

Cosmological Perturbation Theory

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Abstract. This is a review on cosmological perturbation theory. After an introduction, it presents the problem of gauge transformation. Gauge invariant variables are introduced and the Einstein and conservation equations are written in terms of these variables. Some examples, especially perfect fluids and scalar fields are presented in detail. The generation of perturbations during inflation is studied. Lightlike geodesics and their relevance for CMB anisotropies are briefly discussed. Perturbation theory in braneworlds is also introduced.

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1 Introduction

The idea that the large scale structure of our Universe might have grown out of small initial fluctuations via gravitational instability goes back to Newton (letter to Bentley, 1692[1]).

The first relativistic treatment of linear perturbations in a Friedmann-Lemaître universe was given by Lifshitz (1946)[2]. There He found that the gravitational potential cannot grow within linear perturbation theory and he concluded that galaxies have not formed by gravitational instability.

Today we know that it is sufficient that matter density fluctuations can grow. Nevertheless, considerable initial fluctuations with amplitudes of the order of 10^{-5} are needed in order to reproduce the cosmic structures observed today. These are much larger than typical statistical fluctuations on scales of galaxies and we have to propose a mechanism to generate them. Furthermore, the measurements of anisotropies in the cosmic microwave background show that the amplitude of fluctuations is constant over a wide range of scales, the spectrum is scale independent.

As we shall see, standard inflation generically produces such a spectrum of nearly scale invariant fluctuations.

In this course I present gauge invariant cosmological perturbation theory. I shall start by defining gauge invariant perturbation variables. Then I present the basic perturbation equations. As examples for the matter equations we shall consider perfect fluids and scalar fields. Then we briefly discuss lightlike geodesics and CMB anisotropies (this section will be very brief since it is complemented by the course on CMB anisotropies by A. Challinor). Finally, I shall make some brief comments on perturbation theory for braneworlds, a topic which is still wide open in my opinion.

2 The background

I shall not come back to the homogeneous universe which has been discussed in depth in the course by K. Tamvakis. I just specify our notation which is as follows:

A Friedmann-Lemaître universe is a homogeneous and isotropic solution of Einstein's equations. The hyper-surfaces of constant time are homogeneous and isotropic, *i.e.*, spaces of constant curvature with metric $a^2(\eta)\gamma_{ij}dx^i dx^j$, where γ_{ij} is the metric of a space with constant curvature κ . This metric can be expressed in the form

$$\gamma_{ij}dx^i dx^j = dr^2 + \chi^2(r) (d\vartheta^2 + \sin^2\vartheta d\varphi^2) \quad (1)$$

$$\chi(r) = \begin{cases} r & , \kappa = 0 \\ \sin r & , \kappa = 1 \\ \sinh r & , \kappa = -1, \end{cases} \quad (2)$$

where we have rescaled $a(\eta)$ such that $\kappa = \pm 1$ or 0. (With this normalization the scale factor a has the dimension of a length and η and r are dimensionless for $\kappa \neq 0$.) The four-dimensional metric is then of the form

$$g_{\mu\nu}dx^\mu dx^\nu = -a^2(\eta)d\eta^2 + a^2(\eta)\gamma_{ij}dx^i dx^j. \quad (3)$$

Here η is called the *conformal time*. The *physical* or *cosmological time* is given by $dt = a d\eta$.

Einstein's equations reduce to ordinary differential equations for the function $a(\eta)$ (with $\dot{} \equiv d/d\eta$):

$$\left(\frac{\dot{a}}{a}\right)^2 + \kappa = \mathcal{H}^2 + \kappa = \frac{8\pi G}{3}a^2\rho + \frac{1}{3}\Lambda a^2 \quad (4)$$

$$\left(\frac{\dot{a}}{a}\right)' = \dot{\mathcal{H}} = -\frac{4\pi G}{3}a^2(\rho + 3p) + \frac{1}{3}\Lambda a^2 = \left(\frac{\ddot{a}}{a}\right) - \left(\frac{\dot{a}}{a}\right)^2, \quad (5)$$

where $\rho = -T_0^0$, $p = T_i^i$ (no sum!) and all other components of the energy momentum tensor have to vanish by the requirement of isotropy and homogeneity. Λ is the cosmological constant. We have introduced $\mathcal{H} = \dot{a}/a$. The Hubble parameter is defined by

$$H = \frac{da/dt}{a} = \frac{\dot{a}}{a^2} = \mathcal{H}/a.$$

Energy momentum "conservation" (which is also a consequence of (4) and (5) due to the contracted Bianchi identity) reads

$$\dot{\rho} = -3\left(\frac{\dot{a}}{a}\right)(\rho + p) = -3(1+w)\mathcal{H}, \quad (6)$$

where $w \equiv p/\rho$. Later we will also use $c_s^2 \equiv \dot{p}/\dot{\rho}$. From the definition of w and ρ together with Eq. (6) one finds

$$\dot{w} = 3(1+w)(w - c_s^2)\mathcal{H}. \quad (7)$$

From the Friedmann equations one easily concludes that for $\kappa = \Lambda = 0$ and $w = \text{const.}$ the scale factor behaves like a power law,

$$a \propto \eta^{\frac{2}{1+3w}} \propto t^{\frac{2}{3(1+w)}}. \quad (8)$$

Important examples are

$$a \propto \eta^q \quad \text{with} \quad \begin{cases} q = 2 & \text{for dust,} & w = 0 \\ q = 1 & \text{for radiation,} & w = 1/3 \\ q = -1 & \text{for inflation (or a cosm. const.),} & w = -1 \end{cases} \quad (9)$$

We also define

$$\begin{aligned}
\Omega_\rho &= \left(\frac{8\pi G \rho a^2}{3\mathcal{H}^2} \right)_{\eta=\eta_0} \\
\Omega_\Lambda &= \frac{\Lambda a^2}{3\mathcal{H}^2} \Big|_{\eta=\eta_0} \\
\Omega_\kappa &= \frac{-\kappa}{\mathcal{H}^2} \Big|_{\eta=\eta_0},
\end{aligned} \tag{10}$$

where the index $_0$ indicates the value of a given variable today. Friedmann's equation (4) then requires

$$1 = \Omega_\rho + \Omega_\Lambda + \Omega_\kappa. \tag{11}$$

One often also uses $\Omega = \Omega_\rho + \Omega_\Lambda = 1 - \Omega_\kappa$.

3 Gauge invariant perturbation variables

The observed Universe is not perfectly homogeneous and isotropic. Matter is arranged in galaxies and clusters of galaxies and there are large voids in the distribution of galaxies. Let us assume, however, that these inhomogeneities lead only to small variations of the geometry which we shall treat in first order perturbation theory. For this we define the perturbed geometry by

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \varepsilon a^2 h_{\mu\nu} \tag{12}$$

$\bar{g}_{\mu\nu}$ being the unperturbed Friedmann metric. We conventionally set (absorbing the ‘‘smallness’’ parameter ε into $h_{\mu\nu}$)

$$\begin{aligned}
g_{\mu\nu} &= \bar{g}_{\mu\nu} + a^2 h_{\mu\nu}, & \bar{g}_{00} &= -a^2, & \bar{g}_{ij} &= a^2 \gamma_{ij} & |h_{\mu\nu}| &\ll 1 \\
T_\nu^\mu &= \bar{T}_\nu^\mu + \theta_\nu^\mu, & \bar{T}_0^0 &= -\bar{\rho}, & \bar{T}_j^i &= \bar{p} \delta_j^i & |\theta_\nu^\mu|/\bar{\rho} &\ll 1.
\end{aligned} \tag{13}$$

3.1 Gauge transformation, gauge invariance

The first fundamental problem we want to discuss is the choice of gauge in cosmological perturbation theory:

For linear perturbation theory to apply, the spacetime manifold \mathcal{M} with metric g and the energy momentum tensor T of the real, observable universe must be in some sense close to a Friedmann universe, *i.e.*, the manifold \mathcal{M} with a Robertson–Walker metric \bar{g} and a homogeneous and isotropic energy momentum tensor \bar{T} . It is an interesting, non-trivial unsolved problem how to construct ‘the best’ \bar{g} and \bar{T} from the physical fields g and T in practice. There are two main difficulties: First, spatial averaging procedures depend on the choice of a hyper-surface of constant time and they do not commute with derivatives, so that averaged fields \bar{g} and \bar{T} will in general not satisfy

Einstein's equations. Secondly, averaging is in practice impossible over super-horizon scales.

Even though we cannot give a constructive prescription, we now assume that there exists an averaging procedure which leads to a Friedmann universe with spatially averaged tensor fields \bar{Q} , such that the deviations $(T_{\mu\nu} - \bar{T}_{\mu\nu}) / \max_{\{\alpha\beta\}} \{|\bar{T}_{\alpha\beta}|\}$ and $(g_{\mu\nu} - \bar{g}_{\mu\nu}) / \max_{\{\alpha\beta\}} \{\bar{g}_{\alpha\beta}\}$ are small, and \bar{g} and \bar{T} satisfy Friedmann's equations. Let us call such an averaging procedure 'admissible'. There may be many different admissible averaging procedures (e.g. over different hyper-surfaces) leading to slightly different Friedmann backgrounds. But since $|g - \bar{g}|$ is small of order ϵ , the difference of the two Friedmann backgrounds must also be small of order ϵ and we can regard it as part of the perturbation.

We consider now a fixed admissible Friedmann background (\bar{g}, \bar{T}) as chosen. Since the theory is invariant under diffeomorphisms (coordinate transformations), the perturbations are not unique. For an arbitrary diffeomorphism ϕ and its push forward ϕ^* , the two metrics g and $\phi^*(g)$ describe the same geometry. Since we have chosen the background metric \bar{g} we only allow diffeomorphisms which leave \bar{g} invariant *i.e.* which deviate only in first order from the identity. Such an 'infinitesimal' diffeomorphism can be represented as the infinitesimal flow of a vector field X , $\phi = \phi_\epsilon^X$. Remember the definition of the flow: For the integral curve $\gamma_x(s)$ of X with starting point x , *i.e.*, $\gamma_x(s=0) = x$ we have $\phi_s^X(x) = \gamma_x(s)$. In terms of the vector field X , to first order in ϵ , its pullback is then of the form

$$\phi^* = id + \epsilon L_X$$

(L_X denotes the Lie derivative in direction X). The transformation $g \rightarrow \phi^*(g)$ is equivalent to $\bar{g} + \epsilon a^2 h \rightarrow \bar{g} + \epsilon(a^2 h + L_X \bar{g}) + \mathcal{O}(\epsilon^2)$, *i.e.* under an 'infinitesimal coordinate transformation' the metric perturbation h transforms as

$$h \rightarrow h + a^{-2} L_X \bar{g} . \quad (14)$$

In the context of cosmological perturbation theory, infinitesimal coordinate transformations are called 'gauge transformations'. The perturbation of a arbitrary tensor field $Q = \bar{Q} + \epsilon Q^{(1)}$ obeys the gauge transformation law

$$Q^{(1)} \rightarrow Q^{(1)} + L_X \bar{Q} . \quad (15)$$

Since every vector field X generates a gauge transformation $\phi = \phi_\epsilon^X$, we can conclude that only perturbations of tensor fields with $L_X \bar{Q} = 0$ for all vector fields X , *i.e.*, with vanishing (or constant) 'background contribution' are gauge invariant. This simple result is sometimes referred to as the 'Stewart-Walker Lemma' [3].

The gauge dependence of perturbations has caused many controversies in the literature, since it is often difficult to extract the physical meaning of gauge dependent perturbations, especially on super-horizon scales. This has

led to the development of gauge invariant perturbation theory which we are going to use throughout this review. The advantage of the gauge-invariant formalism is that the variables used have simple geometric and physical meanings and are not plagued by gauge modes. Although the derivation requires somewhat more work, the final system of perturbation equations is usually simple and well suited for numerical treatment. We shall also see, that on sub-horizon scales, the gauge invariant matter perturbation variables approach the usual, gauge dependent ones. Since one of the gauge invariant geometrical perturbation variables corresponds to the Newtonian potential, the Newtonian limit can be performed easily.

First we note that since all relativistic equations are covariant (i.e. can be written in the form $Q = 0$ for some tensor field Q), it is always possible to express the corresponding perturbation equations in terms of gauge invariant variables [4, 5, 6].

3.2 Harmonic decomposition of perturbation variables

Since the $\{\eta = \text{const}\}$ hyper-surfaces are homogeneous and isotropic, it is sensible to perform a harmonic analysis: A (spatial) tensor field Q on these hyper-surfaces can be decomposed into components which transform irreducibly under translations and rotations. All such components evolve independently. For a scalar quantity f in the case $\kappa = 0$ this is nothing else than its Fourier decomposition:

$$f(\mathbf{x}, \eta) = \int d^3k \hat{f}(\mathbf{k}) e^{i\mathbf{k}\mathbf{x}}. \quad (16)$$

(The exponentials $Y_{\mathbf{k}}(\mathbf{x}) = e^{i\mathbf{k}\mathbf{x}}$ are the unitary irreducible representations of the Euclidean translation group.) For $\kappa = 1$ such a decomposition also exists, but the values k are discrete, $k^2 = \ell(\ell + 2)$ and for $\kappa = -1$, they are bounded from below, $k^2 > 1$. Of course, the functions $Y_{\mathbf{k}}$ are different for $\kappa \neq 0$.

They form the complete orthogonal set of eigenfunctions of the Laplacian,

$$\Delta Y_{\mathbf{k}}^{(S)} = -k^2 Y_{\mathbf{k}}^{(S)}. \quad (17)$$

In addition, a tensorial variable (at fixed position \mathbf{x}) can be decomposed into irreducible components under the rotation group $SO(3)$.

For a vector field, this is its decomposition into a gradient and a rotation,

$$V_i = \nabla_i \varphi + B_i, \quad (18)$$

where

$$B_{|i}^i = 0, \quad (19)$$

where we used $X_{|i}$ to denote the three-dimensional covariant derivative of X . Here φ is the spin 0 and \mathbf{B} is the spin 1 component of the vector field V .

For a symmetric tensor field we have

$$H_{ij} = H_L \gamma_{ij} + \left(\nabla_i \nabla_j - \frac{1}{3} \Delta \gamma_{ij} \right) H_T + \frac{1}{2} \left(H_{i|j}^{(V)} + H_{j|i}^{(V)} \right) + H_{ij}^{(T)}, \quad (20)$$

where

$$H_i^{(V)|i} = H_i^{(T)i} = H_{i|j}^{(T)j} = 0. \quad (21)$$

Here H_L and H_T are spin 0 components, $H_i^{(V)}$ is a spin 1 component and $H_{ij}^{(T)}$ is a spin 2 component.

We shall not need higher tensors (or spinors). As a basis for vector and tensor modes we use the vector and tensor type eigenfunctions of the Laplacian,

$$\Delta Y_j^{(V)} = -k^2 Y_j^{(V)} \quad \text{and} \quad (22)$$

$$\Delta Y_{ji}^{(T)} = -k^2 Y_{ji}^{(T)}, \quad (23)$$

where $Y_j^{(V)}$ is a transverse vector, $Y_j^{(V)|j} = 0$ and $Y_{ji}^{(T)}$ is a symmetric transverse traceless tensor, $Y_j^{(T)j} = Y_{ji}^{(T)i} = 0$.

According to Eqs. (18) and (20) we can construct scalar type vectors and tensors and vector type tensors. To this goal we define

$$Y_j^{(S)} \equiv -k^{-1} Y_{|j}^{(S)} \quad (24)$$

$$Y_{ij}^{(S)} \equiv k^{-2} Y_{|ij}^{(S)} + \frac{1}{3} \gamma_{ij} Y^{(S)} \quad (25)$$

$$Y_{ij}^{(V)} \equiv -\frac{1}{2k} (Y_{i|j}^{(V)} + Y_{j|i}^{(V)}). \quad (26)$$

In the following we shall extensively use this decomposition and write down the perturbation equations for a given mode k .

The decomposition of a vector field is then of the form

$$B_i = B Y_i^{(S)} + B^{(V)} Y_i^{(V)}. \quad (27)$$

The decomposition of a tensor field is given by (compare Eq. (20))

$$H_{ij} = H_L Y^{(S)} \gamma_{ij} + H_T Y_{ij}^{(S)} + H^{(V)} Y_{ij}^{(V)} + H^{(T)} Y_{ij}^{(T)}. \quad (28)$$

Here B , $B^{(V)}$, H_L , H_T , $H^{(V)}$ and $H^{(T)}$ are functions of η and \mathbf{k}

3.3 Metric perturbations

Perturbations of the metric are of the form

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + a^2 h_{\mu\nu}. \quad (29)$$

We parameterize them as

$$h_{\mu\nu}dx^\mu dx^\nu = -2Ad\eta^2 - 2B_i d\eta dx^i + 2H_{ij}dx^i dx^j, \quad (30)$$

and we decompose the perturbation variables B_i and H_{ij} according to (27) and (28).

Let us consider the behavior of $h_{\mu\nu}$ under gauge transformations. We set the vector field defining the gauge transformation to

$$\mathbf{X} = T\partial_\eta + L^i\partial_i. \quad (31)$$

Using simple identities from differential geometry like $L_{\mathbf{X}}(df) = d(L_{\mathbf{X}}f)$ and $(L_{\mathbf{X}}\gamma)_{ij} = X_{i|j} + X_{j|i}$, we obtain

$$\begin{aligned} L_{\mathbf{X}}\bar{g} = a^2 & \left[-2 \left(\frac{\dot{a}}{a}T + \dot{T} \right) d\eta^2 + 2 \left(\dot{L}_i - T_{,i} \right) d\eta dx^i \right. \\ & \left. + \left(2\frac{\dot{a}}{a}T\gamma_{ij} + L_{i|j} + L_{j|i} \right) dx^i dx^j \right]. \end{aligned} \quad (32)$$

Comparing this with (30) and using (14) we obtain the following behavior of our perturbation variables under gauge transformations (we decompose the vector $L_i = LY_i^{(S)} + L^{(V)}Y_i^{(V)}$):

$$A \rightarrow A + \frac{\dot{a}}{a}T + \dot{T} \quad (33)$$

$$B \rightarrow B - \dot{L} - kT \quad (34)$$

$$B^{(V)} \rightarrow B^{(V)} - \dot{L}^{(V)} \quad (35)$$

$$H_L \rightarrow H_L + \frac{\dot{a}}{a}T + \frac{k}{3}L \quad (36)$$

$$H_T \rightarrow H_T - kL \quad (37)$$

$$H^{(V)} \rightarrow H^{(V)} - kL^{(V)} \quad (38)$$

$$H^{(T)} \rightarrow H^{(T)}. \quad (39)$$

Two scalar and one vector variable can be set to zero by a cleverly chosen gauge transformations.

One often chooses $kL = H_T$ and $kT = B + \dot{L}$, so that the variables H_T and B vanish. In this gauge (longitudinal gauge), scalar perturbations of the metric are of the form ($H_T = B = 0$):

$$h_{\mu\nu}^{(S)} = -2\Psi d\eta^2 - 2\Phi\gamma_{ij}dx^i dx^j. \quad (40)$$

Ψ and Φ are the so called *Bardeen* potentials. In a generic gauge they are given by

$$\Psi = A - \frac{\dot{a}}{a}k^{-1}\sigma - k^{-1}\dot{\sigma} \quad (41)$$

$$\Phi = -H_L - \frac{1}{3}H_T + \frac{\dot{a}}{a}k^{-1}\sigma \quad (42)$$

with $\sigma = k^{-1}\dot{H}_T - B$. A short calculation using Eqs. (33) to (37) shows that Ψ and Φ are gauge invariant.

In a Friedmann universe the Weyl tensor vanishes. It therefore is a gauge invariant perturbation. For scalar perturbations one finds

$$C^0_{i0j} = \frac{1}{2} \left[(\Psi + \Phi)_{|ij} - \frac{1}{3} \Delta(\Psi + \Phi) \gamma_{ij} \right] \quad (43)$$

All other components vanish.

For vector perturbations it is convenient to set $kL^{(V)} = H^{(V)}$ so that $H^{(V)}$ vanishes and we have

$$h_{\mu\nu}^{(V)} dx^\mu dx^\nu = 2\sigma^{(V)} Y_i^{(V)} d\eta dx^i. \quad (44)$$

We shall call this gauge the ‘‘vector gauge’’. In general $\sigma^{(V)} = k^{-1}\dot{H}^{(V)} - B^{(V)}$ is gauge invariant¹. The Weyl tensor from vector perturbation is given by

$$C^0_{i0j} = \frac{1}{2} \dot{\sigma}^{(V)} Y_{ij}^{(V)} \quad (45)$$

$$C^0_{jlm} = \frac{1}{2} \sigma^{(V)} [Y_{j|l|m}^{(V)} - Y_{jm|l}^{(V)} - \frac{1}{3} \gamma^{jl} Y_{m|k}^{(V)} + \frac{1}{3} \gamma_{jm} Y_{l|k}^{(V)}] \quad (46)$$

Clearly there are no tensorial (spin 2) gauge transformation and hence $H_{ij}^{(T)}$ is gauge invariant. The expression for the Weyl tensor from tensor perturbation is identical to the one for vector perturbation upon replacement of $\sigma_{ij}^{(V)}$ by $\dot{H}_{ij}^{(T)}$.

3.4 Perturbations of the energy momentum tensor

Let $T_\nu^\mu = \bar{T}_\nu^\mu + \theta_\nu^\mu$ be the full energy momentum tensor. We define its energy density ρ and its energy flow 4-vector u as the time-like eigenvalue and eigenvector of T_ν^μ :

$$T_\nu^\mu u^\nu = -\rho u^\mu, \quad u^2 = -1. \quad (47)$$

We then parameterize their perturbations by

$$\rho = \bar{\rho}(1 + \delta), \quad u = u^0 \partial_t + u^i \partial_i. \quad (48)$$

u^0 is fixed by the normalization condition,

$$u^0 = \frac{1}{a}(1 - A). \quad (49)$$

We further set

$$u^i = \frac{1}{a} v^i = v Y^{(S)i} + v^{(V)} Y^{(V)i}. \quad (50)$$

¹ $Y_{ij}^{(V)} \sigma^{(V)}$ is the shear of the hyper-surfaces of constant time.

We define $P_\nu^\mu \equiv w^\mu u_\nu + \delta_\nu^\mu$, the projection tensor onto the part of tangent space normal to u and set the stress tensor

$$\tau^{\mu\nu} = P_\alpha^\mu P_\beta^\nu T^{\alpha\beta}. \quad (51)$$

In the unperturbed case we have $\tau_0^0 = 0, \tau_j^i = \bar{p}\delta_j^i$. Including perturbations, to first order we still obtain

$$\tau_0^0 = \tau_i^0 = \tau_0^i = 0. \quad (52)$$

But τ_j^i contains in general perturbations. We set

$$\tau_j^i = \bar{p} [(1 + \pi_L) \delta_j^i + \Pi_j^i], \quad \text{with } \Pi_i^i = 0. \quad (53)$$

We decompose Π_j^i as

$$\Pi_j^i = \Pi^{(S)} Y_j^{(S)i} + \Pi^{(V)} Y_j^{(V)i} + \Pi^{(T)} Y_j^{(T)i}. \quad (54)$$

We shall not derive the gauge transformation properties of these perturbation variables in detail, but just state some results which can be obtained as an exercise (see also [5]):

- Of the variables defined above only the $\Pi^{(S,V,T)}$ are gauge invariant; they describe the anisotropic stress tensor, $\Pi_\nu^\mu = \tau_\nu^\mu - 1/3\tau_\alpha^\alpha \delta_\nu^\mu$. They are gauge invariant due to the Stewart–Walker lemma, since $\bar{\Pi} = 0$. For perfect fluids $\Pi_\nu^\mu = 0$.
- A second gauge invariant variable is

$$\Gamma = \pi_L - \frac{c_s^2}{w} \delta, \quad (55)$$

where $c_s^2 \equiv \dot{p}/\dot{\rho}$ is the adiabatic sound speed and $w \equiv p/\rho$ is the enthalpy. One can show that Γ is proportional to the divergence of the entropy flux of the perturbations. Adiabatic perturbations are characterized by $\Gamma = 0$.

- Gauge invariant density and velocity perturbations can be found by combining δ, v and $v_i^{(V)}$ with metric perturbations.

We shall use

$$V \equiv v - \frac{1}{k} \dot{H}_T = v^{(\text{long})} \quad (56)$$

$$D_s \equiv \delta + 3(1+w) \frac{\dot{a}}{a} (k^{-2} \dot{H}_T - k^{-1} B) \equiv \delta^{(\text{long})} \quad (57)$$

$$D \equiv \delta^{(\text{long})} + 3(1+w) \left(\frac{\dot{a}}{a} \right) \frac{V}{k} \quad (58)$$

$$D_g \equiv \delta + 3(1+w) \left(H_L + \frac{1}{3} H_T \right) = \delta^{(\text{long})} - 3(1+w) \Phi \quad (59)$$

$$V^{(V)} \equiv v^{(V)} - \frac{1}{k} \dot{H}^{(V)} = v^{(\text{vec})} \quad (60)$$

$$\Omega \equiv v^{(V)} - B^{(V)} = v^{(\text{vec})} - B^{(V)} \quad (61)$$

$$\Omega - V^{(V)} = \sigma^{(V)}. \quad (62)$$

Here $v^{(\text{long})}$, $\delta^{(\text{long})}$ and $v_i^{(\text{vec})}$ are the velocity (and density) perturbations in the longitudinal and vector gauge respectively, and $\sigma^{(V)}$ is the metric perturbation in vector gauge (see Eq. (44)). These variables can be interpreted nicely in terms of gradients of the energy density and the shear and vorticity of the velocity field [7].

Here I just want to show that on scales much smaller than the Hubble scale, $k\eta \gg 1$, the metric perturbations are much smaller than δ and v and we can thus “forget them” (which will be important when comparing experimental results with calculations in this formalism):

The perturbations of the Einstein tensor are given by second derivatives of the metric perturbations. Einstein’s equations yield the following order of magnitude estimate:

$$\mathcal{O}\left(\frac{\delta T}{T}\right) \underbrace{\mathcal{O}(8\pi GT)}_{\mathcal{O}\left(\frac{\dot{a}}{a}\right)^2 = \mathcal{O}(\eta^{-2})} = \mathcal{O}\left(\frac{1}{\eta^2}h + \frac{k}{\eta}h + k^2h\right) \quad (63)$$

$$\mathcal{O}\left(\frac{\delta T}{T}\right) = \mathcal{O}(h + k\eta h + (k\eta)^2h). \quad (64)$$

For $k\eta \gg 1$ this gives $\mathcal{O}(\delta, v) = \mathcal{O}\left(\frac{\delta T}{T}\right) \gg \mathcal{O}(h)$. On sub-horizon scales the difference between δ , $\delta^{(\text{long})}$, D_g and D is negligible as well as the difference between v and V or $v^{(V)}$, $V^{(V)}$ and $\Omega^{(V)}$.

4 Einstein’s equations

We do not derive the first order perturbations of Einstein’s equations. This can be done by different methods, for example with Mathematica. We just write down those equations which we shall need later.

4.1 Constraint equations

$$\left. \begin{aligned} 4\pi Ga^2 \rho D &= -(k^2 - 3\kappa)\Phi \quad (00) \\ 4\pi Ga^2(\rho + p)V &= k \left(\left(\frac{\dot{a}}{a}\right) \Psi + \dot{\Phi} \right) \quad (0i) \end{aligned} \right\} \text{(scalar)} \quad (65)$$

$$8\pi Ga^2(\rho + p)\Omega = \frac{1}{2}(2\kappa - k^2)\sigma^{(V)} \quad (0i) \quad \text{(vector)} \quad (66)$$

4.2 Dynamical equations

$$k^2 (\Phi - \Psi) = 8\pi G a^2 p \Pi^{(S)} \quad (\text{scalar}) \quad (67)$$

$$k \left(\dot{\sigma}^{(V)} + 2 \left(\frac{\dot{a}}{a} \right) \sigma^{(V)} \right) = 8\pi G a^2 p \Pi^{(V)} \quad (\text{vector}) \quad (68)$$

$$\ddot{H}^{(T)} + 2 \left(\frac{\dot{a}}{a} \right) \dot{H}^{(T)} + (2\kappa + k^2) H^{(T)} = 8\pi G a^2 p \Pi_{ij}^{(T)} \quad (\text{tensor}) \quad (69)$$

There is a second dynamical scalar equation, which is somewhat cumbersome and not really needed, since we may use one of the conservation eqs. given below instead. Note that for perfect fluids, where $\Pi_j^i \equiv 0$, we have $\Phi = \Psi$, $\sigma^{(V)} \propto 1/a^2$ and $H^{(T)}$ obeys a damped wave equation. The damping term can be neglected on small scales (over short time periods) when $\eta^{-2} \lesssim 2\kappa + k^2$, and H_{ij} represents propagating gravitational waves. For vanishing curvature, these are just the sub-horizon scales, $k\eta \gtrsim 1$. For $\kappa < 0$, waves oscillate with a somewhat smaller frequency, $\omega = \sqrt{2\kappa + k^2}$, while for $\kappa > 0$ the frequency is somewhat larger than k .

4.3 Energy momentum conservation

The conservation equations, $T_{;\nu}^{\mu\nu} = 0$ lead to the following perturbation equations.

$$\left. \begin{aligned} \dot{D}_g + 3(c_s^2 - w) \left(\frac{\dot{a}}{a} \right) D_g + (1+w)kV + 3w \left(\frac{\dot{a}}{a} \right) \Gamma = 0 \\ \dot{V} + \left(\frac{\dot{a}}{a} \right) (1 - 3c_s^2) V = k \left(\Psi + 3c_s^2 \Phi \right) + \frac{c_s^2 k}{1+w} D_g \\ + \frac{wk}{1+w} \left[\Gamma - \frac{2}{3} \left(1 - \frac{3\kappa}{k^2} \right) \Pi \right] \end{aligned} \right\} \quad (\text{scalar}) \quad (70)$$

$$\dot{\Omega}_i + (1 - 3c_s^2) \left(\frac{\dot{a}}{a} \right) \Omega_i = \frac{p}{2(\rho + p)} \left(k - \frac{2\kappa}{k} \right) \Pi_i^{(V)} \quad (\text{vector}) \quad (71)$$

These can of course also be obtained from the Einstein equations since they are equivalent to the contracted Bianchi identities. For scalar perturbations we have 4 independent equations and 6 variables. For vector perturbations we have 2 equations and 3 variables, while for tensor perturbations we have 1 equations and 2 variables. To close the system we must add some matter equations. The simplest prescription is to set $\Gamma = \Pi_{ij} = 0$. This matter equation, which describes adiabatic perturbations of a perfect fluid gives us exactly two additional equations for scalar perturbations and one each for vector and tensor perturbations.

Another simple example is a universe with matter content given by a scalar field. We shall discuss this case in the next section. More complicated examples are those of several interacting particle species of which some have to be described by a Boltzmann equation. This is the actual universe at late times, say $z \lesssim 10^7$.

4.4 A special case

Here we want to rewrite the scalar perturbation equations for a simple but important special case. We consider adiabatic perturbations of a perfect fluid. In this case $\Pi = 0$ since there are no anisotropic stresses and $\Gamma = 0$. Eq. (67) then implies $\Phi = \Psi$. Using the first equation of (65) and Eqs. (59,58) to replace D_g in the second of Eqs. (70) by Ψ and V , finally replacing V by (65) one can derive a second order equation for Ψ , which is, in this case the only dynamical degree of freedom

$$\ddot{\Psi} + 3\mathcal{H}(1 + c_s^2)\dot{\Psi} + [(1 + 3c_s^2)(\mathcal{H}^2 - \kappa) - (1 + 3w)(\mathcal{H}^2 + \kappa) + c_s^2 k^2]\Psi = 0. \quad (72)$$

Another interesting case (especially when discussing inflation) is the scalar field case. There, as we shall see in Section 6, $\Pi = 0$, but in general $\Gamma \neq 0$ since $\delta p/\delta\rho \neq \dot{p}/\dot{\rho}$. Nevertheless, since this case again has only one dynamical degree of freedom, we can express the perturbation equations in terms of one single second order equation for Ψ . In Section 6 we shall find the following equation for a perturbed scalar field cosmology

$$\ddot{\Psi} + 3\mathcal{H}(1 + c_s^2)\dot{\Psi} + [(1 + 3c_s^2)(\mathcal{H}^2 - \kappa) - (1 + 3w)(\mathcal{H}^2 + \kappa) + k^2]\Psi = 0. \quad (73)$$

The only difference between the perfect fluid and scalar field perturbation equation is that the latter is missing the factor c_s^2 in front of the oscillatory k^2 term. Note also that for $\kappa = 0$ and $w = c_s^2 = \text{constant}$ the time dependent mass term $m^2(\eta) = -(1 + 3c_s^2)(\mathcal{H}^2 - \kappa) + (1 + 3w)(\mathcal{H}^2 + \kappa)$ vanishes. It is useful to define also the variable [11]

$$u = a \left[4\pi G(\mathcal{H}^2 - \dot{\mathcal{H}} + \kappa) \right]^{-1/2} \Psi, \quad (74)$$

which satisfies the equation

$$\ddot{u} + (\Upsilon k^2 - \ddot{\theta}/\theta)u = 0, \quad (75)$$

where $\Upsilon = c_s^2$ or $\Upsilon = 1$ for a perfect fluid or a scalar field background respectively, and

$$\theta = \frac{3\mathcal{H}}{2a\sqrt{\mathcal{H}^2 - \dot{\mathcal{H}} + \kappa}}. \quad (76)$$

Another interesting variable is

$$\zeta \equiv \frac{2(\mathcal{H}^{-1}\dot{\Psi} + \Psi)}{3(1 + w)} + \Psi. \quad (77)$$

For the rest of this section we set $\kappa = 0$. Using Eqs. (72) and (73) respectively one then finds

$$\dot{\zeta} = -k^2 \frac{\Upsilon\mathcal{H}}{\mathcal{H}^2 - \dot{\mathcal{H}}} \Psi, \quad (78)$$

hence on super horizon scales, $k/\mathcal{H} \ll 1$, this variable is conserved.

The evolution of ζ is closely related to the canonical variable v defined by

$$v = -\frac{a\sqrt{\mathcal{H}^2 - \dot{\mathcal{H}}}}{\sqrt{4\pi\bar{G}\Upsilon\mathcal{H}}}\zeta. \quad (79)$$

which satisfies the equation

$$\ddot{v} + (\Upsilon k^2 - \dot{z}/z)v = 0, \quad (80)$$

for

$$z = \frac{a\sqrt{\mathcal{H}^2 - \dot{\mathcal{H}} + \kappa}}{\Upsilon\mathcal{H}}. \quad (81)$$

More details on the significance of the canonical variable v will be found in sections 6 and 7.

5 Simple examples

We first discuss two simple applications which are important to understand the CMB anisotropy spectrum.

5.1 The pure dust fluid for $\kappa = 0, \Lambda = 0$

We assume the dust to have $w = c_s^2 = p = 0$ and $\Pi = \Gamma = 0$. Equation (72) then reduces to

$$\ddot{\Psi} + \frac{6}{\eta}\dot{\Psi} = 0, \quad (82)$$

with the general solution

$$\Psi = \Psi_0 + \Psi_1 \frac{1}{\eta^5} \quad (83)$$

with arbitrary constants Ψ_0 and Ψ_1 . Since the perturbations are supposed to be small initially, they cannot diverge for $\eta \rightarrow 0$, and we have therefore to choose the growing mode, $\Psi_1 = 0$. Another way to argue is as follows: If the mode Ψ_1 has to be small already at some early initial time η_{in} , it will be even much smaller later and may hence be neglected at late times. But also the Ψ_0 mode is only constant. This fact led Lifshitz who was the first to analyze cosmological perturbations to the conclusions that linear perturbations do not grow in a Friedman universe and cosmic structure cannot have evolved by gravitational instability [2]. However, the important point to note here is that, even if the gravitational potential remains constant, matter density fluctuations do grow on sub-horizon scales and therefore inhomogeneities, structure can evolve on scales which are smaller than the Hubble scale. To see that we consider the conservation equations (70), (67) and the Poisson equation (65). For the pure dust case, $w = c_s^2 = \Pi = \Gamma = 0$, they reduce to

$$\dot{D}_g = -kV \quad (\text{energy conservation}) \quad (84)$$

$$\dot{V} + \left(\frac{\dot{a}}{a}\right)V = k\Psi \quad (\text{gravitational acceleration}) \quad (85)$$

$$-k^2\Psi = 4\pi G a^2 \rho \left(D_g + 3 \left(\Psi + \left(\frac{\dot{a}}{a}\right) k^{-1}V \right) \right) \quad (\text{Poisson}), \quad (86)$$

where we have used the relation

$$D = D_g + 3(1+w) \left(-\Phi + \left(\frac{\dot{a}}{a}\right) k^{-1}V \right). \quad (87)$$

The Friedmann equation for dust gives $4\pi G \rho a^2 = 3/2(\dot{a}/a)^2 = 6/\eta^2$. Setting $k\eta = x$ and $' = d/dx$, the system (84-86) becomes

$$D'_g = -V \quad (88)$$

$$V' + \frac{2}{x}V = \Psi \quad (89)$$

$$\frac{6}{x^2} \left(D_g + 3 \left(\Psi + \frac{2}{x}V \right) \right) = -\Psi. \quad (90)$$

We use (90) to eliminate Ψ and (88) to eliminate D_g , leading to

$$(18 + x^2) V'' + \left(\frac{72}{x} + 4x \right) V' - \left(\frac{72}{x^2} + 4 \right) V = 0. \quad (91)$$

The general solution of Eq. (91) is

$$V = V_0 x + \frac{V_1}{x^4} \quad (92)$$

The V_1 mode is the decaying mode (corresponding to Ψ_1) which we neglect. The perturbation variables are then given by

$$V = V_0 x \quad (93)$$

$$D_g = -15V_0 - \frac{1}{2}V_0 x^2 \quad (94)$$

$$V_0 = \Psi_0/3. \quad (95)$$

We distinguish two regimes:

i) super-horizon, $x \ll 1$ where we have

$$V = \frac{1}{3}\Psi_0 x \quad (96)$$

$$D_g = -5\Psi_0 \quad (97)$$

$$\Psi = \Psi_0. \quad (98)$$

Note that even though V is growing, it always remains much smaller than Ψ or D_g on super-horizon scales. Hence the largest fluctuations are of order Ψ

which is constant.

ii) Sub-horizon, $x \gg 1$ where the solution is dominated by the terms

$$V = \frac{1}{3}\Psi_0 x \quad (99)$$

$$D_g = -\frac{1}{6}\Psi_0 x^2 \quad (100)$$

$$\Psi = \Psi_0 = \text{constant}. \quad (101)$$

Note that for dust

$$D = D_g + 3\Psi + \frac{6}{x}V = -\frac{1}{6}\Psi_0 x^2 .$$

In the variable D the constant term has disappeared and we have $D \ll \Psi$ on super-horizon scales, $x \ll 1$.

On sub-horizon scales, the density fluctuations grow like the scale factor $\propto x^2 \propto a$. Nevertheless, Lifshitz' conclusion [2] that pure gravitational instability cannot be the cause for structure formation has some truth: If we start from tiny thermal fluctuations of the order of 10^{-35} , they can only grow to about 10^{-30} due to this mild, power law instability during the matter dominated regime. Or, to put it differently, if we want to form structure by gravitational instability, we need initial fluctuations of the order of at least 10^{-5} , much larger than thermal fluctuations. One possibility to create such fluctuations is quantum particle production in the classical gravitational field during inflation. The rapid expansion of the universe during inflation quickly expands microscopic scales at which quantum fluctuations are important to cosmological scales where these fluctuations are then "frozen in" as classical perturbations in the energy density and the geometry. We will discuss the induced spectrum on fluctuations in Section 7.

5.2 The pure radiation fluid, $\kappa = 0, \Lambda = 0$

In this limit we set $w = c_s^2 = 1/3$ and $II = \Gamma = 0$ so that $\Phi = -\Psi$. We conclude from $\rho \propto a^{-4}$ that $a \propto \eta$. For radiation, the u -equation (75) becomes

$$\ddot{u} + \left(\frac{1}{3}k^2 - \frac{2}{\eta^2}\right)u = 0, \quad (102)$$

with general solution

$$u(x) = A \left(\frac{\sin(x)}{x} - \cos(x) \right) + B \left(\frac{\cos(x)}{x} - \sin(x) \right), \quad (103)$$

where we have set $x = k\eta/\sqrt{3} = c_s k\eta$. For the Bardeen potential we obtain with (74), up to constant factors,

$$\Psi(x) = \frac{u(x)}{x^2}. \quad (104)$$

On super-horizon scales, $x \ll 1$, we have

$$\Psi(x) \simeq \frac{A}{3} + \frac{B}{x^3}. \quad (105)$$

We assume that the perturbations have been initialized at some early time $x_{\text{in}} \ll 1$ and that at this time the two modes have been comparable. If this is the case then $B \ll A$ and we may neglect the B -mode at later times.

To determine the density and velocity perturbations and for illustration, we also solve the radiation equations using the conservation and Poisson equations like for dust. In the radiation case the perturbation equations become (with the same notation as above, $x = c_s k \eta$)

$$D'_g = -\frac{4}{\sqrt{3}}V \quad (106)$$

$$V' = 2\sqrt{3}\Psi + \frac{\sqrt{3}}{4}D_g \quad (107)$$

$$-2x^2\Psi = D_g + 4\Psi + \frac{4}{\sqrt{3}x}V. \quad (108)$$

The general solution of this system is

$$D_g = D_2 \left[\cos(x) - \frac{2}{x} \sin(x) \right] + D_1 \left[\sin(x) + \frac{2}{x} \cos(x) \right] \quad (109)$$

$$V = -\frac{\sqrt{3}}{4}D'_g \quad (110)$$

$$\Psi = -\frac{D_g + \frac{4}{\sqrt{3}x}V}{4 + 2x^2}. \quad (111)$$

Again, regularity at $x = 0$ requires $D_1 = 0$. Comparing with Eqs. (103,104) gives $D_2 = 2A$. In the **super-horizon regime**, $x \ll 1$, we obtain

$$\Psi = \frac{A}{3}, \quad D_g = -2A - \frac{A}{3\sqrt{3}}x^2, \quad V = \frac{A}{2\sqrt{3}}x. \quad (112)$$

On **sub-horizon scales**, $x \gg 1$, we find oscillating solutions with constant amplitude and with frequency of $k/\sqrt{3}$:

$$V = \frac{\sqrt{3}A}{2} \sin(x) \quad (113)$$

$$D_g = 2A \cos(x), \quad \Psi = -A \cos(x)/x^2. \quad (114)$$

Note that also for radiation perturbations

$$D = -\frac{A}{3\sqrt{3}}x^2 \ll \Psi$$

is small on super horizon scales, $x \ll 1$. The perturbation amplitude is given by the largest gauge invariant perturbation variable. We conclude therefore that perturbations outside the Hubble horizon are frozen to first order. Once they enter the horizon they start to collapse, but pressure resists the gravitational force and the radiation fluid fluctuations oscillate at constant amplitude. The perturbations of the gravitational potential oscillate and decay like $1/a^2$ inside the horizon.

5.3 Adiabatic initial conditions

Adiabaticity requires that the perturbations of all contributions to the energy density are initially in thermal equilibrium. This fixes the ratio of the density perturbations of different components. There is no entropy flux and thus $\Gamma = 0$. Here we consider as a simple example non relativistic matter and radiation perturbations. Since the matter and radiation perturbations behave in the same way on super-horizon scales,

$$D_g^{(r)} = A + Bx^2, \quad D_g^{(m)} = A' + B'x^2, \quad V^{(r)} \propto V^{(m)} \propto x, \quad (115)$$

we may require a constant ratio between matter and radiation perturbations. As we have seen in the previous section, inside the horizon ($x > 1$) radiation perturbations start to oscillate while matter perturbations keep following a power law. On sub-horizon scales a constant ratio can thus no longer be maintained. There are two interesting possibilities: adiabatic and isocurvature perturbations. Here we concentrate on adiabatic perturbations which seem to dominate the observed CMB anisotropies.

From $\Gamma = 0$ one easily derives that two components with $p_i/\rho_i = w_i = \text{constant}$, $i = 1, 2$, are adiabatically coupled if $(1 + w_1)D_g^{(2)} = (1 + w_2)D_g^{(1)}$. Energy conservation then implies that also their velocity fields agree, $V^{(1)} = V^{(2)}$. This result is also a consequence of the Boltzmann equation in the strong coupling regime. We therefore require

$$V^{(r)} = V^{(m)}, \quad (116)$$

so that the energy flux in the two fluids is coupled initially.

We restrict ourselves to a matter dominated backgrounds, the situation relevant in the observed universe after equality. We first have to determine the radiation perturbations during a *matter dominated era*. Since Ψ is dominated by the matter contribution (it is proportional to the background density of a given component), we have $\Psi \simeq \text{const} = \Psi_0$. We neglect the contribution from the sub-dominant radiation to Ψ . Energy-momentum conservation for radiation then gives, with $x = k\eta$,

$$D_g^{(r)'} = -\frac{4}{3}V^{(r)} \quad (117)$$

$$V^{(r)'} = 2\Psi + \frac{1}{4}D_g^{(r)}. \quad (118)$$

Now Ψ is just a constant given by the matter perturbations, and it acts like a constant source term. The general solution of this system is then

$$D_g^{(r)} = A \cos(c_s x) - \frac{4}{\sqrt{3}} B \sin(c_s x) + 8\Psi [\cos(c_s x) - 1] \quad (119)$$

$$V^{(r)} = B \cos(c_s x) + \frac{\sqrt{3}}{4} A \sin(c_s x) + 2\sqrt{3}\Psi \sin(c_s x), \quad (120)$$

where $c_s = 1/\sqrt{3}$ is the sound speed of radiation. Our adiabatic initial conditions require

$$\lim_{x \rightarrow 0} \frac{V^{(r)}}{x} = V_0 = \lim_{x \rightarrow 0} \frac{V^{(m)}}{x} < \infty. \quad (121)$$

Therefore $B = 0$ and $V_0 = A/4 - 2\Psi$. Using in addition $\Psi = 3V_0$ (see (101)) we obtain

$$D_g^{(r)} = \frac{4}{3}\Psi \cos\left(\frac{x}{\sqrt{3}}\right) - 8\Psi \quad (122)$$

$$V^{(r)} = \frac{1}{\sqrt{3}}\Psi \sin\left(\frac{x}{\sqrt{3}}\right) \quad (123)$$

$$D_g^{(m)} = -\Psi\left(5 + \frac{1}{6}x^2\right) \quad (124)$$

$$V^{(m)} = \frac{1}{3}\Psi x. \quad (125)$$

On super-horizon scales, $x \ll 1$ we have

$$D_g^{(r)} \simeq -\frac{20}{3}\Psi \quad \text{and} \quad V^{(r)} \simeq \frac{1}{3}x\Psi, \quad (126)$$

note that $D_g^{(r)} = (4/3)D_g^{(m)}$ and $V^{(r)} = V^{(m)}$ for adiabatic initial conditions.

Another possibility for the initial condition would be iso-curvature initial conditions, where you have non-vanishing $D^{(r)}$, $D^{(m)}$ and $V^{(r)}$, $V^{(m)}$ which compensate each other in such a way that $\Psi = 0$ on super-horizon scales. The simplest inflationary models do not lead to such perturbations and the observations imply that they are not dominating the observed anisotropies in the CMB even though they may contribute which could seriously hamper the determination of cosmological parameters with CMB anisotropies (see e.g. [8, 9])

6 Scalar field cosmology

We now consider the special case of a Friedmann universe filled with self interacting scalar field matter. The action is given by

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{|g|} R + \int d^4x \sqrt{|g|} \left(\frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - W(\varphi) \right) \quad (127)$$

where φ denotes the scalar field and W is the potential. The energy momentum tensor is obtained by varying the action wrt the metric $g_{\mu\nu}$,

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \left[\frac{1}{2}\partial_\lambda\varphi\partial^\lambda\varphi + W \right] g_{\mu\nu} \quad (128)$$

The energy density ρ and the energy flux u are defined by

$$T_\nu^\mu u^\nu = -\rho u^\mu . \quad (129)$$

For the Friedmann background this gives

$$\rho = \frac{1}{2a^2}\dot{\varphi}^2 + W \quad (u^\mu) = \frac{1}{a}(1, \mathbf{0}) . \quad (130)$$

The pressure is given by

$$T_j^i = p\delta_j^i \quad p = \frac{1}{2a^2}\dot{\varphi}^2 - W . \quad (131)$$

We now define the scalar field perturbation,

$$\varphi = \bar{\varphi} + \delta\varphi . \quad (132)$$

Clearly, the scalar field only generates scalar perturbations (to first order). Inserting Eq. (132) in the definition of the energy velocity perturbation v ,

$$(u^\mu) = \frac{1}{a}(1 - A, -v,{}_i) \quad (133)$$

and the energy density perturbation $\delta\rho$,

$$\rho = \bar{\rho} + \delta\rho , \quad (134)$$

we obtain

$$\delta\rho = \frac{1}{a^2}\dot{\bar{\varphi}}\delta\dot{\varphi} - \frac{1}{a^2}\dot{\bar{\varphi}}^2 A + W_{,\varphi}\delta\varphi \quad (135)$$

and

$$v = \frac{k}{\dot{\bar{\varphi}}}(\delta\varphi + \dot{\bar{\varphi}}k^{-1}B) . \quad (136)$$

From the stress tensor, $T_{ij} = \varphi_{,i}\varphi_{,j} - [\frac{1}{2}\partial_\lambda\varphi\partial^\lambda\varphi + W]g_{ij}$ we find

$$p\pi_L = \frac{1}{a^2}\dot{\bar{\varphi}}\delta\dot{\varphi} - \frac{1}{a^2}\dot{\bar{\varphi}}^2 A - W_{,\varphi}\delta\varphi \quad \text{and} \quad \Pi = 0 . \quad (137)$$

We now define the gauge invariant scalar field perturbation as the value of $\delta\varphi$ in longitudinal gauge.

$$\delta\varphi^{(gi)} = \delta\varphi^{(long)} = \delta\varphi + \dot{\bar{\varphi}}\left(B - k^{-1}\dot{H}_T\right) = \delta\varphi - \dot{\bar{\varphi}}\sigma . \quad (138)$$

The last expression gives $\delta\varphi^{(gi)}$ in a generic gauge. It is clear that this combination is gauge-invariant. This variable is very simply related to the other gauge-invariant scalar variables. Short calculations give

$$V = k\delta\varphi^{(gi)}/\dot{\varphi} \quad (139)$$

$$D_g = -(1+w) \left[4\Psi + 2\frac{\dot{a}}{a}k^{-1}V - k^{-1}\dot{V} \right] \quad (140)$$

$$D_s = D_g + 3(1+w)\Psi \quad (141)$$

$$\Gamma = \frac{2W_{,\varphi}}{p\dot{\rho}} \left[\dot{\varphi}\rho D_s - \dot{\rho}\delta\varphi^{(gi)} \right] \quad (142)$$

$$\Pi = 0 . \quad (143)$$

The last equation shows that the two Bardeen potentials are equal for scalar field perturbations, $\Phi = \Psi$. With this we can write the perturbed Einstein equations fully in terms of the Bardeen potential Ψ and V . Since we will need them mainly to discuss inflation where curvature plays a minor role, we write them down here only for the case of vanishing spatial curvature. From Eqs. (65) and (67) one can easily generalize to the case with curvature.

$$-3\mathcal{H}\dot{\Psi} - (\dot{\mathcal{H}} + \mathcal{H}^2 - k^2)\Psi = 4\pi G\dot{\varphi}k^{-1}(\dot{\varphi}V + a^2W_{,\varphi}V) \quad (144)$$

$$\dot{\Psi} + \mathcal{H}\Psi = 4\pi G\dot{\varphi}^2k^{-1}V \quad (145)$$

$$\ddot{\Psi} + 3\mathcal{H}\dot{\Psi} + (\dot{\mathcal{H}} + 2\mathcal{H}^2)\Psi = 4\pi G\dot{\varphi}k^{-1}(\dot{\varphi}\dot{V} - \ddot{\varphi}V - a^2W_{,\varphi}V) \quad (146)$$

These lead to the following second order equation for the Bardeen potential which we have discussed above:

$$\ddot{\Psi} + 2(\mathcal{H} - \ddot{\varphi}/\dot{\varphi})\dot{\Psi} + (2\dot{\mathcal{H}} - 2\mathcal{H}\ddot{\varphi}/\dot{\varphi} + k^2)\Psi = 0 \quad (147)$$

or, using the definition $c_s^2 = \dot{p}/\dot{\rho}$,

$$\ddot{\Psi} + 3\mathcal{H}(1 + c_s^2)\dot{\Psi} + (2\dot{\mathcal{H}} + (1 + 3c_s^2)\mathcal{H}^2 + k^2)\Psi = 0 . \quad (148)$$

As already mentioned above, this equation differs from the Ψ equation for a perfect fluid only in the last term proportional to k^2 . This comes from the fact that the scalar field is not in a thermal state with fixed entropy, but it is in a fully coherent state ($\Gamma \neq 0$) and field fluctuations propagate with the speed of light. On large scales, $k\eta \ll 1$ this difference is not relevant, but on sub-horizon scales it does play a certain role.

7 Generation of perturbations during inflation

So far we have simply assumed some initial fluctuation amplitude A , without investigating where it came from or what the k -dependence of A might be. In this section we discuss the most common idea about the generation of

cosmological perturbations, namely their production from the quantum vacuum fluctuations during an inflationary phase. The treatment here is focused mainly on getting the correct result with as little effort as possible; we ignore several subtleties related, *e.g.* to the transition from quantum fluctuations of the field to classical fluctuations in the energy momentum tensor. The idea is of course that the source of metric fluctuations are the expectation values of the energy momentum tensor operator of the scalar field.

The basic idea is simple: A time dependent gravitational field very generically leads to particle production, analogously to the electron positron production in a classical, time dependent electromagnetic field.

7.1 Scalar perturbations

The main result is the following: During inflation, the produced particles induce a perturbed gravitational field with a (nearly) scale invariant spectrum,

$$k^3 |\Psi(k, \eta)|^2 = k^{n-1} \times \text{const.} \quad \text{with} \quad n \simeq 1. \quad (149)$$

The quantity $k^3 |\Psi(k, \eta)|^2$ is the squared amplitude of the metric perturbation at comoving scale $\lambda = \pi/k$. To insure that this quantity is small on a broad range of scales, so that neither black holes are formed on small scales nor there are large deviation from homogeneity and isotropy on large scales, we must require $n \simeq 1$. These arguments have been put forward for the first time by Harrison and Zel'dovich [10] (still before the advent of inflation), leading to the name 'Harrison-Zel'dovich spectrum' for a scale invariant perturbation spectrum.

To derive the above result we consider a scalar field background dominated by a potential, hence $a \propto |\eta|^q$ with $q \sim -1$. Looking at the action of this system,

$$S = \int dx^4 \sqrt{|g|} \left(\frac{R}{16\pi G} + \frac{1}{2} (\nabla\varphi)^2 \right),$$

it can be shown (see [11]) that the second order perturbation of this action around the Friedmann solution is given by

$$\delta S = \int dx^4 \sqrt{|\bar{g}|} \frac{1}{2} (\partial_\mu v)^2 \quad (150)$$

up to some total differential. Here v is the perturbation variable

$$v = -\frac{a\sqrt{\mathcal{H}^2 - \dot{\mathcal{H}}}}{\sqrt{4\pi G\mathcal{H}}} \zeta \quad (151)$$

introduced in Eq. (79). Via the Einstein equations, this variable can also be interpreted as representing the fluctuations in the scalar field. Therefore, we quantize v and assume that initially, on small scales, $k|\eta| \ll 1$, v is in

the (Minkowski) quantum vacuum state of a massless scalar field with mode function

$$v_{\text{in}} = \frac{v_0}{\sqrt{k}} \exp(ik\eta) . \quad (152)$$

The pre-factor v_0 is a k -independent constant which depends on convention, but is of order unity. From (78) we can derive

$$(v/z)' = \frac{k^2 u}{z} ,$$

where $z \propto a$ is defined in Eq. (81) and $u \propto a\eta\Psi$ is given in Eq. (74). On small scales, $k|\eta| \ll 1$, this results in the initial condition for u

$$u_{\text{in}} = \frac{-iv_0}{k^{3/2}} \exp(ik\eta) . \quad (153)$$

The evolution equation for u , (102), reduces in the case of power law expansion, $a \propto |\eta|^q$ to

$$\ddot{u} + \left(k^2 - \frac{q(q+1)}{\eta^2}\right)u = 0. \quad (154)$$

The solutions to this equation are of the form $(k|\eta|)^{1/2}H_\mu^{(i)}(k\eta)$, where $\mu = q + 1/2$ and $H_\mu^{(i)}$ is the Hankel function of the i th kind ($i = 1$ or 2) of order μ . The initial condition (153) requires that only $H_\mu^{(2)}$ appears, so that we obtain

$$u = \frac{\alpha}{k^{3/2}} (k|\eta|)^{1/2} H_\mu^{(2)}(k\eta) ,$$

where again α is a constant of order unity. We define the value of the Hubble constant during inflation, which is nearly constant by H_i . With $H = \mathcal{H}/a \simeq 1/(|\eta|a)$ we then obtain $a \sim 1/H_i|\eta|$. Eq. (74) with the Planck mass defined by $8\pi G = M_P^{-2}$ then gives

$$\Psi = \frac{H_i}{2M_P} u \simeq \frac{H_i}{M_P} k^{-3/2} (k|\eta|)^{1/2} H_\mu^{(2)}(k\eta) . \quad (155)$$

On small scales this is a simple oscillating function while on large scales $k|\eta| \gg 1$ it can be approximated by a power law,

$$\Psi \simeq \frac{H_i}{M_P} k^{-3/2} (k|\eta|)^{1+q} \simeq \frac{H_i}{M_P} k^{-3/2} , \quad \text{for } k|\eta| \gg 1 . \quad (156)$$

Here we have used $\mu = 1/2 + q < 0$ and $q \sim -1$. This yields

$$k^3 |\Psi|^2 \simeq \left(\frac{H_i}{M_P}\right)^2 , \quad (157)$$

hence $n = 1$. Detailed studies have shown that even though the amplitude of Ψ can still be severely affected by the transition from inflation to the subsequent radiation era, the obtained spectrum is very stable. Simple deviations from de Sitter inflation (like *e.g.* power law inflation), $q > -1$ lead to slightly blue spectra, $n \gtrsim 1$.

7.2 Vector perturbations

In the simplest models of inflation where the only degrees of freedom are the scalar field and the metric, no vector perturbations are generated. But even if they are, subsequent evolution after inflation will lead to their decay. In a perfect fluid background, $\Pi_{ij} = 0$, vector perturbations evolve according to Eq.(71) which implies

$$\Omega \propto a^{3c_s^2-1}. \quad (158)$$

For a radiation fluid, $\dot{p}/\dot{\rho} = c_s^2 \leq 1/3$, this leads to a non-growing vorticity. The dynamical Einstein equation (68) gives

$$\sigma^{(V)} \propto a^{-2}, \quad (159)$$

and the constraint (66) reads (at early times, so that we can neglect curvature)

$$\Omega \sim (k\eta)^2 \sigma^{(V)}. \quad (160)$$

Therefore, even if they are created in the very early universe on super-horizon scales during an inflationary period, vector perturbations of the metric decay and become soon entirely negligible. Even if Ω_i remains constant in a radiation dominated universe, it has to be so small on relevant scales at formation ($k\eta_{in} \ll 1$) that we may safely neglect it.

Vector perturbations are irrelevant if perturbations have been created at some early time, *e.g.* during inflation. This result changes completely when considering 'active perturbations' like for example topological defects where vector perturbations contribute significantly to the CMB anisotropies on large scales, see Ref. [12]. Furthermore, it is interesting to note that vector perturbations do not satisfy a wave equation and therefore will in no case show oscillations. Vorticity simply decays with time.

7.3 Tensor perturbations

The situation is different for tensor perturbations. Again we consider the perfect fluid case, $\Pi_{ij}^{(T)} = 0$. Eq. (69) implies, if κ is negligible,

$$\ddot{H}_{ij} + \frac{2\dot{a}}{a}\dot{H}_{ij} + k^2 H_{ij} = 0. \quad (161)$$

If the background has a power law evolution, $a \propto \eta^q$ this equation can be solved in terms of Bessel or Hankel functions. The less decaying mode solution to Eq. (161) is $H_{ij} = e_{ij} x^{1/2-\beta} J_{1/2-q}(x)$, where J_ν denotes the Bessel function of order ν , $x = k\eta$ and e_{ij} is a transverse traceless polarization tensor. This leads to

$$H_{ij} = \text{const} \quad \text{for } x \ll 1 \quad (162)$$

$$H_{ij} = \frac{1}{a} \quad \text{for } x \gtrsim 1. \quad (163)$$

One may also quantize the tensor fluctuations which represent gravitons. Doing this, one obtains (up to log corrections a scale invariant spectrum of tensor fluctuations from inflation: For tensor perturbations the canonical variable is simple given by $h_{ij} = M_P a H_{ij}$. The evolution equation for $h_{ij} = h e_{ij}$ is of the form

$$\ddot{h} + (k^2 + m^2(\eta))h = 0, \quad (164)$$

where $m^2(\eta) = -\ddot{a}/a$. During inflation $m^2 = -q(q-1)$ is negative, leading to particle creation. Like for scalar perturbations, the vacuum initial conditions are given on scales which are inside the horizon, $k^2 \gg |m^2|$,

$$h_{\text{in}} = \frac{1}{\sqrt{k}} \exp(k\eta) \quad \text{for } k|\eta| \gg 1.$$

Solving Eq. (164) with this initial condition, gives

$$h = \frac{1}{\sqrt{k}} (k|\eta|)^{1/2} H_{q-1/2}^{(2)}(k\eta),$$

where $H_\nu^{(2)}$ is the Hankel function of degree ν of the second kind. On super horizon scales, $H_{q-1/2}^{(2)}(k\eta) \propto (k|\eta|)^{q-1/2}$ this results in $|h|^2 \simeq |\eta|(k|\eta|)^{2q-1}$. Using the relation between $h_{ij} = h e_{ij}$ and H_{ij} one obtains the spectrum of tensor perturbation generated during inflation. For exponential inflation, $q \simeq -1$ one finds again a scale invariant spectrum for H_{ij} on super-horizon scales.

$$k^3 |H_{ij} H^{ij}| \simeq (H_{\text{in}}/M_P)^2 \propto k^{n_T} \quad \text{with } n_T \simeq 0. \quad (165)$$

8 Lightlike geodesics and CMB anisotropies

After decoupling, $\eta > \eta_{\text{dec}}$, photons follow to a good approximation light-like geodesics. The temperature shift of a Planck distribution of photons is equal to the energy shift of any given photon, which is independent of the photon energy (gravity is 'achromatic').

The unperturbed photon trajectory follows

$$(x^\mu(\eta)) \equiv (\eta, \int_\eta^{\eta_0} \mathbf{n}(\eta') d\eta' + \mathbf{x}_0),$$

where \mathbf{x}_0 is the photon position at time η_0 and \mathbf{n} is the (parallel transported) photon direction. With respect to a geodesic basis $(\mathbf{e})_{i=1}^3$, the components of \mathbf{n} are constant. If $\kappa = 0$ we may choose $\mathbf{e}_i = \partial/\partial x^i$; if $\kappa \neq 0$ these vector fields are no longer parallel transported and therefore do not form a geodesic basis ($\nabla_{\mathbf{e}_i} \mathbf{e}_j = 0$).

Our metric is of the form

$$d\tilde{s}^2 = a^2 ds^2, \quad \text{with} \quad (166)$$

$$ds^2 = (\gamma_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu, \quad \gamma_{00} = -1, \gamma_{i0} = 0, \gamma_{ij} = \gamma_{ji} \quad (167)$$

as before.

We make use of the fact that light-like geodesics are conformally invariant. More precisely, ds^2 and $d\tilde{s}^2$ have the same light-like geodesics, only the corresponding affine parameters are different. Let us denote the two affine parameters by λ and $\tilde{\lambda}$ respectively, and the tangent vectors to the geodesic by

$$n = \frac{dx}{d\lambda}, \quad \tilde{n} = \frac{1}{a} n = \frac{dx}{d\tilde{\lambda}}, \quad n^2 = \tilde{n}^2 = 0, \quad n^0 = 1, \quad \mathbf{n}^2 = 1. \quad (168)$$

We set $n^0 = 1 + \delta n^0$. The geodesic equation for the perturbed metric

$$ds^2 = (\gamma_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu \quad (169)$$

yields, to first order,

$$\frac{d}{d\lambda} \delta n^\mu = -\delta \Gamma_{\alpha\beta}^\mu n^\alpha n^\beta. \quad (170)$$

For the energy shift, we have to determine δn^0 . Since $g^{0\mu} = -1 \cdot \delta_{0\mu} +$ first order, we obtain $\delta \Gamma_{\alpha\beta}^0 = -1/2(h_{\alpha 0|\beta} + h_{\beta 0|\alpha} - \dot{h}_{\alpha\beta})$, so that

$$\frac{d}{d\lambda} \delta n^0 = h_{\alpha 0|\beta} n^\beta n^\alpha - \frac{1}{2} \dot{h}_{\alpha\beta} n^\alpha n^\beta. \quad (171)$$

Integrating this equation we use $h_{\alpha 0|\beta} n^\beta = \frac{d}{d\lambda}(h_{\alpha 0} n^\alpha)$, so that the change of n^0 between some initial time η_i and some final time η_f is given by

$$\delta n^0|_i^f = [h_{00} + h_{0j} n^j]_i^f - \frac{1}{2} \int_i^f \dot{h}_{\mu\nu} n^\mu n^\nu d\lambda. \quad (172)$$

On the other hand, the ratio of the energy of a photon measured by some observer at t_f to the energy emitted at t_i is

$$\frac{E_f}{E_i} = \frac{(\tilde{n} \cdot u)_f}{(\tilde{n} \cdot u)_i} = \frac{T_f}{T_i} \frac{(n \cdot u)_f}{(n \cdot u)_i}, \quad (173)$$

where u_f and u_i are the four-velocities of the observer and emitter respectively, and the factor T_f/T_i is the usual (unperturbed) redshift, which relates n and \tilde{n} . The velocity field of observer and emitter is given by

$$u = (1 - A)\partial_\eta + v^i \partial_i. \quad (174)$$

An observer measuring a temperature T_0 receives photons that were emitted at the time η_{dec} of decoupling of matter and radiation, at the fixed temperature T_{dec} . In first-order perturbation theory, we find the following relation

between the unperturbed temperatures T_f , T_i , the measurable temperatures $T_0 = T_f + \delta T_f$, $T_{dec} = T_i + \delta T_i$, and the photon density perturbation:

$$\frac{T_f}{T_i} = \frac{T_0}{T_{dec}} \left(1 - \frac{\delta T_f}{T_f} + \frac{\delta T_i}{T_i} \right) = \frac{T_0}{T_{dec}} \left(1 - \frac{1}{4} \delta^{(r)}|_i^f \right), \quad (175)$$

where $\delta^{(r)}$ is the intrinsic density perturbation in the radiation and we have used $\rho^{(r)} \propto T^4$ in the last equality. Inserting the above equation and Eq. (172) into Eq. (173), and using Eq. (30) for the definition of $h_{\mu\nu}$, one finds, after integration by parts [6] the following result for scalar perturbations:

$$\frac{E_f}{E_i} = \frac{T_0}{T_{dec}} \left\{ 1 - \left[\frac{1}{4} D_g^{(r)} + V_j^{(b)} n^j + \Psi + \Phi \right]_i^f + \int_i^f (\dot{\Psi} + \dot{\Phi}) d\lambda \right\}. \quad (176)$$

Here $D_g^{(r)}$ denotes the density perturbation in the radiation fluid, and $V^{(b)}$ is the peculiar velocity of the baryonic matter component (the emitter and observer of radiation).

Evaluating Eq. (176) at final time η_0 (today) and initial time η_{dec} , we obtain the temperature difference of photons coming from different directions \mathbf{n} and \mathbf{n}'

$$\frac{\Delta T}{T} \equiv \frac{\Delta T(\mathbf{n})}{T} - \frac{\Delta T(\mathbf{n}')}{T} \equiv \frac{E_f}{E_i}(\mathbf{n}) - \frac{E_f}{E_i}(\mathbf{n}'). \quad (177)$$

Direction independent contributions to $\frac{E_f}{E_i}$ do not contribute to this difference. We also do not want to include the term $V_j(\eta_0)n^j$ which simply describes the dipole due to our motion with respect to the emission surface and which is much larger than the contributions from the higher multipoles. Therefore we can set

$$\frac{\Delta T(\mathbf{n})}{T} = \left[\frac{1}{4} D_g^{(r)} + V_j^{(b)} n^j + \Psi + \Phi \right] (\eta_{dec}, \mathbf{x}_{dec}) + \int_{\eta_{dec}}^{\eta_0} (\dot{\Psi} + \dot{\Phi})(\eta, \mathbf{x}(\eta)) d\eta, \quad (178)$$

where $\mathbf{x}(\eta)$ is the unperturbed photon position at time η for an observer at \mathbf{x}_0 , and $\mathbf{x}_{dec} = \mathbf{x}(\eta_{dec})$ (If $\kappa = 0$ we simply have $\mathbf{x}(\eta) = \mathbf{x}_0 - (\eta_0 - \eta)\mathbf{n}$). The first term in Eq. (178) is the one we have discussed in the previous chapter. It describes the intrinsic inhomogeneities on the surface of last scattering, due to acoustic oscillations prior to decoupling. Depending on the initial conditions, it can contribute significantly also on super-horizon scales. This is especially important in the case of adiabatic initial conditions. As we have seen in Eq. (126), in a dust + radiation universe with $\Omega = 1$, adiabatic initial conditions imply $D_g^{(r)}(k, \eta) = -\frac{2\Omega}{3}\Psi(k, \eta)$ and $V^{(b)} = V^{(r)} \ll D_g^{(r)}$ for $k\eta \ll 1$. With $\Phi = \Psi$ the the square bracket of Eq. (178) therefore gives for adiabatic perturbations

$$\left(\frac{\Delta T(\mathbf{n})}{T} \right)_{\text{adiabatic}}^{(OSW)} = \frac{1}{3} \Psi(\eta_{dec}, \mathbf{x}_{dec})$$

on super-horizon scales. The contribution to $\frac{\delta T}{T}$ from the last scattering surface on very large scales is called the 'ordinary Sachs–Wolfe effect' (OSW). It has been derived for the first time by Sachs and Wolfe [13] in 1967. For isocurvature perturbations, the initial condition $D_g^{(r)}(k, \eta) \rightarrow 0$ for $\eta \rightarrow 0$ is satisfied and the contribution of D_g to the ordinary Sachs–Wolfe effect can be neglected.

$$\left(\frac{\Delta T(\mathbf{n})}{T}\right)_{\text{isocurvature}}^{(OSW)} = 2\Psi(\eta_{dec}, \mathbf{x}_{dec}) .$$

The second term in (178) describes the relative motion of emitter and observer. This is the Doppler contribution to the CMB anisotropies. It appears on the same angular scales as the acoustic term; we call the sum of the acoustic and Doppler contributions ‘‘acoustic peaks’’.

The last two terms are due to the inhomogeneities in the spacetime geometry; the first contribution determines the change in the photon energy due to the difference of the gravitational potential at the position of emitter and observer. Together with the part contained in $D_g^{(r)}$ they represent the ‘‘ordinary’’ Sachs–Wolfe effect. The integral accounts for red-shift or blue-shift caused by the time dependence of the gravitational field along the path of the photon, and represents the so-called integrated Sachs–Wolfe (ISW) effect. In a $\Omega = 1$, pure dust universe, the Bardeen potentials are constant and there is no integrated Sachs–Wolfe effect; the blue-shift which the photons acquire by falling into a gravitational potential is exactly canceled by the redshift induced by climbing out of it. This is no longer true in a universe with substantial radiation contribution, curvature or a cosmological constant.

The sum of the ordinary Sachs–Wolfe term and the integral is the full Sachs–Wolfe contribution (SW).

For **vector** perturbations $\delta^{(r)}$ and A vanish and Eq. (173) leads to

$$(E_f/E_i)^{(V)} = (a_i/a_f)[1 - V_j^{(m)}n^j|_i^f + \int_i^f \dot{\sigma}_j n^j d\lambda] . \quad (179)$$

We obtain a Doppler term and a gravitational contribution. For **tensor** perturbations, i.e. gravitational waves, only the gravitational part remains:

$$(E_f/E_i)^{(T)} = (a_i/a_f)[1 - \int_i^f \dot{H}_{lj} n^l n^j d\lambda] . \quad (180)$$

Equations (176), (179) and (180) are the manifestly gauge invariant results for the energy shift of photons due to scalar, vector and tensor perturbations. Disregarding again the dipole contribution due to our proper motion, Eqs. (179,180) imply the vector and tensor temperature fluctuations

$$\left(\frac{\Delta T(\mathbf{n})}{T}\right)^{(V)} = V_j^{(m)}(\eta_{dec}, \mathbf{x}_{dec})n^j + \int_i^f \dot{\sigma}_j(\eta, \mathbf{x}(\eta))n^j d\lambda \quad (181)$$

$$\left(\frac{\Delta T(\mathbf{n})}{T}\right)^{(T)} = - \int_i^f \dot{H}_{lj}(\eta, \mathbf{x}(\eta))n^l n^j d\lambda . \quad (182)$$

Note that for models where initial fluctuations have been laid down in the very early universe, vector perturbations are irrelevant as we have already pointed out. In this sense Eq. (181) is here mainly for completeness. However, in models where perturbations are sourced by some inherently inhomogeneous component (*e.g.* topological defects, see Ref. [12]) vector perturbation can be important.

9 Power spectra

One of the basic tools to compare models of large scale structure having stochastic initial fluctuations with observations are power spectra. They are the “harmonic transforms” of the two point correlation functions. If the perturbations of the model under consideration are Gaussian (a relatively generic prediction from inflationary models), then the power spectra contain the full statistical information of the model.

Let us first consider the power spectrum of dark matter,

$$P_D(k) = \left\langle \left| D_g^{(m)}(\mathbf{k}, \eta_0) \right|^2 \right\rangle. \quad (183)$$

Here $\langle \rangle$ indicates a statistical average, ensemble average, over “initial conditions” in a given model. $P_D(k)$ is usually compared with the observed power spectrum of the galaxy distribution. This is clearly problematic since it is by no means evident what the ratio of these two spectra should be. This problem is known under the name of ‘biasing’ and it is very often simply assumed that the dark matter and galaxy power spectra differ only by a constant factor. The hope is also that on sufficiently large scales, since the evolution of both, galaxies and dark matter is governed by gravity, their power spectra should not differ too much. This hope seems to be reasonably justified [14].

The power spectrum of velocity perturbations satisfies the relation

$$P_V(k) = \left\langle |\mathbf{V}(\mathbf{k}, \eta_0)|^2 \right\rangle \simeq H_0^2 \Omega^{1.2} P_D(k) k^{-2}. \quad (184)$$

For \simeq we have used that $|kV|(\eta_0) = \dot{D}_g^{(m)}(\eta_0) \sim H_0 \Omega^{0.6} D_g$ on sub-horizon scales (see *e.g.* [15]).

The spectrum we can be both, measured and calculated to the best accuracy is the CMB anisotropy power spectrum. It is defined as follows: $\Delta T/T$ is a function of position \mathbf{x}_0 , time η_0 and photon direction \mathbf{n} . We develop the \mathbf{n} -dependence in terms of spherical harmonics. We will suppress the argument η_0 and often also \mathbf{x}_0 in the following calculations. All results are for today (η_0) and here (\mathbf{x}_0). By statistical homogeneity statistical averages over an ensemble of realisations (expectation values) are supposed to be independent of position. Furthermore, we assume that the process generating the initial

perturbations is statistically isotropic. Then, the off-diagonal correlators of the expansion coefficients $a_{\ell m}$ vanish and we have

$$\frac{\Delta T}{T}(\mathbf{x}_0, \mathbf{n}, \eta_0) = \sum_{\ell, m} a_{\ell m}(\mathbf{x}_0) Y_{\ell m}(\mathbf{n}), \quad \langle a_{\ell m} \cdot a_{\ell' m'}^* \rangle = \delta_{\ell\ell'} \delta_{mm'} C_\ell \quad (185)$$

The C_ℓ 's are the CMB power spectrum.

The two point correlation function is related to the C_ℓ 's by

$$\begin{aligned} \left\langle \frac{\Delta T}{T}(\mathbf{n}) \frac{\Delta T}{T}(\mathbf{n}') \right\rangle_{\mathbf{n} \cdot \mathbf{n}' = \mu} &= \sum_{\ell, \ell', m, m'} \langle a_{\ell m} \cdot a_{\ell' m'}^* \rangle Y_{\ell m}(\mathbf{n}) Y_{\ell' m'}^*(\mathbf{n}') = \\ &= \sum_{\ell} C_\ell \underbrace{\sum_{m=-\ell}^{\ell} Y_{\ell m}(\mathbf{n}) Y_{\ell m}^*(\mathbf{n}')}_{\frac{2\ell+1}{4\pi} P_\ell(\mathbf{n} \cdot \mathbf{n}')} = \frac{1}{4\pi} \sum_{\ell} (2\ell+1) C_\ell P_\ell(\mu), \end{aligned} \quad (186)$$

where we have used the addition theorem of spherical harmonics for the last equality; the P_ℓ 's are the Legendre polynomials.

Clearly the $a_{\ell m}$'s from scalar, vector and tensor perturbations are uncorrelated,

$$\langle a_{\ell m}^{(S)} a_{\ell' m'}^{(V)} \rangle = \langle a_{\ell m}^{(S)} a_{\ell' m'}^{(T)} \rangle = \langle a_{\ell m}^{(V)} a_{\ell' m'}^{(T)} \rangle = 0. \quad (187)$$

Since vector perturbations decay, their contributions, the $C_\ell^{(V)}$, are negligible in models where initial perturbations have been laid down very early, *e.g.*, after an inflationary period. Tensor perturbations are constant on super-horizon scales and perform damped oscillations once they enter the horizon.

Let us first discuss in somewhat more detail scalar perturbations. We specialize to the case $\kappa = 0$ for simplicity. We suppose the initial perturbations to be given by a spectrum,

$$\langle |\Psi|^2 \rangle k^3 = A^2 k^{n-1} \eta_0^{n-1}. \quad (188)$$

We multiply by the constant η_0^{n-1} , the actual comoving size of the horizon, in order to keep A dimensionless for all values of n . A then represents the amplitude of metric perturbations at horizon scale today, $k = 1/\eta_0$.

On *super-horizon scales* we have, for *adiabatic* perturbations:

$$\frac{1}{4} D_g^{(r)} = -\frac{5}{3} \Psi + \mathcal{O}((k\eta)^2), \quad V^{(b)} = V^{(r)} = \mathcal{O}(k\eta) \quad (189)$$

The dominant contribution on super-horizon scales (neglecting the integrated Sachs–Wolfe effect $\int \dot{\Phi} - \dot{\Psi}$) is then

$$\frac{\Delta T}{T}(\mathbf{x}_0, \mathbf{n}, \eta_0) = \frac{1}{3} \Psi(x_{\text{dec}}, \eta_{\text{dec}}). \quad (190)$$

The Fourier transform of (190) gives

$$\frac{\Delta T}{T}(\mathbf{k}, \mathbf{n}, \eta_0) = \frac{1}{3}\Psi(k, \eta_{\text{dec}}) \cdot e^{i\mathbf{k}\mathbf{n}(\eta_0 - \eta_{\text{dec}})}. \quad (191)$$

Using the decomposition

$$e^{i\mathbf{k}\mathbf{n}(\eta_0 - \eta_{\text{dec}})} = \sum_{\ell=0}^{\infty} (2\ell + 1) i^\ell j_\ell(k(\eta_0 - \eta_{\text{dec}})) P_\ell(\hat{\mathbf{k}} \cdot \mathbf{n}),$$

where j_ℓ are the spherical Bessel functions, we obtain

$$\begin{aligned} & \left\langle \frac{\Delta T}{T}(\mathbf{x}_0, \mathbf{n}, \eta_0) \frac{\Delta T}{T}(\mathbf{x}_0, \mathbf{n}', \eta_0) \right\rangle \quad (192) \\ &= \frac{1}{V} \int d^3x_0 \left\langle \frac{\Delta T}{T}(\mathbf{x}_0, \mathbf{n}, \eta_0) \frac{\Delta T}{T}(\mathbf{x}_0, \mathbf{n}', \eta_0) \right\rangle \\ &= \frac{1}{(2\pi)^3} \int d^3k \left\langle \frac{\Delta T}{T}(\mathbf{k}, \mathbf{n}, \eta_0) \left(\frac{\Delta T}{T} \right)^*(\mathbf{k}, \mathbf{n}', \eta_0) \right\rangle \\ &= \frac{1}{(2\pi)^{39}} \int d^3k \left\langle |\Psi|^2 \right\rangle \sum_{\ell, \ell'=0}^{\infty} (2\ell + 1)(2\ell' + 1) i^{\ell - \ell'} \\ & \quad \cdot j_\ell(k(\eta_0 - \eta_{\text{dec}})) j_{\ell'}(k(\eta_0 - \eta_{\text{dec}})) P_\ell(\hat{\mathbf{k}} \cdot \mathbf{n}) \cdot P_{\ell'}(\hat{\mathbf{k}} \cdot \mathbf{n}'). \quad (193) \end{aligned}$$

In the second equal sign we have used the unitarity of the Fourier transformation. Inserting $P_\ell(\mathbf{k}\mathbf{n}) = \frac{4\pi}{2\ell+1} \sum_m Y_{\ell m}^*(\hat{\mathbf{k}}) Y_{\ell m}(\mathbf{n})$ and

$P_{\ell'}(\hat{\mathbf{k}}\mathbf{n}') = \frac{4\pi}{2\ell'+1} \sum_{m'} Y_{\ell' m'}^*(\hat{\mathbf{k}}) Y_{\ell' m'}(\mathbf{n}')$, integration over the directions $d\Omega_{\hat{\mathbf{k}}}$ gives $\delta_{\ell\ell'} \delta_{mm'} \sum_m Y_{\ell m}^*(\mathbf{n}) Y_{\ell m}(\mathbf{n}')$.

Using as well $\sum_m Y_{\ell m}^*(\mathbf{n}) Y_{\ell m}(\mathbf{n}') = \frac{2\ell+1}{4\pi} P_\ell(\mu)$, where $\mu = \mathbf{n} \cdot \mathbf{n}'$, we find

$$\begin{aligned} & \left\langle \frac{\Delta T}{T}(\mathbf{x}_0, \mathbf{n}, \eta_0) \frac{\Delta T}{T}(\mathbf{x}_0, \mathbf{n}', \eta_0) \right\rangle_{\mathbf{nn}'=\mu} = \\ & \sum_{\ell} \frac{2\ell+1}{4\pi} P_\ell(\mu) \frac{2}{\pi} \int \frac{dk}{k} \left\langle \frac{1}{9} |\Psi|^2 \right\rangle k^3 j_\ell^2(k(\eta_0 - \eta_{\text{dec}})). \quad (194) \end{aligned}$$

Comparing this equation with Eq. (186) we obtain for *adiabatic perturbations* on scales $2 \leq \ell \ll \chi(\eta_0 - \eta_{\text{dec}})/\eta_{\text{dec}} \sim 100$

$$C_\ell^{(SW)} \simeq C_\ell^{(OSW)} \simeq \frac{2}{\pi} \int_0^\infty \frac{dk}{k} \left\langle \left| \frac{1}{3} \Psi \right|^2 \right\rangle k^3 j_\ell^2(k(\eta_0 - \eta_{\text{dec}})). \quad (195)$$

If Ψ is a pure power law as in Eq. (188) and we set $k(\eta_0 - \eta_{\text{dec}}) \sim k\eta_0$, the integral (195) can be performed analytically. For the ansatz (188) one finds

$$C_\ell^{(SW)} = \frac{A^2}{9} \frac{\Gamma(3-n)\Gamma(\ell - \frac{1}{2} + \frac{n}{2})}{2^{3-n}\Gamma^2(2 - \frac{n}{2})\Gamma(\ell + \frac{5}{2} - \frac{n}{2})} \quad \text{for } -3 < n < 3. \quad (196)$$

Of special interest is the *scale invariant* or Harrison–Zel’dovich spectrum, $n = 1$ (see Section 7). It leads to

$$\ell(\ell + 1)C_\ell^{(SW)} = \text{const.} \simeq \left\langle \left(\frac{\Delta T}{T}(\vartheta_\ell) \right)^2 \right\rangle, \quad \vartheta_\ell \equiv \pi/\ell. \quad (197)$$

This is precisely (within the accuracy of the experiment) the behavior observed by the DMR experiment aboard the satellite COBE [16].

Inflationary models predict very generically a HZ spectrum (up to small corrections). The DMR discovery has therefore been regarded as a great success, if not a proof, of inflation. There are other models like topological defects [17, 18, 19] or certain string cosmology models [20] which also predict scale-invariant, *i.e.*, Harrison–Zel’dovich spectra of fluctuations. These models do however not belong to the class investigated here, since in these models perturbations are induced by seeds which evolve non-linearly in time.

For isocurvature perturbations, the main contribution on large scales comes from the integrated Sachs–Wolfe effect and (195) is replaced by

$$C_\ell^{(ISW)} \simeq \frac{8}{\pi} \int \frac{dk}{k} k^3 \left\langle \left| \int_{\eta_{\text{dec}}}^{\eta_0} \dot{\Psi}(k, \eta) j_\ell^2(k(\eta_0 - \eta)) d\eta \right|^2 \right\rangle. \quad (198)$$

Inside the horizon Ψ is roughly constant (matter dominated). Using the ansatz (188) for Ψ inside the horizon and setting the integral in (198) $\sim 2\Psi(k, \eta = 1/k) j_\ell^2(k\eta_0)$, we obtain again (196), but with $A^2/9$ replaced by $4A^2$. For a fixed amplitude A of perturbations, the Sachs–Wolfe temperature anisotropies coming from isocurvature perturbations are therefore about a factor of 6 times larger than those coming from adiabatic perturbations.

On smaller scales, $\ell \gtrsim 100$ the contribution to $\Delta T/T$ is usually dominated by acoustic oscillations, the first two terms in Eq. (178). Instead of (198) we then obtain

$$C_\ell^{(AC)} \simeq \frac{2}{\pi} \int_0^\infty \frac{dk}{k} k^3 \left\langle \left| \frac{1}{4} D^{(r)}(k, \eta_{\text{dec}}) j_\ell(k\eta_0) + V^{(r)}(k, \eta_{\text{dec}}) j_\ell'(k\eta_0) \right|^2 \right\rangle. \quad (199)$$

To remove the SW contribution from $D_g^{(r)}$ we have simply replaced it by $D^{(r)}$ which is much smaller than Ψ on super-horizon scales and therefore does not contribute to the SW terms. On sub-horizon scales $D^{(r)} \simeq D_g^{(r)}$ and $V^{(r)}$ are oscillating like sine or cosine waves depending on the initial conditions. Correspondingly the $C_\ell^{(AC)}$ will show peaks and minima. On very small scales they are damped by the photon diffusion which takes place during the recombination process (see contribution by A. Challinor).

For gravitational waves (tensor fluctuations), a formula analogous to (196) can be derived,

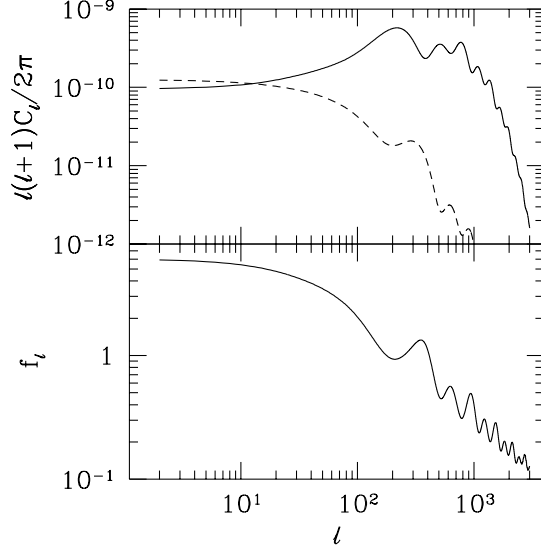


Fig. 1. A COBE normalized sample adiabatic (solid line) and isocurvature (dashed line) CMB anisotropy spectrum, $\ell(\ell+1)C_\ell$, are shown on the top panel. The quantity shown in the bottom panel is the ratio of temperature fluctuations for fixed value of A (from Kanazawa et al. [21]).

$$C_\ell^{(T)} = \frac{2}{\pi} \int dk k^2 \left\langle \left| \int_{\eta_{\text{dec}}}^{\eta_0} d\eta \dot{H}(\eta, k) \frac{j_\ell(k(\eta_0 - \eta))}{(k(\eta_0 - \eta))^2} \right|^2 \right\rangle \frac{(\ell+2)!}{(\ell-2)!}. \quad (200)$$

To a very crude approximation we may assume $\dot{H} = 0$ on super-horizon scales and $\int d\eta \dot{H} j_\ell(k(\eta_0 - \eta)) \sim H(\eta = 1/k) j_\ell(k\eta_0)$. For a pure power law,

$$k^3 \left\langle |H(k, \eta = 1/k)|^2 \right\rangle = A_T^2 k^{n_T} \eta_0^{-n_T}, \quad (201)$$

one obtains

$$\begin{aligned} C_\ell^{(T)} &\simeq \frac{2}{\pi} \frac{(\ell+2)!}{(\ell-2)!} A_T^2 \int \frac{dx}{x} x^{n_T} \frac{j_\ell^2(x)}{x^4} \\ &= \frac{(\ell+2)!}{(\ell-2)!} A_T^2 \frac{\Gamma(6-n_T) \Gamma(\ell-2+\frac{n_T}{2})}{2^{6-n_T} \Gamma^2(\frac{7}{2}-n_T) \Gamma(\ell+4-\frac{n_T}{2})}. \end{aligned} \quad (202)$$

For a scale invariant spectrum ($n_T = 0$) this results in

$$\ell(\ell+1)C_\ell^{(T)} \simeq \frac{\ell(\ell+1)}{(\ell+3)(\ell-2)} A_T^2 \frac{8}{15\pi}. \quad (203)$$

The singularity at $\ell = 2$ in this crude approximation is not real, but there is some enhancement of $\ell(\ell+1)C_\ell^{(T)}$ at $\ell \sim 2$ see Fig. 2).

Since tensor perturbations decay on sub-horizon scales, $\ell \gtrsim 60$, they are not very sensitive to cosmological parameters.

Again, inflationary models (and topological defects) predict a scale invariant spectrum of tensor fluctuations ($n_T \sim 0$).

On very small angular scales, $\ell \gtrsim 800$, fluctuations are damped by collisional damping (Silk damping). This effect has to be discussed with the Boltzmann equation for photons which is presented in detail in the course by A. Challinor.

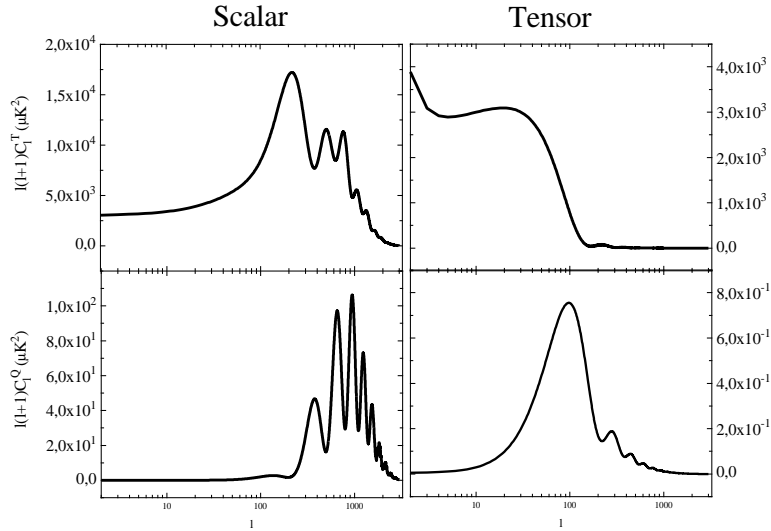


Fig. 2. Adiabatic scalar and tensor CMB anisotropy spectra are shown (top panels). The bottom panels show the corresponding polarization spectra (see course by A. Challinor). (from [22]).

10 Some remarks on perturbation theory in braneworlds

Since there has been so much interest in them recently, let me finally make some remarks on perturbation theory of five dimensional braneworlds. I shall just present some relatively simple aspects without derivation. A thorough discussion of braneworlds is given in the course by R. Maartens. Different aspects of the perturbation theory of braneworlds can be found in the growing literature on the subject [23, 24].

The bulk background metric of a five dimensional braneworld has three dimensional spatial slices which homogeneous and isotropic, hence spaces of constant curvature. Its line element is therefore of the form

$$ds^2 = -n^2(t, y)dt^2 + a^2(t, y)\gamma_{ij}dx^i dx^j + b^2(t, y)dy^2 = g_{AB}dx^A dx^B \quad (204)$$

Perturbations of such a spacetime can be decomposed into scalar, vector and tensor modes with respect to the three dimensional spatial slices of constant curvature. One can always choose the so-called **generalized longitudinal gauge** such that the perturbations of the metric are given as follows:

$$ds^2 = -n^2(1 + 2\Psi)dt^2 + a^2 [(1 - 2\Phi)\gamma_{ij} + 2H_{ij}] dx^i dx^j + b^2(1 + 2C)dy^2 - 2nbBdtdy - 2na\Sigma_i dx^i dt + ab2\mathcal{E}_i dx^i dy . \quad (205)$$

Here Σ_i and \mathcal{E}_i are divergence free vector fields, vector perturbations and H_{ij} is a divergence free traceless symmetric tensor, the tensor perturbation. Ψ, Φ, C and B are four scalar perturbations.

It can be shown that this choice determines the gauge completely. One can actually define gauge invariant perturbation variables which reduce to the ones above in the generalized longitudinal gauge [24]. Writing down the perturbed Einstein equations for these variables in the most general case is quite involved. These equations can be found in Ref. [24], but I don't want to repeat them here. I just discuss their general structure in the case of an empty bulk. It is clear that Ψ and Φ correspond to the Bardeen potentials of four dimensional cosmology, Σ_i is the four dimensional vector perturbation and H_{ij} represents four dimensional gravitational waves. C and B as well as \mathcal{E}_i are new degrees of freedom which are not present in the four dimensional theory.

If we assume vanishing perturbations of the bulk energy momentum tensor e.g. if the bulk is anti de Sitter like in the Randall Sundrum model [25] called RSII in what follows (see course by R. Maartens), the perturbation equations reduce to

$$\square_5(\Psi + \Phi) = 0 \quad (206)$$

$$\square_5 \Sigma_i = 0 \quad (207)$$

$$\square_5 H_{ij} = 0 \quad (208)$$

and all the other perturbation variables are determined by constraint equations. Here \square_5 is the five dimensional d'Alembertian with respect to the background metric (204). This structure of the equations is to be expected: Gravitational waves in d spacetime dimensions are a spin 2 field with respect to the group of rotations $SO(d-2)$ since they are massless (see e.g. [26]). For $d = 5$, $d - 2 = 3$ they therefore have 5 degrees of freedom. These correspond

exactly to the one scalar (206), two vector (207) and two tensor (208) degrees of freedom with respect to the 3-dimensional slices of constant curvature. These free massless degrees of freedom obey the wave equations above.

The perturbed Israel junction conditions (see course by R. Maartens) then determine boundary conditions for the behavior of the perturbations at the brane position(s).

As an example, I write down the vector and tensor perturbation equations and their bulk solutions for the RSII model which has only one brane. There the bulk is a five dimensional anti-de Sitter spacetime, and we can choose coordinates so that the background metric has the form

$$ds^2 = \left(\frac{L}{y}\right)^2 [-dt^2 + \delta_{ij}dx^i dx^j + dy^2] \quad (209)$$

and

$$\square_5 = -\partial_t^2 + \Delta + \partial_y^2 - \frac{3}{y}\partial_y, \quad (210)$$

where Δ denotes the three dimensional spatial Laplacian. For an arbitrary mode which satisfies this wave equation we make the ansatz

$$\phi(t, \mathbf{x}, y) = \exp(i(\mathbf{k} \cdot \mathbf{x} - \omega t))\phi(\omega, \mathbf{k}, y).$$

We then obtain a Bessel differential equation for $\phi(\omega, \mathbf{k}, y)$ with general solution

$$\begin{aligned} \phi &= A(\omega, \mathbf{k})(ym)^2 J_2(my) + B(\omega, \mathbf{k})(ym)^2 Y_2(my), \quad m^2 = \omega^2 - k^2, \\ &= \phi_A + \phi_B. \end{aligned} \quad (211)$$

These modes are normalizable in the sense that

$$\int_{y_b}^{\infty} |\phi|^2 \sqrt{-g} dy < \infty.$$

Here $y_b > 0$ is the brane position. The Israel junction condition for the vector and tensor modes for RSII become

$$-\partial_y H_{ij}(y = y_b) = \kappa_5^2 \Pi_{ij}(y_b) \quad (212)$$

$$-\Sigma_i(y_b) = \kappa_5^2 \Pi_i(y_b), \quad (213)$$

where Π_{ij} respectively Π_i are the tensor resp. vector contribution to the anisotropic stresses on the brane. If the latter vanish, the junction condition simply requires

$$B(\omega, \mathbf{k}) = A(\omega, \mathbf{k}) J_2(my_b) / Y_2(my_b).$$

This result has been derived for the tensor mode by Randall and Sundrum [25]. For scalar perturbations the situation is somewhat more complicated since there is an additional degree of freedom which is the perturbed

position of the brane, $y_b \rightarrow y_b + \epsilon$. It is rather subtle to take this brane bending correctly into account. A very interesting work showing that this effect is actually most relevant to obtain the correct Newtonian limit in the RSII model can be found in Ref. [27]

Let us also briefly discuss the zero-mode. From the brane point of view, the modes discussed here represent waves (particles) which couple only to the energy momentum tensor of the brane and which obey a dispersion relation $\omega^2 - k^2 = m^2$, hence the parameter m of the solutions (213) is their mass. In the limit $m \rightarrow 0$ the solutions turn into power laws in y ,

$$\phi_0(y) = Ay^4 + B . \quad (214)$$

Of these modes, for tensor and vector perturbations, the B mode is normalizable. For scalar perturbations the situation is more complicated. For a mode to be 'normalizable' we want all the perturbations to be normalizable,

$$\int_{y_b}^{\infty} |\phi|^2 \sqrt{-g} dy < \infty , \quad \text{for } \phi = \Phi , \Psi , C \text{ and } B .$$

But from the constraints one obtains for $m = 0$ $C \propto y^2$ which is not normalizable. This mode diverges logarithmically and only converges due to the oscillations of the Bessel functions if $m \neq 0$.

If all the five graviton zero-modes are normalizable, like in all compact braneworlds, e.g. in models with two branes, the vector and scalar mode lead to three additional degrees of freedom (to the usual two four-dimensional graviton modes) which couple via the junction conditions to the brane energy momentum tensor and spoil the phenomenology of the model. The resulting four dimensional gravity is not Einstein gravity but Brans-Dicke or even more complicated. As an example, in Ref. [28] the contribution from the scalar zero-mode (206) to the gravity wave emission from a binary pulsar is calculated and shown to be in contradiction with observations. This problem is well-known from Kaluza Klein theories, it is the so called moduli problem. The way out of it is usually to render the modes massive. There are different suggestions how this can be achieved for the scalar mode. One possibility is the so called Goldberger-Wise mechanism [29] which shows that under certain circumstances a bulk scalar field can do the job. But certainly, a physically acceptable braneworld is defined only together with its mechanism how to get rid of such unwanted modes (see also Ref. [30]).

The advantage of the RSII model is that the scalar gravity wave zero-mode is not normalizable in this model and therefore it does not contribute. This very promising property of the RSII model has led to its popularity. It seems to induce the correct four dimensional Einstein gravity on the brane, in the cosmological context this property is still maintained at sufficiently low energies.

11 Conclusions

In this course I have given an introduction to cosmological perturbation theory. Perturbation theory is an important tool especially to calculate CMB anisotropies and polarisation since these are very small and can be determined reliably within linear cosmological perturbation theory. To determine the evolution of the cosmic matter density, linear perturbation theory has to be complemented with the theory of weakly non-linear Newtonian gravity and with N-body simulations. To finally understand the formation of galaxies non-gravitational highly non-linear physics, like heating and cooling mechanisms, dissipation, nuclear reactions etc. have to be taken into account. This very difficult subject is still in its infancy.

To make progress in our understanding of braneworlds, linear perturbation theory can also be most helpful. We can use it to determine e.g. the propagating modes of the gravitational field on the brane, light deflection and redshift in weak gravitational fields and the Newtonian limit. The condition that linear perturbations on the brane at low energy and large distances reduce to those resulting from Einstein gravity is non-trivial and has, to my knowledge, not yet been fully explored to limit braneworld models.

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