Deriving the Einstein-Infeld-Hoffman (EIH) Lagrangian
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The Lagrangian for a system of gravitating N point-masses, in the first post-Newtonian (1PN) approximation, known as the EIH Lagrangian, can be derived by summing the Lagrangians which would give geodesic motion for each point-mass in the appropriate external/regularized metric, *but one must also add to these the contribution from the Einstein-Hilbert action for the gravitational field.* The necessity of the latter contribution is evident even at Newtonian (0PN) order.

The 1PN metric in (conformally Cartesian) coordinates $x^\alpha = (t, x^i)$ [not (ct, x^i) in this note] can be written as

$$ds^2 = \left(-c^2 + 2V - \frac{2V^2}{c^2}\right) dt^2 - \frac{8V_i}{c^2}\, dt\, dx^i + \left(1 + \frac{2V}{c^2}\right) \delta_{ij}\, dx^i dx^j + O(c^{-4}), \tag{1}$$

where $V(t, \boldsymbol{x})$ is a scalar potential [containing both $O(c^0)$ and $O(c^{-2})$ parts] and $V_i(t, \boldsymbol{x})$ is a vector potential. In harmonic gauge,

$$\partial_\alpha(\sqrt{-g} g^{\alpha\beta}) = 0 \qquad \Rightarrow \qquad \dot{V} + \partial_i V_i = O(c^{-2}), \tag{2}$$

the components of the Ricci tensor and of the field equation $R_{\alpha\beta} = (8\pi G/c^4)(T_{\alpha\beta} - g_{\alpha\beta} T_\gamma{}^\gamma/2)$ are

$$-\left(1 + \frac{4V}{c^2}\right) R_{00} + O(c^{-4}) \;=\; \nabla^2 V - \frac{\ddot{V}}{c^2} \;=\; -4\pi G \left(T^{00} + \frac{T^{ii}}{c^2}\right) + O(c^{-4}), \tag{3}$$

$$\frac{c^2}{2} R_{0i} + O(c^{-2}) \;=\; \nabla^2 V_i \;=\; -4\pi G\, T^{0i} + O(c^{-2}), \tag{4}$$

$$-c^2 R_{ij} + O(c^{-2}) \;=\; \nabla^2 V \delta_{ij} \;=\; -4\pi G\, T^{00} \delta_{ij} + O(c^{-2}). \tag{5}$$

The metric determinant is given by

$$\sqrt{-g} = c\left[1 + \frac{2V}{c^2} + O(c^{-4})\right], \tag{6}$$

and its $O(c^{-4})$ corrections involve the $O(c^{-4})$ corrections to g_{ij}. The relative $O(c^{-4})$ part of the Ricci scalar R also depends on $g^{(4)}_{ij}$, but the combination $\sqrt{-g}R$ can be written as [see e.g. V. Brumberg, Celestial Mech Dyn Astr (2007) 99:245–252]

$$\begin{aligned} c\sqrt{-g}R \;&=\; -2\nabla^2 V + \text{other total derivatives } [\partial_i(\ldots), \partial_t(\ldots)] \\ &\quad + \frac{2}{c^2} V \nabla^2 V - \frac{2}{c^4} V \ddot{V} - \frac{8}{c^4} V_i \nabla^2 V_i + O(c^{-6}), \end{aligned} \tag{7}$$

Thus, up to boundary terms, the Hilbert action can be written as

$$S_{\rm g} = \frac{c^3}{16\pi G} \int d^4x \sqrt{-g} R \;=\; \frac{1}{16\pi G} \int dt\, d^3x \left[2V\nabla^2 V - \frac{2}{c^2} V\ddot{V} - \frac{8}{c^2} V_i \nabla^2 V_i + O(c^{-4})\right] \tag{8}$$

$$\;=\; \int dt\, d^3x \left[-\frac{V}{2}\left(T^{00} + \frac{T^{ii}}{c^2}\right) + \frac{2}{c^2} V_i T^{0i} + O(c^{-4})\right], \tag{9}$$

where the second line has used the field equations.

Consider a point particle with rest mass m, worldline $z^\alpha(\tau)$, and velocity $u^\alpha = \gamma(1, v_i)$. The "matter action" and stress tensor are

$$S_{\rm m} = -mc^2 \int d\tau = -mc^2 \int dt\, \gamma^{-1}, \tag{10}$$

$$T^{\alpha\beta}(x) = m \int d\tau\, u^\alpha u^\beta \frac{\delta^4(x - z)}{\sqrt{-g}} = m\gamma^{-1} u^\alpha u^\beta \frac{\delta^3(\boldsymbol{x} - \boldsymbol{z})}{\sqrt{-g}}. \tag{11}$$

It follows from $g_{\alpha\beta}u^\alpha u^\beta = -c^2$ that

$$\gamma = 1 + \frac{1}{c^2}\left(\frac{v^2}{2} + V\right) + \frac{1}{c^4}\left(\frac{V^2}{2} + \frac{5Vv^2}{2} + \frac{3v^4}{8} - 4v_iV_i\right) + O(c^{-6}), \tag{12}$$

$$\Rightarrow \quad \gamma^{-1} = 1 - \frac{1}{c^2}\left(\frac{v^2}{2} + V\right) + \frac{1}{c^4}\left(\frac{V^2}{2} - \frac{3Vv^2}{2} - \frac{v^4}{8} + 4v_iV_i\right) + O(c^{-6}), \tag{13}$$

and the components of $T^{\alpha\beta}$ are then given by

$$T^{00} = m\left(1 + \frac{v^2}{2c^2} - \frac{V}{c^2}\right)\delta^3(\boldsymbol{x} - \boldsymbol{z}) + O(c^{-4}), \tag{14}$$

$$T^{0i} = m\, v_i \delta^3(\boldsymbol{x} - \boldsymbol{z}) + O(c^{-2}), \tag{15}$$

$$T^{ij} = m\, v_i v_j \delta^3(\boldsymbol{x} - \boldsymbol{z}) + O(c^{-2}). \tag{16}$$

Now add the matter actions and stress tensors for particles $A = 1, \ldots, N$. Using (10) and (13) for $S_{\rm m}$ [dropping the constant terms], and (9) and (14)–(16) for $S_{\rm g}$, we find

$$S_{\rm m} = \int dt \sum_A m_A\left[\frac{v_A^2}{2} + V_A + \frac{1}{c^2}\left(-\frac{V_A^2}{2} + \frac{3V_Av_A^2}{2} + \frac{v_A^4}{8} - 4v_A^iV_A^i\right) + O(c^{-4})\right], \tag{17}$$

$$S_{\rm g} = \int dt \sum_A m_A\left[\quad -\frac{V_A}{2} + \frac{1}{c^2}\left(+\frac{V_A^2}{2} - \frac{3V_Av_A^2}{4} \quad + 2v_A^iV_A^i\right) + O(c^{-4})\right] \tag{18}$$

$$\Rightarrow \quad S_{\rm m} + S_{\rm g} = \int dt \sum_A m_A\left[\frac{v_A^2}{2} + \frac{V_A}{2} + \frac{1}{c^2}\left(\quad + \frac{3V_Av_A^2}{4} + \frac{v_A^4}{8} - 2v_A^iV_A^i\right) + O(c^{-4})\right], \tag{19}$$

where $V_A = V(z_A)$ and $V_A^i = V_i(z_A)$, but with the contributions to these potentials from body A itself (which blow up at $x = z_A$) dropped.

Note that the effect of adding $S_{\rm g}$ is to halve all of the terms linear in the potentials, and to completely cancel the term quadratic in V.

Also note that the field equations follow directly from varying $S_{\rm m}$ as in (17) plus $S_{\rm g}$ as in (8) [before inserting the field equations] with respect to V and V_i.

The field equation

$$\nabla^2 V - \frac{\ddot{V}}{c^2} = -4\pi G\left(T^{00} + \frac{T^{ii}}{c^2}\right) = -4\pi G \sum_B m_B\left(1 + \frac{3v_B^2}{2c^2} - \frac{V_B}{c^2}\right)\delta^3(\boldsymbol{x} - \boldsymbol{z}_B)$$

$+O(c^{-4})$ has the solution

$$V(t, \boldsymbol{x}) = \sum_B m_B\left[\frac{1}{r_B}\left(1 + \frac{3v_B^2}{2c^2} - \frac{V_B}{c^2}\right) + \frac{\ddot{r}_B}{2c^2}\right]$$

$$= \sum_B \frac{m_B}{r_B}\left[1 + \frac{1}{c^2}\left(2v_B^2 - V_B - \frac{1}{2}(v_B \cdot n_B)^2 - \frac{r_B}{2}a_B \cdot n_B\right)\right],$$

where $r_B = |\boldsymbol{x} - \boldsymbol{z}_B|$ and $\boldsymbol{n}_B = (\boldsymbol{x} - \boldsymbol{z}_B)/r_B$. Similarly, the solution for the vector potential is

$$V_i(t, \boldsymbol{x}) = \sum_B \frac{m_B v_B^i}{r_B}. \tag{20}$$

Evaluating these at $\boldsymbol{x} = \boldsymbol{z}_A$, dropping $B = A$ terms, and plugging into $S_{\rm g} + S_{\rm m}$, integrating by parts to get rid of the acceleration term, using

$$\partial_t(v_B \cdot n_{AB}) = a_B \cdot n_{AB} + \frac{v_B \cdot v_{AB} - (v_B \cdot n_{AB})(v_{AB} \cdot n_{AB})}{r_{AB}}, \tag{21}$$

the action becomes

$$
\begin{aligned}
S_{\rm g} + S_{\rm m} &= \int dt \sum_A m_A \Bigg\{ \frac{v_A^2}{2} + \frac{1}{2} \sum_{B \neq A} \frac{G m_B}{r_{AB}} \\
&\quad + \frac{v_A^4}{8c^2} + \frac{1}{4c^2} \sum_{B \neq A} \frac{G m_B}{r_{AB}} \left[3 v_A^2 + 3 v_B^2 - 7 v_A \cdot v_B - (v_A \cdot n_{AB})(v_B \cdot n_{AB}) - 2 \sum_{C \neq B} \frac{G m_C}{r_{BC}} \right] \Bigg\},
\end{aligned}
$$

which is the EIH action.