Limits on stochastic magnetic fields: A defense of our paper [1]

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In their recent paper "Faraday rotation of the cosmic microwave background polarization by a stochastic magnetic field", Kosowsky *et al.* [2] have commented about our paper [1], in which we derived very strong limits on the amplitude of a primordial magnetic field from gravitational wave production. They argue that our limits are erroneous. In this short comment we defend our result.

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In Ref. [1] we have shown that, if a magnetic field is present on super-horizon scales in the early universe, its power is very efficiently converted into gravitational waves during its evolution from super- to sub-horizon scales. We used this fact to derive stringent limits on the amplitude of a magnetic field created before the nucleosynthesis epoch.

In their recent paper [2], Kosowsky *et al.* state that our limits are not valid. In the discussion section, they argue "... the expansion rate of the universe is the same whether energy density is converted from magnetic fields into gravity waves or not, since the energy density of both scale the same way with the expansion of the universe. So the actual constraint is on the total radiation energy density in the magnetic field, which is constrained to be about 1% of the total energy density in the usual manner... The corresponding limit on the total comoving mean magnetic field strength is around 10^{-8}Gauss, not the 10^{-27}Gauss claimed in [1]."

We now explain why this conclusion is wrong. We employ the same notation convention as [1], and we always consider the comoving amplitude of the magnetic field. For magnetic fields with spectral index $n > -3$, the magnetic field energy at wave number k is given by

$$\frac{d\Omega_B(k)}{d\log(k)} = \frac{B_\lambda^2}{8\pi\rho_c}\frac{(k\lambda)^{n+3}}{2^{(n+3)/2}\Gamma(\frac{n+3}{2})} \ , \qquad (1)$$

where B_λ is the magnetic field amplitude at some fixed reference scale λ. This energy spectrum is always blue, and therefore dominated by its value at the upper cutoff k_c. This cutoff scale is time dependent, $k_c(\eta)$. We set the magnetic field to zero on scales which are already sub-horizon at the time η_* of formation of the magnetic field, since we cannot be sure that its spectrum is a power law on these very small scales. Therefore, the upper cutoff at the time of formation of the magnetic field is given by $k_* = \eta_*^{-1}$ (where η denotes conformal time). This assumption is a conservative one for the derivation of our result.

At later times, the magnetic field is damped on scales smaller than a time dependent damping scale, which gives us the cutoff $k_D(\eta)$ [3, 4]. We therefore obtain the cutoff function

$$k_c(\eta) = \min\left(k_*, k_D(\eta)\right) \ . \qquad (2)$$

Of course at formation $k_* = 1/\eta_* \ll k_D(\eta_*)$, while at later time $k_D(\eta)$ is decreasing, and eventually becomes smaller that k_*. The magnetic field energy density at a given time η is therefore given by

$$\begin{aligned}\Omega_B(\eta) = \Omega_B(k_c(\eta)) &= \int_0^{k_c(\eta)} \frac{dk}{k}\frac{d\Omega_B(k)}{d\log(k)} \\ &= \frac{B_\lambda^2}{8\pi\rho_c}\frac{(k_c(\eta)\lambda)^{n+3}}{2^{(n+5)/2}\Gamma(\frac{n+5}{2})} \ . \end{aligned} \qquad (3)$$

In our paper [1], we have shown that at the time a given scale crosses the horizon, and for the maximally allowed magnetic fields which are such that $\Omega_B \sim \Omega_{\rm rad}$, a considerable fraction[1] of the magnetic field energy density is converted into gravitational wave energy density (see Eqs. (24) and (26) in Ref. [1]):

$$\left.\frac{d\Omega_G(k,\eta)}{d\log(k)}\right|_{k=1/\eta} \sim \frac{d\Omega_B(k)}{d\log(k)} \qquad (4)$$

(for a more precise estimate, *cf* Eqs. (23) and (25) of [1]).

The order of magnitude of this surprising result can also be obtained as follows: On super-Hubble scales, $k \ll \mathcal{H}$ (here $\mathcal{H} = aH$ denotes the conformal Hubble parameter), Einstein's equations to first order perturbation theory $\delta G_{\mu\nu} = 8\pi G T^B_{\mu\nu}$ reduce to

$$\mathcal{H}^2 h \sim 8\pi G B^2 a^2 \, .$$

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[1] For some values of the spectral index we obtain more energy in gravity waves than in the magnetic field. This comes from the fact that we linearize the problem and therefore do not take into account back-reaction. We expect the correct fraction of the energy in gravity waves to lie between 30% and 100% of the magnetic field energy density.

Here h is the (tensor) metric perturbation (we drop the indices for simplicity), $T^B_{\mu\nu}$ is the magnetic field energy momentum tensor, and B^2 is the energy density of the magnetic field. The energy density in gravitational waves is given by $\rho_G \sim \dot{h}^2/8\pi G$. On large scales, $\dot{h} \sim \mathcal{H}h$ so that

$$\rho_G \sim \frac{(\mathcal{H}h)^2}{8\pi G} \sim \frac{8\pi G}{\mathcal{H}^2} B^4 a^2 \simeq \frac{\rho_B^2}{\rho_{\rm rad}} \ . \quad (5)$$

Dividing both sides by ρ_c we obtain

$$\Omega_G \sim \frac{1}{\Omega_{\rm rad}} \Omega_B^2 \quad (6)$$

which, for $\Omega_B \sim \Omega_{\rm rad}$ agrees with Eq.(4). In the exact result presented in Ref. [1] there is an additional numerical factor 24 and a considerable logarithm due to the logarithmic build up of gravity wave since the generation of the magnetic field until horizon entry.

As time goes on, the magnetic field is damped by plasma viscosity on sub-horizon scales $k > k_D(\eta) \gg 1/\eta$, while the gravitational waves are not damped, since after formation they no longer interact with the matter and the radiation in the universe. This is the main point which Kosowsky *et al.* have missed. The magnetic field density parameter at nucleosynthesis is given by

$$\Omega_B(\eta_{\rm nuc}) = \frac{B_\lambda^2}{8\pi\rho_c} \frac{(k_D(\eta_{\rm nuc})\lambda)^{n+3}}{2^{(n+5)/2}\Gamma(\frac{n+5}{2})} \ , \quad (7)$$

where we have integrated up to the cutoff at nucleosynthesis, $k_D(\eta_{\rm nuc})$; while the gravitational wave density parameter is

$$\Omega_G \simeq \Omega_B(\eta_*) = \frac{B_\lambda^2}{8\pi\rho_c} \frac{(k_*\lambda)^{n+3}}{2^{(n+5)/2}\Gamma(\frac{n+5}{2})} \ , \quad (8)$$

where we integrate up to the cutoff corresponding to the time of creation of the magnetic field: gravitational wave production for a magnetic mode k takes place before horizon crossing, before the magnetic field is damped by interaction with the cosmic plasma. A considerable part of the magnetic energy is therefore converted into gravitational waves.

After their generation, gravitational waves are decoupled from matter and radiation: consequently, the amount of gravitational wave energy density present on sub-horizon scales at nucleosynthesis is not affected by the process of dissipation of the magnetic field, and by the subsequent injection of energy in the plasma [5]. Even in the limiting case of complete dissipation (all the magnetic energy is converted into heat by the time of nucleosynthesis), one would still obtain roughly the same amount of gravity waves Ω_G at nucleosynthesis : by the time a given mode enters the horizon and gets dissipated, the gravity waves have already been created. The energy dissipated into heat can contribute significantly to the radiation density but always by a factor of less than 2, since we start with a magnetic field energy which is less than the radiation energy density (otherwise we could not apply a perturbative treatment), and we loose a significant fraction of it by gravity wave production.

From (7) and (8), one can see that the ratio between the two energy densities is $\Omega_B(\eta_{\rm nuc})/\Omega_G \simeq (k_D(\eta_{\rm nuc})/k_*)^{n+3}$. Only for $n \simeq -3$ or $\eta_* \simeq \eta_{\rm nuc}$ this factor is of order unity; in this case the nucleosynthesis bounds for Ω_G or Ω_B lead to the same constraint. In most cases considered in the literature, however, the ratio is huge. Let us consider the example of magnetic field generation at the electroweak phase transition. In [1] we calculate $k_D(\eta_{\rm nuc}) \simeq 6 \times 10^{-7}{\rm sec}^{-1} \sim 60 \ {\rm pc}^{-1}$, and $\eta_{\rm ew} \simeq 4 \times 10^4$ sec. Taking into account that electroweak magnetic field generation is causal (not inflationary), and therefore $n = 2$ (see [6]), we obtain

$$\left.\frac{\Omega_B(\eta_{\rm nuc})}{\Omega_G}\right|_{\eta_*=\eta_{\rm ew}} \simeq (k_D(\eta_{\rm nuc})\eta_{\rm ew})^5 \simeq 8 \times 10^{-9} \ , \quad (9)$$

and by no means one! If the magnetic field is generated during inflation, one is no longer forced to have $n = 2$, but can have arbitrary values of $n > -3$. If we take $n \simeq 0$, for an inflation scale of 10^{15} GeV, we have $k_* = 1/\eta_{\rm inf} \simeq 10^{13}/\eta_{\rm ew}$, and we obtain

$$\left.\frac{\Omega_B(\eta_{\rm nuc})}{\Omega_G}\right|_{\eta_*=\eta_{\rm ew}} \simeq (k_D(\eta_{\rm nuc})\eta_{\rm inf})^3 \simeq 10^{-43} \quad !! \quad (10)$$

We can conclude that, applying the nucleosynthesis bound on Ω_G, we find much stronger constraints on B_λ, due to the fact that a seizable fraction of the magnetic field energy is converted into gravitational waves before the damping process.

Apart from not taking into account this damping, there is a second point that has been missed in Kosowsky *et al.* We specifically pronounce a limit for the amplitude of the stochastic magnetic field smoothed over a scale $\lambda \sim 0.1$ Mpc. On the contrary, they talk about 'the total comoving mean magnetic field', which is largely dominated by its value on small scales, hence $B(k_D(\eta_{\rm nuc}))$. The value of the field at this scale is limited to a few 10^{-8}Gauss by the constraint $\Omega_B < 0.1\,\Omega_{\rm rad}$ at nucleosynthesis. But this field value has no relevance at late times, for two reasons. First of all, because the damping scale will grow, which means that $B(k_D(\eta_{\rm nuc}))$ will be damped away before it can ever give rise to magnetic fields in galaxies. Typically, the highest mode which survives damping is $k_D(\eta_{\rm rec}) \simeq 10\,{\rm Mpc}^{-1}$, much smaller than $k_D(\eta_{\rm nuc})$ [3, 4]. Secondly, the scale relevant for magnetic fields in galaxies and clusters is $\lambda \sim 0.1$—1 Mpc, and we have thus formulated limits for this scale. If $B(k_D(\eta_{\rm nuc}))k_D(\eta_{\rm nuc})^{3/2} \equiv B_{k_D} \lesssim 10^{-8}$Gauss, the limit on the scale $\lambda \gg 1/k_D(\eta_{\rm nuc})$ is much smaller, namely $B_\lambda = B(k = 1/\lambda)\lambda^{-3/2} = B_{k_D}(k_D(\eta_{\rm nuc})\lambda)^{-(n+3)/2} \lesssim 10^{-8}{\rm Gauss} \times 10^{-6(n+3)/2}$. For a spectral index $n = 2$, for example, B_λ is smaller than the maximal field by a factor of about 10^{15}, namely $B_\lambda \lesssim 10^{-23}$ Gauss.

In conclusion, even if the magnetic field and gravitational wave energy densities scale in the same way with the expansion of the universe, applying the nucleosynthesis bound on the induced gravitational wave energy density gives a much stronger constraint on the amplitude of the magnetic field. Both the magnetic field and gravitational wave energy spectra are blue, and therefore dominated by their value at the upper cutoff. However, the upper cutoff for the gravitational wave spectrum is much higher than the one for the magnetic field spectrum at the epoch of nucleosynthesis: $\eta_*^{-1} \gg k_D(\eta_{\rm nuc})$. The reason for this being, that the conversion of magnetic field energy into gravitational waves takes place when a given mode enters the horizon, before the magnetic field is dissipated by interaction with the cosmic fluid.

Furthermore, the interesting limit is not the one on the 'mean magnetic field' which is dominated by the value at the smallest scale, but the limit on the field amplitude at some scale λ which is relevant for galactic magnetic fields, and certainly has to be larger than the damping scale at the redshift of galaxy formation.

Of course, these limits apply for magnetic fields generated before the epoch of nucleosynthesis. Moreover, they are valid in the context of linear perturbation theory: we have neglected back-reaction effects of the generated gravitational waves on the source magnetic field (in [1] we present a very qualitative discussion of some possible consequences of back-reaction). The mutual interaction of a magnetic field and a pre-existing background of gravitational waves (from inflation) is studied in Refs .[7, 8], where it is found that this coupling may lead to an amplification of the seed magnetic field. If this mechanism can be seen as a possible back-reaction effect for our case, then it would go in the direction of strengthening our limits on the seed magnetic field. We suppose that the combination of the two processes, conversion of magnetic energy in gravitational waves leading to a reduction of the magnetic field on the one hand and amplification of the magnetic field by this gravitational radiation, should reach at some point an equilibrium, in which a considerable fraction of magnetic energy is converted into gravity waves.

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