corresponding observer should then conclude that the mass μ does not radiate, contrary to the findings of the observer at infinity.

This apparent paradox can be understood recalling that the equivalence principle holds only locally, i.e. in a region around the mass μ much smaller than the typical scales of spatial variation of the gravitational field. One such scale is the length $\lambdabar$, over which retardation effects become important (and which determines the wavelength of the GWs detected by the observer at infinity.) Then, conclusions based on the equivalence principle can be valid only up to a distance r from the mass μ, much smaller than $\lambdabar$.[23] This means that the equivalence principle at most gives us informations on what happens in the near zone $r \ll \lambdabar$; GWs rather appear in the far zone $r \gg \lambdabar$, so there is no paradox in the fact that, using arguments valid only for $r \ll \lambdabar$, one does not see them. The presence of gravitational radiation at infinity is reflected, in the near zone, in the existence of the force given by eqs. (3.114) and (3.115). However, in the near region retardation effects are negligible, so this term just gives a correction to the static gravitational force, which furthermore is hidden behind other, much larger, corrections. We will see in fact in Chapter 5 that, in an expansion in v/c, the radiation-reaction force is of order $(v/c)^5$ (as it is already clear from the factor $1/c^5$ in eq. (3.114)), while the Newtonian gravitational field receives general-relativistic corrections, corresponding to conservative forces, already at orders $(v/c)^2$ and $(v/c)^4$. All these tidal-like terms, however, in the far region decrease much faster than $1/r$, leaving us with the radiation field only.

[23] In fact, r must also be much smaller than the scale of spatial variation of the quasi-static tidal gravitational fields near the mass μ, which in turn is much smaller than $\lambdabar$.

3.3.5 Radiation from a closed system of point masses

For a free point-like particle moving on a trajectory $x_0(t)$ in flat space-time, the energy–momentum tensor is[24]

$$T^{\mu\nu}(t,\mathbf{x}) = \frac{p^\mu p^\nu}{\gamma m}\,\delta^{(3)}(\mathbf{x}-\mathbf{x}_0(t))\,, \tag{3.120}$$

where $\gamma = (1-v^2/c^2)^{-1/2}$, and $p^\mu = \gamma m(dx_0^\mu/dt) = (E/c, \mathbf{p})$ is the four-momentum. If we have a set of free point particles labeled by an index A, moving on trajectories $x_A^\mu(t)$, the total energy–momentum tensor of the system is therefore

$$\begin{aligned} T^{\mu\nu}(t,\mathbf{x}) &= \sum_A \frac{p_A^\mu p_A^\nu}{\gamma_A m_A}\,\delta^{(3)}(\mathbf{x}-\mathbf{x}_A(t)) \\ &= \sum_A \gamma_A m_A \frac{dx_A^\mu}{dt}\frac{dx_A^\nu}{dt}\,\delta^{(3)}(\mathbf{x}-\mathbf{x}_A(t))\,, \end{aligned} \tag{3.121}$$

and in particular

$$T^{00}(t,\mathbf{x}) = \sum_A \gamma_A m_A c^2 \delta^{(3)}(\mathbf{x}-\mathbf{x}_A(t))\,. \tag{3.122}$$

[24] See, e.g. Landau and Lifshitz, Vol. II (1979), eq. (33.5), or Weinberg (1972), Section 2.8. The generalization to curved space is more easily obtained writing $p^\mu = m dx_0^\mu/d\tau$ where τ is the proper time of the particle. In flat space $c^2 d\tau^2 = -\eta_{\mu\nu}dx_0^\mu dx_0^\nu$, so $d/d\tau = \gamma d/dt$ and we get $p^\mu = \gamma m(dx_0^\mu/dt)$. In curved space, instead, $c^2 d\tau^2 = -g_{\mu\nu}dx_0^\mu dx_0^\nu$. Furthermore, in flat space $(1/\gamma)\delta^{(3)}(\mathbf{x}-\mathbf{x}_0(t)) = (d\tau/dt)\delta^{(3)}(\mathbf{x}-\mathbf{x}_0(t))$, which can be rewritten as $\int d\tau \delta^{(4)}(x-x_0(\tau))$. In curved space $\delta^{(4)}(x-x_0(\tau))$ becomes $(1/\sqrt{-g})\delta^{(4)}(x-x_0(\tau))$, so in the end in curved space $(1/\gamma)\delta^{(3)}(\mathbf{x}-\mathbf{x}_0(t))$ becomes $(d\tau/dt)(1/\sqrt{-g})\delta^{(3)}(\mathbf{x}-\mathbf{x}_0(t))$. In this way we obtain the curved-space expression given in eq. (5.47).

Observe that this energy-momentum tensor is conserved only if the particles follow the geodesics of flat space-time, $\dot{p}^\mu_A = 0$ for all A. Thus, a priori it is not legitimate to use it to compute the radiation emitted by a system of interacting particles moving on generic trajectories $\mathbf{x}_A(t)$. In a consistent treatment we should include in the energy-momentum tensor all the interaction terms among the particles, and possibly with external sources, that cause them to deflect from a straight-line trajectory. However, for a non-relativistic self-gravitating system it is still possible to use the energy-momentum tensor (3.121) to compute both the leading term in eq. (3.34) (i.e. the mass quadrupole radiation) as well as the next-to-leading term, i.e. the term proportional to $\dot{S}^{kl,m}$ (which, as we will discuss in more detail in Sect. 3.4, is the sum of mass octupole and current quadrupole radiation). In this way, using only linearized theory, we can obtain the correct lowest-order results that will be derived, with much more effort, with the full non-linear formalism described in Chapter 5. To this purpose we need to observe that the full energy-momentum tensor also has interaction terms responsible for binding the particles in a orbit. For a self-gravitating system these terms are $O(v^2/c^2)$, as it is clear from the fact the gravitational potential energy $-Gm_1m_2/r$ is of order v^2/c^2.[25] Therefore, for a gravitationally-bound two-particle system in the non-relativistic limit,

$$T^{\mu\nu} = \sum_{A=1,2} m_A \frac{dx^\mu_A}{dt}\frac{dx^\nu_A}{dt}\,\delta^{(3)}(\mathbf{x}-\mathbf{x}_A(t)) + O(v^2/c^2)\,. \tag{3.123}$$

Since $T^{00} = O(v^0)$, $T^{0i} = O(v/c)$ and $T^{ij} = O(v^2/c^2)$, T^{00} and T^{0i} can be computed consistently, to lowest order, ignoring the interaction term, while T^{ij} cannot. Observe also that the use of the lowest-order expressions

$$T^{00} = \sum_A m_A c^2 \delta^{(3)}(\mathbf{x}-\mathbf{x}_A(t))\,, \tag{3.124}$$

$$T^{0i} = \sum_A m_A c\,\dot{x}^i(t)\delta^{(3)}(\mathbf{x}-\mathbf{x}_A(t))\,, \tag{3.125}$$

is consistent with the conservation equation $\partial_0 T^{00} + \partial_i T^{0i} = 0$ since, with these expressions, $\partial_0 T^{00} = -\sum_A m_A c\,\dot{x}^i_A \partial_i \delta^{(3)}(\mathbf{x}-\mathbf{x}_A(t))$ and $\partial_i T^{0i} = +\sum_A m_A c\,\dot{x}^i_A \partial_i \delta^{(3)}(\mathbf{x}-\mathbf{x}_A(t))$. So, to lowest order $\partial_0 T^{00} = -\partial_i T^{0i}$, independently of the trajectory used. In contrast, the conservation equation $\partial_0 \tau^{0j} + \partial_i \tau^{ij} = 0$ (with τ^{ij} given in Note 25 and $\tau^{0j} = T^{0j}$ to lowest order) is only satisfied if $\mathbf{x}_A(t)$ satisfies the equation of motion in the potential V, as can be checked with the explicit computation.

Thus, if we want to compute directly S^{kl} or $\dot{S}^{kl,m}$ in eq. (3.34), we need T^{ij}, and therefore we need to include also the interaction terms, which are $O(v^2/c^2)$, just as the leading term in the T^{ij} components of the free-particle energy-momentum tensor. However, using energy-momentum conservation, we can transform S^{kl} into $\ddot{M}^{kl}$, see eq. (3.52). Similarly, using eq. (3.54), we can trade $\dot{S}^{ij,k}$ for $\dddot{M}^{ijk}$ and $\ddot{P}^{i,jk}$. The derivation of eq. (3.54) uses the conservation of energy-momentum tensor so, when

[25] The explicit form of these terms will be given in Section 5.3.2, when we study systematically the relativistic corrections. We will see in eqs. (5.111)–(5.113) that, including the first post-Newtonian correction, T^{00} must be replaced by

$$\tau^{00} = \left(1+\frac{4V}{c^2}\right)T^{00} - \frac{7}{8\pi G}\partial_k V \partial_k V\,,$$

while T^{0i} is unchanged and T^{ij} is replaced by

$$\tau^{ij} = T^{ij} + \frac{1}{4\pi G}\left(\partial_i V \partial_j V - \frac{1}{2}\delta_{ij}\partial_k V \partial_k V\right),$$

where V is given in eq. (5.39) and, to lowest order, reduces to the Newtonian potential.

we write the GW amplitude in terms of $\dddot{M}^{ijk}$ and $\ddot{P}^{i,jk}$, we are already implicitly using the correct $T^{\mu\nu}$, with all the necessary interaction terms. The important point is that, to trade $\dot{S}^{ij,k}$ for $\dddot{M}^{ijk}$ and $\ddot{P}^{i,jk}$, we do not need to know the explicit form of $T^{\mu\nu}$, including the interaction terms, but only that it satisfies energy-momentum conservation. The advantage of this procedure is that M^{ij} and M^{ijk} only depend on T^{00}, and $P^{i,jk}$ only depends on T^{0i}. Therefore, to lowest order they can be consistently computed neglecting the potential terms in $T^{\mu\nu}$. In conclusion, both the leading term (i.e. the quadrupole radiation) and the next-to-leading term (i.e. the mass octupole plus current quadrupole radiation) can be consistently computed using eq. (3.124) to evaluate M^{ij} and M^{ijk}, and eq. (3.125) to evaluate $P^{i,jk}$, and then using eqs. (3.52) and (3.54) to evaluate S^{ij} and $\dot{S}^{ij,k}$. In contrast, if one wants to compute S^{ij} and $\dot{S}^{ij,k}$ directly from T^{ij}, which is $O(v^2)$, even to lowest order one needs to include the interaction terms, and one cannot use the free-particle energy-momentum tensor.

As will be shown in Section 5.1, the relativistic corrections to the Newtonian orbit start from order v^2/c^2. Therefore, the computation of the GW amplitude to leading and next-to-leading order in v/c can be performed evaluating the components T^{00} and T^{0i} of free-particle energy momentum tensor on the Newtonian orbit, and using them to compute M^{ij}, M^{ijk} and $P^{i,jk}$. We will perform these computations explicitly in Problems 3.2 and 3.3, in the Solved Problems section. Observe however that, in the radiated power, the corrections to the quadrupole amplitude give corrections to the leading term in the power of order $|1 + O(v^2/c^2)|^2 = 1 + O(v^2/c^2)$, while the mass octupole and current quadrupole give a correction $|O(v/c)|^2$, which is again $O(v^2/c^2)$, see the discussion below eq. (3.156).

For a non-relativistic two-body system it is convenient to define as usual the relative coordinate $\mathbf{x}_0 = \mathbf{x}_1 - \mathbf{x}_2$ and the center-of-mass coordinate

$$\mathbf{x}_{\rm CM} = \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2}{m_1 + m_2} \,. \tag{3.126}$$

(Starting from $O(v^2/c^2)$ this must actually be replaced by a center-of-energy, which also receives contributions from the interaction potential). We also denote by $m = m_1+m_2$ the total mass and by $\mu = m_1m_2/m$ the reduced mass. For a non-relativistic system, the second mass moment can be written as

$$M^{ij} = m_1 x_1^i x_1^j + m_2 x_2^i x_2^j = m x_{\rm CM}^i x_{\rm CM}^j + \mu x_0^i x_0^j \,. \tag{3.127}$$

If the system is isolated, $x_{\rm CM}^i$ is not accelerating and does not contribute to the GW production. It is then convenient to choose the frame such that $x_{\rm CM}^i = 0$, and we are left with a single effective particle of mass μ and coordinate $x_0^i(t)$. In the CM frame, the mass density is then

$$\rho(t,\mathbf{x}) = \mu\, \delta^{(3)}\left(\mathbf{x} - \mathbf{x}_0(t)\right) \,, \tag{3.128}$$

the second mass moment is

$$M^{ij}(t) = \mu\, x_0^i(t) x_0^j(t) \,, \tag{3.129}$$

and the mass quadrupole is

$$Q^{ij}(t) = \mu \left(x_0^i(t) x_0^j(t) - \frac{1}{3} r_0^2(t) \delta^{ij} \right). \tag{3.130}$$

We can now study the radiation emitted by a two-body system whose relative coordinate performs a given periodic motion, say simple harmonic oscillations. Suppose that the relative coordinate $\mathbf{x}_0(t)$ performs a simple periodic motion with frequency ω_s, say along the z direction,[26]

$$z_0(t) = a_1 \cos \omega_s t\,. \tag{3.131}$$

Then

$$\begin{aligned} M^{ij}(t) &= \delta^{i3}\delta^{j3}\, \mu z_0^2(t) \\ &= \delta^{i3}\delta^{j3}\, \frac{\mu a_1^2}{2} \left(1 + \cos 2\omega_s t\right). \end{aligned} \tag{3.132}$$

Since the GW amplitude depends on the second derivative of M^{ij}, the constant term does not contribute and the only contribution to $h_{ij}^{\rm TT}$ comes from the term proportional to $\cos 2\omega_s t$. From eq. (3.55), we see that the corresponding waveform $h_{ij}^{\rm TT}$ oscillates as $\cos 2\omega_s t$. This shows that *a non-relativistic source performing simple harmonic oscillations with a frequency ω_s emits monochromatic quadrupole radiation at $\omega = 2\omega_s$.*

However, the fact that the quadrupole radiation is at twice the source frequency is only true if the source performs a simple harmonic motion. For instance, if the motion of the source is a superposition of a periodic motion and of its higher harmonics, e.g. if

$$z_0(t) = a_1 \cos \omega_s t + a_2 \cos 2\omega_s t + \ldots\,, \tag{3.133}$$

then $z_0^2(t)$ contains the term

$$a_1^2 \cos^2 \omega_s t = a_1^2\, \frac{1 + \cos 2\omega_s t}{2}\,, \tag{3.134}$$

considered above, plus a term

$$a_2^2 \cos^2 2\omega_s t = a_2^2\, \frac{1 + \cos 4\omega_s t}{2}\,, \tag{3.135}$$

which gives radiation at $\omega_{\rm gw} = 4\omega_s$, etc., but also mixed terms such as

$$2a_1 a_2 \cos(\omega_s t) \cos(2\omega_s t) = a_1 a_2 \left(\cos \omega_s t + \cos 3\omega_s t\right). \tag{3.136}$$

Therefore in this case quadrupole radiation is emitted at all frequencies $n\omega_s$ for all integers n, both even and odd, including $n = 1$. We will see an example of this type in Section 4.1.2, when we study the radiation emitted from a mass in a Keplerian elliptic orbit.

An even simpler example is given by a system of two masses connected by a spring with rest length L (see Problem 3.1). In this case the relative coordinate satisfies

$$z_0(t) = L + a \cos \omega_s t\,, \tag{3.137}$$

and in $z_0^2(t)$, besides a constant and a term $(a^2/2)\cos 2\omega_s t$, we also have a term $2La \cos \omega_s t$, so the spectrum of gravitational radiation has two lines, one at $\omega = \omega_s$, and one at $\omega = 2\omega_s$.

[26] In a one-dimensional motion this example would be quite unrealistic, since the two bodies would go through each other whenever $\cos \omega_s t = 0$. However, this is just an example to illustrate what happens to a typical oscillatory mode of a system. For instance, one can consider an elliptic motion on a plane, which is the combination of two simple oscillations along the two axes.