Problem 3.3. Mass octupole and current quadrupole radiation from a mass in circular orbit

In this problem we compute the mass octupole and current quadrupole radiation generated by a binary system of reduced mass μ, whose center-of-mass coordinate describes a circular trajectory. As discussed in Section 3.3.5, to lowest order in v/c the mass octupole and the current quadrupole radiation emitted by a self-gravitating system can be consistently computed taking the free-particle energy-momentum tensor, evaluating it on a Newtonian circular orbit, and plugging the resulting values of T^{00} and T^{0i} into the expression for $\dddot{M}^{ijk}$ and $\ddot{P}^{i,jk}$ (while the same procedure is not correct if applied to $\dot{S}^{ij,k}$). Defining as usual the center-of-mass coordinate $\mathbf{x}_{\rm CM}$ by $m\mathbf{x}_{\rm CM} = m_1\mathbf{x}_1 + m_2\mathbf{x}_2$ (where $m = m_1 + m_2$) and the relative coordinate $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$, we have $\mathbf{x}_1 = \mathbf{x}_{\rm CM} + (m_2/m)\mathbf{x}$ and $\mathbf{x}_2 = \mathbf{x}_{\rm CM} - (m_1/m)\mathbf{x}$. Thus, in the CM frame where $\mathbf{x}_{\rm CM} = 0$, $T^{00}(t,\mathbf{x}) = m_1 c^2 \delta^{(3)}\left(\mathbf{x} - \frac{m_2}{m}\mathbf{x}_0(t)\right) + m_2 c^2 \delta^{(3)}\left(\mathbf{x} + \frac{m_1}{m}\mathbf{x}_0(t)\right)$, and

$$M^{ijk}(t) = \frac{1}{c^2}\int d^3x\, T^{00}(t,\mathbf{x}) x^i(t) x^j(t) x^k(t) = \mu\,\frac{\delta m}{m} x_0^i(t) x_0^j(t) x_0^k(t)\,,$$

where $\delta m = m_2 - m_1$. To compute the radiation emitted from the star in the direction of the observer it is simpler to use the geometrical setting of Fig. 3.6 (labeling now the axes of this figure as (x, y, z) rather than (x', y', z')), in which the observer is along the z axis. The equation of the orbit of the relative coordinate in the center-of-mass frame is then

$$\begin{aligned} x_0(t) &= R\cos\omega_s t\,, \\ y_0(t) &= R\cos\iota\sin\omega_s t\,, \\ z_0(t) &= R\sin\iota\sin\omega_s t\,, \end{aligned} \tag{3.341}$$

and is obtained from an orbit lying in the (x, y) plane performing a rotation by an angle ι around the x axis. We set the observer in the z direction, so we compute the radiation emitted along $\hat{\mathbf{n}} = (0, 0, 1)$. For the octupole radiation, eq. (3.141) gives

$$\left(h_{ij}^{\rm TT}\right)_{\rm oct} = \frac{1}{r}\,\frac{2G}{3c^5}\Lambda_{ij,kl}(\hat{\mathbf{n}})\dddot{M}_{kl3}\,. \tag{3.342}$$

(As usual, in actual computations, it is more convenient to use M^{ijk} rather than $\mathcal{O}^{ijk}$.) We found in eq. (3.63) that, when $\hat{\mathbf{n}} = (0, 0, 1)$, in the multiplication of a matrix $\dddot{M}_{kl3}$ by the Lambda tensor the components of $\dddot{M}_{kl3}$ with $k = 3$ or $l = 3$ do not contribute, so we just need to compute M_{ab3} with $a, b = 1, 2$. With the trajectory given in eq. (3.341) we find, for $a, b = 1, 2$

$$M_{ab3} = \mu\frac{\delta m}{m}R^3\sin\iota\sin\omega_s t\begin{pmatrix} \cos^2\omega_s t & \cos\iota\sin\omega_s t\cos\omega_s t \\ \cos\iota\sin\omega_s t\cos\omega_s t & \cos^2\iota\sin^2\omega_s t \end{pmatrix}_{ab}. \tag{3.343}$$

Computing $\Lambda_{ab,cd}(\hat{\mathbf{n}})M_{cd3}$ using eq. (3.63) we get

$$\begin{aligned} \Lambda_{ab,cd}(\hat{\mathbf{n}})M_{cd3} &= \mu\frac{\delta m}{m}R^3\sin\iota\sin\omega_s t\,\times \\ &\times\begin{pmatrix} \frac{1}{2}(\cos^2\omega_s t - \cos^2\iota\sin^2\omega_s t) & \cos\iota\sin\omega_s t\cos\omega_s t \\ \cos\iota\sin\omega_s t\cos\omega_s t & -\frac{1}{2}(\cos^2\omega_s t - \cos^2\iota\sin^2\omega_s t) \end{pmatrix}_{ab}, \end{aligned} \tag{3.344}$$

and taking the third time derivative we find

$$\begin{aligned} (h_+)_{\rm oct} &= \frac{1}{r}\,\frac{G\mu R^3\omega_s^3}{12c^5}\,\frac{\delta m}{m}\sin\iota\,\left[(3\cos^2\iota - 1)\cos\omega_s t - 27(1 + \cos^2\iota)\cos 3\omega_s t\right]\,, \\ (h_\times)_{\rm oct} &= \frac{1}{r}\,\frac{G\mu R^3\omega_s^3}{12c^5}\,\frac{\delta m}{m}\,\sin 2\iota\left[\sin\omega_s t - 27\sin 3\omega_s t\right]\,. \end{aligned} \tag{3.345}$$

As expected, we have radiation both at $\omega = \omega_s$ and at $\omega = 3\omega_s$, since the mass octupole is trilinear in $x_0^i(t)$. The current quadrupole radiation can be computed similarly. Plugging $T^{0i}(t, \mathbf{x}) = m_1 c v_1^i \delta^{(3)}\left(\mathbf{x} - \frac{m_2}{m}\mathbf{x}_0(t)\right) + m_2 c v_2^i \delta^{(3)}\left(\mathbf{x} + \frac{m_1}{m}\mathbf{x}_0(t)\right)$ into eqs. (3.148) and (3.149) we get

$$J^{i,j}(t) = \mu\,\frac{\delta m}{m}\,\epsilon^{ikl} x_0^k(t)\dot{x}_0^l(t) x_0^j(t)\,. \tag{3.346}$$

Observe that $l^i \equiv \epsilon^{ikl} x_0^k(t)\dot{x}_0^l(t)$ is the angular momentum of a unit mass moving on the circular orbit $x_0^i(t)$, and is therefore a constant vector of modulus $R^2\omega_s$ and direction normal to the plane of the orbit. Therefore $J^{i,j}(t) = \mu\,(\delta m/m) l^i x_0^j(t)$ depends on time only through $x_0^j(t)$ and oscillates only at the frequency ω_s, rather than at ω_s and $3\omega_s$ as the mass octupole. Inserting this expression for $J^{i,j}(t)$ into eq. (3.151) and setting $\hat{\mathbf{n}} = (0, 0, 1)$ we get the current quadrupole radiation

$$\left(h_{ij}^{\rm TT}\right)_{\rm cq} = \frac{1}{r}\,\frac{4G}{3c^5}\Lambda_{ij,kl}(\hat{\mathbf{n}})\left(\epsilon^{3kp}\ddot{J}^{p,l} + \epsilon^{3lp}\ddot{J}^{p,k}\right)\,. \tag{3.347}$$

Performing the contraction with the Lambda tensor we get

$$(h_+)_{\rm cq} = \frac{1}{r}\,\frac{4G\mu R^3\omega_s^3}{3c^5}\,\frac{\delta m}{m}\,\sin\iota\,\cos\omega_s t\,,$$
$$(h_\times)_{\rm cq} = \frac{1}{r}\,\frac{2G\mu R^3\omega_s^3}{3c^5}\,\frac{\delta m}{m}\,\sin 2\iota\,\sin\omega_s t\,.$$

Summing the mass octupole and current quadrupole radiation, using Kepler's law $\omega_s^2 R^3 = Gm$, and introducing the notation $x = (Gm\omega_s/c^3)^{2/3}$ that is useful to make contact with the results of Chapter 5, we finally get

$$(h_+)_{\rm oct+cq} = \frac{G\mu}{4rc^2}\,\frac{\delta m}{m}\,x^{3/2}\sin\iota\,\left[(\cos^2\iota + 5)\cos\omega_s t - 9(1+\cos^2\iota)\cos 3\omega_s t\right]\,,$$
$$(h_\times)_{\rm oct+cq} = \frac{3G\mu}{4rc^2}\,\frac{\delta m}{m}\,x^{3/2}\sin 2\iota\,\left[\sin\omega_s t - 3\sin 3\omega_s t\right]\,. \tag{3.348}$$

This result agrees with the one that we will find in Chapter 5 from a complete post-Newtonian treatment, see eqs. (5.262), (5.266) and (5.267).(*)

(*) The apparent difference by an overall minus sign is due to a sign convention explained in Note 19 on page 250. Observe also that, in the full treatment of the problem, the argument of the trigonometric function is not $\omega_s t$ but $\phi(t) = \int^t dt'\,\omega_s(t')$, since the frequency ω_s changes because of the energy loss due to GW emission. Further higher-order effects (tail terms) are included in the phase, see eq. (5.265), as we will see in Chapter 5.

The contribution to the total radiated power from the mass octupole and the current quadrupole is

$$P_{\rm oct+cq} = \frac{r^2c^3}{16\pi G}\,2\pi\int_{-1}^{1} d\cos\iota\,\langle \dot{h}_+^2 + \dot{h}_\times^2\rangle_{\rm oct+cq}$$
$$= \frac{62}{7}\,\frac{G\mu^2}{c^7}\,\left(\frac{\delta m}{m}\right)^2 R^6\omega_s^8\,. \tag{3.349}$$

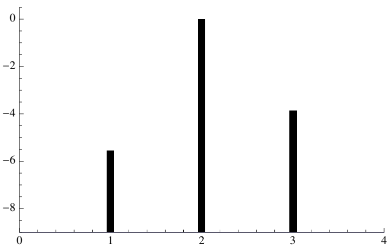

Fig. 3.8 $\log_{10}[P(\omega)/P(2\omega_s)]$, as a function of ω/ω_s, for $v/c = 10^{-2}$ and $\delta m/m = 1$, including the contributions of the mass quadrupole, of the mass octupole, and of the current quadrupole. The line at $\omega = 2\omega_s$ is due to the mass quadrupole, the line at $\omega = \omega_s$ is due to the mass octupole and current quadrupole, while that at $\omega = 3\omega_s$ is due only to the mass octupole.

It is interesting to compare the power at $\omega = \omega_s, 3\omega_s$ (generated by the mass octupole plus the current quadrupole), with the power at $\omega = 2\omega_s$, generated by the mass quadrupole. From eq. (3.348) we find

$$P(\omega_s) = \frac{25}{896}\,\left(\frac{v}{c}\right)^2\left(\frac{\delta m}{m}\right)^2 P(2\omega_s)\,, \tag{3.350}$$

and

$$P(3\omega_s) = \frac{1215}{896}\,\left(\frac{v}{c}\right)^2\left(\frac{\delta m}{m}\right)^2 P(2\omega_s)\,, \tag{3.351}$$

where $P(2\omega_s)$ is the leading-order quadrupole result, eq. (3.339). In Fig. 3.8 we show the relative intensity of the three spectral lines at $\omega = \omega_s, 2\omega_s$ and $3\omega_s$, for $v/c = 10^{-2}$ and an extreme mass ratio $m_2/m_1 \to 0$ and therefore $\delta m/m = 1$. Observe that the vertical scale is logarithmic.