\ &= (L_{T\xi} - L_{(T\xi)^\perp}) f(p^\mu e_\mu) \,, \end{aligned}$$

where

$$(T\xi)^\perp = p^\mu (\tilde{e}_\mu - \bar{e}_\mu)^\nu \frac{\partial}{\partial p^\nu} \,. \tag{2.58}$$

We write this result as follows:

$$\delta_\xi f^{(1)} = L_{(T\xi)^\parallel} f \tag{2.59}$$

with

$$(T\xi)^\parallel = T\xi - (T\xi)^\perp \,. \tag{2.60}$$

Remark:

There are, of course, many possible choices for perturbed orthonormal tetrads $\{\tilde{e}_\mu\}$ which deviate only in first order from $\{e_\mu\}$. They have the generic form

$$\tilde{e}_\mu = e_\mu + L_\mu{}^\nu e_\nu \,. \tag{2.61}$$

Since

$$\tilde{g}(e_\mu, e_\nu) = g(e_\mu, e_\nu) \,,$$

we find

$$L_{\mu\nu} + L_{\nu\mu} = -\delta g_{\mu\nu} \,. \tag{2.62}$$

The orthogonality condition fixes thus only the symmetric contribution to L. The choice of $\{\tilde{e}_\mu\}$ determines $(T\xi)^\perp$, i.e., determines the direction along which we project $T\xi$ onto the mass bundle P_m. $(T\xi)^\parallel$ and, therefore, the transformation properties of $f^{(1)}$ depend on $\{\tilde{e}_\mu\}$. Let us verify that for an arbitrary choice of $\{\tilde{e}_\mu\}$

$$L_{(T\xi)^{\parallel}} g(p,p) = 0 \,, \tag{2.63}$$

so that $(T\xi)^{\parallel}$ is in fact parallel to P_m, and the Lie-derivative in equation (2.59) is well defined.

$$L_{(T\xi)^{\parallel}} g(p,p) = L_{(T\xi)} g(p,p) - L_{(T\xi)^{\perp}} g(p,p) \,. \tag{2.64}$$

For the second term on the right-hand side, we obtain by (2.58), (2.61), (2.62) and $\delta\bar{g} - \delta g = L_\xi g$

$$\begin{aligned} L_{(T\xi)^{\perp}} g(p,p) &= 2g_{\mu\nu} p^\mu p^\lambda (\tilde{e}_\lambda - \bar{e}_\lambda)^\nu \\ &= 2g_{\mu\nu} p^\mu p^\lambda (L_\lambda{}^\nu - \bar{L}_\lambda{}^\nu) \\ &= p^\mu p^\lambda (\delta\bar{g}_{\mu\lambda} - \delta g_{\mu\lambda}) \\ &= p^\mu p^\lambda (L_\xi g)_{\mu\lambda} \end{aligned}$$

For the first term one obtains the same result immediately from the definitions. Hence, the two terms on the r.h.s. of (2.64) cancel and (2.63) is valid.

In order to work out the explicit form of $(T\xi)^{\parallel}$ for our choice of (1.43) for $\{\tilde{e}_\mu\}$, we must calculate $\bar{e}_\mu - \tilde{e}_\mu$ for a gauge transformation given by ξ. From the transformation properties of the lapse function and the shift vector given in equation (2.52), we find:

$$\begin{aligned} \bar{e}_0 &= \bar{n} = \bar{\alpha}^{-1}(\partial_t - \bar{\beta}) = \tilde{e}_0 + (\tilde{\alpha} - \bar{\alpha})\partial_t + ((\tilde{\beta}^i - \bar{\beta}^i)e_i \\ &= \tilde{e}_0 - [(\dot{\xi}^0 + (\dot{a}/a)\xi^0)e_0 + (\dot{\xi}^i - \xi^{0|i})e_i] \,. \end{aligned} \tag{2.65}$$

For the horizontal basis vector fields we have the following transformation law:

$$\begin{aligned} \bar{e}_i &= \tilde{e}_i + (L_\xi \tilde{e}_i)^{\text{hor}} = \tilde{e}_i + [\xi, e_i]^{\text{hor}} \\ &= \tilde{e}_i + [\xi^0 (a^{-1})^{\cdot} a e_i - \xi^\mu_{,i} e_\mu]^{\text{hor}} \\ &= \tilde{e}_i - [(\dot{a}/a)\xi^0 e_i + \xi^j_{,i} e_j] \,. \end{aligned} \tag{2.66}$$

The second term on the right-hand side of (2.66) is of first order. It is therefore sufficient that the square bracket is horizontal with respect to the background metric.

With the help of (2.65) and (2.66), equation (2.58) yields

$$(T\xi)^{\perp} = ((\dot a/a)\xi^0 + \dot\xi^0)p^0\frac{\partial}{\partial p^0} + (\dot\xi^i - \xi^{0,i})p^0\frac{\partial}{\partial p^i} + (\dot a/a)\xi^0 p^i\frac{\partial}{\partial p^i} + \xi^j_{,i}p^i\frac{\partial}{\partial p^j}\,.$$

Furthermore, in a coordinate frame $\{\partial_\mu\}$ we have:

$$T\xi = \xi^\mu\partial_\mu + \xi^\mu_{,\nu}p^\nu\frac{\partial}{\partial p^\mu}\,.$$

This leads to

$$(T\xi)^{\parallel} = \xi^\mu\partial_\mu - ((\dot a/a)\xi^0 p^0 - \xi^0_{,i}p^i)\frac{\partial}{\partial p^0} - ((\dot a/a)\xi^0 p^i - \xi^{0,i}p^0)\frac{\partial}{\partial p^i}\,. \quad (2.67)$$

Since for a vector-type gauge transformation $\xi^0 = 0$ (see (2.54)), vector perturbations of the distribution function are gauge-invariant. We have thus only to consider scalar perturbations. For a scalar perturbation ξ is of the form given in (2.50), and we obtain

$$(T\xi)^{\parallel} = TY\partial_t + LY^i\partial_i - T[((\dot a/a)Yp^0 + kY_{,i}p^i)\frac{\partial}{\partial p^0} + ((\dot a/a)Yp^i + kY^{,i}p^0)\frac{\partial}{\partial p^i}]\,. \quad (2.68)$$

Let us again choose the variables $(t, \boldsymbol{x}, v, \gamma)$ on P_m. Since f depends only on v, we find

$$(\partial_t f)_{p^i} = 2(\dot a/a)v\frac{df}{dv}\,, \quad (2.69)$$

$$(\frac{\partial f}{\partial p^i})_t = (v/p)\gamma^i\frac{df}{dv}\,, \quad (2.70)$$

where the variables which are kept constant are indicated by subscripts. With

$$q := (a^2/m_X)p^0 = (v^2 + a^2)^{1/2}\,, \quad (2.71)$$

equation (2.68) yields

$$L_{(T\xi)^{\parallel}}f = T\frac{df}{dv}[(\dot a/a)vY - kq\gamma^i Y_{,i}]\,. \quad (2.72)$$

In the expansion (2.26), this leads to the following transformation laws:

$$\begin{aligned} F_0 &\rightarrow F_0 + (\tfrac{\dot{a}}{a})Tv\tfrac{df}{dv} , \\ F_1 &\rightarrow F_1 - kTq\tfrac{df}{dv} , \\ F_r &\rightarrow F_r , \quad \text{for } r > 1. \end{aligned} \tag{2.73}$$

In the case $K = 0$, we use

$$\boldsymbol{Y}_j = -i\frac{\boldsymbol{k}_j}{k}Y \quad \text{and (2.28).}$$

We then find the transformation law

$$F \rightarrow F + \frac{df}{dv}\{(\frac{\dot{a}}{a})v + iqk\mu\}T . \tag{2.74}$$

2.3.2 Geometrical quantities and gauge-invariant variables

Scalar perturbations

For scalar perturbations the 4-velocity field u is hypersurface orthogonal. Indeed, it can easily be seen that u is proportional to the gradient of the function

$$t_m = t + k^{-1}(v - B)Y . \tag{2.75}$$

This function defines another foliation of spacetime, which will be very convenient for the derivation of the conservation equations.

We compute first the second fundamental forms for the two families of slices $\{t = \text{const}\}$ and $\{t_m = \text{const}\}$. This is conveniently done with the use of (1.61). For the original slicing $\{t = \text{const}\}$ we obtain in zeroth order

$$\boldsymbol{K} = -\dot{a}\gamma_{ij}dx^i dx^j , \ \text{tr}(\boldsymbol{K}) = -3\dot{a}/a^2 . \tag{2.76}$$

Using (2.18) and (2.14), one finds in the linearized approximation

$$\begin{aligned} \widetilde{\boldsymbol{K}}^{\text{aniso}} &= -ak\sigma_g \boldsymbol{Y}_{ij}dx^i dx^j , \\ \text{tr}(\widetilde{\boldsymbol{K}}) &= -3(\dot{a}/a^2)(1 + \kappa_g \boldsymbol{Y}) , \end{aligned} \tag{2.77}$$

where $\widetilde{\boldsymbol{K}}^{\text{aniso}}$ is the trace-free part of $\widetilde{\boldsymbol{K}}$ and

$$\kappa_g = -A + \frac{1}{3}(\dot{a}/a)^{-1}kB + (\dot{a}/a)^{-1}\dot{H}_L\ , \tag{2.78}$$

$$\sigma_g = k^{-1}\dot{H}_T - B\ . \tag{2.79}$$

Let us calculate the second fundamental form $\widetilde{\boldsymbol{K}}^m$ of the slices $\{t_m = \text{const.}\}$ somewhat more explicitly. Using (1.61) we have

$$\widetilde{\boldsymbol{K}}^m = -\frac{1}{\alpha_m}[\partial_{t_m}\tilde{\boldsymbol{g}} - L_{\tilde{\boldsymbol{\beta}}_m}\tilde{\boldsymbol{g}}]\ . \tag{2.80}$$

Using

$$dt_m = [1 + k^{-1}(\dot{v} - \dot{B}]dt - (v - B)Y_i dx^i$$

and

$$\tilde{\theta}^{(m)0} = -u^\flat = a[(1 + A)dt - (v - B)Y_i dx^i = \alpha_m dt_m\ ,$$

we obtain

$$\alpha_m = a(1 + [A - k^{-1}(\dot{v} - \dot{B})]Y)\ . \tag{2.81}$$

With the help of

$$\tilde{\boldsymbol{e}}_i^{(m)} = \tilde{\boldsymbol{e}}_i + \frac{1}{a}\epsilon(v - B)Y_i\frac{\partial}{\partial t}$$

as an orthonormal basis of the hypersurfaces orthogonal to $\tilde{u}$, we find, up to first order,

$$\begin{aligned}\tilde{\theta}^{(m)i} &= \tilde{\theta}^i - a(v - B)Y^i dt_m \\ &= \tilde{\boldsymbol{\vartheta}}^i - avdt_m\ .\end{aligned} \tag{2.82}$$

Hence, by (1.43),

$$\tilde{\boldsymbol{\beta}}^{(m)} = -vY^i\tilde{\boldsymbol{e}}_i\ . \tag{2.83}$$

From

$$u = \alpha_m^{-1}(\partial_{t_m} - \tilde{\boldsymbol{\beta}}^{(m)})\ ,$$

we finally obtain

$$\partial_{t_m} = [1 - k^{-1}(\dot{v} - \dot{B}Y]\partial_t\ . \tag{2.84}$$

Using these results in (2.80) leads to

$$\begin{aligned}\widetilde{\boldsymbol{K}}^{(m)\text{aniso}} &= -ak\sigma_m \boldsymbol{Y}_{ij}dx^i dx^j\ , \\ \text{tr}(\boldsymbol{K}^{(m)}) &= -3(\dot{a}/a^2)(1 + \kappa_m Y)\ ,\end{aligned} \tag{2.85}$$

with

$$\kappa_m = -A + \frac{1}{3}(\dot{a}/a)^{-1}kv + (\dot{a}/a)^{-1}\dot{H}_L \,, \tag{2.86}$$

$$\sigma_m = k^{-1}\dot{H}_T - v \,. \tag{2.87}$$

We shall need also the perturbation of the scalar Riemann curvature **R** of the slices $\{t = \text{const}\}$. It is derived in Appendix B with the result (B.11)

$$\delta\mathbf{R} = 4a^{-2}(k^2 - 3K)\mathcal{R}Y \,, \tag{2.88}$$

where

$$\mathcal{R} = H_L + \frac{1}{3}H_T \,. \tag{2.89}$$

From the transformation laws given in Section 3, we deduce

$$\begin{aligned}
\mathcal{R} &\rightarrow \mathcal{R} + (\dot{a}/a)T \,, \\
\kappa_g &\rightarrow \kappa_g - (\dot{a}/a)^{-1}[(\dot{a}/a)^2 + (k^2/3) - (\dot{a}/a)^{\cdot}]T \,, \\
\sigma_g &\rightarrow \sigma_g + kT \,, \\
\kappa_m &\rightarrow \kappa_m - (\dot{a}/a)^{-1}[(\dot{a}/a)^2 - (\dot{a}/a)^{\cdot}]T \,, \\
\sigma_m &\rightarrow \sigma_m \,.
\end{aligned} \tag{2.90}$$

Hence, the following combinations are gauge-invariant:

$$\mathcal{A} = A - a^{-1}(\frac{a^2}{\dot{a}}\mathcal{R})^{\cdot} \,, \tag{2.91}$$

$$\mathcal{B} = k(\dot{a}/a)^{-1}\mathcal{R} - \sigma_g \,. \tag{2.92}$$

One of the four geometric perturbation variables A, B, H_L and H_T can be fixed by the choice of coordinates on the $\{t = \text{const.}\}$ hypersurfaces. Therefore, there are only three independent perturbation variables which do not change under gauge transformations which leave these hypersurfaces invariant (i.e., gauge transformations with $T = 0$). Thus, the variables $A, \mathcal{R}, \kappa_g$ and σ_g can not be independent. Indeed, one finds

$$\kappa_g = -A + (\dot{a}/a)^{-1}[\dot{\mathcal{R}} - (k/n)\sigma_g] \,. \tag{2.93}$$

Of course, any combination of $\mathcal{A}$ and $\mathcal{B}$ and their derivatives is again gauge-invariant. Later on we shall work with the following combinations:

$$\Phi = k^{-1}(\dot{a}/a)\mathcal{B} \,, \tag{2.94}$$

$$\Psi = \mathcal{A} + (ka)^{-1}(a\mathcal{B})^{\cdot} \,. \tag{2.95}$$

One finds

$$\Phi = \mathcal{R} - k^{-1}(\dot{a}/a)\sigma_g \,, \tag{2.96}$$

$$\Psi = A - (ka)^{-1}(a\sigma_g)^{\cdot} \,. \tag{2.97}$$

Remark: In a shearfree time slicing ($\sigma_g = 0$) which is called 'Newtonian slicing', we have $\Psi = A$ which is the well-known gravitational potential in the Newtonian limit. In the next subsection we shall also see that the field equation for Ψ is most similar to the Poisson equation of the Newtonian theory.

From the matter variables alone one finds quickly by (2.53) the following gauge-invariant combinations:

$$\Pi = \pi_T \,, \tag{2.98}$$

$$\Gamma = \pi_L - (c_s^2/w)\delta \,, \tag{2.99}$$

where Π is the amplitude of the anisotropy of the stress tensor. As we shall see in the next section, Γ represents the amplitude of the entropy production rate of the perturbation.
The shear σ_m of the hypersurfaces $\{t_m = \text{const.}\}$ is gauge-invariant. We define

$$V = -\sigma_m = v - k^{-1}\dot{H}_T = v - B - \sigma_g \tag{2.100}$$

as our gauge-invariant velocity variable. This velocity variable has its most natural interpretation in the Newtonian slicing, where

$$V = v - B \,.$$

This is the velocity of matter relative to an observer with constant space coordinates in the hypersurfaces $\{t = \text{const.}\}$. Furthermore, we have

$$3(\dot{a}/a^2)(\kappa_m - \kappa_g) = (k/3)V - \sigma_g \,. \tag{2.101}$$

In the Newtonian slicing, this reduces to

$$3(\dot{a}/a^2)(\kappa_m - \kappa_g)Y = (1/3)(VY^i)_{|i} \,.$$

Thus, in this slicing the difference between the expansion rates of matter and space is equal to the divergence of the velocity vector, VY^i.

For the density perturbation there are several possibilities of gauge-invariant perturbation amplitudes. The simplest of them are

$$\Delta_g = \delta + 3(1+w)\mathcal{R} , \tag{2.102}$$

$$\Delta_s = \delta + 3(1+w)(\dot{a}/a)k^{-1}\sigma_g , \tag{2.103}$$

$$\Delta_s = \Delta_g - 3(1+w)\Phi . \tag{2.104}$$

Δ_s reduces to the ordinary density contrast in the Newtonian slicing. Δ_g is the density perturbation of the slicing with no curvature perturbation, the 'flat slicing'.

Another gauge-invariant quantity is the acceleration of $\tilde{u}$:

$$\tilde{a} = \nabla_{\tilde{u}}\tilde{u}$$

A short calculation leads to

$$\tilde{a} = -akA_m Y^i e_i , \tag{2.105}$$

with

$$A_m = A - (ka)^{-1}[a(v-B)]^{\cdot} = \Psi - (ka)^{-1}(aV)^{\cdot} . \tag{2.106}$$

From this equation, we see again that V is the natural choice for a velocity variable: In the Newtonian slicing, Ψ represents the Newtonian potential. The acceleration is due to the gravitational attraction (first term of (2.107)) and the change in the expansion (second term of (2.107)):

$$-kA_m Y_j = (\Psi Y)_{|j} + a^{-1}(aVY_j)^{\cdot} . \tag{2.107}$$

From the transformation properties of the distribution function, equation (2.73) and equations (2.90), we find the following gauge-invariant combinations:

$$\mathcal{F}_0 = F_0 - \mathcal{R}v\frac{df}{dv} , \tag{2.108}$$

$$\mathcal{F}_1 = F_1 + \sigma_g q\frac{df}{dv} , \tag{2.109}$$

$$\mathcal{F}_r = F_r , \quad \text{for } r > 1 . \tag{2.110}$$

(2.111)

In the case $K = 0$, equation (2.74) leads to the gauge-invariant combination

$$\mathcal{F} = F - \{\mathcal{R}v + iq\mu\sigma_g\}\frac{df}{dv} . \tag{2.112}$$

Vector perturbations

For vector perturbations, the second fundamental form of the slices $\{t = \text{const.}\}$ is given by

$$\widetilde{\boldsymbol{K}} = (\dot{a}\gamma_{ij} + ak\sigma_g\boldsymbol{X}_{ij})dx^i \otimes dx^j , \tag{2.113}$$

with

$$\sigma_g = k^{-1}\dot{H}_T - B . \tag{2.114}$$

From equation (2.52) for $T = 0$, one immediately sees that for vector perturbations, σ_g is gauge-invariant by itself. Now, $\tilde{u}$ is no longer hypersurface-orthogonal, its vorticity does not vanish. One finds for the shear and vorticity of the vector field $\tilde{u}$

$$\tilde{u}_{(i;j)} = ak\sigma_m X_{ij} , \tag{2.115}$$

$$\tilde{u}_{[i;j]} = ak\omega(X_{i;j} - X_{j;i}) , \tag{2.116}$$

with

$$\sigma_m = k^{-1}\dot{H}_T - v = -V_s , \tag{2.117}$$

$$\omega = B - v = -V_v , \tag{2.118}$$

$$V_v = V_s + \sigma_g . \tag{2.119}$$

V_s and V_v are gauge-invariant velocity variables. They describe the shear and vorticity of the matter velocity field.

As already mentioned, the perturbations of the distribution function $\{F_r\}$, or F in the case $K = 0$, defined in equations (2.38) and (2.40) are already gauge-invariant.

2.4 Perturbation equations

2.4.1 The field equations

The perturbed field equations can be directly obtained from the general formulae (1.58) to (1.61).

a) Constraint equations

Scalar perturbations

By compairing $u = a^{(-1)}\partial_t$ and $\tilde{u}$ given in (2.23) with $\tilde{n} = \tilde{\alpha}^{-1}(\partial_t - \tilde{\beta})$, one sees that u is equal to n up to a first order horizontal contribution. Hence, we obtain to first order from our expression for T in Section 2 :

$$T(n,n) = \tilde{\rho} = \rho(1+\delta Y) \; , \; T(n,\partial_i) = -a(\rho+p)(v-B)Y_i \, .$$

For the first order term on the right-hand side of equation (1.58), we use

$$\delta[(\mathrm{tr}\boldsymbol{K})^2 - \mathrm{tr}(\boldsymbol{K}^2)] = 12(\frac{\dot{a}}{a^2})^2\kappa_g Y$$

and (2.88). Then, we can write down the perturbation of the constraint equation (1.58):

$$6(\dot{a}/a)^2\kappa_g + 2a^{-2}(k^2-3K)\mathcal{R} = 8\pi G\rho\delta \, .$$

With (2.102) and the zero-order relation

$$4\pi G\rho(1+w) = a^{-2}[(\dot{a}/a)^2 - (\dot{a}/a)^{\cdot} + K] \, , \tag{2.120}$$

we can write this as

$$4\pi G\rho\Delta_g = 3(\dot{a}/a)^2\kappa_g + a^{-2}[k^2 + 3((\dot{a}/a)^2 - (\dot{a}/a)^{\cdot})]\mathcal{R} \, .$$

Expressing, finally, the right-hand side in terms of gauge-invariant quantities, one finds

$$4\pi G a^2\rho\Delta_g = -3(\dot{a}/a)^2\mathcal{A} + k(\dot{a}/a)\mathcal{B} \, . \tag{2.121}$$

In order to write down (1.59), we note that

$$(\mathrm{tr}\boldsymbol{K})_{|i} - \boldsymbol{K}^j{}_{i|j} = 2k(\dot{a}/a^2)\kappa_g Y_i + \frac{2}{3}a^{-1}(k^2-3K)\sigma_g Y_i \, .$$

Thus, (1.59) reads

$$-8\pi G\rho a(1+w)(v-B) = 2k(\dot{a}/a^2)\kappa_g + \frac{2}{3}a^{-2}(k^2-3K)\sigma_g$$

or with (2.100) and (2.120),

$$-4\pi Ga^2\rho(1+w)V = k(\dot{a}/a)\kappa_g + [\frac{1}{3}k^2 + (\dot{a}/a)^2 - (\dot{a}/a)^{\cdot}]\sigma_g \,.$$

Expressing again the right-hand side in terms of $\mathcal{A}$ and $\mathcal{B}$ leads to

$$4\pi Ga^2\rho(1+w)V = k(\dot{a}/a)\mathcal{A} + [(\dot{a}/a)^2 - (\dot{a}/a)^{\cdot}]\mathcal{B} \,. \qquad (2.122)$$

With the two constraint equations (2.121) and (2.122), one can express Φ algebraically in terms of V and Δ_g :

$$\Phi = \frac{4\pi Ga^2\rho}{k^2-3K+12\pi G(1+w)a^2\rho}(\Delta_g + 3(1+w)(\dot{a}/a)k^{-1}V) \,. \quad (2.123)$$

Vector perturbations

For vector modes, only the constraint equation (1.59) remains, and leads after similar calculations as in the preceding paragraph to

$$8\pi G\rho a^2(1+w)V_v = (K - \frac{1}{2}k^2)\sigma_g \,. \qquad (2.124)$$

b) Dynamical equations

In order to work out the content of the dynamical equation (1.60), it is convenient to start from the trace-free part of this equation. With the help of (1.63) we obtain

$$\begin{aligned}\partial_t\widetilde{\boldsymbol{K}}^{\text{aniso}} = &\ \boldsymbol{L}_{\tilde{\boldsymbol{\beta}}}\widetilde{\boldsymbol{K}}^{\text{aniso}} - \mathbf{Hess}^{\text{aniso}}(\tilde{\alpha}) \\ &+\tilde{\alpha}[\widetilde{\mathbf{Ric}}^{\text{aniso}} - 2\widetilde{\boldsymbol{K}}\cdot\widetilde{\boldsymbol{K}}^{\text{aniso}} + \widetilde{\boldsymbol{K}}^{\text{aniso}}\text{tr}(\widetilde{\boldsymbol{K}}) - \widetilde{\boldsymbol{T}}^{\text{aniso}}] \,.\end{aligned} \qquad (2.125)$$

Since $\widetilde{\boldsymbol{K}}^{\text{aniso}}$ is of first order, $\boldsymbol{L}_{\tilde{\boldsymbol{\beta}}}\widetilde{\boldsymbol{K}}^{\text{aniso}}$ is of second order, and we can thus neglect this contribution.

Scalar perturbations: If we use the result that for scalar perturbations $\widetilde{\mathbf{Ric}}^{\text{aniso}} = -k^2\mathcal{R}Y_{ij}$ (see Appendix B, equation (B.12)), we find immediately

$$8\pi Ga^2p\Pi = k^2[-(A-(ka)^{-1}(a\sigma_g)^{\cdot}) + ((\dot{a}/a)k^{-1}\sigma_g - \mathcal{R})] ,$$

or with(2.96) and (2.97),

$$-8\pi Ga^2k^{-2}p\Pi = \Phi + \Psi . \tag{2.126}$$

This enables us to express also Ψ algebraically in terms of matter variables.

Vector perturbations: Using the result of Section 2 of Appendix B, we obtain by similar calculations as those above

$$8\pi Ga^2p\Pi = 2k\frac{\dot{a}}{a}\sigma_g + k\dot{\sigma}_g . \tag{2.127}$$

Tensor perturbations: With the help of (B.17) we obtain for tensor perturbations

$$8\pi Ga^2p\Pi = \ddot{H}_T + \frac{\dot{a}}{a}\dot{H}_T + (k^2+2K)H_T . \tag{2.128}$$

2.4.2 Energy-momentum conservation

In order to work out the conservation laws $\nabla \cdot T = 0$, we start with the 3+1 split of them carried out in Section 1.3 .

Scalar perturbations

Up to first order, the quantities in (1.55) and (1.56) are

$$\tilde{\epsilon} = \rho(1+\delta Y) ,$$

$$\widetilde{\boldsymbol{S}} = (\rho+p)v^i\partial_i , \qquad v^i = a^{-1}(v-B)Y^i ,$$

$$\widetilde{\boldsymbol{T}} = a^{-2}p[(1+\pi_L Y)\gamma^{ij} + \pi_T Y^{ij}]\partial_i \otimes \partial_j ,$$

$$\tilde{\alpha} = a(1+AY) , \qquad \beta = -BY^i\partial_i .$$

Inserting this in (1.55) and using the zero-order relation (2.5),

$$\dot{\rho} = -3(\dot{a}/a)(\rho + p)\ , \text{ gives}$$

$$(\rho + p)3(\dot{a}/a)(\kappa_m + A) + \rho\dot{\delta} - p3(\dot{a}/a)(\delta - \pi_L) = 0\ .$$

This can be expressed in terms of gauge-invariant quantities. Making use also of the unperturbed relation

$$\dot{w} = -3(c_s^2 - w)(1 + w)\dot{a}/a\ , \tag{2.129}$$

we find

$$\dot{\Delta}_g + 3(c_s^2 - w)(\dot{a}/a)\Delta_g + (1 + w)kV + 3w(\dot{a}/a)\Gamma = 0\ . \tag{2.130}$$

Similarly, equation (1.56) gives

$$(\dot{a}/a)(1 + w)(1 - 3c_s^2)(v - B) + (1 + w)(\dot{v} - \dot{B}) = (1 + w)kA + wk\pi_L$$
$$-wk\tfrac{2}{3}(1 - \tfrac{3K}{k^2})\pi_T\ .$$

With the help of (2.96) - (2.100), (2.103) and (2.104), this leads to the gauge-invariant equation

$$\begin{aligned} \dot{V} + (\dot{a}/a)(1 - 3c_s^2)V &= k(\Psi - 3c_s^2\Phi) \\ &\quad + \tfrac{c_s^2}{1+w}k\Delta_g + \tfrac{kw}{1+w}(\Gamma - \tfrac{2}{3}(1 - \tfrac{3K}{k^2})\Pi)\ . \end{aligned} \tag{2.131}$$

Vector Perturbations

For vector perturbations, (1.56) yields, instead of (2.131),

$$\dot{V}_v + (\dot{a}/a)(1 - 3c_s^2)V_v = \frac{kw}{w+1}(1/2 - \frac{K}{k^2})\Pi\ . \tag{2.132}$$

A short discussion of vector and tensor perturbations is given in [28].

2.4.3 Entropy perturbation and heat flux

In this section we want to show that the entropy production rate of the perturbation is given by the amplitude Γ. (Vector and tensor perturbations are thus always adiabatic.) For our derivations we need some notions from relativistic thermodynamics. One can find them, e.g., in Appendix B of [52] or in [61].

For a generic fluid (not necessarily in thermodynamical equilibrium), let us introduce the following notation for the scope of this section:

$$\begin{aligned} v &= \text{4-velocity of particle transport,} \\ N &= \text{particle current,} \\ \sigma &= \text{entropy flux,} \\ s &= \text{entropy per particle ,} \\ n &= \text{particle density,} \\ \theta &= \text{temperature,} \\ \mu &= \text{chemical potential.} \end{aligned}$$

Furthermore, we use as before the energy-momentum tensor T, the energy transport velocity u, the energy density ρ and the pressure p. In general, we thus have

$$T \cdot u = -\rho u \tag{2.133}$$

and, of course,

$$\nabla \cdot T = 0 \ . \tag{2.134}$$

In thermodynamical equilibrium, which is attained, for example, in an unperturbed Friedman universe, the energy flux vanishes in the matter rest frame:

$$T \cdot v = -\rho v \ , \quad \text{thus} \ \ v = u \ . \tag{2.135}$$

In addition, we have the following Euler relation and Gibbs' equation:

$$ns = \theta^{-1}(\rho + p) - \frac{\mu}{\theta} n \ , \tag{2.136}$$

$$\theta ds = d(\rho/n) + p d(1/n) \ ,$$

or, equivalently,

$$\theta n ds = d\rho - \frac{\rho + p}{n} dn \ . \tag{2.137}$$

In thermodynamical equilibrium, $\sigma = snv$. Equations (2.136) and (2.135) thus yield

$$\sigma = \frac{\mu}{\theta}N - \theta^{-1}T \cdot v + \frac{p}{\theta}v \,. \tag{2.138}$$

If v is kept fixed, we then find by (2.136), (2.135) and (2.137)

$$d\sigma = -\theta^{-1}(dT) \cdot v - \frac{\mu}{\theta}dN \,. \tag{2.139}$$

Following this diversion, let us consider a perturbed Friedman universe. In general, the perturbed state is no longer a thermodynamical equilibrium state. If one accepts the postulate of "release of variations" (no extra differentials in (2.139)), which can be justified by kinetic theories, one can show that, for an arbitrary state (N, T, σ) near some equilibrium state $(\mu, v, \theta, p, \cdots)$, (2.138) remains valid in first order perturbation theory. We want now to fit an equilibrium state to our actual, perturbed state. Of course, we could take the unperturbed Friedman universe as a neighboring equilibrium state. But we shall follow the fitting procedure of Eckart and assume that

$$\widetilde{N} = \tilde{n}\tilde{v} \quad \text{and} \quad \tilde{\rho} = \rho(1 + \delta Y)$$

are equal in our equilibrium state and in the perturbed state (Remember: Perturbed quantities are indicated by a twiddle.). The pressure of this equilibrium state is thus

$$\tilde{p}_{eq} = p + c_s^2 \rho \delta Y = p(1 + \frac{c_s^2}{w}\delta Y) \,. \tag{2.140}$$

We fit the actual energy-momentum tensor in the following way:

$$\widetilde{T} = (\tilde{\rho} + \tilde{p}_{eq})v \otimes v + \tilde{p}_{eq}\tilde{g}^{\#} + \Delta T \,. \tag{2.141}$$

Let us define $\tilde{\boldsymbol{q}}$ by

$$\tilde{v} = \tilde{u} - \tilde{\boldsymbol{q}} \,.$$

From the normalization condition, $v^2 = u^2 = -1$, we conclude $u \cdot \tilde{\boldsymbol{q}} = 0$ up to first order, and thus, $\tilde{\boldsymbol{q}}$ is horizontal with respect to the matter slicing, $\{t_m = \text{const.}\ \}$. (In this section we denote by boldface letters quantities which are horizontal with respect to the <u>matter slicing</u>.) The perturbed energy-momentum tensor is also given by equation (2.20):

$$\widetilde{T} = \tilde{\rho}\tilde{e}_0 \otimes \tilde{e}_0 + \tilde{\boldsymbol{\tau}} = \tilde{\rho}\tilde{e}_0 \otimes \tilde{e}_0 + \tilde{p}\tilde{\boldsymbol{g}}^{\#} + \tilde{\boldsymbol{\pi}} \,, \tag{2.142}$$

with $\mathrm{tr}(\widetilde{\boldsymbol{\pi}}) = 0$. Equation (2.141) expressed in $\tilde{e}_0$ and $\tilde{\boldsymbol{q}}$ yields up to first order

$$\widetilde{T} = \tilde{\rho}\tilde{e}_0 \otimes \tilde{e}_0 - (\rho + p)[\tilde{e}_0 \otimes \tilde{\boldsymbol{q}} + \tilde{\boldsymbol{q}} \otimes \tilde{e}_0] + \tilde{p}_{eq}\tilde{\boldsymbol{g}}^{\#} + \Delta T \,. \tag{2.143}$$

Comparing (2.142) and (2.143) and inserting (2.99) leads to

$$\Delta T = \Gamma Y \boldsymbol{g}^{\#} + \widetilde{\boldsymbol{\pi}} + (\rho + p)[e_0 \otimes \tilde{\boldsymbol{q}} + \tilde{\boldsymbol{q}} \otimes e_0] \,. \tag{2.144}$$

With the help of (2.138) and (2.136), we find

$$\sigma = nsv + \theta^{-1}\Delta T \cdot v \,. \tag{2.145}$$

For the perturbed Friedman universe with ΔT given by (2.144), this leads to

$$(\rho + p)\tilde{\boldsymbol{q}} = \theta(\tilde{\sigma} - \tilde{n}\tilde{s}\tilde{v}) \,, \tag{2.146}$$

which is just the heat flux (see [52]). Thus, $(\rho + p)\tilde{\boldsymbol{q}}$ represents the heat flux of the perturbation. Since the zero order contributon of $\tilde{\boldsymbol{q}}$ vanishes, this quantity is of course gauge-invariant.

For the entropy production rate one obtains from (2.145) after a short calculation

$$\nabla \cdot \tilde{\sigma} = -\theta^{-2}\Delta T^{\flat}(\nabla\theta, v) + \theta^{-1}\Delta T \cdot (\nabla v^{\flat}) \,. \tag{2.147}$$

Since $\Delta T^{\flat}(e_0, e_0) = 0$, the first term on the r.h.s. of (2.147) does not contribute in first order. For the second term we need only the zero order contribution of $\nabla\tilde{v}^{\flat}$,

$$\nabla v^{\flat} = -\boldsymbol{K} = \frac{\dot{a}}{a^2}\boldsymbol{g} \,.$$

Using this together with (2.141) in (2.147) results in

$$\nabla\tilde{\sigma} = \frac{3(\dot{a}/a^2)}{\theta}\Gamma Y \,, \tag{2.148}$$

which shows that, up to a normalization factor, Γ represents the entropy production rate of the perturbation.

2.4.4 The Liouville equation

Liouville's equation provides the equation of motion for the amplitudes $\mathcal{F}_r$, which we derive now. Using equation (1.48) we can write the Liouville equation in the form

$$\widetilde{X}_g \tilde{f} = 0 \,, \tag{2.149}$$

with

$$\widetilde{X}_g = \frac{p^0}{\tilde{\alpha}}\partial_t + \boldsymbol{p} - \frac{p^0}{\alpha}\tilde{\boldsymbol{\beta}} - [\widetilde{\omega}^i{}_j(\boldsymbol{p} - \frac{p^0}{\tilde{\alpha}}\tilde{\boldsymbol{\beta}})p^j + (p^0)^2(\ln\tilde{\alpha})^{|i} + \tilde{\alpha}^{-1}(\tilde{\boldsymbol{\beta}}_j{}^{|i} - \tilde{c}_j{}^i)p^0p^j]\frac{\partial}{\partial p^i}. \tag{2.150}$$

Here, the momentum components are those with respect to the tetrad $\{\tilde{e}_0 = \tilde{n}, \tilde{e}_i\}$ adapted to the sclicing $\{t = \text{const.}\}$. Let us consider, as before, $\tilde{f}$ as a function of the variables $(t, \boldsymbol{x}, v, \gamma)$. Since

$$v = (a/m_X)p = T_X^{-1}p\ ,$$

we have

$$(\frac{\partial \tilde{f}}{\partial t})_p = (\frac{\partial \tilde{f}}{\partial t})_v + (\dot{a}/a)v\frac{\partial \tilde{f}}{\partial v}\ ,$$

where the subscripts p and v indicate which variable is kept constant while evaluating the t-derivative. Using in addition

$$\boldsymbol{p} = T_X v\gamma\ \ ,\ \ p^0 = T_X q\ ,$$

we can write (2.149) in the form

$$\frac{q}{\tilde{\alpha}}(\partial_t + (\dot{a}/a)v\partial_v)\tilde{f} + (v\gamma - \frac{q}{\tilde{\alpha}}\tilde{\beta})\tilde{f} - T_X[\ \widetilde{\omega}^i{}_j(v\gamma - \frac{q}{\tilde{\alpha}}\tilde{\beta})\gamma^j v + q^2(\frac{\tilde{\alpha}^{|i}}{\tilde{\alpha}}) + \tilde{\alpha}^{-1}(\tilde{\boldsymbol{\beta}}_j{}^{|i} - \tilde{c}_j{}^i)qv\gamma^j]\frac{\partial}{\partial p^i}\tilde{f} = 0\ . \tag{2.151}$$

To find the background and the first order contribution to (2.151), we use the decomposition

$$\tilde{f} = f + f^{(1)}\ ,$$

and make use of the background identities

$$\boldsymbol{\beta} = 0\ \ ,\ \ \alpha = a\ \ ,\ \ c^i{}_j = (\dot{a}/a)\delta^i_j\ . \tag{2.152}$$

Taking into account the homogeneity and isotropy of f,

$$\partial_i f = 0\ \ ,\ \ \frac{\partial f}{\partial p^i} = \gamma^i\frac{\partial f}{\partial p} = \gamma^i T_X^{-1}\frac{\partial f}{\partial v}\ ,$$

as well as the antisymmetry of the ω_{ij}, we obtain the background contribution to (2.151),

$$\partial_t f = 0\ . \tag{2.153}$$

Hence, as we already obtained in Section 1, f is a function of v alone.

To find the first order contribution, we use (2.18) and

$$\tilde{c}^i_{\ j} = (\dot{a}/a)\delta^i_j + (\dot{H}_L Y \delta^i_j + \dot{H}_T Y^i_j)$$

as well as (2.152). Equation (2.151) then yields the first order equation

$$\begin{aligned}&[q\partial_t + v\gamma^i\partial_i - v\gamma^i\omega^k_{\ i}(\gamma)\tfrac{\partial}{\partial\gamma^k}]f^{(1)} = \\ &-[AY^i\gamma^i q^2 k + BqvY_{\|l\|i}\gamma^l\gamma^i - (\dot{H}_L Y - \dot{H}_T Y_{ij})\gamma^i\gamma^j vq]\tfrac{df}{dv} = 0\,.\end{aligned} \tag{2.154}$$

For the third term on the left-hand side of (2.154) we made use of the antisymmetry of ω_{ij}, which leads to

$$\gamma^i\omega^k_{\ i}\frac{\partial}{\partial p^k} = p^{-1}\gamma^i\omega^k_{\ i}\frac{\partial}{\partial\gamma^k}\,.$$

If we now insert the decomposition (2.26) for $f^{(1)}$,

$$f^{(1)} = \sum_{r=0}^{\infty} F_{(r)}(t,v)\gamma^{i_1}\cdots\gamma^{i_r}Y_{i_1\cdots i_r}\,,$$

and use

$$\gamma^l\gamma^i\omega^k_{\ i}(e_l)\frac{\partial f^{(1)}}{\partial\gamma^k} = \sum_{r=0}^{\infty} F_{(r)}(t,v)r\Gamma^k_{li}\gamma^{j_1}\cdots\gamma^{j_{r-1}}\gamma^i\gamma^l Y_{j_1\cdots i_{r-1},k}\,,$$

the second and the third terms on the l.h.s. of (2.154) add up to the covariant derivative of $Y_{j_1\cdots i_r}$, and we obtain

$$\begin{aligned}&\Sigma_{r=0}^{\infty}(q\partial_t F_{(r)}Y_{j_1\cdots j_r} + F_{(r)}v\gamma^k Y_{j_1\cdots j_r|j_k})\gamma^{j_1}\cdots\gamma^{j_r} = \\ &\tfrac{df}{dv}[vq(\dot{H}_L - \tfrac{1}{3}Bk)Y - kq^2AY_i\gamma^i + vq(\dot{H}_T - kB)Y_{ij}\gamma^i\gamma^j]\,.\end{aligned} \tag{2.155}$$

Equation (2.155) can be used for the discussion of scalar, vector and tensor perturbations. For vector perturbations we only have to replace $Y_{j_1\cdots j_r}$ by $X_{j_1\cdots j_r}$ for $r \geq 1$ and set $A = H_L = F_{(0)} = 0$. For tensor perturbations we have to replace $Y_{j_1\cdots j_r}$ by $Z_{j_1\cdots j_r}$ for $r \geq 2$ and set $A = H_L = B = F_{(0)} = F_{(1)} = 0$.

Scalar perturbations

With the help of (2.13) and $Y^j_{l_2\cdots l_r j} = 0$, we find

$$\begin{aligned}&Y_{l_1\cdots l_r|j}\gamma^{l_1}\cdots\gamma^{l_r}\gamma^j = \\ &-kY_{l_1\cdots l_r j}\gamma^{l_1}\cdots\gamma^{l_r}\gamma^j + \tfrac{k}{3}(1 - r(r-1)\tfrac{K}{k^2})Y_{l_1\cdots l_{r-1}}\gamma^{l_1}\cdots\gamma^{l_{r-1}}\,.\end{aligned}$$

A comparison of the coefficients of $Y_{l_1\cdots l_r}\gamma^{l_1}\cdots\gamma^{l_r}$ yields, thus,

$$q\partial_t F_{(0)} - \frac{k}{3}vF_{(1)} = \frac{df}{dv}qv(\dot{H}_L + \frac{k}{3}B) , \quad (2.156)$$

$$q\partial_t F_{(1)} - kvF_{(0)} - \frac{k}{3}v(1-\frac{2K}{k^2})F_{(2)} = -\frac{df}{dv}q^2kA , \quad (2.157)$$

$$q\partial_t F_{(2)} - kvF_{(1)} - \frac{k}{3}v(1-\frac{6K}{k^2})F_{(3)} = \frac{df}{dv}vq(\dot{H}_T - kB) , \quad (2.158)$$

and

$$q\partial_t F_{(r)} - kvF_{(r-1)} - \frac{k}{3}(1-\frac{r(r+1)K}{k^2})F_{(r+1)} = 0 \text{ for } r > 2. \quad (2.159)$$

Equation (2.159) is already gauge-invariant. Inserting $\mathcal{F}_{(0)} = F_{(0)} - \mathcal{R}v\frac{df}{dv}$ and $\mathcal{F}_{(1)} = F_{(1)} + \sigma_g q\frac{df}{dv}$ leads to

$$\partial_t \mathcal{F}_{(0)} - \frac{kv}{3q}\mathcal{F}_{(1)} = 0 , \quad (2.160)$$

$$\partial_t \mathcal{F}_{(1)} - \frac{kv}{q}\mathcal{F}_{(0)} - \frac{kv}{3q}(1-\frac{2K}{k^2})\mathcal{F}_{(2)} = \frac{df}{dv}k[\frac{v^2}{q}\Phi - q\Psi] , \quad (2.161)$$

and

$$\partial_t \mathcal{F}_{(r)} - \frac{kv}{q}\mathcal{F}_{(r-1)} - \frac{kv}{3q}(1-\frac{r(r+1)K}{k^2})\mathcal{F}_{(r+1)} = 0, \text{ for } r > 2. \quad (2.162)$$

Inserting $f^{(1)} = FY$ in (2.155) for the case $K = 0$, leads by similar manipulations (see [14]) to

$$\partial_t \mathcal{F} + \frac{ikv\mu}{q}\mathcal{F} = -i\mu k\frac{df}{dv}[\frac{v^2}{q}\Phi - q\Psi] . \quad (2.163)$$

Vector perturbations

With the help of (2.33) we find

$$X_{l_1\cdots l_r|s}\gamma^{l_1}\cdots\gamma^{l_r}\gamma^s =$$

$$-kX_{l_1\cdots l_r s}\gamma^{l_1}\cdots\gamma^{l_r}\gamma^s + \tfrac{k}{3}(1-[r(r-1)-2]K)X_{l_1\cdots l_{r-1}}\gamma^{l_1}\cdots\gamma^{l_{r-1}} .$$

A comparison of the coefficients of $X_{l_1 \cdots l_r}\gamma^{l_1}\cdots\gamma^{l_r}$ yields

$$q\partial_t F_{(1)} - \frac{k}{3}vF_{(2)} = 0\,, \tag{2.164}$$

$$q\partial_t F_{(2)} - kvF_{(1)} - \frac{k}{3}v(1-\frac{4K}{k^2})F_{(3)} = -\frac{df}{dv}vqk\sigma_g\,, \tag{2.165}$$

and

$$q\partial_t F_{(r)} - kvF_{(r-1)} - \frac{kv}{3}(1-\frac{[r(r+1)-2]K}{k^2})F_{(r+1)} = 0, \text{ for } r>2. \tag{2.166}$$

Since the amplitudes $F_{(r)}$ and σ_g of vector-type perturbations are gauge-invariant, these equations are already gauge-invariant.

Inserting $f^{(1)} = FX_j\gamma^j$ in (2.155) for the case $K=0$ leads to

$$\partial_t F + \frac{ikv\mu}{q}F = +i\mu k\frac{df}{dv}v\sigma_g\,. \tag{2.167}$$

Tensor perturbations

For tensor perturbations, (2.155) together with (2.43) yields

$$\partial_t F_{(2)} - \frac{kv}{3q}F_{(3)} = -\frac{df}{dv}v\dot{H}_T\,, \tag{2.168}$$

and

$$\partial_t F_{(r)} - k(v/q)F_{(r-1)} - \frac{kv}{3q}(1-\frac{[r(r+1)-6]K}{k^2})F_{(r+1)} = 0\,, \quad \text{for } r>2. \tag{2.169}$$

In the case $K=0$, we find

$$\partial_t F + \frac{ikv\mu}{q}F = i\mu\frac{df}{dv}v\dot{H}_T\,. \tag{2.170}$$

2.4.5 Momentum integrals

In this section we calculate the gauge-invariant fluid perturbation amplitudes by integration of the perturbations of the distribution function over the mass shell.

In an orthonormal frame, the invariant volume element $\pi(x)$ of the mass shell $P_m(x)$ looks like in special relativity:

$$\pi(x) = \frac{p^2}{p^0} dp d\Omega \ .$$

Using the definitions of v and q and T_X,

$$T_X = m_X/a \ , \tag{2.171}$$

we therefore obtain

$$\pi(x) = T_X^2 \frac{v^2}{q} dv d\Omega \ . \tag{2.172}$$

Keeping in mind equation (1.31), this leads to

$$\rho \ = \ T_X^4 4\pi \int f v^2 q dv \ , \tag{2.173}$$

$$p \ = \ \frac{T_X^4 4\pi}{3} \int f(v^4/q) dv \ . \tag{2.174}$$

Scalar perturbations

For scalar perturbations, the calculation of the energy-momentum tensor from $\tilde{f}$ yields the following equations for the gauge-invariant fluid variables defined in equations (2.102), (2.100), (2.98), and (2.99):

$$\Delta_g \ = \ \frac{T_X^4}{\rho} \int v^2 q \mathcal{F}_{(0)} dv d\Omega \ , \tag{2.175}$$

$$V \ = \ \frac{T_X^4}{3(\rho + p)} \int v^3 \mu \mathcal{F}_{(1)} dv d\Omega \ , \tag{2.176}$$

$$\Pi \ = \ \frac{2T_X^4}{5p} \int (v^4/q) \mathcal{F}_{(2)} dv d\Omega \ , \tag{2.177}$$

$$\Gamma \ = \ \frac{T_X^4}{p} \int (v^4/3q - c_s^2 v^2 q) \mathcal{F}_{(0)} dv d\Omega \ . \tag{2.178}$$

Let us briefly derive these equations: The energy-momentum tensor is given by (1.31),

$$T^\mu_\nu = T_X^2 \int p^\mu p_\nu f(t, \boldsymbol{x}, v, \gamma)(v^2/q) dv d\Omega \ .$$

Here the p^μ's and p_ν's are considered as functions of $(t, \boldsymbol{x}, v, \gamma)$. In first order we find

$$T^0_0 = -T_X^4 \int q v^2 (f + f_{(1)}) dv d\Omega \ .$$

On the other hand, we have by (2.22)

$$T^0_0 = -\rho(1+\delta Y)\ ,$$

and thus,

$$\rho\delta = T^4_X \int q v^2 f^{(1)} dv d\Omega\ .$$

If we insert here the definition (2.26) and take into account that the $F_{i_1\cdots i_r}$ are traceless, we find that only F_0 contributes to $\delta\rho$:

$$T^4_X \int q v^2 \mathcal{F}_{(0)} dv d\Omega = \rho\delta + \mathcal{R} T^4_X \int \frac{d}{dv}(qv^3) f dv d\Omega\ . \qquad (2.179)$$

The integral on the right-hand side of (2.179) is

$$T^4_X \int \frac{d}{dv}(qv^3) f dv d\Omega = 3(\rho + p)\ . \qquad (2.180)$$

Using finally the definition (2.102) of Δ_g, we obtain (2.175).

Next, we calculate

$$T^0_j = Y T^4_X \int \tilde{f} \gamma_j v^3 dv d\Omega\ . \qquad (2.181)$$

Since the $Y_{i_1\cdots i_r}$ are traceless, only $F_{(1)}$ contributes to $T_j^{\ 0}$. Using again (2.180) and the definition of $\mathcal{F}_{(1)}$, we obtain (2.176).

By the trace argument, and since $\int(1/3-\gamma^2)d\Omega = 0$, only $F_{(2)}$ contributes to (2.177). The numerical coefficient is obtained by direct calculation.

To find (2.178), we note that, again by the traceless condition of the $Y_{i_1\cdots i_r}$, only $F_{(0)}$ can contribute to π_L. From

$$\pi_L = \frac{T^4_X}{3p} \int (v^4/q) f^{(1)} d\Omega dv\ ,$$

we obtain

$$\frac{T^4_X}{3p} \int (v^4/q) F_{(0)} d\Omega dv = \pi_L - \mathcal{R}\frac{4\pi T^4_X}{3p} \int (v^5/q) \frac{df}{dv} dv\ . \qquad (2.182)$$

Using the background equation (2.5) for $\dot\rho$ and performing an integration by parts, we can write the second term in the form

$$\mathcal{R}\frac{4\pi T^4_X}{3p} \int (v^5/q)\frac{df}{dv} dv = -\mathcal{R}\frac{3(1+w)(\dot a/a)}{w\dot\rho}\frac{4\pi T^4_X}{3}\int (\frac{5v^4}{q} - \frac{v^6}{q^3})\frac{df}{dv} dv\ .$$

On the other hand, a simple calculation using $\dot{T}_X = -(\dot{a}/a)T_X$ and $\dot{q} = (\dot{a}/a)(q - \frac{v^2}{q})$ yields

$$\dot{p} = -\frac{4\pi T_X^4}{3}(\dot{a}/a)\int(\frac{5v^4}{q} - \frac{v^6}{q^3})\frac{df}{dv}dv\ .$$

Inserting these results in (2.182) leads to

$$\frac{T_X^4}{3p}\int(v^4/q)F_{(0)}d\Omega dv = \pi_L + 3(1+w)\frac{c_s^2}{w}\mathcal{R} = \Gamma + \frac{c_s^2}{w}\Delta_g\ ,$$

where we use the definitions (2.99) and (2.102) for the second equality. Subtracting $\frac{c_s^2}{w}\Delta_g$ on both sides leads finally, with the help of (2.175), to (2.178).

Let us also treat the simplification which is possible in the case $\underline{K=0}$, since this case will be relevant for the applications in Chapters 3 and 4. Instead of equations (2.175) to (2.178), one finds

$$\Delta_g = \frac{T_X^4}{\rho}\int v^2 q\mathcal{F}dv d\Omega\ , \tag{2.183}$$

$$V = \frac{iT_X^4}{(\rho+p)}\int v^3\mu\mathcal{F}dv d\Omega\ , \tag{2.184}$$

$$\Pi = \frac{T_X^4}{2p}\int(v^4/q)(1-3\mu^2)\mathcal{F}dv d\Omega\ , \tag{2.185}$$

$$\Gamma = \frac{T_X^4}{p}\int(v^4/3q - c_s^2 v^2 q)\mathcal{F}dv d\Omega\ , \tag{2.186}$$

where μ is defined by (2.27). We derive, as an illustration, equation (2.184).

In flat space we have to first order for a general mode

$$k^{-1}k^j T^0_j = (\rho+p)(v-B)Y_j k^{-1}k^j = -i(\rho+p)(v-B)Y\ , \tag{2.187}$$

so that by the definition (2.100) of V,

$$VY = i(\rho+p)^{-1}k^{-1}k^j T^0_j - \sigma_g Y\ .$$

Inserting (2.181) gives

$$V = -\sigma_g + i(\rho+p)^{-1}T_X^4\int v^3\mu F dv d\Omega$$

and thus, with (2.112), we obtain (2.184). The other equations are similarly derived.

Vector perturbations

By similar manipulations as before, one finds

$$V_v = \frac{T_X^4}{3(\rho + p)} \int v^3 \mu \mathcal{F}_{(1)} dv d\Omega \,, \tag{2.188}$$

$$\Pi = \frac{2T_X^4}{5p} \int (v^4/q) \mathcal{F}_{(2)} dv d\Omega \,. \tag{2.189}$$

Tensor perturbations

Here one obtains for the only gauge-invariant fluid variable Π

$$\Pi = \frac{2T_X^4}{5p} \int (v^4/q) \mathcal{F}_{(2)} dv d\Omega \,. \tag{2.190}$$

Chapter 3

Simple Applications of Cosmological Perturbation Theory

For all the applications presented in this and the following chapters, we specialize to the case $K = 0$. For physical applications, this is a good approximation for $z \geq 5$, since (including dark matter) we have good reasons to limit Ω by $0.2 \leq \Omega \leq 1$ as mentioned already in the introduction. Therefore, deviations from an Einstein-de Sitter Universe play a role only for $z \leq \Omega^{-1}$ (see [42]). Taking into account the instability of the value $\Omega = 1$, it seems very unlikely to us, that Ω is so close to 1, if there is no physical reason requiring $\Omega \equiv 1$.

3.1 Perfect fluids

As a first simple application of the theory developed in the preceding chapter, we treat perturbations of perfect fluids in thermodynamical equilibrium. We then have

$$\Gamma = \Pi = 0 \, , \tag{3.1}$$

and

$$w = c_s^2 = \text{const.} \quad .$$

(3.1) provides the matter equation for perfect fluids. The equations of motion for Δ and V are then given by equations (2.130) and (2.131). With (3.1) they result in

$$\dot{\Delta} = -(1+w)kV \, , \tag{3.2}$$

$$\dot{V} + (\dot{a}/a)(1-3w)V = k(\Psi - 3w\Phi) + \frac{w}{1+w}k\Delta \, , \tag{3.3}$$

where we have set $\Delta_g \equiv \Delta$, since we will use only this variable for the density perturbation from now on. The field equations (2.123) and (2.126) yield

$$\Phi = -\Psi = \frac{4\pi G a^2 \rho}{k^2 + 12\pi G(1+w)a^2\rho}(\Delta + 3(1+w)(\frac{\dot{a}}{a})k^{-1}V) . \qquad (3.4)$$

For $w = c_s^2 =$ const., the scale factor exhibits the well-known power law behaviour

$$a = (\nu\sqrt{C}t)^\nu ,$$

where

$$\nu = \frac{2}{3w+1} , \quad (1/2 \le \nu \le 2) ,$$

and

$$C = \frac{8\pi G}{3}\rho a^{\frac{2\nu+2}{\nu}} = \text{const.} .$$

Thus,

$$\frac{\dot{a}}{a} = \nu t^{-1} ,$$

and

$$\rho a^3 = C a^{\frac{\nu-2}{\nu}} .$$

With $\eta := kt$, equations (3.2), (3.3) and (3.4) then yield

$$\Delta' = -(\frac{2\nu+2}{3\nu})V , \qquad (3.5)$$

$$V' + 2(\nu-1)\eta^{-1}V = \frac{2}{\nu}\Psi + \frac{2-\nu}{2\nu+2}\Delta , \qquad (3.6)$$

$$-\Psi = \frac{3\nu^2}{2\eta^2 + 6\nu(\nu+1)}[\Delta + 2(\nu+1)\eta^{-1}V] , \qquad (3.7)$$

where $'$ denotes the derivative with respect to η.

Setting

$$f = -\frac{2}{3\nu^2}\eta^\nu\Psi = \eta^{\nu-2}(\Delta + 2(\nu+1)\eta^{-1}V + 2\frac{\nu+1}{\nu}\Psi) , \qquad (3.8)$$

we find

$$V' + \frac{\nu}{\eta}V = [\frac{2-\nu}{2+2\nu}\eta^{2-\nu} - \frac{3\nu^2}{2}\eta^{-\nu}]f , \qquad (3.9)$$

and

$$f' = -\frac{2\nu+2}{3\nu}\eta^{\nu-2}V . \qquad (3.10)$$

This leads to the following second order equation for f:

$$f'' + \frac{2}{\eta}f' + (w - \frac{\nu(\nu+1)}{\eta^2})f = 0 . \tag{3.11}$$

For $w = c_s^2 \neq 0$, this is a well known-Bessel equation (see, e.g., [2], p437ff) with the solution

$$f = Aj_\nu(c_s\eta) + Bn_\nu(c_s\eta) =: Z_\nu(c_s\eta) , \tag{3.12}$$

where j_ν and n_ν denote the spherical Bessel functions of order ν. With (3.8) and (3.10) this yields the perturbation amplitudes

$$\Psi = -\frac{3\nu^2}{2}\eta^{-\nu}Z_\nu(c_s\eta) , \tag{3.13}$$

$$V = \frac{3\nu}{2}\eta^{1-\nu}[Z_\nu(c_s\eta) - \frac{c_s}{\nu+1}\eta Z_{\nu-1}(c_s\eta)] , \tag{3.14}$$

$$\Delta = \eta^{2-\nu}Z_\nu(c_s\eta) - 3\nu c_s\eta^{1-\nu}Z_{\nu-1}(c_s\eta) , \tag{3.15}$$

where we use the identity

$$Z_\nu(c_s\eta)' = c_sZ_{\nu-1}(c_s\eta) - (\nu+1)\eta^{-1}Z_\nu(c_s\eta)$$

for equation (3.14).

From the asymptotic behaviour of Bessel functions (see [2]), we obtain

$$Z_\nu \approx C\eta^\nu + D\eta^{-(\nu+1)} \quad \text{for} \quad c_s\eta \ll 1 ,$$

and

$$Z_\nu \approx \frac{A}{\eta}\cos(c_s\eta - \alpha_\nu) + \frac{B}{\eta}\sin(c_s\eta - \alpha_\nu) \quad \text{for} \quad c_s\eta \gg 1 ,$$

with $\alpha_\nu = \frac{\pi}{2}(\nu+1)$.

Thus, the 'growing' mode is of the order

$$\left.\begin{array}{rcl} \Delta & \approx & \Delta_0 \\ V & \approx & V_0\eta \\ \Psi & \approx & \Psi_0 \end{array}\right\} \text{for } c_s\eta \ll 1 , \tag{3.16}$$

and

$$\left.\begin{array}{rcl} \Delta & \approx & \Delta_0\eta^{1-\nu}\cos(c_s\eta - \alpha_\nu) \\ V & \approx & V_0\eta^{1-\nu}\cos(c_s\eta - \alpha_{\nu-1}) \\ \Psi & \approx & \Psi_0\eta^{-1-\nu}\cos(c_s\eta - \alpha_\nu) \end{array}\right\} \text{for } c_s\eta \gg 1 . \tag{3.17}$$

Hence, the density perturbations grow on sub-horizon scales if $\nu < 1$, i.e., $w > 1/3$ and they decay if $\nu > 1$, i.e., $w < 1/3$. Radiation, $w = 1/3$, provides the limiting case where the perturbation amplitudes remain constant. The interpretation of super-horizon perturbations is discussed in Section 4. There we will find that the physical amplitude of super-horizon perturbations is given by Ψ and Φ. Hence, super-horizon perturbations of ideal fluids neither grow nor decay, irrespective of the value of w (see also [28]).

Fortunately, this analysis is only valid for $w \neq 0$. If $w = 0$ (3.11) is elementarily integrable and we obtain

$$f = A\eta^2 + B\eta^{-3} ,$$

$$\Psi = -6A - 6B\eta^{-5} , \tag{3.18}$$

$$V = -2A\eta + 3B\eta^{-4} , \tag{3.19}$$

$$\Delta = A(\eta^2 + 24) + B\eta^{-3} , \tag{3.20}$$

Thus, on sub-horizon scales, $\eta \gg 1$, Δ grows according to the law

$$\Delta \propto \eta^2 \propto a . \tag{3.21}$$

Compared to the very limited growth for $K = \pm 1$ universes (see [32]), the growth (3.21) can become quite considerable depending on the redshift z_{eq} when the universe becomes matter dominated. (Concrete examples are given in the next section and in Chapter 4.)

3.2 The Sachs-Wolfe effect

We consider now an isotropic and homogenous radiation background in a dust universe. Let us assume that at some initial time, t_i , the perturbation amplitudes of the radiation are negligible compared to those of the matter component, the dust. We want to calculate the perturbations induced in the radiation by gravitational interaction with the matter perturbations. This was done first in a gauge-dependent way by Sachs and Wolfe [46].

For radiation, the conservation equations (2.130) and (2.131) reduce to

$$\dot{\Delta}_\gamma = -(4/3)kV_\gamma \;, \tag{3.22}$$

$$\dot{V}_\gamma = (1/4)k\Delta_\gamma + 2k\Psi \;. \tag{3.23}$$

Since the matter perturbations are much larger than the perturbations of the radiation, we neglect the contribution of the latter to Ψ. For the growing mode Ψ is thus given by the first term of (3.18), $\Psi = -6A =$ const. . Hence, equations (3.22) and (3.23) provide an inhomogenous linear system of differential equations with constant coefficients. This can be solved analytically by standard methods with the result

$$\begin{aligned} \Delta_\gamma(t) &= \Delta_\gamma(t_i)\cos(\omega(t-t_i)) - (4/\sqrt{3})V_\gamma(t_i)\sin(\omega(t-t_i)) \\ &\quad +8\Psi[\cos(\omega(t-t_i)) - 1] \;, \end{aligned} \tag{3.24}$$

$$\begin{aligned} V_\gamma(t) &= V_\gamma(t_i)\cos(\omega(t-t_i)) + (4/\sqrt{3})^{-1}\Delta_\gamma(t_i)\sin(\omega(t-t_i)) \\ &\quad +2\sqrt{3}\Psi\sin(\omega(t-t_i)) \;, \end{aligned} \tag{3.25}$$

with $\omega = k/\sqrt{3}$. If the initial perturbation amplitudes of the photons are negligible, (3.24) and (3.25) yield

$$\begin{aligned} \Delta_\gamma &\approx -8\Psi[1 - \cos(\frac{\eta - \eta_i}{\sqrt{3}})] \;, \\ V_\gamma &\approx 2\sqrt{3}\Psi\sin(\frac{\eta - \eta_i}{\sqrt{3}}) \;. \end{aligned}$$

On <u>super-horizon scales</u>, $\eta \ll 1$, $\eta_i \ll 1$, this leads to

$$\begin{aligned} \Delta_\gamma &= -\frac{4}{\sqrt{3}}\Psi(\eta - \eta_i)^2 \;, \\ V_\gamma &= 2\Psi(\eta - \eta_i) \;. \end{aligned}$$

(The gauge-invariant treatment of the Sachs-Wolfe effect on super-horizon scales for more general matter components is presented in [40].)

On sub-horizon scales, $(\eta - \eta_i) \gg 1$, Δ_γ oscillates with the constant amplitude 8Ψ.

Let us apply this last result to the cosmic microwave background radiation (MBR). On scales of supercluster size, $l_s \approx 30Mpc$, we know $\Delta_{\text{matter}} \approx 1$. Since this scale is far within the horizon already at recombination time t_R (see next section), we can set, applying (3.21),

$$-8\Psi = \frac{48\Delta(t_0)_{\text{matter}}}{(k_s t_0)^2} \approx |\Delta_\gamma| \;, \tag{3.26}$$

where k_s denotes the wavenumber whose wavelength corresponds to the size of a supercluster and t_0 is the present time. Thus,

$$k_s t_0 = 2\pi(\frac{l_h(t_0)}{l_s}) .$$

(l_h = horizon distance.)
With $l_s \approx 30Mpc$ and $l_h(t_0) \approx 6000h_{50}^{-1}Mpc$ (h_{50} denotes the Hubble constant in units of $50\frac{km}{s\cdot Mpc}$), we find on scales of $30Mpc$ for $\Delta_{\text{matter}}(l_s) \geq 1$

$$|\Delta_\gamma| \geq 10^{-5}h_{50}^2 . \quad (3.27)$$

A short calculation using standard methods which are presented, e.g., in [53] shows that the size of $30Mpc$ corresponds to an angular separation of $\delta \approx 4'$ on the microwave sky. (We set $z_R = 1500$.) Since $\rho_\gamma \propto T_\gamma^4$, we have $\frac{\Delta T_\gamma}{T_\gamma} = \frac{1}{4}\Delta_\gamma$. Hence, we find

$$\frac{\Delta T_\gamma(\delta = 4')}{T_\gamma} \geq 2.5 \cdot 10^{-6}h_{50}^2 . \quad (3.28)$$

The best observational limits on these angular scales [44,63] are of the order of 10^{-5}. Thus, the upper limit, $h_{50} = 2$, is only marginally consistent with (3.28).

We want to emphasize that for the result (3.28), we do not need any specific assumptions on the galaxy formation process. We just assume that the MBR freely propagated since recombination (no reheating). But it is surely difficult to find a physical process which would damp the inhomogeneities of the MBR down to such tiny values.

3.3 The impossibility of a purely baryonic scenario of galaxy formation

In this section we present simple arguments which show that the traditional, purely baryonic scenario is ruled out because of the isotropy of the MBR.

Let us assume that our universe consists only of the traditional matter components: The baryons and electrons which make up the galaxies, the stars and ourselves, the photons which today provide the MBR and massless neutrinos.

Since before recombination the baryons were, via Thompson scattering of the electrons, tightly coupled to the photons, it is natural to assume

$$\Delta_\gamma(t_R) \approx \Delta_B(t_R) ,$$

where t_R denotes the recombination time and Δ_B is the density perturbation of the baryons. Around recombination, the universe with the above matter content becomes matter dominated (see, e.g., [61]). Thus, according to (3.21),

$$\Delta_B(t_0) = \Delta_B(t_R)(a_0/a_R) = \Delta_B(t_R)z_R , \tag{3.29}$$

for perturbations which lie within the horizon already at t_R , i.e., $kt_R \gg 1$. Using the values $l_h(t_0) \approx 6000h_{50}^{-1}Mpc$, $z_R \approx 1500$, we can conclude $l_h(t_R) = l_h(t_0)z_R^{-3/2} \approx 0.1h_{50}^{-1}Mpc$. Hence, perturbations which are smaller than the horizon size at t_R correspond to wavelengths $\lambda < z_R l_h(t_R) \approx 150h_{50}^{-1}Mpc$ today. Especially, equation (3.29) is applicable to all stuctures up to superclusters.

Since we know that today the baryonic perturbations on these scales must be of the order of 1 or larger, (3.29) is incompatible with the observational limits on the inhomogeneity of the cosmic MBR: From observations (see [44,12,34,63]) we know

$$\Delta_\gamma(t_R) \le 10^{-5} , \quad \text{for} \;\; \lambda(t_0) \approx 30Mpc , \tag{3.30}$$

and

$$\Delta_\gamma(t_R) \le 10^{-4} , \quad \text{for} \;\; \lambda(t_0) \ll 30Mpc . \tag{3.31}$$

This leads, with (3.29) and $\Delta_\gamma(t_R) \approx \Delta_B(t_R)$, to the contradiction

$$\Delta_B(t_0) \le 2(\cdot 10^{-2} \; - \; \cdot 10^{-1})$$

on scales where we observe $\Delta_B(t_0) \ge 1$.

Hence, if we want to keep our beliefe in the origin of the MBR, and if we assume that gravity is the interaction that induced the formation of galaxies, some other matter component must be present and must have played a major role for the development of structure.

In the next chapter we will present a candidate for this missing matter component which would at the same time provide the non-baryonic contribution to the dark matter of the universe: a weakly interacting, collisionless fermion.

3.4 Perturbations of massless, collisionless particles on super-horizon scales

For massless particles we can set $q = v$ in Liouville's equation, (2.163), and the momentum integrals, (2.183) to (2.186). For the unperturbed equations, (2.173) and (2.174), we obtian

$$\rho_X = 4\pi T_X^4 \int f_X v^3 dv = \alpha T_X^4 , \quad (3.32)$$

where α is the Stefan-Boltzmann constant. (3.33)

$$p_X = \frac{4\pi}{3} T_X^4 \int v^3 f_X dv = \frac{1}{3}\rho . \quad (3.34)$$

Thus, $w = c_s^2 = 1/3$, as we of course know.

For massless particles it is convenient to define the momentum integrated perturbation amplitude

$$\mathcal{M} = \frac{4\pi}{\alpha} \int v^3 \mathcal{F}_X(\mu, v, t) dv . \quad (3.35)$$

Liouville's equation, (2.163), leads to the following equation of motion for $\mathcal{M}$:

$$\partial_t \mathcal{M} + ik\mu\mathcal{M} = 4ik\mu[\Phi - \Psi] . \quad (3.36)$$

From (3.35) and (2.183) to (2.186) one finds the fluid variables for massless particles:

$$\Delta_X = 1/2 \int_{-1}^{1} \mathcal{M} d\mu , \quad (3.37)$$

$$V_X = i3/8 \int_{-1}^{1} \mu \mathcal{M} d\mu , \quad (3.38)$$

$$\Pi_X = 3/4 \int_{-1}^{1} (1 - 3\mu^2) \mathcal{M} d\mu , \quad (3.39)$$

$$\Gamma_X = 0 . \quad (3.40)$$

Equation (3.40) implies that in a universe consisting of purely massless particles, the perturbations are always adiabatic, independent of the equation of motion of the massless particles. This is not very surprising, since in a purely massless scenario, $p = \frac{1}{3}\rho$ is fixed and there is no possibility for $\delta p \neq \frac{1}{3}\delta\rho$.

We want to emphasize that in the presence of a massive background contribution with $c_s^2 \neq 1/3$, we have

$$\begin{aligned}\Gamma &= \pi_L - \tfrac{c_s^2}{w}\delta = \tfrac{p_X}{p}[(\pi_L)_X - \tfrac{c_X^2}{w_X}\delta_X] + (c_X^2 - c_s^2)\tfrac{\rho_X}{p}\delta_X \\ &= \tfrac{p_X}{p}\Gamma_X + (c_X^2 - c_s^2)\tfrac{\rho_X}{p}\delta_X \\ &= \tfrac{p_X}{p}\Gamma_X + \Gamma_{\rm rel}\,.\end{aligned} \tag{3.41}$$

(Here $\pi_L = \frac{\delta p_X}{p}$, $(\pi_L)_X = \frac{\delta p_X}{p_X}$ and so on.)
For massless X-particles the first term vanishes due to (3.40), but, all the same, the second term can provide a nonvanishing entropy production rate. Γ_X and $\Gamma_{\rm rel}$ are both gauge-invariant. The general treatment of gauge-invariant cosmological perturbation theory for mixed systems is presented in [28].

Let us write down the field equations (2.123) and (2.126) for massless particles. With $\eta = kt$ we find

$$\Phi = \frac{3}{2\eta^2+12}(\Delta_X + \frac{4}{\eta}V_X)\,, \tag{3.42}$$

$$\Phi + \Psi = -\eta^{-2}\Pi\,. \tag{3.43}$$

Equations (3.36) to (3.39) and (3.42), (3.43) provide a closed system, which can easily be solved numerically. The results for massless neutrinos in the presence of photons are shown in Figures 6 to 8[1]. In synchronous gauge, this is worked out in [41].

Since we will deal with the interpretation of the numerical results in the next chapter, let us discuss here only the limiting case $\eta \ll 1$, i.e., perturbations with wavelengths much larger than the horizon size.

To lowest order in η, (3.42) reduces then to

$$\Phi = \frac{1}{4}\Delta_X + \eta^{-1}V_X\,, \tag{3.44}$$

[1] These Figures are listed at the end of Chapter 4.

so that

$$-\Psi = \frac{1}{4}\Delta_X + \frac{1}{\eta}V_X + \frac{1}{\eta^2}\Pi_X + \text{higher order terms.} \tag{3.45}$$

Since we are mainly interested in the growing mode, let us make the ansatz

$$\mathcal{M} = c_0 + ic_1\eta\mu + c_2(\eta\mu)^2 + O(\eta^3)\ . \tag{3.46}$$

We then obtain with the help of (3.37) to 3.39)

$$\Delta = c_0 + \frac{1}{3}c_2\eta^2 + O(\eta^4)\ , \tag{3.47}$$

$$V = -\frac{1}{4}c_1\eta + O(\eta^3)\ , \tag{3.48}$$

$$\Pi = -\frac{2}{5}c_2\eta^2 + O(\eta^4)\ . \tag{3.49}$$

Thus,

$$\Phi - \Psi = (\frac{1}{2}c_0 - \frac{1}{2}c_1 - \frac{2}{5}c_2) + O(\eta^2)\ . \tag{3.50}$$

We insert (3.50) in Liouville's equation , (3.36), and compare the first powers of μ with the result

$$c_0 = 3c_1 + \frac{8}{5}c_2\ . \tag{3.51}$$

The coefficient of μ^2 yields

$$c_2 = \frac{1}{2}c_1\ . \tag{3.52}$$

These equations can, of course, also be obtained from the conservation equations (2.130) and (2.131) which are a consequence of Liouville's equation (see Section 1.2.4). From (3.51) and (3.52) we find

$$\mathcal{M} = A[\frac{19}{5} + i\eta\mu + \frac{1}{2}(\eta\mu)^2 + O(\eta^3)]\ . \tag{3.53}$$

To lowest order in η this yields

$$\Delta = \frac{19}{5}A + O(\eta^2)\ , \tag{3.54}$$

$$V = -\frac{1}{4}A\eta + O(\eta^3)\ , \tag{3.55}$$

$$\Pi = -\frac{1}{5}A\eta^2 + O(\eta^4)\ , \tag{3.56}$$

$$\Phi = \frac{7}{10}A + O(\eta^2) , \tag{3.57}$$

$$\Psi = -\frac{1}{2}A + O(\eta^2) . \tag{3.58}$$

We want to show now that A really represents the amplitude of the perturbation, i.e., if $A \ll 1$, linear perturbation theory is valid, and if $A \geq 1$, linear perturbation theory breaks down.

If we can find any gauge in which all deviations from Friedman background are small, linear perturbation theory is of course a good approximation. (Large perturbation amplitudes in any other gauge are then only coordinate effects.) Let us thus define the amplitude of a perturbation as the maximum value of all perturbation variables in a gauge in which this maximum is made as small as possible:

$$\mathrm{Amp} = \min_{\{\mathrm{gauges}\}} (\max\{A, B, H_L, H_T, \delta, v, \pi_L, \Pi_T\}) .$$

On super-horizon scales, the field equations require that the matter perturbations and the perturbations in the geometry are of the same order of magnitude[2]. It is therefore sufficient if the geometrical perturbations are small. (Footnote 2 shows also that on sub-horizon scales geometrical perturbations are much smaller than matter perturbations. On these scales the difference between gauge-invariant and gauge-dependent amplitudes becomes thus very small, namely $O(kt)^{-2}$.)

Let us choose the following four gauge-dependent geometrical perturbation amplitudes:
$\mathcal{R}$, B, κ_g, σ_g. One easily verifies that A, B, H_L and H_T have to be small in order to keep all of them small. By a purely spatial gauge transformation we can attain $B = 0$ without changing $\mathcal{R}$, σ_g and κ_g (see equations (2.52) and (2.90)). From equations (2.93), (2.96) and (2.97) one obtains

$$\begin{aligned} \mathcal{R} &= \Phi + \eta^{-1}\sigma_g , \\ \kappa_g &= -\Psi + \eta\Phi' - \tfrac{2}{\eta}\sigma_g + \text{ smaller terms} . \end{aligned} \tag{3.59}$$

[2]By a simple counting of derivatives using Palatini's identity (see [52]), the perturbations of the field equations yield the following order of magnitude equation: $O(\frac{\delta T}{T})O(8\pi G t^2 T_{\mu\nu}) = O(\frac{\delta g}{g} + kt\frac{\delta g}{g} + (kt)^2\frac{\delta g}{g})$. With the help of Friedman's equation, we find $O(t^{-2}) = O(8\pi G T_{\mu\nu})$. The second factor of the l.h.s. is thus of order 1 and we obtain $\frac{\delta T}{T} \approx \frac{\delta g}{g}$ for $kt \ll 1$ (super-horizon scales), and $\frac{\delta T}{T} \approx (kt)^2\frac{\delta g}{g}$ for $kt \gg 1$ (sub-horizon scales).

If we perform a gauge transformation such that in addition to B also the largest of the variables $\mathcal{R}$, σ_g and κ_g vanishes, one finds that the amplitude of the perturbation is given by the maximum of Φ and Ψ. (It is not possible to find a gauge transformation such that simultanously $\mathcal{R}$, κ_g and σ_g are much smaller than $\max\{\Phi, \Psi\}$.)

This general argumentation shows that, on super-horizon scales, the amplitude of the perturbation is determined by Ψ and Φ. This is independent of the matter content of the Friedman universe as long as $\dot{a}/a$ is of the order t^{-1}, since then, (3.59) remains valid as an order of magnitude equation.

For our example of massless collisionless particles, the perturbation amplitude is therefore given by A.

This discussion shows also that the example given in Section 1 exhibits no growth of perturbations on super-horizon scales for arbitrary values of w.

3.5 An integral representation of Liouville's equation and its nonrelativistic limit

Here, we present an integral representation of the Liouville equation, (2.163),

$$\partial_t \mathcal{F} + i\mu k \mathcal{F} = ik\mu \frac{df}{dv}[q\Psi - \frac{v^2}{q}\Phi] \ , \tag{3.60}$$

and discuss its nonrelativistic limit. To this end, we introduce the quantity

$$\Delta(t_1, t_2, v) = \int_{t_1}^{t_2} (v/q) dt \ , \tag{3.61}$$

which is the comoving distance travelled by a particle of momentum v in the time intervall t_1 to t_2. Then, (3.60) has the following integral representation:

$$\begin{aligned} \mathcal{F}(t, v, \mu) = \ & \tfrac{df}{dv}[ik\mu \textstyle\int_{t_*}^{t} (q\Psi - (v^2/q)\Phi) \exp(-ik\mu\Delta(t', t, v)) dt' \\ & + \mathcal{F}(t_*, v, \mu) \exp(-ik\mu\Delta(t_*, t, v)] \ . \end{aligned} \tag{3.62}$$

$\mathcal{F}(t_*, v, \mu)$ is the initial value .

Let us proceed now to the nonrelativistic limit, for which

$$q \approx a >> v \ , \tag{3.63}$$

and therefore,

$$\Delta(t_*, t, v) \approx v \int_{t_*}^{t} a^{-1} =: vs \ . \tag{3.64}$$

We then obtain

$$\mathcal{F}(s, v, \mu) = \mathcal{F}(0, v, \mu)e^{-is\boldsymbol{k}\cdot\boldsymbol{v}} + i\boldsymbol{k}\cdot\boldsymbol{v}v^{-1}\frac{df}{dv}\int_0^s a^2\Psi e^{-i\boldsymbol{k}\cdot\boldsymbol{v}(s-s')}ds' \ . \tag{3.65}$$

In the nonrelativistic approximation ($p\Pi << \rho\Delta_g$, $kt >> 1$), (2.123) and (2.126) give

$$a^2\Psi = -\frac{8\pi G a^4 \rho}{2k^2}\Delta_g \ .$$

Recalling that ρa^3 is constant, this leads to

$$\begin{aligned}\mathcal{F}(s, v, \mu) = \ & \mathcal{F}(0, v, \mu)e^{-is\boldsymbol{k}\cdot\boldsymbol{v}} \\ & -i\boldsymbol{k}\cdot\tfrac{\partial f}{\partial \boldsymbol{v}} 4\pi G\rho(0)a(0)^3 \textstyle\int_0^s a\Delta_g k^{-2} e^{-i\boldsymbol{k}\cdot\boldsymbol{v}(s-s')}ds' \ .\end{aligned} \tag{3.66}$$

After multiplication by q and integration over d^3v, we obtain

$$\begin{aligned}\tfrac{\rho}{T_X^4}\Delta_g = \ & a(s) \textstyle\int \mathcal{F}(o, \mu, v)e^{-i\boldsymbol{k}\cdot\boldsymbol{v}s}d^3v \\ & +a(s)4\pi G\rho(0)a^3(0) \textstyle\int_0^s a(s')\Delta_g(s')\phi(\boldsymbol{k}(s-s'))(s-s')ds' \ ,\end{aligned} \tag{3.67}$$

with

$$\phi(\boldsymbol{x}) = \int f(v)e^{-i\boldsymbol{v}\cdot\boldsymbol{x}}d^3v \ . \tag{3.68}$$

For the left-hand side of (3.67), we used (2.183). On the right-hand side, we applied (3.63), and in the second term we integrated by parts. With the help of $T_X = m_X/a$, we finally bring (3.67) into the form:

$$\begin{aligned}\Delta_g = \ & \tfrac{m_X^4}{\rho(0)a^3(0)} \textstyle\int \mathcal{F}(o, \mu, v)e^{-i\boldsymbol{k}\cdot\boldsymbol{v}s}d^3v \\ & +4\pi(m_X^2/m_{pl})^2 \textstyle\int_0^s a(s')\Delta_g(s')\phi(\boldsymbol{k}(s-s'))(s-s')ds' \ .\end{aligned} \tag{3.69}$$

This integral equation coincides with Gilbert's equation (See [21], equations (32) and (33)).

Chapter 4

Numerical Results for Collisionless Matter Scenarios

In this chapter we present numerical solutions of the gauge-invariant cosmological perturbation equations for a universe with $K = 0$ whose matter consists of massive collisionless particles (dark matter), massless neutrinos and radiation. Within the gauge-invariant treatment we have no growth of the perturbation amplitude on super-horizon scales. The physical interpretation of the resulting spectra is compatible with the results of [5] and [43]: For hot dark matter the perturbation spectrum peaks at $M_\nu^{\max} \approx m_{pl}^3/m_\nu^2$ and then decays very steeply.

For cold dark matter the spectrum is relatively flat between $M_X^{\max} \approx m_{pl}^3/m_X^2$ and the horizon mass at equal energy density ($\rho_X = \rho_\gamma + \rho_{\nu_0}$), and then again decays steeply.

Of course, the best candidate for hot dark matter is a massive neutrino with $10eV \leq m_\nu \leq 100eV$ and, therefore, $M_\nu^{\max} \approx 10^{16} M_\odot$. Candidates for cold dark matter are hypothetical particles like photinos, gravitinos, higgsinos and so on, with masses around 1MeV. Thus, $M_X^{\max} \approx 10^6 M_\odot$.

4.1 The collisionless component

As explained in the introduction, we assume that a large fraction (at least 80%) of the mass of the universe is due to a non-baryonic, collisionless component. Massive neutrinos or any other weakly-interacting massive particles like photinos, gravitinos or axions could provide such a collisionless relic. (For additional information on the dark matter problem, see

[29,49,55] and references therein.)

We assume, as in the case of neutrinos, that there exists a very early epoch during which these so-called X-particles were in thermal equilibrium with the rest (radiation, leptons, quarks ...). We call g_* the 'effective' number of degrees of freedom in extremely relativistic particles which are in thermal equilibrium at the time $t_{\rm dec}$ when the X-particles decouple. Let us explain shortly what we mean by 'effective':
The entropy density of extremely relativistic particles of type i in thermal equilibrium (neglecting chemical potentials) is given by

$$s_i = \frac{1}{T}\int p f_i d^3p \,,$$

where

$$f_i = \begin{cases} \frac{N_i}{(2\pi)^3(\exp(p/T)+1)} & \text{for fermions,} \\ \frac{N_i}{(2\pi)^3(\exp(p/T)-1)} & \text{for bosons.} \end{cases}$$

Thus

$$s_i = \begin{cases} \frac{7}{8}\frac{6\zeta(4)T^3N_i}{2\pi^2} & \text{for fermions,} \\ \frac{6\zeta(4)T^3N_i}{2\pi^2} & \text{for bosons.} \end{cases}$$

ζ denotes the Riemannian zeta-function.

Hence, the total entropy in extremely relativistic particles is given by

$$s = (\sum_{i=\text{boson}} N_i + \frac{7}{8}\sum_{j=\text{fermion}} N_j)T^3 \cdot \text{const.} = g_* T^3 \cdot \text{const.} \;, \qquad (4.1)$$

where g_* = #bosonic degrees of freedom +(7/8)#fermionic degrees of freedom. (By 'effective' we therefore mean that fermionic degrees of freedom count only 7/8.)

Massive neutrinos decouple at a temperature of about 1.5 MeV where only γ's and $e^\pm$ are still relativistic. Hence, in this case we have $g_* = 2 + 4\cdot 7/8 = 11/2$.

If we assume that the annihilation process of these relativistic particles is adiabatic, we find from the annihilation of $e^\pm$, by balancing the entropy (4.1) before and after the process (see, e.g., [61]),

$$\frac{11}{2}T_\nu^3 = 2T_\gamma^3 \;,\; T_\gamma/T_\nu \equiv \alpha_\nu = (11/4)^{1/3} \,. \qquad (4.2)$$

For an X-particle decoupling earlier than the neutrino, we have to take (4.2) into account. We therefore obtain:

$$g_* T_X^3 = (2 + 6 \cdot 7/8(11/4)^{-1}) T_\gamma^3 \ , \ (T_\gamma/T_X) \equiv \alpha_X \approx (g_*/3.9)^{1/3} \ . \quad (4.3)$$

Let us present some realistic values for g_*: For $\gamma, e^\pm, 3\nu\bar{\nu}$ only in thermal equilibrium ($2MeV \leq T \leq 100MeV$) we find $g_* = \frac{43}{4}$. Above the quark hadron phase transition ($T \geq 300MeV$), $\gamma, e^\pm, 3\nu\bar{\nu}, \mu^\pm, u\bar{u}, d\bar{d}, s\bar{s}$ and 8 gluons are in thermal equilibrium and thus $g_* \approx 60$. In minimal $SU(5)$ GUT models ($T \geq 10^{15}GeV$) one obtains $g_* \approx 150$, namely, 30 fermionic degrees of freedom per family, $2 \cdot 24$ gauge bosonic degrees of freedom and about 24 Higgs degees of freedom.

If g_* is large, the photons experience an intensive reheating after X-decoupling and, hence, dominate the energy density of the universe long after the X's have become nonrelativistic. Therefore, the growth of X-perturbations is retarded by a Mészáros effect [37] until t_{eq}, where

$$\rho_X(t_{eq}) = \rho_\gamma(t_{eq}) + \rho_{\nu_0}(t_{eq}) \ .$$

We assume further that the X-particles decouple when they are still extremely relativistic. Since after decoupling the distribution function changes only by the redshift of the physical momenta due to the expansion of the universe, it is then given by

$$f_X = \frac{1}{\exp(p/T_X) + 1} \ , \quad (4.4)$$

where p denotes the physical momentum $p = \sqrt{g(\boldsymbol{p}, \boldsymbol{p})}$, and T_X is defined by
$T_X(t) = T_X(t_{\rm dec}) a(t_{\rm dec})/a(t)$. With this definition, the relations (4.2) and (4.3) remain true for all $t \geq t_{\rm dec}$. (Note that after decoupling, T_X is no longer a temperature in a thermodynamical sense, but denotes merely a parameter of the distribution function.)

We do not treat the baryons separately, since their qualitative behaviour in the linear regime is well known: Before recombination, practically all baryonic perturbations are erased due to Silk damping [42,48]. After recombination, the baryons fall very soon into the potential of the collisionless component and then closely follow the X-perturbations. Of course, the treatment of the baryons in the nonlinear regime, when one has to take into account hydrodynamics, is very difficult and different from that of collisionless particles. But this lies beyond the scope of this work (see [19] and [20]).

4.2 The perturbation equations

4.2.1 The background

In the scenario which we want to treat in this chapter, the matter consists of a collisionless component (massive neutrinos, gravitinos, photinos, ...) which we denote by X-particles if not specified, massless neutrinos (ν_0) and radiation (γ).

The scale factor evolves according to Friedman's equation for $K = 0$,

$$(\dot{a}/a)^2 = \frac{8\pi G}{3}(\rho_X + \rho_{\nu_0} + \rho_\gamma)a^2 \,. \tag{4.5}$$

We choose the normalization of a such that $a = (m_X/T_X)$, where m_X denotes the mass of the X-particles and T_X their temperature. The unperturbed energy densities and pressures are then given by:

$$\rho_\gamma = (\pi^2/15)T_\gamma^4 \,, \tag{4.6}$$

$$\rho_{\nu_0} = T_{\nu_0}^4 4\pi \int f_{\nu_0} v^3 dv = (7/16)N_{\nu_0}\alpha_\nu^{-4}\rho_\gamma \,, \tag{4.7}$$

$$\rho_X = T_X^4 4\pi \int f_X v^2 (v^2 + a^2)^{1/2} dv \,, \tag{4.8}$$

with

$$f_{(\cdot)} = N_{(\cdot)}(2\pi)^{-3}\frac{1}{e^v + 1} \,. \tag{4.9}$$

Here v denotes the dimensionless, time independent velocity:

$$v = ap/m_X = p/T_X \quad \text{, respectively} \quad v = p/T_\nu \tag{4.10}$$

for the physical momentum $p = \sqrt{g(\boldsymbol{p}, \boldsymbol{p})}$. Further,

$$\rho = \rho_\gamma + \rho_X + \rho_{\nu_0} \,, \tag{4.11}$$

$$p_\gamma = (1/3)\rho_\gamma \,,\; p_{\nu_0} = (1/3)\rho_{\nu_0} \,, \tag{4.12}$$

$$p_X = T_X^4 \frac{4\pi}{3} \int \frac{v^4}{(v^2 + a^2)^{1/2}} f_X dv \,, \tag{4.13}$$

$$p = p_\gamma + p_X + p_{\nu_0} \,. \tag{4.14}$$

4.2.2 Scalar perturbations

The perturbed field equations are given by (2.123) and (2.126) for $K = 0$:

$$\Phi = \frac{4\pi G a^2 \rho}{k^2 + 12\pi G(1+w)a^2\rho}(\Delta + 3(1+w)(\dot{a}/a)k^{-1}V) , \qquad (4.15)$$

$$\Phi + \Psi = -8\pi G a^2 k^{-2} p\Pi , \qquad (4.16)$$

where

$$\Delta = \rho^{-1}(\rho_X \Delta_X + \rho_\gamma \Delta_\gamma + \rho_{\nu_0}\Delta_{\nu_0}) , \qquad (4.17)$$
$$V = (\rho + p)^{-1}[(\rho_X + p_X)V_X + (\rho_\gamma + p_\gamma)V_\gamma + (\rho_{\nu_0} + p_{\nu_0})V_{\nu_0}], \qquad (4.18)$$
$$\Pi = p^{-1}(p_X \Pi_X + p_\gamma \Pi_\gamma + p_{\nu_0}\Pi_{\nu_0}) . \qquad (4.19)$$

For the photons we set

$$\Gamma_\gamma = 0 , \qquad (4.20)$$

which is always true for massless particles as we discussed in section 3.4 .

Furthermore, we make the assumption that

$$\Pi_\gamma = 0 , \qquad (4.21)$$

which is justified, for example, in [28]. Also from our numerical investigations, we conclude that the anisotropic stress perturbation Π is unimportant for the growth and decay of density perturbations in all treated cases (see for example Figure 11)[1].

With the help of these matter equations, the time development of Δ_γ and V_γ are determined by the energy-momentum 'conservation' equation, $\nabla \cdot \widetilde{T}_\gamma = 0$, which leads to (see Section 3.4 , equations (3.22) and (3.23))

$$\dot{\Delta}_\gamma = -(4/3)kV_\gamma , \qquad (4.22)$$

$$\dot{V}_\gamma = (k/4)\Delta_\gamma + k(\Psi - \Phi) . \qquad (4.23)$$

Since the X-particles are collisonless, we have to describe them by their distribution function $\tilde{f}$. The corresponding gauge-invariant perturbation variable, $\mathcal{F}_X$, is defined in Chapter 2, equation (2.28). Liouville's equation

[1] All Figures corresponding to this chapter are shown at the end of the chapter.

provides the time evolution of $\tilde{f}$ and leads to the following perturbation equation for $\mathcal{F}_X$ (2.163):

$$\partial_t \mathcal{F}_X + ik\mu(q/v)\mathcal{F}_X = ik\mu \frac{df}{dv}[q\Psi - (v^2/q)\Phi] , \tag{4.24}$$

with

$$q = (v^2 + a^2)^{1/2} = E/T_X \quad \text{and} \quad \mu = \frac{\sum_{i=1}^3 v^i k^i}{vk} . \tag{4.25}$$

The calculations of the fluid variables from $\mathcal{F}_X$ are given in equations (2.183) to (2.186). For the sake of completeness we repeat them here:

$$\begin{aligned}
\Delta_X &= \frac{T_X^4}{\rho_X} \int_{\mathrm{R}^3} v^2 q \mathcal{F}_X dv d\Omega , & (4.26)\\
V_X &= \frac{iT_X^4}{(\rho_X + p_X)} \int v^3 \mu \mathcal{F}_X dv d\Omega , & (4.27)\\
\Pi_X &= \frac{T_X^4}{2p_X} \int \frac{v^4}{q}(1 - 3\mu^2)\mathcal{F}_X dv d\Omega , & (4.28)\\
\Gamma_X &= \frac{T_X^4}{p_X} \int (\frac{v^4}{3q} - c_X^2 v^2 q)\mathcal{F}_X dv d\Omega . & (4.29)
\end{aligned}$$

The perturbations of the massless neutrinos can be described by the perturbation of the 'brightness function', $\mathcal{M}$, which is defined in Section 3.4 . Liouville's equation leads to the following equation of motion for $\mathcal{M}$ (see equation (3.36), Section 3.4):

$$\partial_t \mathcal{M} + ik\mu\mathcal{M} = 4ik\mu[\Phi - \Psi] . \tag{4.30}$$

The fluid variables for the massless neutrinos are given by (3.37) to (3.40):

$$\begin{aligned}
\Delta_{\nu_0} &= 1/2 \int_{-1}^{1} \mathcal{M} d\mu , & (4.31)\\
V_{\nu_0} &= i3/8 \int_{-1}^{1} \mu \mathcal{M} d\mu , & (4.32)\\
\Pi_{\nu_0} &= 3/4 \int_{-1}^{1} (1 - 3\mu^2)\mathcal{M} d\mu , & (4.33)\\
\Gamma_{\nu_0} &= 0 . & (4.34)
\end{aligned}$$

The gravitational coupling of different matter components through Ψ and Φ is determined by the equations (4.15), (4.16) and (4.17) to (4.19).

4.2.3 The numerical treatment of the perturbation equations

In this subsection we describe briefly how we solve the system (4.15) to (4.33) numerically.

First we have to specify initial values for the perturbations $\mathcal{F}_X$, $\mathcal{M}$, Δ_γ , V_γ , Φ and Ψ, such that the algebraic relations (4.15) and(4.16) are fulfilled. To do this, we choose an initial time t_* where the X-particles are still extremely relativistic, (i.e., $a \ll 1$, $\langle v\rangle \approx \langle q\rangle$,) and where the wavelength of the perturbation is much larger than the horizon scale (i.e., $kt_* \ll 1$). At this initial time we do not have to distinguish between X-particles and massless neutrinos. We can describe them both by a total distribution function

$$\tilde{f} = \tilde{f}_X + (\alpha_X/\alpha_{\nu_0})^4 \tilde{f}_{\nu_0} \,. \tag{4.35}$$

The corresponding gauge-invariant perturbation variable is

$$\mathcal{F} = \mathcal{F}_X + (\alpha_X/\alpha_{\nu_0})^4 \mathcal{F}_{\nu_0} \,. \tag{4.36}$$

Since we want to treat extremely relativistic particles, we can work with the energy integrated variable $\mathcal{M}$, which is defined in the preceding subsection. (4.30) provides the perturbation equation for $\mathcal{M}$. Let us now assume, following [41], that initially the relative perturbation amplitudes in each matter component are approximately equal:

$$\Delta = \Delta_X = \Delta_{\nu_0} = \Delta_\gamma \,, \tag{4.37}$$

$$V = V_X = V_{\nu_0} = V_\gamma \quad \text{and} \tag{4.38}$$

$$\Pi_X = \Pi_{\nu_0} \;\; , \;\; \Pi_\gamma = 0 \;\; , \;\; \Pi = \frac{Y}{Y+1}\Pi_X \,, \tag{4.39}$$

with

$$Y = (\frac{\rho_X + \rho_{\nu_0}}{\rho_\gamma})_{e.r.} = 7/16[N_X \alpha_X^{-4} + N_{\nu_0}\alpha_{\nu_0}^{-4}] \,. \tag{4.40}$$

The suffix e.r. in (4.40) indicates that this quotient is taken in the extremely relativistic regime.

Proceeding as in Section 3.4, we obtain now

$$\mathcal{M} = A[\beta + ikt\mu + 1/2(kt\mu)^2 + O(kt\mu)^3] \,, \tag{4.41}$$

with

$$\beta = \frac{15 + 19Y}{5 + 5Y} \qquad (3 \leq \beta < 4) . \tag{4.42}$$

A is the unknown primordial perturbation amplitude. In the absence of photons, $Y = \infty$ and (4.41) reproduces (3.53). To lowest order in kt, we find the following initial values:

$$\Delta = \Delta_X = \Delta_{\nu_0} = \Delta_\gamma = A\beta , \tag{4.43}$$

$$V = V_X = V_{\nu_0} = V_\gamma = -\frac{A}{4}kt , \tag{4.44}$$

$$\Pi_X = \Pi_{\nu_0} = -\frac{1}{5}A(kt)^2 , \tag{4.45}$$

$$\Pi = \frac{Y}{Y+1}\Pi_X = -\frac{\beta - 3}{4}A(kt)^2 , \tag{4.46}$$

$$\Phi = \frac{\beta - 1}{4}A \quad \text{and} \tag{4.47}$$

$$\Psi = -0.5A . \tag{4.48}$$

From the results derived in Section 3.4, we know that A really represents the amplitude of the perturbation.

Following the arguments of [13] (the primordial perturbation consists of a differing amount of expansion in different points leading to differences in the temperature and thus to a perturbation of the type $\frac{\partial f}{\partial T}$), we choose the momentum dependence of $\mathcal{F}_X(t_*)$ as $-v\frac{df}{dv}$. Hence, we start with the following set of initial values at t_* :

$$\mathcal{F}_X(t_*, v, \mu) = -v\frac{df}{dv}\frac{A}{4}[\beta + ikt_*\mu + 1/2(kt_*\mu)^2] , \tag{4.49}$$

$$\mathcal{M}_{\nu_0}(t_*, \mu) = A[\beta + ikt_*\mu + 1/2(kt_*\mu)^2] . \tag{4.50}$$

$\Delta_\gamma(t_*)$, $V_\gamma(t_*)$, $\Phi(t_*)$ and $\Psi(t_*)$ are given by (4.43), (4.44) for $t = t_*$, (4.47) and (4.48).

The numerical problem consists now in solving the coupled system of first order ordinary differential equations (4.22), (4.23), (4.24) and (4.30) with source terms determined by (4.15) to (4.19), (4.26) to (4.28) and (4.31) to (4.33). The initial values are given by (4.43), (4.44) and (4.47) to (4.50).

The integrations over the mass shell $P_m(x)$ to obtain the fluid variables from the distribution function are performed using a 12- to 24-point

Gauss-Laguerre method for the integration over v, and a 6- to 12-point Gauss-Legendre method for the one over μ. Calculating $\mathcal{F}$ and $\mathcal{M}$ at all the required base points, leads to a coupled linear system of 158 to 602 real first order ordinary differential equations, which we solve numerically by application of a variable order (up to 12) Adams predictor-corrector method. We performed our calculations on an IBM machine, which was then equipped with a 3083 processor. The computer programs are written in FORTRAN77. For the Adams predictor-corrector method we applied the IMSL-routine DGEAR.

This procedure is for us much less computer-time consuming than the method of [5], where Bond and Szalay deal with the angle-integrated variables to avoid the harmless μ-integration, accepting the cost of an additional time-integration. The coupling of all different angular moments, which is present in the Liouville equation, is thereby transformed into a coupling of all different times. (In [5] an integral representation of Liouville's equation similar to the one given in Section 3.5 is used.)

The computer time, t_c, needed for the integration of the differential equation depends strongly on the wavenumber, k. For very small wavenumbers, $k \leq 10^{-2}\frac{m_X^2}{m_{pl}}$, t_c was a few seconds. For large wavenumbers, $k \geq 10\frac{m_X^2}{m_{pl}}$, t_c amounted to 2 minutes and more. Also the accuracy of the results decreases with increasing wavenumber: The resulting spectra (Figs. 4 and 5) for different numbers of interpolation points (and also for slightly different integration methods) coincide for $k \leq 10m_X^2/m_{pl}$ within 2% and for $k \leq 4m_X^2/m_{pl}$ even within 0.1%. For $28m_X^2/m_{pl} \geq k \geq 10m_X^2/m_{pl}$, the coincidence is only within about 10%.

This decay of accuracy is plausible: Perturbations belonging to high k-values enter the horizon while the X-particles are still relativistic. These perturbations, therefore, perform several damped oscillations before they start growing. An accurate numerical mapping of the oscillations is, of course, difficult. In Section 4 we derive an analytic approximation formula for the behaviour of the spectrum in this regime.

The scale factor, a, is obtained by analytic approximation formulas in the extremely relativistic ($a \ll 1$) and nonrelativistic ($a \gg 1$) regimes, so that the relative error never exceeds 0.2% . For cold dark matter ($g_* \geq 30$), these approximation formulas can be connected to an approximation which lies within 0.5% of the true value over the whole range $a \ll 1$ to

$a \gg 1$. For massive neutrinos, a is obtained by numerical integration of Friedman's equation in the semi-relativistic regime.

4.3 Discussion of the numerical results

4.3.1 Characteristic scales

Before trying to understand the resulting spectra (Figures 4 and 5), we introduce three characteristic scales of the problem.

The Jeans scale

Let us forget for a moment the time dependence of the background quantities, i.e., we set $a =$ const. and $\dot{a} = 0$, and think for simplicity of a universe where only the X-component of the matter is relevant. Then, the algebraic equations for Φ and Ψ, (4.15) and (4.16) reduce to

$$\Phi = \frac{4\pi G a^2 \rho_X}{k^2 + 12\pi G(1+w_X)a^2\rho_X}\Delta \quad \text{and} \tag{4.51}$$

$$\Psi = -8\pi G a^2 k^{-2} p_X \Pi - \Phi \, . \tag{4.52}$$

Using (4.26) and (4.28) for Δ and Π, we find

$$\frac{df}{dv}[q\Psi - \frac{v^2}{q}\Phi] = -G8\pi^2 a^2 T_X^4 \tfrac{df}{dv}[\ \frac{q + v^2/q}{k^2 + 12G\pi a^2 \rho_X(1+w_X)}\int v^2 q\mathcal{F} dv d\mu$$
$$+\frac{q}{k^2}\int \frac{v^4}{q}(1-3\mu^2)\mathcal{F} dv d\mu] \, . \tag{4.53}$$

As in Section 1.1, Liouville's equation (4.24) can now be turned into a dispersion relation. To do this, we make a plane wave ansatz for $\mathcal{F}$:

$$\mathcal{F} = e^{i(kx-\omega t)}\mathrm{F} \, , \tag{4.54}$$

and enter with it into (4.24). With the help of (4.53), this leads to

$$\mathrm{F} = \ -\tfrac{8G\pi^2 a^2 T_X^4}{v/q - \omega/k\mu}\tfrac{df}{dv}[\ \frac{q + v^2/q}{k^2 + 12G\pi a^2 \rho_X(1+w_X)}\int v^2 q\mathrm{F} dv d\mu$$
$$+\frac{q}{k^2}\int \frac{v^4}{q}(1-3\mu^2)\mathrm{F} dv d\mu] \, . \tag{4.55}$$

After multiplication of this equation with $v^2 q$ and integration over $dv d\mu$, we obtain the dispersion relation

$$1=-8G\pi^2a^2T_X^4\int[\frac{q^2+v^2}{k^2+12G\pi a^2\rho_X(1+w_X)}+2\frac{q^2w_X}{k^2}\frac{\Pi}{\Delta}]\frac{v^2qk\mu}{vk\mu-q\omega}\frac{df}{dv}dvd\mu\ . \tag{4.56}$$

In general, (4.56) leads to different solutions $\omega(k)$ for different values of Π/Δ, i.e., for different F's. But we assume now that the anisotropy of the perturbation is small, i.e., $p_X\Pi \ll \rho_X\Delta$. (This is fulfilled in all calculations done in this context, as one sees, for example, in Figure 11.) Thus, we can neglect the second term on the right hand side of (4.56) and obtain the F-independent integral equation

$$1=\frac{8G\pi^2a^2T_X^4}{k^2+12G\pi a^2\rho_X(1+w_X)}\int(q^2+v^2)\frac{v^2qk\mu}{vk\mu-q\omega}\frac{df}{dv}dvd\mu\ . \tag{4.57}$$

This is an implicit equation for $\omega(k)$. The instability of $\mathcal{F}$ sets in when the imaginary part of ω changes its sign. As in Section 1.1, this happens at the wavenumber k_J with $\omega(k_J)=0$. Setting $\omega=0$ in (4.57) and solving for k leads to

$$k_J^2=4\pi Ga^2(3\rho+I)\ , \tag{4.58}$$

with

$$I=4\pi T_X^4\int q^3fdv\ . \tag{4.59}$$

From (4.58) we calculate the Jeans mass, the mass inside a ball of diameter $\lambda_J=a\frac{2\pi}{k_J}$:

$$M_J(t)=\frac{4\pi}{3}\rho(\frac{\pi}{4G(3\rho+I)})^{3/2}\ . \tag{4.60}$$

In the extremely relativistic case, where $a\ll 1$, $v\approx q$, $I\approx\rho$, we find

$$M_J^{e.r.}=\frac{\pi^{7/2}}{42(21\zeta(4))^{1/2}}a^2\frac{m_{pl}^3}{m_X^2}\approx 0.5a^2\frac{m_{pl}^3}{m_X^2}\ . \tag{4.61}$$

In the nonrelativistic case, where $a\gg 1$, $v\ll q$, $I\gg\rho$, we can neglect ρ in k_J and obtain

$$M_J^{n.r.}=\frac{\zeta(3)\pi^{7/2}}{4(\ln 2)^{3/2}}a^{-3/2}(m_{pl}^3/m_X^2)\approx 30a^{-3/2}(m_{pl}^3/m_X^2)\ , \tag{4.62}$$

which coincides with the Newtonian result (1.15). The curves $M_J(a)$ for massive neutrinos without and with radiation and massless ν-modes, and for X-particles with $g_*=100$ are shown in Figures 1, 2 and 3.

Only perturbations with wavelengths larger than the Jeans length can become unstable. Those with $\lambda < \lambda_J(t)$ oscillate and decay by directional and velocity dispersion. Of course, our discussion is valid only provided that the typical time scale of the problem is smaller than the expansion time. This means $\lambda \ll l_h$, where l_h denotes the horizon distance, or, equivalently, $kt \gg 1$. In the extremely relativistic case, $\lambda_J \approx l_h$. Hence, in this situation no perturbation can grow at all.

Another important scale of the problem is, therefore,

the horizon scale

$$l_h = at \approx a(\dot{a}/a)^{-1} = (\frac{3}{8\pi G\rho})^{1/2} . \tag{4.63}$$

The corresponding mass is given by

$$M_h = \frac{4\pi}{3}\rho l_h^3 = 1/4(\frac{3}{2\pi})^{1/2}\rho^{-1/2}m_{pl}^3 . \tag{4.64}$$

Also the curve $M_h(a)$ is drawn in Figures 1, 2 and 3. The meaning of this scale is clear: Perturbations with wavelength $\lambda > l_h$ are not causally connected, so they can neither grow nor decay by physical processes.

The third scale which is important for the treatment of collisionless matter is

the free-streaming scale.

This is the length that a particle with some mean velocity has traveled up to time t.

$$l_{fs} = a\int_0^t \langle c\rangle_{t'} dt' , \tag{4.65}$$

with

$$\langle c\rangle_t = \frac{\int v^3/q f dv}{\int v^2 f dv} . \tag{4.66}$$

The free-streaming mass is defined like the horizon mass above:

$$M_{fs} = \frac{4\pi}{3}\rho l_{fs}^3 . \tag{4.67}$$

In the extremely relativistic limit, we have $\langle c \rangle = 1$ and, therefore, $M_{fs}^{e.r.} = M_h$. As soon as the particles become nonrelativistic, $\langle c \rangle$ decreases rapidly and the free-streaming mass converges to a constant.

$$\lim_{t \to \infty} M_{fs} = M_{\infty} ,$$

with

$$M_{\infty} = \frac{4\pi}{3}\rho_{n.r.}[l_{fs}(t_{n.r.}) + a_{n.r.}^{-1/2}\frac{7\zeta(4)}{6\zeta(3)}(\frac{2\pi}{N_X\zeta(3)})^{1/2}\frac{m_{pl}}{m_X^2}]^3 ,$$

where $t_{n.r.}$ denotes a nonrelativistic time, when the X-particles are cold and dominate the energy density of the universe, so that a scales as t^2 for $t > t_{n.r.}$.

It is clear, that on scales larger than l_{fs}, nothing significant can happen. Perturbations neither decay nor can they grow.

4.3.2 The perturbation spectrum

With the help of the scales introduced in the preceding subsection, we arrive at the following qualitative picture:

- On scales smaller than l_{fs} and smaller than l_J, the perturbations decay (diagonally lined region in Figs. 1 to 3).
- On scales smaller than l_{fs} but larger than l_J, the perturbations grow.
- On scales larger than l_{fs}, the perturbation amplitude remains constant.

The first scale to collapse is, therefore, given by $l_{fs} = l_J$. The critical mass M_c, which is defined by $M_{fs} = M_J$, is (for massive ν's and for cold dark matter) of the order m_{pl}^3/m_X^2. The corresponding wavenumber lies always in the range $2m_X^2/m_{pl} < k_c < 3m_X^2/m_{pl}$. This is also the region where the output spectrum peaks (see Figures 4 and 5). Let us define $k_{\max}$ as the wavenumber where the output spectrum is maximal. Then $|k_{\max} - k_c|/k_{\max} \leq 0.3$ in all cases.

The difference between the relative positions of the M_J-curve in the Figures 1, 2 and 3 is due to the Mészáros effect [37], which is important for large values of g_*, and lifts the M_J-curve. Hence, the typical behaviour of a solution with wavelength λ can be described as follows:

- As long as $\lambda > l_{fs}$, the perturbation amplitude remains constant. (Recall that on super-horizon scales it is given by Ψ and Φ.)
- As soon as $\lambda < l_{fs}$, we have to distinguish between two cases:
  - If $\lambda < \lambda_J$, the perturbation decays quickly due to directional dispersion as long as the particles are extremely relativistic, and slowly due to velocity dispersion as soon as they become nonrelativistic (see Figure 6 for the first case).
  - If $\lambda_J < \lambda$, the perturbation grows due to the Jeans instability (see Figure 9).

Let us finally discuss the difference in the spectra for hot (massive ν's) and cold dark matter. (Since we have no good candidate for the latter, we still call it X-particles.) The ν-spectrum (Fig. 4) is narrower than the X-spectrum (Fig. 5). This is due to the fact that those ν-perturbations which cross the horizon when the ν's become nonrelativistic can more or less immediately start growing, while the growth of the corresponding X-perturbations is retarded by the dominant, extremely relativistic background until t_{eq}, where

$$\rho_X(t_{eq}) = \rho_\gamma(t_{eq}) + \rho_{\nu_0}(t_{eq}) \; .$$

This Mészáros effect is also the reason why we have $M_J^{\max} \approx M_c$ for massive ν's, whereas for cold dark matter M_c is substantially less than $M_J^{\max}$ (compare Figures 2 and 3). The qualitative interpretation of the spectra is in agreement with [5]. Their behaviour is even more pronounced:
The density perturbation spectrum of hot dark matter is very steep and peaks at $k_{\max} \approx 2.4 m_\nu^2/m_{pl}$. Hence,

$$M_{\max} = 3.2\pi^2 k_{\max}^{-3} m_\nu^4 \approx 2.2 \frac{m_{pl}^3}{m_\nu^2} \approx 4 \cdot 10^{15} (\frac{m_\nu}{30eV})^{-2} \cdot M_\odot \; ,$$

so that superclusters would be the first nonlinear structures of the universe.
The spectrum of cold dark matter is rather flat between $0.5k_{\max}$ and $2k_{\max}$, i.e., $0.1M_{\max}$ and $10M_{\max}$, and peaks at $k_{\max} = 2.1 m_X^2/m_{pl}$. Thus,

$$M_{\max} = 3.2\pi^2 k_{\max}^{-3} m_X^4 \approx 3.2 \frac{m_{pl}^3}{m_X^2} \approx 5.4 \cdot 10^6 (\frac{m_X}{1MeV})^{-2} \cdot M_\odot \; .$$

Therefore, in this model small objects like globular clusters would form first.

The input spectra are of the Zel'dovich type: The perturbation amplitudes for different wavenumbers coincide at the horizon, i.e., $\Delta(t_k) \approx \Delta(t < t_k) = A\beta$ for all k's, where t_k denotes the time when the wave with wavenumber k enters the horizon, i.e., $t_k = \pi/k$. Of course, the output of any other primordial spectrum can be obtained by multiplication with the spectra shown in Figures 4 and 5.

4.4 The small wavelength behaviour of the transfer function

Let us assume now that $k \gg k_{\max}$ ($k_{\max}$ denotes the wavenumber at the maximum of the spectrum in Figure 4). Since the corresponding small wavelength enters the horizon when the X-particles are still relativistic, the perturbation undergoes several damped oscillations before it starts growing. Hence, the numerical integration is difficult, as mentioned in Section 2. Therefore, we try to find an analytic approximation for the transfer function, $\frac{\Delta_X(t)}{\Delta_X(t_*)}$ (t_* denotes an early time when the perturbation is still larger than the horizon scale.).[2]

When a perturbation enters the horizon, at $t = \pi/k := t_k$, its amplitude is still approximately the primeval amplitude (see Figure 6),

$$\frac{\Delta_X(t)}{\Delta_X(t_*)} \approx 1 \, .$$

Inside the horizon ($t \geq t_k$), the perturbation suffers collisionless damping (see Figure 6), which we now estimate. Since in this regime Φ and Ψ are very small (see Fig. 8), we can neglect them for a qualitative discussion. Hence, Liouville's equation reduces to

$$\partial_t \mathcal{F} + ik\mu(v/q)\mathcal{F} = 0 \, , \qquad (4.68)$$

with the solution

$$\mathcal{F}(t) = \mathcal{F}(t_k)\exp(-ik\mu \int_{t_k}^{t}(v/q)dt') \, . \qquad (4.69)$$

[2] Such an approximation formula is also presented in [13]. Since we can not quite follow the arguments given there, we re-derive it in this place. The disagreement of the results is only due to the coordinate dependence of their formalism.

Using equation (4.26), we find

$$\Delta_X = \frac{2\pi T_X^4}{\rho_X} \int q v^2 \mathcal{F}(t_k, v, \mu) \exp(-ik\mu \int_{t_k}^t (v/q) dt') dv d\mu \,. \qquad (4.70)$$

If we assume that at time t_k $\mathcal{F}$ is still approximately given by its primordial value,

$$\mathcal{F}(t_k, v, \mu) \approx -\frac{A}{4} v \frac{df}{dv} \,,$$

we obtain

$$\Delta_X \approx -\frac{\pi T_X^4}{2\rho_X} A \int q v^3 \frac{df}{dv} \exp(-ik\mu \int_{t_k}^t \frac{v}{q} dt') dv d\mu \,. \qquad (4.71)$$

In order to find the transfer function, we have to evaluate this equation in the far nonrelativistic regime. There we find

$$\begin{aligned} \frac{\Delta_X(t)}{\Delta_X(t_*)} &\approx C \int v^3 \frac{d}{dv}(\frac{1}{e^v+1}) \exp(-ik\mu \int_{t_k}^t \frac{v}{q} dt') dv d\mu \\ &= 2C \int_0^\infty v^2 \frac{d}{dv}(\frac{1}{e^v+1}) \frac{\sin(kvJ)}{kJ} dv \,, \qquad (4.72) \end{aligned}$$

with

$$J = \int_{t_k}^t q^{-1} dt = \int_{a_k}^a (v^2 + a^2)^{-1/2} \dot{a}^{-1} da \,,$$

where we have used the explicit form of the background distribution function f in 4.72. Let us estimate the integral J. As long as the energy density is dominated by relativistic matter, we have $\dot{a} =: \alpha = \text{const.}$. Thus, we obtain

$$J(t_k, t_{eq}) = \alpha^{-1} \ln(\frac{a_{eq} + \sqrt{v^2 + a_{eq}^2}}{a_k + \sqrt{v^2 + a_k^2}}) \,.$$

Since $v^2 \frac{df}{dv}$ reaches its maximum in the range $2 \leq v_{\max} \leq 4$, we can neglect a_k which is much smaller than 1 in (4.72). At the time t_{eq} when the X-particles begin to dominate the energy density, we have $a > 20$, so that we can approximate $J(t_{eq})$ under the integral (4.72) by

$$J(t_k, t_{eq}) \approx \alpha^{-1} \ln(\frac{2a_{eq}}{v}) = \frac{t_k}{a_k} \ln(\frac{2a_{eq}}{v}) \,. \qquad (4.73)$$

After t_{eq} the scale factor evolves according to the matter dominated law, $a = \beta t^2$. Hence,

$$J(t_{eq}, t) = \int_{t_{eq}}^t \frac{dt'}{(v^2 + \beta^2 t^4)^{1/2}} \approx t_{eq}/a_{eq}[1 - (a_{eq}/a)^{1/2}] \approx t_k/a_k[1 - (a_{eq}/a)^{1/2}], \qquad (4.74)$$

where we used $a \gg v$, which is true for the relevant region of integration in (4.72), and $a_k/t_k = \alpha^{-1} \approx a_{eq}/t_{eq}$. Thus, in the far nonrelativistic, X-dominated regime we can approximate J in (4.72) by

$$J(v,t) = t_k/a_k[1 - (a_{eq}/a)^{1/2} + \ln(\frac{2a_{eq}}{v})] \,. \qquad (4.75)$$

(Note that t_k/a_k is a constant, independent of k!)
To get rid of one factor v in (4.72), we rewrite the equation in the form

$$\frac{\Delta_X(t)}{\Delta_X(t_*)} \approx \frac{d}{dx}[\int_0^\infty v(e^{vx}+1)^{-1}\frac{\sin(kvJ)}{kJ}dv]_{x=1} \,, . \qquad (4.76)$$

Since $t_k = \pi/k$ and $a_k \ll 1$, we know $kJ \gg 1$ and, therefore, the $\sin(kJv)$ in the integral of (4.76) varies much faster than the distribition function. Thus, we develop the latter in e^{-vx}. This leads to

$$\frac{\Delta_X(t)}{\Delta_X(t_*)} \approx \frac{d}{dx}[\int_0^\infty ve^{-vx}\frac{\sin(kvJ)}{kJ}dv]_{x=1} \,. \qquad (4.77)$$

Since J varies very slowly with v, we set $J = J_{\max} = J(v_{\max}) = \text{const.}$, and obtain

$$\frac{\Delta_X(t)}{\Delta_X(t_*)} \approx \frac{d}{dx}[\frac{2x}{(x^2+(kJ_{\max})^2)^2}]_{x=1} \approx \text{const.}\cdot k^{-4} \,. \qquad (4.78)$$

For the second approximation we used $kJ_{\max} \gg 1$.
This equation describes the k-dependence of the transfer function at the time when $k > k_J(t)$ but the X-particles already dominate the energy density. As soon as $k = k_J$, we can not neglect the gravitational effects anymore and Δ_X starts growing with the law given in equation (3.21):

$$\Delta_X(t) \propto a(t) \,.$$

Hence, for $k < k_J$ we have

$$\Delta_X(k) = \text{const.}\cdot k^{-4}(a/a(k)) \,,$$

where $a(k)$ denotes the scale factor at the time when $k = k_J$. Since in the nonrelativistic regime $k_J^2 \propto a$ (see equation (4.58)), we finally obtain

$$\Delta_X(k,a) \approx \Delta_0\frac{a}{a_{eq}}(\frac{k}{k_{\max}})^{-6} \,, \quad \text{for } k \gg k_{\max} \text{ and } k < k_J \,, \qquad (4.79)$$

where Δ_0 is of the same order of magnitude as the primordial perturbation amplitude. A comparison of this analytical formula with the numerical calculation is shown in Figure 12.

4.5 Conclusions, outlook

If a dark, collisionless component with mass m_X triggered the formation of inhomogeneities in the baryonic matter, the first nonlinear structure of the universe would consist of masses of the order m_{pl}^3/m_X^2. The smallest masses occuring in Physics would, via this relation, determine the scale of the largest masses. To decide whether cold or hot dark matter is a good candidate for structure formation it is crucial to know if small structures like globular clusters and galaxies are older or younger than superclusters. With nonlinear Newtonian simulations which take the output linear spectrum as input, one can check which candidate provides a more realistic picture of the observed large-scale structure (See [19,20]). We will not treat this problem within this work. But let us estimate the compatibility of the collisionless matter scenario with the isotropy of the MBR.

The perturbations can start growing at a_{eq}, when the energy density in nonrelativistic particles begins to dominate. Since $a_{eq} \approx 10$ for hot dark matter and much larger for warm and cold dark matter, we can approximate ρ_X in

$$\rho_X(a_{eq}) = \rho_{\nu_0}(a_{eq}) + \rho_\gamma(a_{eq})$$

by its nonrelativistic limit. Setting $N_X = 2$, we then obtain

$$a_{eq} = \frac{(2 + (7/8)N_{\nu_0}\alpha_{\nu_0}^{-4})\pi^4\alpha_X^4}{90\zeta(3)} \approx 2.8\alpha_X^4 \,. \tag{4.80}$$

On the other hand, since today the X-particles dominate the energy-density of the universe, we find from the field equations

$$a_0 = \frac{H_0^2\pi m_{pl}^2}{4\zeta(3)T_\gamma^4(t_0)}\alpha_X^4 \approx 4\cdot 10^4\alpha_X^4 \cdot h_{50}^2 \,.$$

Thus,

$$z_{eq} = a_0/a_{eq} \approx 1.4\cdot 10^4 \cdot h_{50}^2 \,. \tag{4.81}$$

Since after a_{eq} the X-perturbations started growing roughly according to the matter-dominated growth law (3.21), the total growth factor up to today is of the order z_{eq}. We do not know which mechanism caused the primordial fluctuations. It seems, therefore, natural to assume that they were of the same order of magnitude for photons and X-particles (see also [41]). We thus have

$$\Delta_\gamma(t_{eq}) \approx \Delta_X(t_{eq}) \; .$$

Since today $\Delta_X \geq 1$ on the relevant scales (see Section 3.2), the perturbations imprinted on the MBR can not be much smaller than

$$\Delta_\gamma \approx \frac{1}{z_{eq}} = 8 \cdot 10^{-5} h_{50}^{-2} \, , \quad \text{i.e.,} \quad \frac{\Delta T}{T} \approx \frac{1}{4}\Delta_\gamma \approx 2 \cdot 10^{-5} h_{50}^{-2} \; .$$

This is only marginally consistent with the observational limits given in [12,34,44,63].

From this result, together with Section 3.2, we must conclude if the MBR limits still go down by a factor 2 or 3, which seems to be within the observational possibilities [35], we run into trouble with most of the currently-investigated theories of galaxy formation.

Are we completely on the wrong track? Was Lifshitz right?

FIGURES

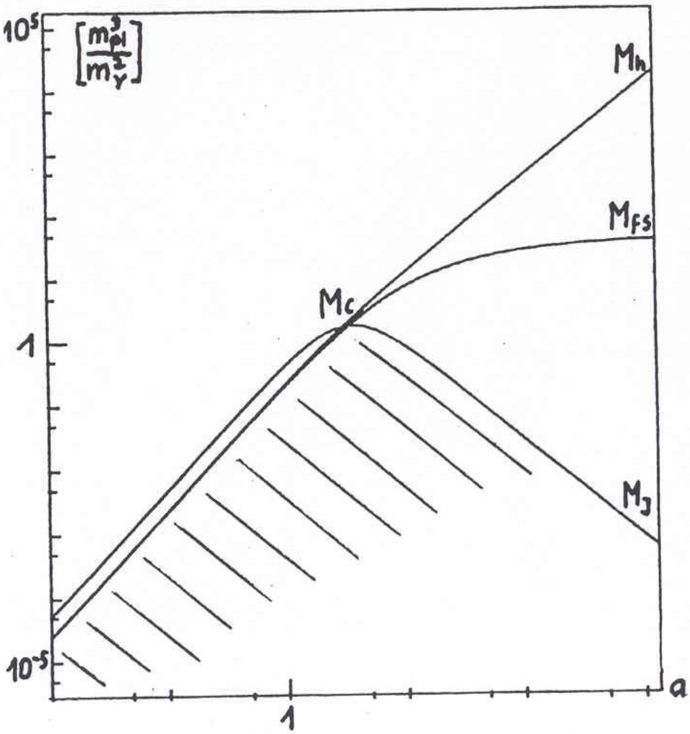


Figure 4.1: The critical scales for a pure ν universe are drawn as functions of the scale factor $a = m_\nu/T_\nu$ ($a_{rec} = 123(m_\nu/30eV)$, $a_0 = 1.8 \cdot 10^5(m_\nu/30eV)$). M_h denotes the horizon mass, M_{fs} the free-streaming mass and M_J the Jeans mass. The numerical value of the critical mass is $M_c = 1.8(m_{pl}^3/m_\nu^2)$. The region of collisionless damping is indicated.

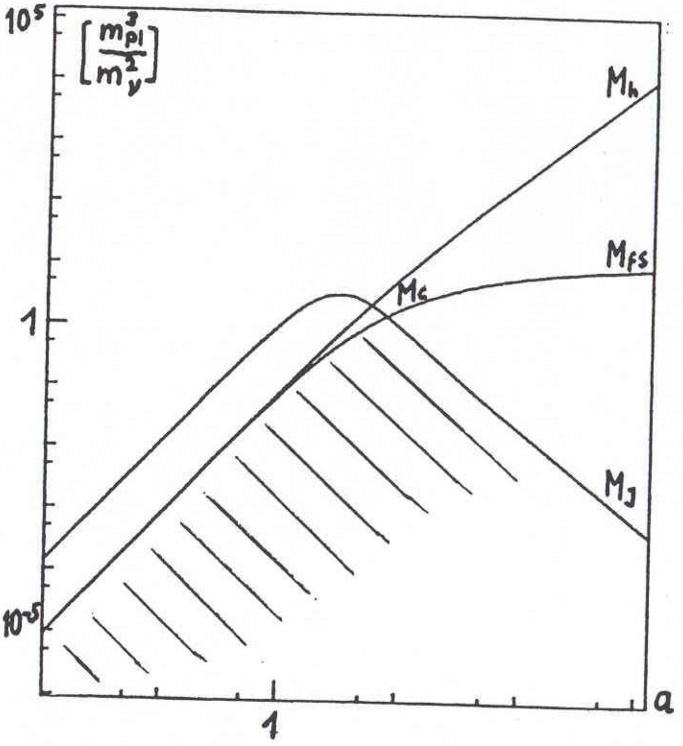


Figure 4.2: The critical scales for a universe with 1 massive and 2 massless ν-flavours are drawn as functions of the scale factor $a = m_\nu/T_\nu$ ($a_{rec} = 130(m_\nu/30eV)$, $a_0 = 1.8 \cdot 10^5(m_\nu/30eV)$). M_h denotes the horizon mass, M_{fs} the free-streaming mass and M_J the Jeans mass. The numerical value of the critical mass is $M_c = 1.6(m_{pl}^3/m_\nu^2)$. The region of collisionless damping is indicated.

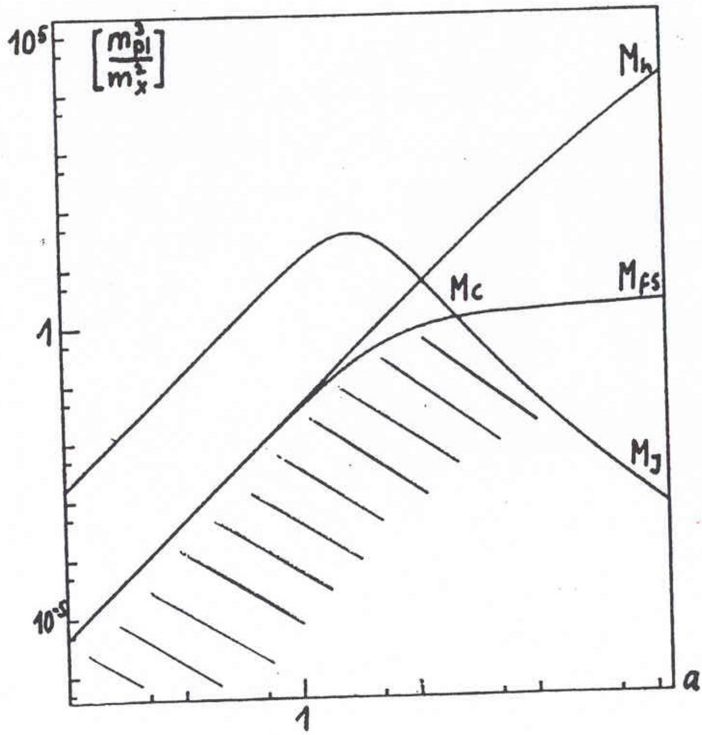


Figure 4.3: The critical scales for X-particles with $g_* = 100$ are drawn as functions of the scale factor $a = m_X/T_X$ ($a_{rec} = 8.7 \cdot 10^3(m_X/1keV)$, $a_0 = 1.3 \cdot 10^7(m_X/1keV)$). M_h denotes the horizon mass, M_{fs} the free-streaming mass and M_J the Jeans-mass. The numerical value of the critical mass is $M_c = 1.2(m_{pl}^3/m_X^2)$. The region of collisionless damping is indicated.

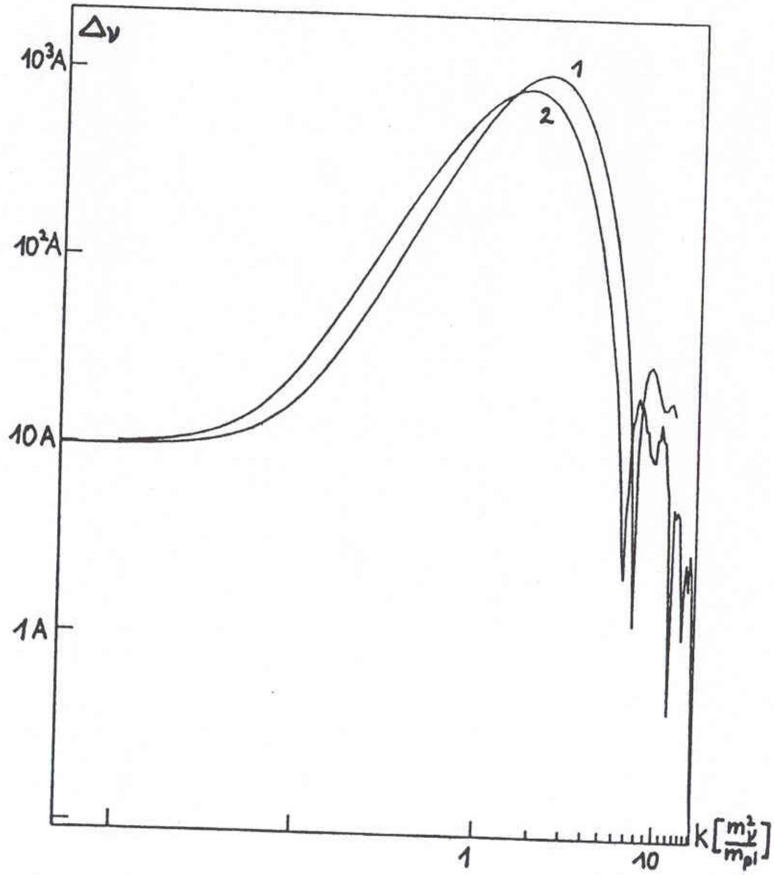


Figure 4.4: The linear spectrum of massive ν-perturbation with (1) and without (2) inclusion of massless ν's, at redshift $z = 100$ for a primordial Zel'dovich spectrum. The maximum occurs at $k_{\max} = 2.4 m_\nu^2 / m_{pl}$.

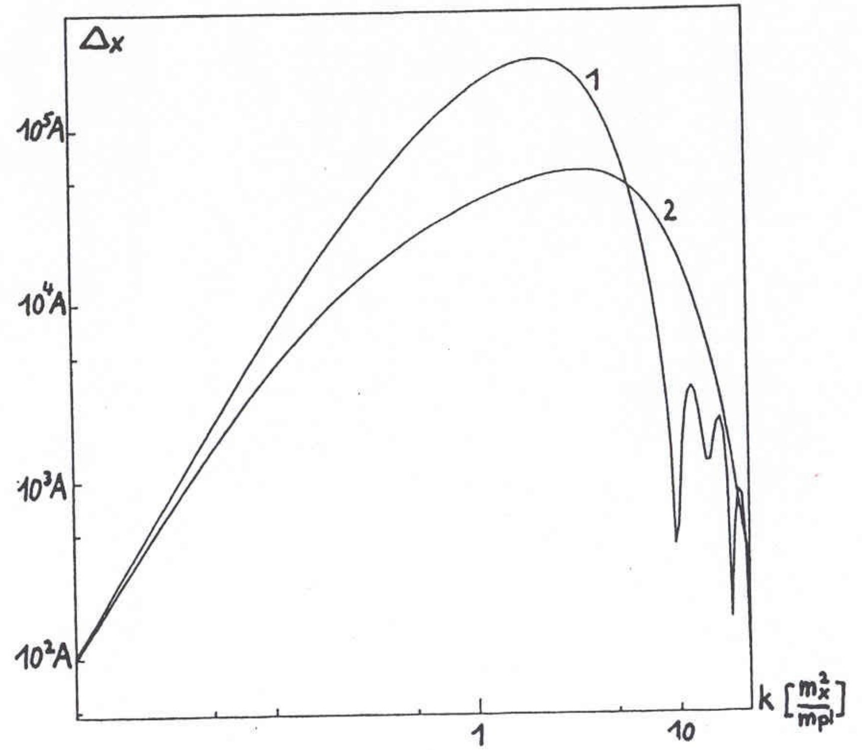
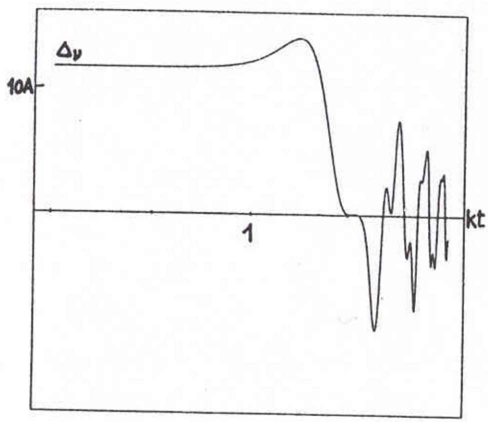


Figure 4.5: The output linear X-spectrum at redshift $z = 10$ of a scenario including massless ν's and photons for $g_* = 30$ (1) and $g_* = 200$ (2), for a primordial Zel'dovich spectrum. The maximum occurs at $k_{\max} = 2.1 m_\nu^2/m_{pl}$.



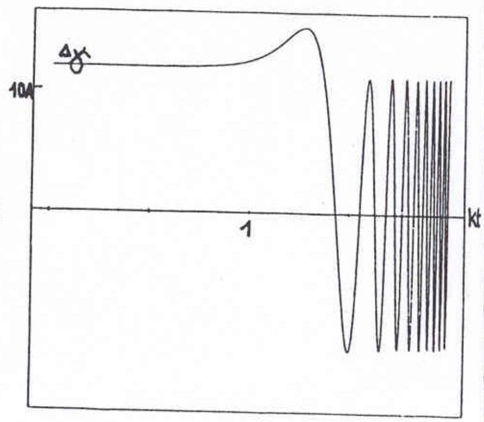


Figure 4.6: The temporal behaviour of ν-density perturbations (left) and γ-density perturbations in the extremely relativistic regime ($T_\nu \gg m_\nu$). The neutrinos undergo directional damping.

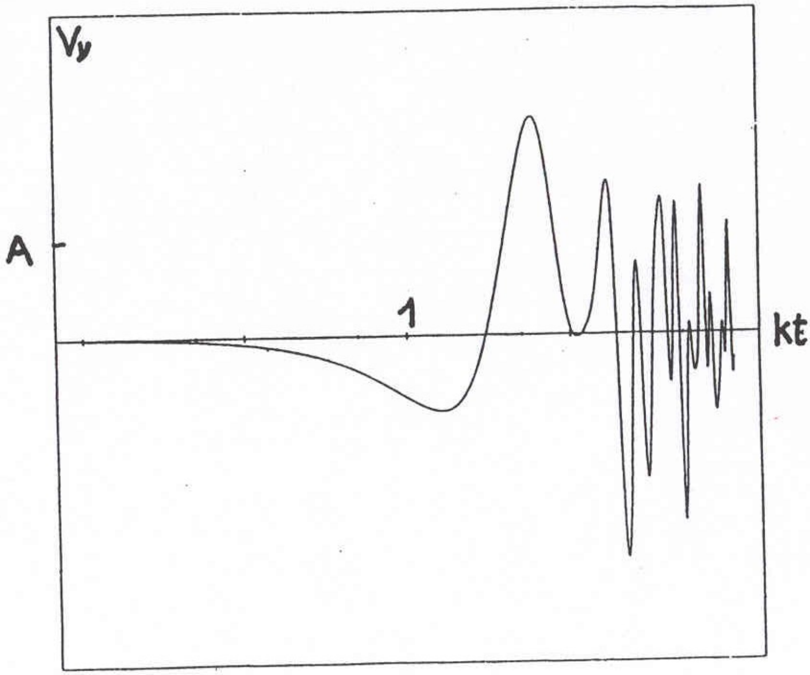


Figure 4.7: The temporal behaviour of the ν velocity perturbations in the extremely relativistic regime ($T_\nu \gg m_\nu$).

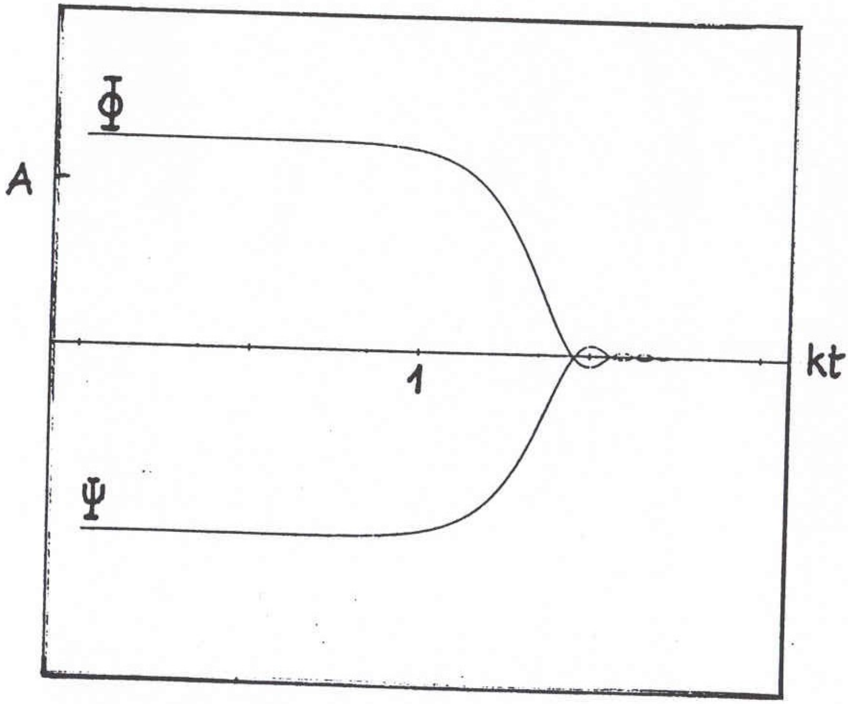


Figure 4.8: The temporal behaviour of the geometrical perturbations Φ and Ψ in the extremely relativistic regime ($T_\nu \gg m_\nu$). As soon as they enter the horizon the amplitude decays.

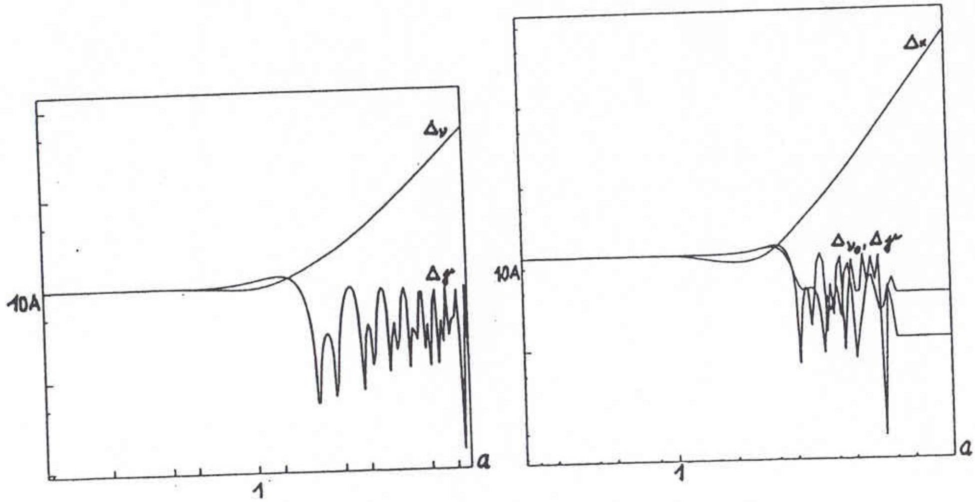


Figure 4.9: Left: The temporal behaviour of the density perturbations for massive ν's and photons with wavenumber $k = k_{\max} \approx 2.4m_\nu^2/m_{pl}$.

Right: The same for X-particles with $g_* = 100$. (After some integration time, the perturbations of the X-particles are so much larger than those of the γ's and ν_0's that we could set the latter equal to a constant without loss of accuracy.)

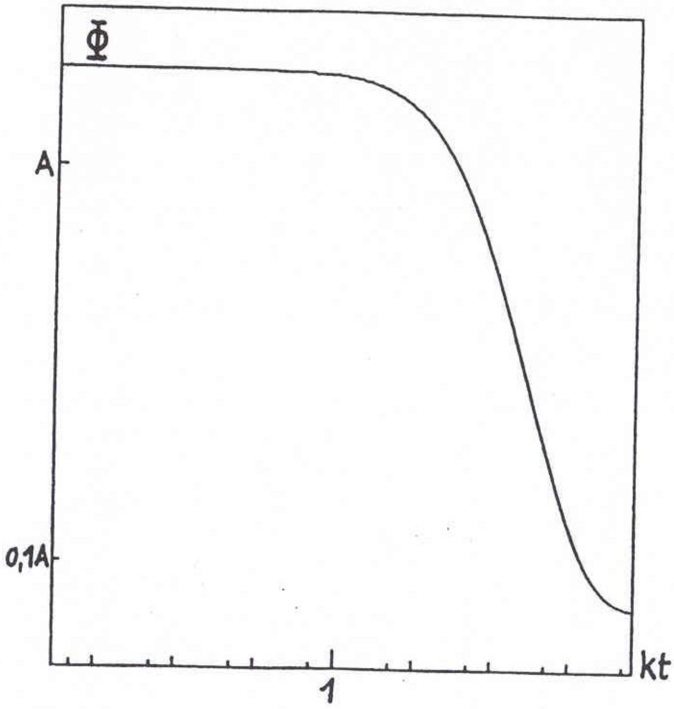


Figure 4.10: The temporal behaviour of the geometrical perturbation, Φ, for $k = k_{\max} = 2.4 m_\nu^2/m_{pl}$. As soon as the wave enters the horizon Φ decays by about 2 orders of magnitude.

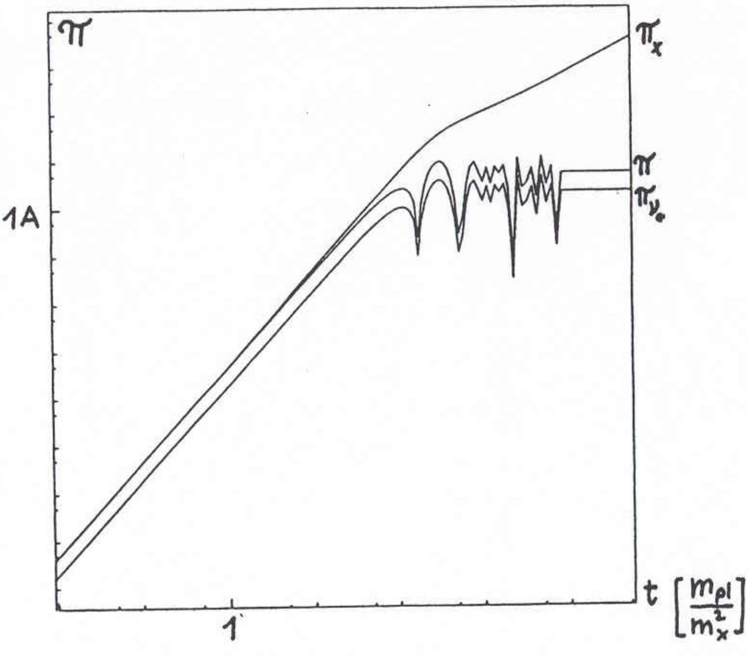


Figure 4.11: The temporal behaviour of the anisotropic pressure perturbation Π, for $k = k_{\max} = 2.1 m_X^2/m_{pl}$. Π_X , Π_{ν_0} and Π are drawn. In the nonrelativistic regime, where Π_X starts growing, the pressure p_X decays so fast that $p_X \Pi_X \ll p_{\nu_0}\Pi_{\nu_0}$, and hence, the total anisotropy Π follows closely Π_{ν_0} which is always less than Δ_{ν_0} and, therefore, completely negligible.

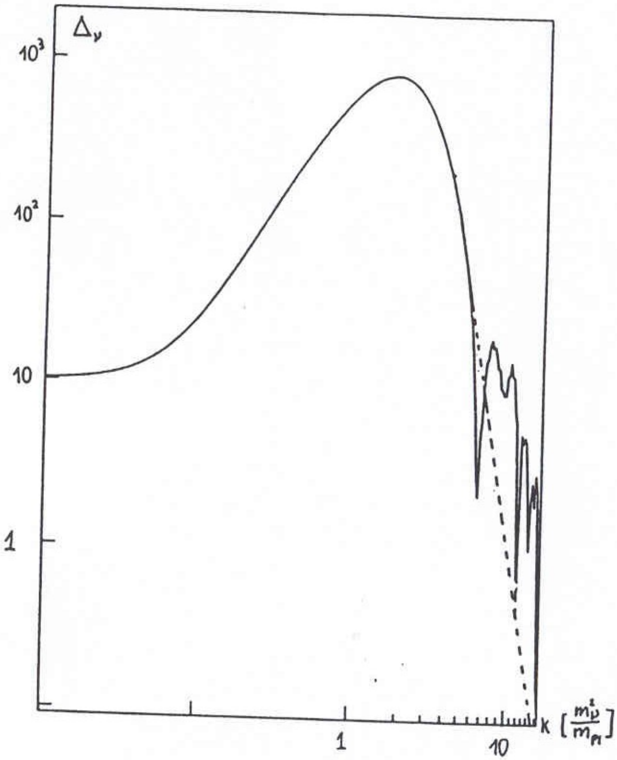


Figure 4.12: The behaviour of the spectrum for $k > k_{\max}$ compared with the analytical fit of Section 4.4: $\Delta(k) \propto k^{-6}$ (broken line).

Appendix A

Connection and curvature forms

In this appendix we are going to calculate the connection and curvature forms in the orthonormal basis introduced in equation (1.43) of Section 1.3.

A.1 The Connection forms

From the first structure equation

$$d\theta^{\mu} + \omega^{\mu}_{\nu} \wedge \theta^{\nu} = 0 \ ,$$

and from the definition of the second fundamental form:

$$K_{ij} = -n_{i;j} \ , \tag{A.1}$$

where n denotes the normal field of the slicing, one finds immediately the Gauss formulas (remember $n = e_0$):

$$\omega^{i}_{\ k}(\mathbf{e}_j) = \boldsymbol{\omega}^{i}_{\ k}(\mathbf{e}_j) \ , \tag{A.2}$$

$$\omega^{0}_{i}(\mathbf{e}_j) = -K_{ij} \ . \tag{A.3}$$

We define the coefficients $c^{i}_{\ j}$ by

$$\partial_t \vartheta^i = c^{i}_{\ j} \vartheta^j \ . \tag{A.4}$$

Now we can calculate the following quantities (details are given below):

$$\omega^{0}_{i}(e_0) = \alpha^{-1}\alpha_{|i} \ , \tag{A.5}$$

$$\omega^{i}_{\ j}(e_0) = -\alpha^{-1}\boldsymbol{\omega}^{i}_{\ j}(\boldsymbol{\beta}) + \frac{1}{2\alpha}(\beta^{i}_{|j} - \beta^{j}_{|i} - c^{i}_{\ j} + c^{j}_{i}) \ , \tag{A.6}$$

$$K_{ij} = \frac{1}{2\alpha}(\beta_{i|j} + \beta_{j|i} - c^{i}_{\ j} - c^{j}_{i}) \ , \tag{A.7}$$

where the vertical bar $_|$ denotes covariant derivation with respect to $\boldsymbol{g}$. The last equation implies

$$\boldsymbol{K} = \frac{1}{2\alpha}[L_{\boldsymbol{\beta}}g - \partial_t g] . \tag{A.8}$$

Using the general relation

$$\partial_t(\det M) = \mathrm{tr}(M^{-1}\partial_t M)\det M ,$$

we obtain

$$\partial_t(\det \boldsymbol{g}) = (\partial_t g)_i^{\;i} \det \boldsymbol{g} .$$

Equation (A.8) yields thus

$$\partial_t \mathrm{vol}(\boldsymbol{g}) = (\mathrm{div}\boldsymbol{\beta} - \alpha \boldsymbol{K}_i^{\;i})\mathrm{vol}(\boldsymbol{g}) . \tag{A.9}$$

The derivation of (A.5), (A.6) and (A.7) proceeds as follows. First we note that

$$d\theta^0 = d(\alpha dt) = \boldsymbol{d}\alpha \wedge dt = \alpha_{|i}\vartheta^i \wedge dt = \alpha^{-1}\alpha_{|i}\theta^i \wedge \theta^0 .$$

From the first structure equation and (A.3) we conclude

$$\begin{aligned} i_{e_l} i_{e_0} d\theta^i &= -i_{e_l} i_{e_0}(\omega^i_{\;0} \wedge \theta^0 + \omega^i_{\;j} \wedge \theta^j) \\ &= -(K_{il} + \omega^i_{\;l}(e_0)) . \end{aligned}$$

The left hand side of this equation is

$$\begin{aligned} i_{e_j} i_{e_0} d\theta^i &= i_{e_j} i_{e_0} d(\boldsymbol{\vartheta}^i + \boldsymbol{\beta}^i dt) \\ &= i_{e_j} i_{e_0}(\boldsymbol{d\vartheta}^i + dt \wedge \partial_t \theta^i + \boldsymbol{d\beta}^i \wedge dt) \\ &= \alpha^{-1} i_{e_j}(i_{\boldsymbol{\beta}}(\boldsymbol{\omega}^i_{\;l} \wedge \boldsymbol{\vartheta}^l) + \partial_t \boldsymbol{\vartheta}^i - \boldsymbol{d\beta}^i) \\ &= \alpha^{-1}[\boldsymbol{\omega}^i_{\;j}(\boldsymbol{\beta}) - \boldsymbol{\omega}^i_{\;k}(\mathrm{e}_j)\boldsymbol{\beta}^k - \boldsymbol{d\beta}^i(\mathrm{e}_j) + \partial_t \boldsymbol{\vartheta}^i(\mathrm{e}_j)] \\ &= \alpha^{-1}[\boldsymbol{\omega}^i_{\;j}(\boldsymbol{\beta}) - \boldsymbol{\beta}^i_{\;|j} + c^i_{\;j}] . \end{aligned}$$

The symmetrical and antisymmetrical contribution of the last identity yield the formulas (A.7) and (A.6) for K_{ij} and $\omega^i_{\;j}(e_0)$ respectively.

A.2 The curvature forms

From the second structure equation

$$\Omega^{\mu}{}_{\nu} = d\omega^{\mu}{}_{\nu} + \omega^{\mu}{}_{\lambda} \wedge \omega^{\lambda}{}_{\nu} ,$$

and equations (A.2) to (A.7), one finds immediately

$$\begin{aligned} \Omega^{i}{}_{j}(\mathbf{e}_k, \mathbf{e}_l) &= \boldsymbol{\Omega}^{i}{}_{j}(e_k, e_l) + \boldsymbol{K}^{i}{}_{k}\boldsymbol{K}_{jl} - \boldsymbol{K}^{i}{}_{l}\boldsymbol{K}_{jk} && \text{(Gauss)} ,\text{(A.10)} \\ \Omega^{0}{}_{j}(\mathbf{e}_k, \mathbf{e}_l) &= \boldsymbol{K}_{jk|l} - \boldsymbol{K}_{jl|k} && \text{(Mainardi)} . \text{(A.11)} \end{aligned}$$

We need also the other components of $\Omega^{0}{}_{j}$. The second structure equation gives, together with (A.3) and (A.5),

$$\Omega^{i}{}_{0} = -d(K_{ij}\theta^{j}) + d(\alpha^{-1}\alpha_{|i}\theta^{0}) + \omega^{i}{}_{l} \wedge (-K_{lj}\theta^{j} + \alpha^{-1}\alpha_{|l}\theta^{0}) .$$

A straightforward calculation leads to

$$\Omega^{i}{}_{0} = \alpha^{-1}\alpha_{|ij}(\theta^{j} \wedge \theta^{0}) - dK^{i}{}_{j} \wedge \theta^{j} + K^{i}{}_{j}(\omega^{j}{}_{l} \wedge \theta^{l} - K^{j}{}_{l}\theta^{l} \wedge \theta^{0}) + \omega^{i}{}_{j}K^{j}{}_{l} \wedge \theta^{l} ,$$

which yields (A.11) and the following components of $\Omega^{i}{}_{0}$:

$$\Omega^{i}{}_{0}(\mathbf{e}_j, e_0) = \alpha^{-1}\alpha^{|i}{}_{|j} + dK^{i}{}_{j}(e_0) - K^{i}_{s}\omega^{s}{}_{j}(e_0) + (K^2)^{i}{}_{j} - \omega^{i}{}_{s}(e_0)K^{s}{}_{j} . \quad \text{(A.12)}$$

Now we can calculate the Ricci tensor, $R_{\beta\sigma} = \Omega^{\alpha}{}_{\beta}(e_\alpha, e_\sigma)$. With (A.12) we find

$$R_{00} = \frac{1}{\alpha}\Delta\alpha + \alpha^{-1}(\partial_t \mathrm{tr}(K) - L_{\boldsymbol{\beta}}\mathrm{tr}(K_{ij})) + \mathrm{tr}\boldsymbol{K}^2 , \quad \text{(A.13)}$$

and (A.11) gives

$$R_{0i} = (trK)_{|i} - K_{i}{}^{j}{}_{|j} . \quad \text{(A.14)}$$

For the spatial components

$$R_{ij} = \Omega^{0}{}_{i}(e_0, e_j) + \Omega^{k}{}_{i}(e_k, e_j)$$

(A.12) and (A.10) lead to

$$R_{ij} = \boldsymbol{R}_{ij} + \mathrm{tr}(K)K_{ij} - \alpha^{-1}\alpha_{|ij} - \alpha^{-1}(\partial_t K_{ij} - L_{\boldsymbol{\beta}}K_{ij}) + K_{is}\omega^{s}{}_{j}(e_0) + K_{js}\omega^{s}{}_{i}(e_0). \quad \text{(A.15)}$$

With help of (A.6), (A.7) and (A.8) one can bring this equation into the form

$$Ric(g) = \mathbf{Ric}(\boldsymbol{g}) + \mathrm{tr}(\boldsymbol{K})\boldsymbol{K} - 2\boldsymbol{K}^2 - \alpha^{-1}(\partial_t\boldsymbol{K} - L_{\boldsymbol{\beta}}\boldsymbol{K}) - \alpha^{-1}\mathbf{Hess}(\alpha) . \quad \text{(A.16)}$$

Using (A.13) and (A.15) we find

$$\begin{aligned} G_{00} &= 1/2(R_{00} + \Sigma_i R_{ii}) \\ &= 1/2[\boldsymbol{R} + (\mathrm{tr}(K))^2 - \mathrm{tr}(K^2)] \,. \end{aligned} \tag{A.17}$$

Appendix B

The Ricci tensor of the spatial slices of a perturbed Friedman universe

To make use of (A.16) we need the Ricci tensor $\mathbf{Ric}(\tilde{g})$ of the $\{t = \text{const.}\}$ slices for the perturbed Friedman universe. This is the Ricci tensor of the induced metric

$$\tilde{g} = [a^2\eta_{ij} + \delta g_{ij}]\tau^i \otimes \tau^j \ , \tag{B.1}$$

where the $\{\tau^i\}$ denote orthonormal 1-forms of the metric γ_{ij} . We set

$$\mathbf{Ric}(\tilde{g}) = (\boldsymbol{R}_{ij} + \delta \boldsymbol{R}_{ij})\tau^i \otimes \tau^j \ ,$$

where (generalizing to n spatial dimensions)

$$\boldsymbol{R}_{ij} = K(n-1)\delta_{ij} \ .$$

By Palatini's identity (see for example [52], p 217)

$$\delta R_{ij}(g + \delta g) = \frac{1}{2}[\delta g^k_{\ i|jk} - \delta g^k_{\ k|ij} + \delta g^k_{\ j|ik} - \Delta \delta g_{ij}] \ . \tag{B.2}$$

The indices are raised and lowered with respect to the background metric g .

B.1 Scalar perturbations

To calculate (B.2) for scalar perturbations which are of the form

$$\tilde{g} = a^2[(1 + 2H_L Y)\delta_{ij} + \epsilon 2H_T Y_{ij}]\tau^i \otimes \tau^j \ , \tag{B.3}$$

we make use of equations (2.9) and (2.14) of Section 2.2 . Furthermore, we have to apply the following identity (which is derived like the equations (2.14))

$$Y^m_{i|jm} = [K(n-1) + \frac{n-1}{n}k^2]Y_{ij} - \frac{(n-1)(n+1)}{n^2}(nK - k^2)Y\delta_{ij} \ . \tag{B.4}$$

One then finds

$$\delta\mathbf{Ric} = [H_L\{2\frac{n-1}{n}k^2Y\delta_{ij} + (2-n)k^2Y_{ij}\} + H_T\{2\frac{n-1}{n^2}(k^2 - nK)Y\delta_{ij} + (\frac{(2-n)}{n}k^2 + 2(n-1)K)Y_{ij}\}]\tau^i \otimes \tau^j \,. \tag{B.5}$$

(B.6)

Hence,

$$\mathbf{Ric}(\tilde{g}) = [K(n-1)\delta_{ij} + (H_L + \tfrac{1}{n}H_T)(2\tfrac{n-1}{n}k^2Y\delta_{ij} + (2-n)k^2Y_{ij}) - H_T(2\tfrac{n-1}{n}KY\delta_{ij} + 2(n-1)KY_{ij})]\tau^i \otimes \tau^j \,. \tag{B.7}$$

We want to write $\mathbf{Ric}(\tilde{g})$ in our adapted basis $\{\tilde{\vartheta}^i\}$, which is orthonormal with respect to the perturbed metric. The transformation matrix is given by

$$\tau^i = \frac{1}{a}[(1 - H_L)\tilde{\vartheta}^i - H_T Y^i_j \tilde{\vartheta}^j] \,. \tag{B.8}$$

A short calculation leads to

$$\mathbf{Ric}(\tilde{g}) = a^{-2} \ [K(n-1)\delta_{ij} + (H_L + \tfrac{1}{n}H_T) (2\tfrac{n-1}{n}(k^2 - nK)Y\delta_{ij} + (2-n)k^2Y_{ij})]\vartheta^i \otimes \vartheta^i \,.$$

Setting

$$\mathcal{R} = H_L + 1/nH_T \,, \tag{B.9}$$

we find the following perturbation of the Ricci curvature with respect to our perturbed orthonormal frame:

$$\delta\mathbf{Ric}(\tilde{g}) = a^{-2}\mathcal{R}[2\frac{n-1}{n}(k^2 - nK)Y\delta_{ij} + (2-n)k^2Y_{ij}]\vartheta^i \otimes \vartheta^i \,. \tag{B.10}$$

From this we extract the trace term and the traceless contribution :

$$\delta\boldsymbol{R} = a^{-2}2(n-1)(k^2 - nK)\mathcal{R}Y \,, \tag{B.11}$$

$$\delta\mathbf{Ric}^{\text{aniso}} = a^{-2}(2-n)k^2\mathcal{R}Y_{ij}\vartheta^i \otimes \vartheta^i \,. \tag{B.12}$$

B.2 Vector perturbations

For vector-type perturbations, $\tilde{g}$ is of the form

$$\tilde{g} = a^2[\delta_{ij} + \epsilon 2H_T X_{ij}]\tau^i \otimes \tau^j \ . \tag{B.13}$$

To calculate $\delta\mathbf{Ric}(\tilde{g})$ from Palatini's identity we make use of

$$X^k_{\ j|ik} + X^k_{\ i|jk} = (2(n+1)K - k^2)X_{ij} \ .$$

After a short calculation we obtain

$$\delta\mathbf{Ric}(\tilde{g}) = 2(n-1)H_T K X_{ij}\tau^i \otimes \tau^j \ . \tag{B.14}$$

For vector perturbations, the transformation into the basis $\{\tilde{\vartheta}^i\}$ is given by

$$\tau^i = a^{-1}(\delta^i_{\ j} - H_T X^i_{\ j})\tilde{\vartheta}^j \ .$$

Hence,

$$\begin{aligned} \mathbf{Ric}(\tilde{g}) &= K(n-1)(\gamma_{ij} + 2H_T X_{ij})\tau^i \otimes \tau^j \\ &= a^{-2}(n-1)K\gamma_{ij}\tilde{\vartheta}^i \otimes \tilde{\vartheta}^j \ . \end{aligned} \tag{B.15}$$

Thus, in the adapted orthonormal frame $\{\tilde{\vartheta}^j\}$, the first order perturbation of the Ricci tensor is absorbed into the perturbations of the basis forms $\{\tilde{\vartheta}^j\}$.

B.3 Tensor perturbations

In this case we obtain from Palatini's identity, simply using $T^{ij}_{\ \ |j} = 0$,

$$\delta\mathbf{Ric}(\tilde{g}) = (k^2 + 2nK)H_T T_{ij}\tau^i \otimes \tau^j \ .$$

With the help of

$$\tau^i = a^{-1}(\delta^i_{\ j} - H_T T^i_{\ j})\tilde{\vartheta}^j \ ,$$

we then find

$$\mathbf{Ric}(\tilde{g}) = a^{-2}[(n-1)K\gamma_{ij} + (k^2 + 2K)H_T T_{ij}]\tilde{\vartheta}^i \otimes \tilde{\vartheta}^j \ . \tag{B.16}$$

Thus,

$$\delta\mathbf{Ric}(\tilde{g}) = a^{-2}(k^2 + 2K)H_T T_{ij}\tilde{\vartheta}^i \otimes \tilde{\vartheta}^j \ . \tag{B.17}$$

Appendix C

Harmonic analysis on spaces of constant curvature

C.1 Preliminaries

Let (Σ, γ) be an n-dimensional space of constant curvature K with isometry group G.

$$G = \begin{cases} SO(n+1) & \text{if } K > 0, \\ E(n) = SO(n) \times \mathbf{R}^n & \text{if } K = 0, \\ SO(n,1) & \text{if } K < 0. \end{cases}$$

A tensor field $\mathbf{t} \in \mathcal{T}(\Sigma)$ of rank s transforms under $R \in G$ according to

$$(D(R)\mathbf{t})_{i_1 \cdots i_s}(x) = R_{i_1}^{\ j_1} \cdots R_{i_s}^{\ j_s} \mathbf{t}_{j_1 \cdots i_s}(R^{-1}x) \, . \tag{C.1}$$

Let $\boldsymbol{\theta}$ be a linear covariant differential operator on (Σ, γ). Then $\boldsymbol{\theta}$ commutes with the representation D of G on the algebra $\mathcal{T}(\Sigma)$ of tensor fields on Σ, i.e., for $\mathbf{t} \in \mathcal{T}(\Sigma)$

$$D(R)(\boldsymbol{\theta}\mathbf{t}) = \boldsymbol{\theta}(D(R)\mathbf{t}) \, .$$

Thus, for a (generally reducible) reduction

$$\mathbf{t} = \sum_l \mathbf{t}_l$$

of $\mathbf{t}$ with respect to the isometry group, an equation of the form

$$\theta t = 0$$

can be solved within each component seperately; the different modes t_l do not couple. This general fact is widely used in linear physical problems.

Let us denote by $P_n(s,k^2)$ the totally symmetric, traceless, divergencefree tensor fields of rank s which are eigenvectors of the Laplace-Beltrami operator with eigenvalue $-k^2$, i.e.,

$$\begin{aligned} \mathbf{t}^{i_1\cdots i_{s-1}}{}_{i_{s-1}} &= 0\,, \\ \mathbf{t}^{i_1\cdots i_s}{}_{|i_s} &= 0\,, \\ (\triangle + k^2)\mathbf{t} &= 0\,. \end{aligned}$$

Clearly, if $\mathbf{t} \in P_n(s,k^2)$, then $D(R)\mathbf{t}$ lies also in $P(s,k^2)$. Furthermore, it is a well-known fact that we can expand every totally symmetric, traceless, divergencefree tensor field of rank s in terms of eigenvectors of the Laplace-Beltrami operator. In addition, the divergence and the trace of a tensor field of rank s are expressible by tensors of smaller rank (see, e.g., [15]). It is thus clear, that every totally symmetric tensor field of rank s can be decomposed into elements of the spaces $P_n(l,k^2)$ with $l \leq s$. This proves the theorem formulated in Chapter 2.

We will now consider this decomposition in more detail. We want to solve the problem of wether this decomposition is irreducible, or can we decompose the space of totally symmetrical tensor fields into lower dimensional subspaces.

In the case $K = 0$, the answer is well-known: The spaces $P(s,k^2)$ are irreducible. This is beautifully described in Varadarajan's book "Geometry of Quantum Theory, Volume II", [58]. Varadarajan finds that an irreducible representaion of the semi-direct product $SO(n) \times \mathbf{R}^n$ acts on the space of divergencefree tensor fields $\mathbf{t}$ which are eigenvectors of the Laplace operator with eigenvalue $-k^2$ and whose value at a point x [$\mathbf{t}(x)$] transforms according to an irreducible representation of the stabilizer of x, $SO(n)$. For totally symmetric tensor fields this yields that $\mathbf{t}(x)$ has to be traceless.

In the next section we shall clarify the situation for $K = +1$. Then, k^2 takes only the discrete values $k^2 = l(l+n-1)$, $l \in \mathbf{N}$. We shall find that the decomposition

$$\mathbf{t} = \sum_{r \leq s,\, l \in \mathbf{N}} \mathbf{t}_{rl}$$

of a symmetric tensor field of rank s with $\mathbf{t}_{rl} \in P_n(r, l(l+n-1))$ is irreducible if n is larger than 3. In the physically most interesting case, $n = 3$, one can decompose $P_3(r, l(l+2))$ into spaces of positive and negative helicity which transform inequivalently under $SO(4)$: $P_3(r, l(l+2))^+ \oplus P_3(r, l(l+2))^-$.

The most difficult case, $K = -1$, will be treated in the last section. Our results for this case are unfortunately incomplete.

C.2 Harmonic analysis on the n-sphere $\mathbf{S}^n$

The 1-valued irreducible representations on $SO(n)$ can all be realized as tensor representations on $\mathbf{R}^n$ (see, e.g., [62]): A Young frame consisting of l rows of $f_1 \geq f_2 \cdots \geq f_l$ squares is admissible for $SO(n)$, if $l \leq [n/2]$. A Young tableau is obtained from a Young frame $[f_1, \cdots, f_l]$ by inserting an arbitrary permutation of the numbers 1 to $f_1 + \cdots + f_l$ into the squares of the frame. A Young tableau T is called standard if the numbers in the squares always increase from left to right and from top to bottom.

$$\textbf{example}: \quad \begin{matrix} 1 & 3 \\ 2 & \end{matrix} = T_1 \,, \qquad \textbf{counter-example}: \quad \begin{matrix} 2 & 3 \\ 1 & \end{matrix} = T_2 \,.$$

In [62], Weyl shows that the space $V_n(T)$ of traceless tensors on $\mathbf{R}^n$ which are invariant under permutations of the indices which correspond to the admissible standard tableau T is invariant under $SO(n)$. For $n = 2p + 1$ all irreducible representations are obtained in this way. Two representations, D_T and $D_{T'}$ on $V_n(T)$ and $V_n(T')$ are equivalent if and only if T and T' belong to the same Young frame $[f_1, \cdots, f_l]$, $l \leq p$. We denote the equivalence class of these representations by $D_n(f_1, \cdots, f_l)$. The p-component vector $(f_1, \cdots, f_l, 0 \cdots 0)$ is the maximal weight of the representation $D_n(f_1, \cdots, f_l)$.

For $n = 2p$, $V_n(T)$ transforms irreducibly if T contains less than p rows. If T contains p rows D_T can be decomposed into two inequivalent representations which we denote by

$$D_T = D_n(f_1, \cdots, f_p) \oplus D_n(f_1, \cdots, -f_p) \,,$$

where $(f_1, \cdots, \pm f_p)$ is the maximum weight of the representation. From the concrete form of the projection operators σ and σ^* onto the irreducible spaces $V_{2p}^+(T)$ and $V_{2p}^-(T)$, one easily concludes that σ is a projection onto a space of definite helicity.

Let now T be a standard tableau admissible to $SO(n)$. For $l \in \mathbf{N}$ we denote by $P_n(T,l)$ the space of divergencefree, traceless tensor fields $\mathbf{t}$ on $\mathbf{S}^n$ which obey $[\triangle + l(l+n-1)]\mathbf{t} = 0$, and whose values $\mathbf{t}(x)$ transform according to the representation D_T of the stabilizer $SO(n)$.

With the help of these notations we can formulate the following theorem:

Theorem 1 *The representation of the symmetry group $SO(n+1)$ on $P_n(T,l)$ is equivalent to D_{T_l}, where T_l is a standard tableau of the Young frame $[f_1, \cdots, f_i, l, f_{i+1}, \cdots, f_r]$ which is obtained from the frame $[f_1, \cdots, f_i, f_{i+1}, \cdots, f_r]$, corresponding to T by inserting a row of l squares at a suitable place, i.e., $f_i \geq l \geq f_{i+1}$. Thus, the representation of $SO(n+1)$ on $P_n(T,l)$ belongs to the equivalence class $D_{n+1}(f_1, \cdots, f_i, l, f_{i+1}, \cdots, f_r)$.*

This theorem, of course, answers also our question concerning the totally symmetric, traceless, divergencefree tensor fields of rank r which are eigenvectors of the Laplace-Beltrami operator with eigenvalue $l(l+n-1)$: They transform under $SO(n+1)$ according to a representation of the class $D_{n+1}(i,j)$, where $i = \max(r,l)$ and $j = \min(r,l)$. From the information given at the begining of this section, we know that this representation is irreducible for $n > 3$. In the physically most interesting case, $n = 3$, $D_4(i,j)$ is the sum of the two inequivalent representations $D_4(i,j)$ and $D_4(i,-j)$ of $SO(4)$.

Proof: The representation D of $SO(n+1)$ on $P_n(T,l)$ is given by equation (C.1) for $R \in SO(n+1)$:

$$(D(R)\mathbf{t})_{i_1\cdots i_s}(x) = R_{i_1}^{\ j_1} \cdots R_{i_s}^{\ j_s} \mathbf{t}_{j_1\cdots j_s}(R^{-1}x) \ .$$

(s is the number of squares in T, $s = |T|$.) We now construct an isomorphism

$$\varphi : P_n(T,l) \to V_{n+1}(T_l) : \mathbf{t} \mapsto \varphi(\mathbf{t}) \tag{C.2}$$

which commutes with rotations, i.e.,

$$\varphi(D(R)\mathbf{t}) = D_{T_l}(R)(\varphi\mathbf{t}).$$

Thus, we show that D and D_{T_l} are equivalent and the theorem holds.

First we expand $\mathbf{t}$ to a tensor field t on $\mathbf{R}^{n+1}$: Let $(e_1, \cdots, e_n)$ be an orthonormal basis of vector fields on $\mathbf{S}^n$. We set $e_{n+1} = \frac{\partial}{\partial r}$. Then $(e_1, \cdots, e_{n+1})$ form an orthonormal basis of vector fields on $\mathbf{R}^{n+1}$ (spherical coordinates). With the help of this basis we define

$$t(x)(e_{i_1}, \cdots, e_{i_s}) = \begin{cases} 0 & \text{if } i_m = n+1 \\ & \text{for any } 1 \leq m \leq s \,, \\ r^l \mathbf{t}(\hat{x})(e_{i_1}, \cdots, e_{i_s}) & \text{if } i_m \neq n+1 \\ & \text{for all } 1 \leq m \leq s \,. \end{cases} \tag{C.3}$$

These requirements uniquely determine t, because $(e_1, \cdots, e_{n+1})$ form a basis of $\mathbf{R}^{n+1}$. Since $\mathbf{t}$ is traceless, so is t. In addition, t is divergencefree: In the basis $(e_1, \cdots, e_{n+1})$ we obtain

$$\begin{aligned} t_{i_1 \cdots i_m \cdots i_s}{}^{|i_m} &= r^l \mathbf{t}_{i_1 \cdots i_m \cdots i_s}{}^{|i_m} + l r^{l-1} t_{i_1 \cdots (n+1) \cdots i_s} \\ &= 0 \end{aligned} \tag{C.4}$$

Here, the first term vanishes since $\mathbf{t}$ is divergencefree and the second term vanishes because of (C.3).

Note: We have to calculate the covariant divergence of t, since the connection coefficients do not vanish in spherical coordinates.

Furthermore, the indices of t clearly exhibit the symmetry corresponding to T. A value $t(x)$ transforms thus according to a representation within the class $D_{n+1}(f_1 \cdots f_r)$ under the stabilizer $SO(n+1)$. (r denotes the number of rows in T.) Since $[\triangle + l(l+n-1)]\mathbf{t} = 0$, the components of $\mathbf{t}$ are spherical harmonics of index l. It is well known (see, e.g., [47]) that these can be obtained by the restriction of homogeneous harmonic polynomials of degree l to the sphere $\mathbf{S}^n$. The components of t are thus harmonic polynomials of degree l, i.e.,

$$\triangle t = 0 \,. \tag{C.5}$$

Here $\triangle$ denotes the Laplace operator on $\mathbf{R}^{n+1}$. Let $x^1, \cdots, x^{n+1}$ be the carthesian coordinates of a point x in $\mathbf{R}^{n+1}$. We then can write t uniquely in the form

$$t_{i_1\cdots i_s}(x) = \tau_{i_1\cdots i_s j_1\cdots j_l} x^{j_1}\cdots x^{j_l} \ , \tag{C.6}$$

where τ is totally symmetric with respect to permutations of the indices $j_1 \cdots j_l$.

We want now to show that the linear map

$$\varphi : \mathbf{t} \mapsto \tau$$

is exactly the isomorphism we are looking for.

It is easy to see that φ is injective, and by direct calculation one checks that φ commutes with rotations. To see that $\tau \in V_{n+1}(T_l)$, we have to show that all $j-j$ traces and all $i-j$ traces of τ vanish and that τ is antisymmetric with respect to mixed indices i and j.

From (C.5) we obtain

$$\sum_j \tau_{i_1\cdots i_s j_1\cdots j\cdots j\cdots j_{l-2}} x_1^{j}\cdots x^{j_{l-2}} = 0$$

for all $x \in \mathbf{R}^{n+1}$. Thus, the $j-j$ traces vanish. Similarly, expressing (C.4) in terms of $\tau_{i_1\cdots i_s j_1\cdots j_l}$, we find that the $i-j$ traces of τ vanish. In addition, since t is tangent to $\mathbf{S}^n$, we have

$$t(x)_{i_1\cdots i_k\cdots i_s} x^{i_k} = 0 \ .$$

Hence,

$$\tau_{i_1\cdots i_k\cdots i_s j_1\cdots j_l} x^{i_k} x^{j_1}\cdots x^{j_l} = 0 \ .$$

This shows that τ is antisymmetric with respect to the permutation of i_k with any of the j_m's for all $1 \le k \le s$ and $1 \le m \le l$, and thus $\tau \in V_{n+1}(T_l)$. On the other hand, for any tensor $\tau \in V_{n+1}(T_l)$ we can construct the tensor field t according to equation (C.6), i.e., φ is a bijection. This completes the proof. □

C.3 Harmonic analysis on the n-dimensional pseudo sphere $\tilde{\mathbb{S}}^n$

In this section we make use of an important theorem in harmonic analysis which one finds, for example, in [24], p417. For the sake of completeness we repeat it here.

Theorem 2 *Let $S = G/K$ be a homogeneous space, K compact. By Ω we denote the set of all equivalence classes ω of representations of G of class 1. Then, there exists a natural bijection between Ω and the positive spherical functions ϕ on S with $\phi(0) = 1$ with the following properties:*

i) For $\pi \in \omega$, e a unit vector in N_π (N_π is the (1-dimensional) space of vectors in the representation space $\mathcal{H}$ which is invariant under K.),

$$\phi(x) = \langle e, \pi(x)e \rangle \ .$$

ii) ω contains the representation associated to ϕ.

Remarks:

- A class 1 representation π is a representation whose representation space $\mathcal{H}$ contains a subspace N_π which transforms trivially under K.
- On homogeneous spaces which can be represented by Riemannian manifolds with isometry group G, the spherical functions are the eigenfunctions of the Laplace-Beltrami operator.
- Since $\tilde{\mathbb{S}}^n = SO(n,1)/SO(n)$, this theorem solves our problem for functions on $\tilde{\mathbb{S}}^n$: To transform irreducibly under $SO(n,1)$, they have to be eigenfunctions of the Laplace-Beltrami operator, i.e., for functions, the decomposition $P_n(0,k^2)$ defined in the begining of this appendix is irreducible.
- In [59] one finds that the eigenvalues of the Laplace-Beltrami operator on $\tilde{\mathbb{S}}^n$ are of the form $-k^2 = \sigma(\sigma + n - 1)$. Since the positivity of $-\Delta$ requires $k^2 \geq 0$, we obtain two possible ranges for σ:

i) $0 \geq \sigma \geq n-1$.

ii) $\sigma = -\frac{n-1}{2} + ip$, $p \in \mathbf{R}$.

The representations of $SO(3,1)$ on the spaces of functions which fulfill i), belong to the supplementary series of irreducible unitary representations of the Lorentz group and the functions which fulfill ii) transform according to representations of the main series.

We now present a method to decompose arbitrary tensor fields on $\tilde{\mathbf{S}}^n$ into irreducible components. To this end, we construct a bundle isomorphism between the isometry group $SO(n,1)$ (interpreted as the bundle $\{SO(n,1)/SO(n), SO(n)\}$ with base space $SO(n,1)/SO(n)$ and structure group $SO(n)$) and the bundle of oriented orthonormal frames on $\tilde{\mathbf{S}}^n$, which we denote by $E^+(\tilde{\mathbf{S}}^n)$. We set $0 = id/SO(n)$ on $\tilde{\mathbf{S}}^n$. Furthermore, we specify an orthonormal basis $(e_1, \cdots, e_n)$ on $T_0\tilde{\mathbf{S}}^n$ which also defines the orientation we choose. Then, we define

$$\varphi \,:\, SO(n,1) \to E^+(\tilde{\mathbf{S}}^n) \,:\, g \mapsto (g(0); g(e_1), \cdots, g(e_n)) \,. \qquad \text{(C.7)}$$

Since $SO(n,1)$ acts transitively and effectively on $\tilde{\mathbf{S}}^n$, and the stabilizer $SO(n)$ acts freely on the oriented orthonormal frames of $T_x(\tilde{\mathbf{S}}^n)$, it is easy to see that φ is a bijection. Furthermore, is is clear that φ commutes with the action of $SO(n)$ on the fibers.

With the help of φ we can lift the tensor fields on $\mathcal{T}^s(\tilde{\mathbf{S}}^n)$ to functions on $SO(n,1)$ by means of

$$\hat{t}(g)_{i_1 \cdots i_s} = t(g(0))(g(e_{i_1}), \cdots, g(e_{i_s})) \,.$$

This map, which we will call φ_s in the sequel, forms a injection from the tensor fields of rank s on $\tilde{\mathbf{S}}^n$ to the n^s-tuples of functions on $SO(n,1)$. To see this, we consider the transformation properties of $\hat{t}_{i_1 \cdots i_s}$ under $g' \in SO(n,1)$:

$$\begin{aligned}
\varphi_s(U_{g'}t)_{i_1 \cdots i_s}) &= U_{g'}(t)(g(0))(g(e_{i_1}), \cdots, g(e_{i_s})) \\
&= g'^{\,j_1}_{i_1} \cdots g'^{\,j_s}_{i_s} t(g'^{-1}g(0))(g(e_{j_1}), \cdots, g(e_{j_s})) \\
&= t(g'^{-1}g(0))(g'^{-1}g(e_{i_1}), \cdots, g'^{-1}g(e_{i_s})) \\
&= \hat{t}(g'^{-1}g)_{i_1 \cdots i_s} \,.
\end{aligned}$$

Hence, the $\hat{t}_{i_1\cdots i_s}$ transform like functions on $SO(n,1)$. Since semi-simple groups like $SO(n,1)$ are Riemannian manifolds, the function $\hat{t}_{i_1\cdots i_s}$ transforms irreducible under $SO(n,1)$, if it is an eigenfunction of the Laplace-Beltrami operator on $SO(n,1)$, i.e.,

$$\triangle_{SO(n,1)}\hat{t}_{i_1\cdots i_s} = -k^2\hat{t}_{i_1\cdots i_s} \ . \tag{C.8}$$

Denoting the intersection1 of the space of eigenfunctions of $\triangle_{SO(n,1)}$ with eigenvalue $-k^2$ with $\varphi_s(\mathcal{T}^s\Sigma)$ by $\mathcal{F}(n,k^2)$, we thus know that the spaces $\varphi_s^{-1}(\mathcal{F}(n,k^2))$ provide the irreducible subspaces of $\mathcal{T}^s(\tilde{\mathbf{S}}^n)$. It would now be very satisfactory to demonstrate, at least for the totally symmetric tensor fields, the following

Conjecture 3 *For totally symmetric tensor fields, the space* $\varphi_s^{-1}(\mathcal{F}(n,k^2))$ *coincides with* $P_n(s,k^2)$.

Unfortunately we could not find a technical proof for this conjecture.

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Lebenslauf

Am 22. Januar 1958 bin ich in Kerns, Obwalden geboren.
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