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An introduction to the unpublished book "Reflections on a Tube" by Mitchell J. Feigenbaum

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ABSTRACT

This paper is an adaptation of the introduction to a book project by the late Mitchell J. Feigenbaum (1944–2019). While Feigenbaum is certainly mostly known for his theory of period doubling cascades, he had a lifelong interest in optics. His book project is an extremely original discussion of the apparently very simple study of anamorphs, that is, the reflections of images on a cylindrical mirror. He observed that there are *two images* to be seen in the tube and discovered that the brain preferentially chooses one of them. I edited and wrote an introduction to this planned book. As the book is still not published, I have now adapted my introduction as a standalone article so that some of Feigenbaum's remarkable work will be accessible to a larger audience.

The late Mitchell J. Feigenbaum (1944–2019) left us with an unfinished book whose title is "Reflections on a Tube." While Feigenbaum is certainly mostly known for his theory of period doubling cascades, he had a lifelong interest in optics. In the book, he starts with the study of the image you can see in a vertically placed cylindrical mirror, usually known as an anamorph. He observed that there are *two images* to be seen in the tube and discovered that the brain preferentially chooses one of them. Fanning out from this observation, he touches on several associated problems: What fish see from under the water, the quality of the fish eye vs the land-animal eye, and many others. As the book is still not published, I have now adapted my introduction to the book as a standalone article so that some of Feigenbaum's remarkable work will be accessible to a larger audience.

I. THE BOOK

When Mitchell J. Feigenbaum passed away in 2019, he left a manuscript of his work on "Reflections on a Tube" to me and, in different versions, to some of his other friends. Since everyone in this group knew that Mitchell and I had discussed many times over the years all aspects of the book, the general feeling was that I should finish the manuscript and publish it as a book under Feigenbaum's name with myself listed as the editor who also completed the missing pieces. The present paper is a modified version of my planned introduction to that book. Unfortunately, the book project had to be given up for difficulties with the copyright, which stays with the heirs. I, therefore, decided to at least make my introduction available to others. I still hope that the book will finally appear in some form or other, but in the meantime, I hope that my "introduction" will make Feigenbaum's ideas known to a larger public. In the meantime, there are two papers available in which Gemunu Gunaratne and I tried to explain some details of Feigenbaum's work (Eckmann, 2021; Eckmann *et al.*, 2022) so that his ideas can be followed on a more technical level.

II. THE SUBJECT OF THE BOOK

This book is about *anamorphs*, reflections of images in a cylindrical tube. They are known to a large public, from first historical examples, such as Fig. 1 (Niceron, 1638), to modern works of art, such as Fig. 2. A drawing is deformed in such a way on a piece of paper so that the observer will see the undistorted image when looking at the tube. In Fig. 1, one sees Louis XIII, and in Fig. 2, a beautiful eye appears.

Given the many anamorphs one can find, one feels that their theory must have been extremely well-studied in optics. Indeed, one can find many programs that allow one to generate the anamorphic picture on the ground from any sample image. The novelty of Feigenbaum's work is that, upon studying the visual properties of the reflections in detail, one finds that the theory of anamorphs requires concepts that go way beyond such a seemingly simple toy problem. Specifically, Feigenbaum worked out an intriguing dichotomy of possible interpretations of what one can see. This dichotomy gives us a glimpse into the inner workings of the human visual system and its connection to the brain. Feigenbaum's observations are largely unexplained from a physiological point of view. What I like about this work is its methodology, which shows how a careful calculation (in this case, in optics) can lead to unexpected observations in another field (in this case, perception).

Mirrors come in many forms: The standard mirror on a wall is flat, but a mirror can also be bent like the cylinder, rippled like the surface of water, or willfully distorted as in Fig. 3. Still, all these mirrors are two-dimensional surfaces. The theory of his book covers visual aspects of such mirrors.

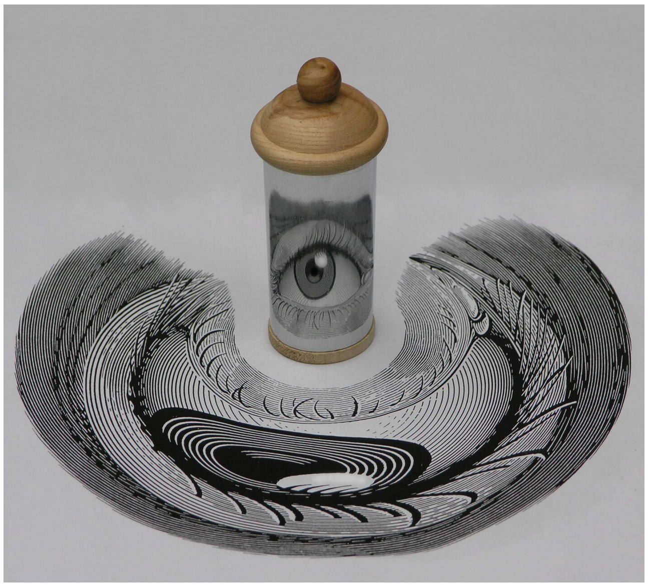

FIG. 2. An anamorph by István Orosz. Reproduced with permission.

III. THREE POSSIBLE ANAMORPHS

The mathematical finding of Feigenbaum is that there are really three possible images to be seen in the tube: namely, a standard one, which he calls "erect," and two others, which he calls "3D" and "flat."

The erect one is shown in Fig. 4. One wraps an image around the cylinder, fixes the position of the eye, and then draws lines from the eye to the cylinder and then to the table using the rules of reflection. This is what is done in the anamorphs of Figs. 1 and 2. While this procedure leads to appealing anamorphs, it is actually not correct because, as is shown in Feigenbaum's book, these views have no power in the sense that they would be what is seen by a pinhole camera, but not by the human eye, which has a non-negligible opening of the pupil.

FIG. 1. An anamorph by Jean-François Niceron (1616–1646). It shows king Louis XIII. Reproduced with permission from Gallerie Nazionali di Arte Antica, Roma (MiC)—Foto Alberto Novelli.

Using the eye, and not a pinhole camera, one can actually see two different images, as suggestively shown in Fig. 5. The first image appears on a surface, which is in the *interior* of the tube, while the second lies *flat* on the table. Thus, *two* different views are presented to the eye.

A poor man's explanation for the two images is understood, indirectly, because a two-dimensional surface has, in every point, *two main curvatures* (the flat mirror is exceptional in this respect since all directions have the same curvature). For example, the tube is flat in the vertical direction and maximally curved in the horizontal direction.

Using the theory of caustics, which will be illustrated below, Feigenbaum showed that there are indeed two images, as in Fig. 5, both of which have more intensity than the erect image sketched in Fig. 4. A further, important, observation shows that the two views appear along the *same* line of sight, as illustrated in Fig. 6. This implies that the two images reach the eye as a superposition.

The intriguing question is then whether one can discriminate between the two superposed images.

We will see that this indeed is possible, and it happens in an unexpected way. This is best understood by looking at Fig. 7. In it, a pattern is seen on the table, which produces a regular set of dots on the tube. The scene is photographed with a camera, but the focal

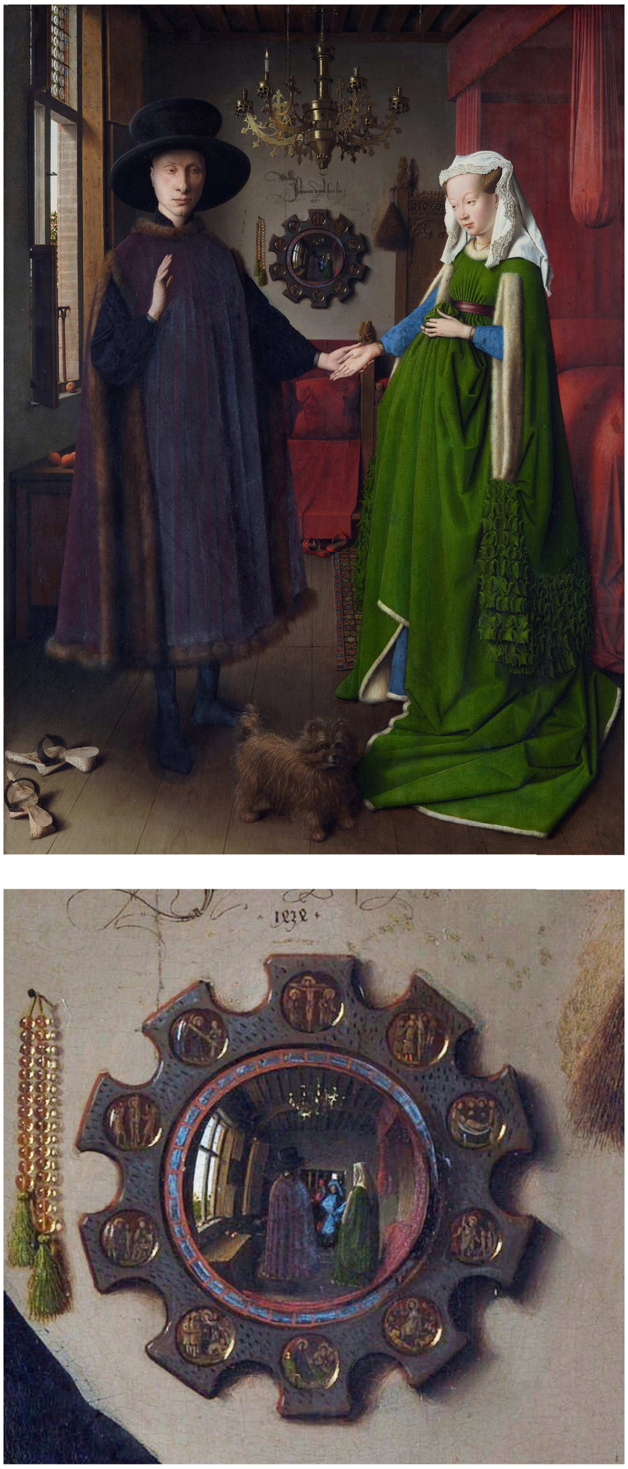

FIG. 3. The Arnolfini portrait of van Eyck 1434 (National Gallery London) is considered the first painting with reflections from a non-flat mirror. Both views were artificially made lighter for better visibility. Source: public domain.

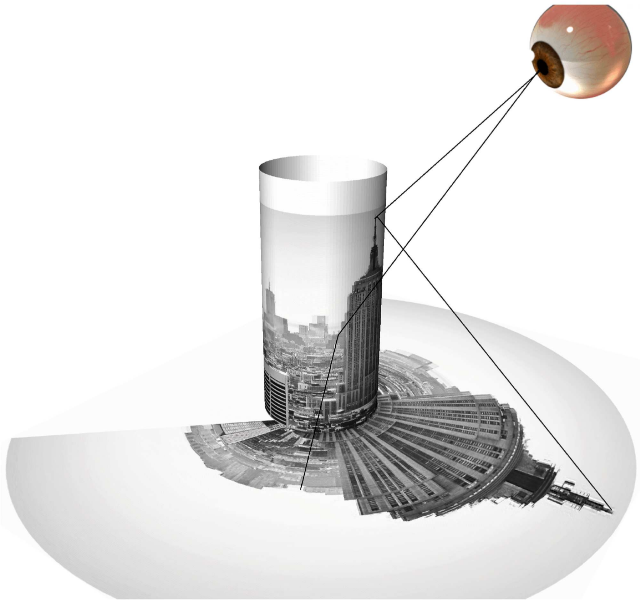

FIG. 4. The "erect" anamorph, where the construction is computed as if the image was wrapped on the cylinder. Photo by Jean-Pierre Eckmann.

distance is changed between the left and right takes. Note that neither of the two choices of focal distance produces a sharp image, as can be seen at the bottom of Fig. 7. Furthermore, no other choice of focus of the camera can make the images of the dots sharp. However, the unsharpness is not arbitrary: Both images are unsharp in a characteristic way: One image is vertically unsharp (called **H** throughout the book), while the other is horizontally unsharp (called **V**). The letter **H** indicates that the line between the two eyes is horizontal. Upon turning the head sideways (as explained later), the line between the eyes will be vertical; thus, **V** is used.

Therefore, the camera produces *no* good image of what is perceived in the mirror. However, the human observer perceives an

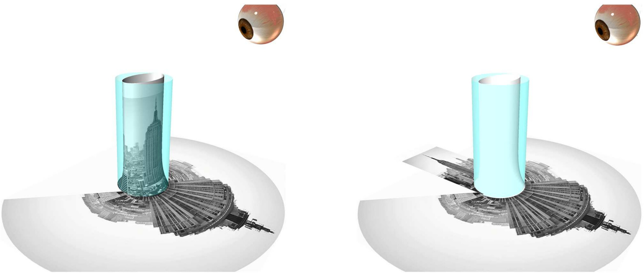

FIG. 5. The 3D and flat anamorphs: Inside the cylinder (left) and flat on the table (right). While, of course, the light rays always get reflected exactly as shown in Fig. 4, the virtual image will appear not *on* the surface of the cylinder, but either on a surface *inside* the cylinder or flat on the table *behind* it. This is what the observer will really see. Photo by Jean-Pierre Eckmann.

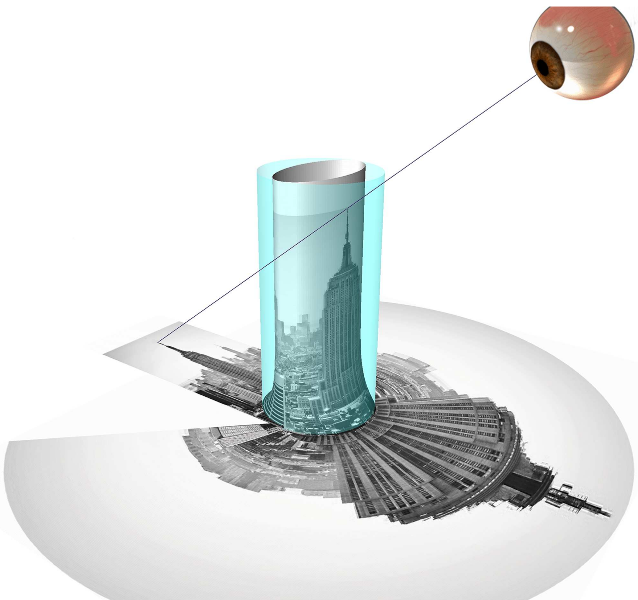

FIG. 6. The two versions of Fig. 5 are visible with exactly the *same* direction of the gaze. Photo by Jean-Pierre Eckmann.

image that seems sharp, and, in fact, there are *two* possible sharp images to be seen, as sketched in Fig. 5.

Since the viewer has two choices of seeing the reflection of the dots in the mirror, the question which Feigenbaum asked is: Which choice is preferred? It turns out that the vertical unsharpness—*vertical relative to the natural orientation of the head*—is preferred by the eye–brain system; that is, our visual system prefers the **H** over the **V**. This seems well-known in ophthalmology: If a patient has vertical astigmatism ("axis" in ophthalmological prescriptions)—called WTR (with the rule)—there is much less need for correction than if the astigmatism is horizontal—called ATR (against the rule). Therefore, the preference seems somehow universal.

In Fig. 7, which is a photograph of the image in a metallized tube, this unsharpness is clearly visible. The uninitiated reader will not notice any difference in the two top figures, but the eye *does*. To be more precise, any photographic image cannot really distinguish between the two possible views, as it will always record a superposition. Only the artifact of focusing at a specific distance, as in the bottom of Fig. 7, indicates at least some difference between the two possible views.

Feigenbaum also shows that the human eye cannot focus simultaneously at the two distances. This means that the viewer must choose (unconsciously) one of the two views, and, as I said, the **H** view is preferred. Furthermore, as the two images are in the tube or on the table, their distance from the eye is not the same, and this allows us to enhance the effect by choosing where to focus. As I said before, the image in the tube (and on the table) is actually unsharp, but we perceive it as sharp because the eye–brain machinery is insensitive to vertical unsharpness (astigmatism).

The difference of the focal distances is actually more pronounced in another experiment shown in Fig. 14, and so many people seem to see the effect better in that case. This one is easy to make with a rectangular box, filled with water, and a ruler (see Fig. 15).

Since we seem to prefer the vertical unsharpness, Feigenbaum suggested that you turn your head 90° sideways as in Fig. 8 (see later for how exactly you are supposed to do this). Then, clearly, the notions of vertical and horizontal get exchanged. And now, suddenly, the *other image*, the **V**, is going to be preferred. You will see the reflection in a different location. With the head in the upright position, the image appears in the tube; with the head turned 90° (and keeping the eye more or less in the same position), the image seems to appear on the table, behind the cylinder (it lies down). Note that the direction in which the image is seen is unchanged, but the distance where it seems to appear depends on the orientation of the head.

To summarize, there are two images, neither of them sharp, and there is *no* sharp image available. In such a case, the eye–brain system will prefer the image, which is unsharp vertically, relative to the orientation of the head.

Since no image is sharp, Feigenbaum and I devised the notion of *non-object* for what is presented to the eye. In contrast to what is seen in a flat mirror, the non-object is not really localized. In a normal perspective, objects just present to the two eyes two different views of something that is fixed in space. The non-objects are not fixed in space so that their image seems to be located in different points in space depending on the vantage point. If you move your head a little bit right–left, the image seems to turn around the cylinder.

(This unsharpness of the non-object has the amusing consequence that autofocusing with digital cameras will be confused by the two unsharp images, which are at different distances, as there is no ideal focal distance available.)

What is the reason for the possibility of seeing two choices? As mentioned before, the eye is not a pinhole camera but has some aperture. This simple, but important fact means that different points on the retina see different caustic points (points of maximal intensity, see Sec. IV). In other words, to really understand what one sees, one has to consider the image, the tube, *and* the eye. This is what Feigenbaum did.

IV. THE MAIN INGREDIENTS

The study of Feigenbaum combines in a clever and certainly completely novel way certain observations, which are well-known to people familiar with optics. I will explain now what the pieces are and how they are combined.

The first ingredient is the role *caustics* play in vision. Every lay person has probably seen caustics without realizing it because they are what one sees in rainbows. What people generally do not know is that the small drops, which form the lenses for the rainbow, are quite terrible lenses. Indeed, as shown in Fig. 10, the light rays that enter the drop do not come out of the drop at a sharp angle, but are rather fanned out. This raises the question of why the rainbow seems reasonably sharp when we look at it. What happens is that the density of the rays in the fan is not uniform, and there is one direction at which the density is the highest, and this is where you

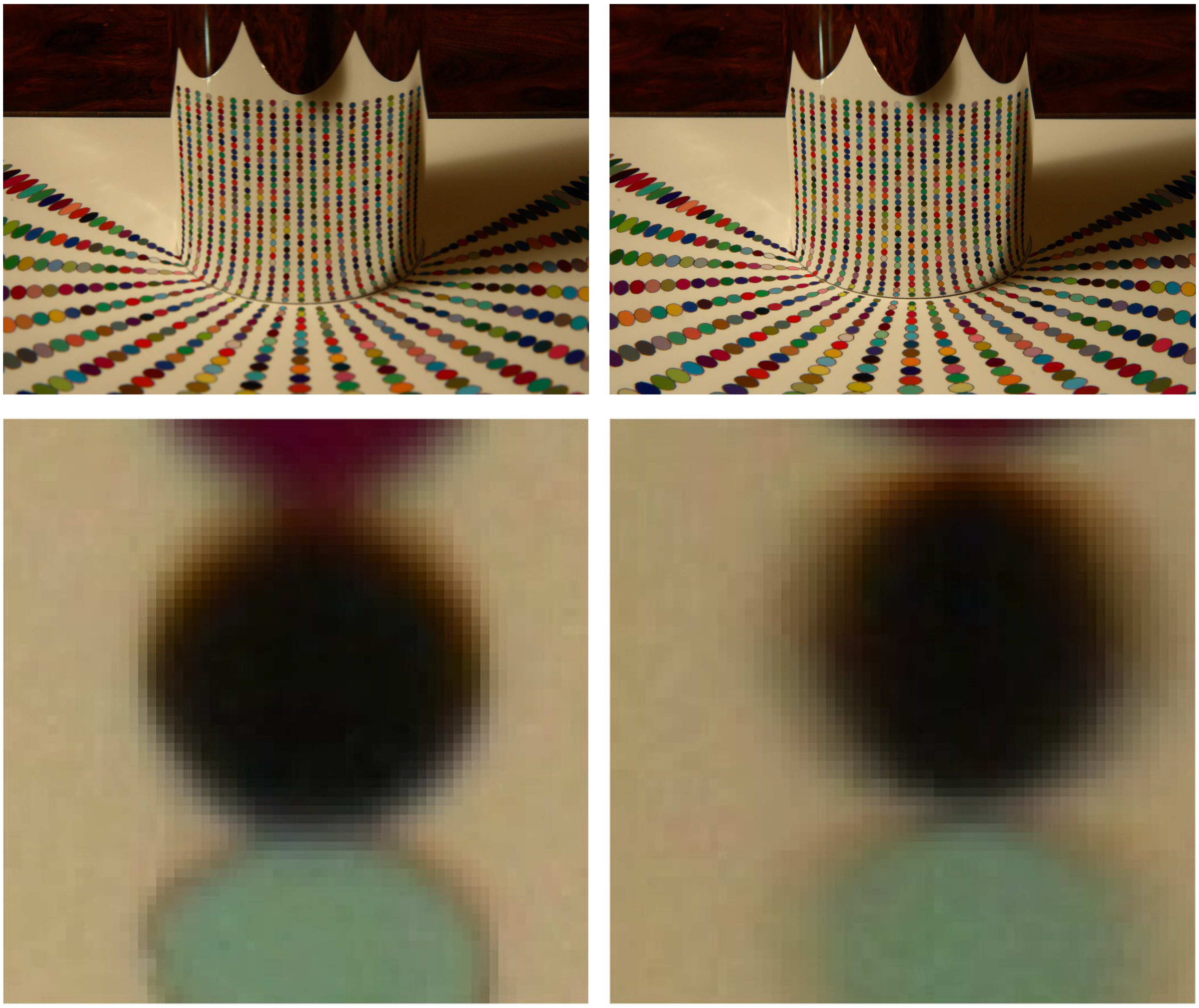

FIG. 7. In both views, the position of the camera and its direction of view are the same. On top: two photographs of exactly the same scene, reflected in the cylindrical mirror. On the bottom, magnification of a few dots. The only difference in making the two pictures is the setting of the focal distance. The pictures were made with a Canon EOS 30D camera and a Canon EF-S17-55mm f/2.8 IS USM lens. On the left, the focus is 1.2 cm in front of the tube's center plane; on the right, the focus is on the fifth horizontal line from the top of the flat image. The f-stop was f/8. The important thing to observe is the change in the direction of unsharpness. On the left, the image is unsharp in the vertical direction, called **H**-astigmatism. On the right, the unsharpness is in the horizontal direction, called **V**-astigmatism. Note that, to the eye, the top photographs seem identical (the anamorphs are of the 3D type). Photographs by Mitchell J. Feigenbaum, August 2006. Reproduced with permission.

see the rainbow, each color at a different angle, but quite sharp. The conclusion to draw is that we do not see the spread-out rays, but rather these highest density regions, which are the caustics.

Another observation you should make is also seen in Fig. 9: namely, the sky around a rainbow seems to be dark on the outside and diffuse on the inside, and this is explained again by Fig. 10, which shows that the fan opens only upward, but not downward. (The reader might be confused by this as in Fig. 10, there are only rays fanning upward from the caustic; however, a little thought shows that we actually look at this figure from below and see images of reflections in drops at different heights.) The conclusion to draw from this discussion is that the correct way to study what one actually sees must go through a study of caustics, and where they are. In particular, the usual ray-tracing methods often found in

FIG. 8. Illustration of how to turn the head 90° sideways. Licensed with permission from Ron Dudley.

FIG. 9. A typical rainbow. Note that the sky on the outside of the rainbow is darker than it is on the inside of the rainbow. Source: https://www.maxpixels.net/Rainbow-Meadow-Sky-Strommast-Landscape-Nature-4285843, open access.

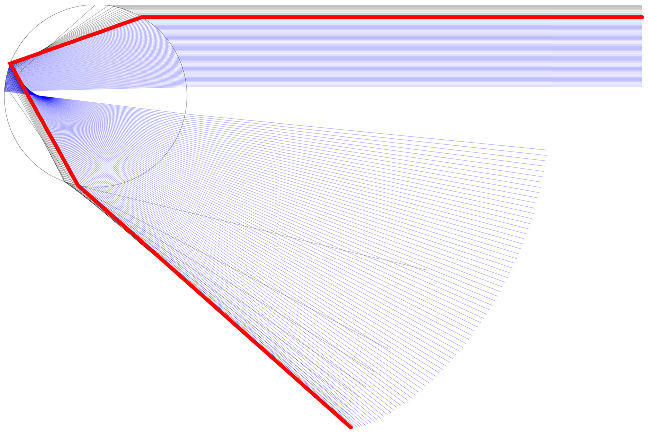

FIG. 10. Illustration of the caustics for a rainbow (for just one color/wavelength) in a spherical (actually cylindrical) raindrop. The rays from the sun reach the drop completely parallel from the top right. They get reflected once on the interior left side of the drop and leave to the right. One can see that many incoming rays get dispersed, but many concentrate near the red line, from both sides, blue and black rays. This gives more intensity, along the red direction. The envelope to which the red line is tangent on the lower right is called the *caustic*. This sketch also explains why the exterior of the rainbow is lighter on the inside: This is because the dispersing rays come out at a flatter angle (and will be seen in drops, which are "below" the rainbow). For more complicated reflection patterns, I like to look at the beautiful photographs in the classical book (Minnaert, 1993). For an accessible theoretical discussion, see Nussenzweig (1977). The Moiré artifacts in the image come from the pixeled nature of reproducing images.

2D-projection graphic programs are not adequate. In the case of reflection from a tube, ray-tracing would do the following, as illustrated in Fig. 4. Take any image and wrap it onto the tube and then fix the position of the observer and assume that the eye is just a point. Draw a line from this "eye" to an image point on the cylinder and continue it down to the table using the rules of reflection from the cylindrical mirror. This is the ray-tracing image of the scene on the mirror. It is *not* the way Feigenbaum constructs his anamorphs because, as shown in the left side of Fig. 5, one of the two images appears on a surface whose cross section is an ellipsoid (of ratio 2 : 1 at infinite height), and therefore, one must wrap the image somehow on this surface (and not on the surface of the tube).

The second ingredient, which is one of his main insights, is the question of what happens if there are *two* caustics presented to the eye. As I alluded to in Sec. III, there are, most often, two caustics to be seen because any mirror is, in every point, curved in two principal directions (this is not to be confused with what earlier authors call two caustics, namely, the two curved pieces of the two-dimensional Fig. 22, which make what here is called one caustic only). (If the mirror is completely flat, the two caustics will coincide.) The tube is a particularly nice example because it is vertically flat, and horizontally just a circle, which allows for an explicit calculation of the caustics. In the paper (Eckmann *et al.*, 2022), the interested reader can find a variant of Feigenbaum's calculation, where the two caustics are determined for reflections from a sphere.

While the book explains that the phenomenon of two caustics is ubiquitous, the most striking example is "Reflections on a Tube," the title of this book. The setup is shown in Figs. 1 and 2 and in many illustrations throughout the book.

After discovering for the first time that there are two disconnected caustic images for the cylinder, the interesting—and in my view, completely novel—question which Feigenbaum asks (and answers) is which of the two options is preferentially chosen. Furthermore, an explanation is given for how this choice is made. And, as will be seen, this finally must be related to how the visual system of animals processes the inputs it gets.

There are several precursors in the literature on viewing objects in water, where the authors knew that there are two images, with different astigmatic directions. As far as I can tell, while some mention a binocular effect, it seems that the cyclopic effect has not been mentioned (see, e.g., Kinsler, 1945; Bartlett *et al.*, 1984; and Horváth *et al.*, 2003). [For a review on classical caustics, see Berry and Upstill, 1980 and the original work on "catastrophe theory" (Thom, 1972; 1976 and Arnol'd, 1974; 1975).]

V. HOW TO LOOK AT THE CYLINDER, THAT IS, THE TUBE

It is easy to make your own cylinder. Best results are obtained for cylinders of diameter of about 5 cm (or 2 in). The cardboard kernel of a kitchen-tissue roll is just about right. One wraps a reflecting (silvery) Mylar sheet tightly around it and places the whole thing in the center of the included figure. (Do not use an aluminum foil, it is not flat enough.)

To make an anamorph of a jpeg file of yours, go to the page https://fiteoweb.unige.ch/eckmannj/a4_shift.html, which also contains the necessary instructions of how to print the new jpeg that it constructs. It is important that the size of the printout is correct [I thank Noé Cuneo for transforming Feigenbaum's program (which was written in Pascal) to the HTML version]. Recall that there are three possible anamorphs, called "erect," "flat," and "3D." The program will produce the 3D version, which distorts the image minimally when seen inside the tube. This is the one that is most natural and is seen in the **H** direction. The flat one is undistorted when viewed in the **V** direction, and the erect one is the ray-tracing. (Note that there is no choice of anamorph that is undistorted in 3D and flat simultaneously.)

The table should be as flat as possible. The eye of the observer is supposed to be positioned at a distance of about 25 cm from the cylinder and about 40 cm above the table. In this position, most people see the image of the drawing as if it were inside the cylinder. On closer inspection, the calculations by Feigenbaum show that the image appears not on the surface of the cylinder, but on an ellipsoid half as thick as the cylinder. Now, one should turn the head 90° sideways, but in such a way that the eye with which one looks (you should look with one eye, see below) remains in the same place. In this configuration, most people see the drawing "lying down," as if it were reflected behind the cylinder. (This rule of how to turn the head is important because you should, as in Fig. 6, not change your line of sight.)

It is important to note that the effect has *nothing to do with binocular vision*, as you can check by covering one eye. However, as Feigenbaum studies in detail, binocular vision into the tube is quite different from binocular vision of true objects because the two eyes see two different non-objects. (This is then an over-determined problem for the brain.)

I showed, in 2009, Feigenbaum's project to a neuro-ophthalmologist friend of mine, Avinoam Safran. He pointed out that turning the head is less good than turning the cylinder (and the "table"). In fact, the inner ear signals the position of the head to the brain, and he told me that the fourth cranial nerve activates a muscle, which rotates the eyes toward the nose. I encourage any interested reader to do the experiment in this more complicated, but cleaner way. Another possibility would be to do this in the space station, where gravitational orientation is missing. Finally, I decided to have a hologram made, which avoids these problems; see Sec. XI.

The book also explains where exactly the image appears. If you watch closely, (with both eyes), you will notice, as I said before, that the reflected image appears glued onto an elliptic surface, inside the cylinder. Furthermore, this surface enlarges toward the bottom to reach the circle where the cylinder touches the table. [The reader can see this illustrated on the left side of Fig. 5. The alternate image, however, is completely flat on the table (right side of Fig. 5).]

Another experiment, related to the different astigmatisms of the two caustics, is to move your head slightly up and down; the picture inside the tube will move with this vertical motion, and sideways motion of the head makes it turn around the cylinder. The roles are exchanged if you focus on the image "behind" the cylinder.

VI. OTHER EXAMPLES OF MULTIPLE CAUSTICS

The aim of Feigenbaum's work is to shed light on the questions these observations raise. I just list some of the issues before I explain some of Feigenbaum's further contributions in his manuscript:

- Is the "pixel" resolution of the eye (i.e., its acuity) good enough to actually distinguish the difference between the two non-objects, i.e., between the two images of Fig. 7?
- When one uses both eyes, the two eyes get two different images from the same scene. What is the magnitude of the effect that both eyes get slightly different images, and how does the brain interpret what is seen if the image is coming from a non-object?

Such questions lead to a discussion of the quality of human eyes vs fish-eyes, which are shown to be vastly better than the human eyes. From this, Feigenbaum draws some conclusions about the relatively bad evolution from fish-eyes to the eyes of land-animals.

The appendixes contain furthermore several beautiful examples based on the **H**–**V** dichotomy where caustics appear in everyday life. I illustrate these in Figs. 11–16, and some explanations are summarized in the captions:

- Looking from air into the water, *from above* (a pool, the sea) as in Fig. 11.
- The bent (broken) pencil as in Figs. 13–15 (one can see again *two* views). (One of the two views was certainly known at the end of the 19th century. I am not aware of any discussion of the second possible view. An example can be found in Watson, 1907, Fig. 316. I worked out some more details in Eckmann, 2021.)
- Looking from water into the air: this is the problem of the archer-fish, which "shoots" at targets in the air from below the surface of the water; see Fig. 16.

FIG. 11. The up-sloping pool floor. In this photograph, the pool floor clearly seems to go up toward the far corner. Please see an example where the slope gets flatter as you look far away, as sketched in Fig. 12. Reproduced with permission of oasistile.com/pages/pool-tile-buying-guide.

It is perhaps useful to add here one of many Feigenbaum sketches for this upsloping effect in Fig. 12 with my own caption added.

VII. AN IMPORTANT COMMENT

As follows from Secs. I–VI, the phenomenon Feigenbaum described is *not* about optical illusions. To see the difference, let me just show, in Fig. 17, a favorite illusion everybody has seen. Here, the effect is that the middle segment looks longer in one case than in the other, and this is triggered, as in many other examples of this kind, by what surrounds the central line. The illusion disappears if one draws vertical lines connecting the tips of the arrows. Other optical illusions, such as Fig. 18, exploit where the eye puts its focus. The apparent motion of the picture disappears if you fix your eyes at a fixed center of the picture. Finally, those like Fig. 19 simply present confusing realities, and the puzzle disappears if one analyzes the way the legs are drawn.

The reflections in the tube are of a completely different nature. Because the mirror is two-dimensional, the viewer is presented with two options (the vertical and horizontal picture), and it is somehow the brain–eye system that has to make a choice: Specifically, your visual apparatus prefers to see that picture, which is unsharp in the current vertical axis of the eye, which depends on the position of your head. No amount of rationalizing what is seen can make the dichotomy disappear because there is simply no adequate best middle ground between the two pictures.

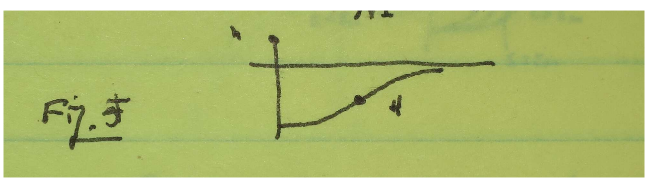

FIG. 12. The apparent bending of the sea floor for an observer standing at height *h* (on the left) with the caustic point *H* indicated. If the viewer looks down at an angle of 35° and the pool is 10′ deep and the viewer's eyes 10′ above the surface, the floor will seem to slope upward by about 10°.

FIG. 13. A standard photograph of the "broken" pencil. However, as shown in Fig. 14, Feigenbaum was interested in what one sees when looking into the water *from the air*, at a shallow angle. This standard image is a view *through* the water. Photo by Jean-Pierre Eckmann.

VIII. THE ORIGIN OF THE BOOK

Mitchell Feigenbaum passed away on June 30, 2019. His interest in anamorphs was originally raised in discussions with Kenneth Brecher in March 2006. By the time of his death, he had worked, continuously, for about 13 years on this project. Many versions of TeX files for the book were created over this period. Several of his colleagues discussed with him his project, helped him typing, and gave input. Perhaps I happened to be the most insistent and enthusiastic follower of his project. The state of the book at Feigenbaum's death was close to finished, but, of course, some things were missing. The last few appendices were incomplete. Feigenbaum certainly wanted to rewrite the preface, which, in the form I had the

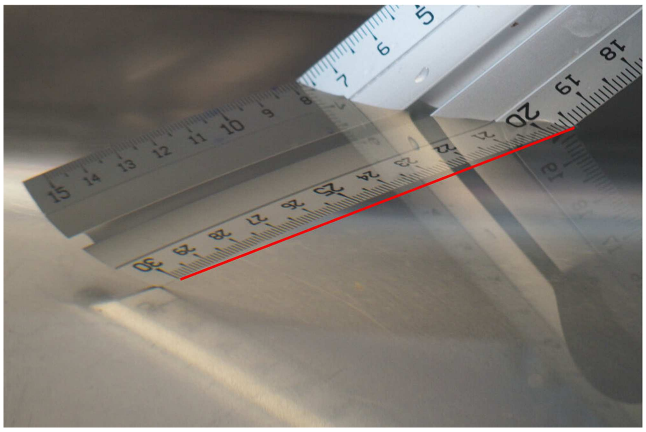

FIG. 14. A photograph of the "broken" pencil (actually a ruler). The point Feigenbaum made is that the part of the pencil *in* the water, when looked at through the top surface of the water, at a very low angle, is *not* straight. This is especially well visible at the lower edge of the ruler and was sketched by Feigenbaum. In Eckmann (2021), I worked out the details of what Feigenbaum had in mind. Note that the ruler is strictly perpendicular to the line of sight from the camera. If you repeat this experiment and tilt your head 90°, you will see that the left bottom corner moves toward you. The left top corner will also seem to move toward you, but less, and therefore, the whole ruler seems to rotate toward you. The interested reader can also see the effect of the ruler in a setup as in Fig. 11: In that case, the vertical bar of the handrail, which goes into the water, seems bent at the bottom toward the viewer, and the distance of the bottom end will change if the viewer rotates the head by 90°. (The effect is the strongest if the eye is close to the surface of the water.) Photo by Jean-Pierre Eckmann.

FIG. 15. A convenient setup for the ruler. It is best to fill water to the rim and to look into the water from above at a flat angle. Photo by Jean-Pierre Eckmann.

manuscript("Manuscript" means here the collection of TeX files and figure files.), did not refer to these last sections. While he had clear ideas of what needed to be done, his health problems did not allow him to finish the task. Given Mitchell's huge investment of time and energy and the originality of his findings, many of his friends felt that it was a major loss to the community if the book project were not completed, as best as possible. However, for the reasons mentioned above regarding publication rights, this has to date not been possible. The present article is an attempt to provide the broader community with some understanding of what Mitchell accomplished. While it is still preferable to produce the book in its amended form, I recognize that Mitchell would probably not have been satisfied with what I have done.

Let me be clear about the following issue: I did not undertake this project as a historian of science, and I do not intend to guess what other thoughts Feigenbaum might have had. One exception to the principle of not being a historian is my treatment of a section "Evolution and Design" that had no text. Since it is an important point, I have added some correspondence from Feigenbaum; see Sec. X.

I think the book should be understood as to how a study of the optics of vision is closely related to questions of how the visual system and the brain interact with the information that comes through the eyes.

My experience is that his unpublished book, as well as other publications by Feigenbaum (1978; 1979) will need some adjustments by the reader. There are two related reasons for this:

First, Mitchell's language is, as noted by John Horgan [in "The End of Science" (Interview 1994) (Horgan, 1997)] "as if English were a second language he (that is, Feigenbaum) had mastered through sheer brilliance."

The second aspect that makes the reading not so easy is Feigenbaum's technique of proving statements. While many scientists are willing to use sentences, such as "the following calculation is left to the reader" or to cite a reference to a known calculation, Mitchell would never allow himself such a liberty, which means that he verified everything from scratch and gave all details. However, of course, some training is required to know which details to gloss over on a first reading of the text. James Joyce is also difficult to read, but, in the end, that makes for a rewarding experience.

I attribute Feigenbaum's style to what I like to call a 19th century mind: He was not only extremely critical of others, but even more rigorous with himself. And so, as I said above, I am not sure that my friend Mitchell would have been happy with how I would

FIG. 16. Illustration of what the archerfish can do. Note that the eye is below the water, and therefore, the fish must "calculate" not only—because of the different indexes of refraction of water and air—in which direction to aim, but also where the object will fall. Feigenbaum has a section in which he does the calculation for this case, which is a variant of the cylinder case. He devotes a chapter "Caustical Imaging at the Air-Water Interface" on this. Archerfish of the species *Toxotes jaculatrix* take down insects in Indonesia. Photograph by A&J Visage, licensed from Alamy appeared in National Geographic, September 4, 2014.

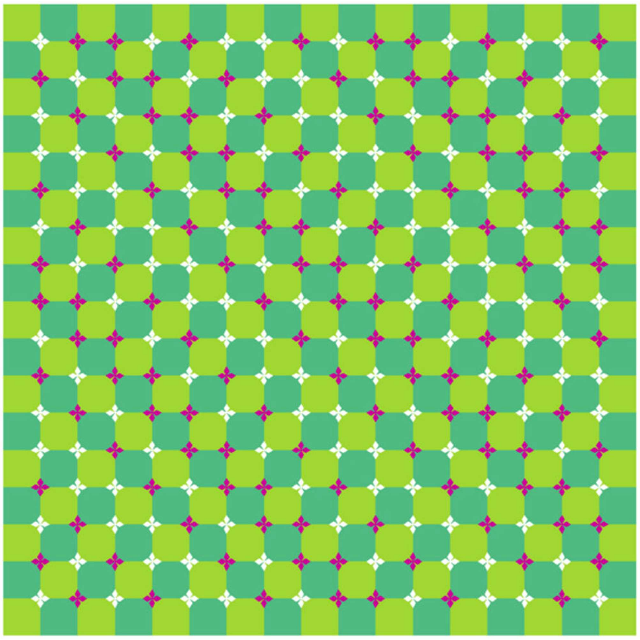

FIG. 18. A moving optical illusion: The pattern seems to move if the head is gently moved up and down. I guess this comes from changing the focus of the eyes. Copyright: Akiyoshi Kitaoka, Ritsumeikan University. Reproduced with permission.

finish the book or even discuss its highlights as in this Introduction. However, I think it is important that those parts of the text that were complete at his death are left "as is" so that they reflect his personal taste and style. As I had checked, criticized, and discussed with him all calculations before his death, my task was reduced to adding missing items, where possible, and giving some explanations we discussed earlier (there were about 50 unresolved questions of mine, and sections I and J were incomplete).

IX. FEIGENBAUM'S INTEREST IN VISION

Let me now come back to the science in Feigenbaum's book and how it is related to his general outlook.

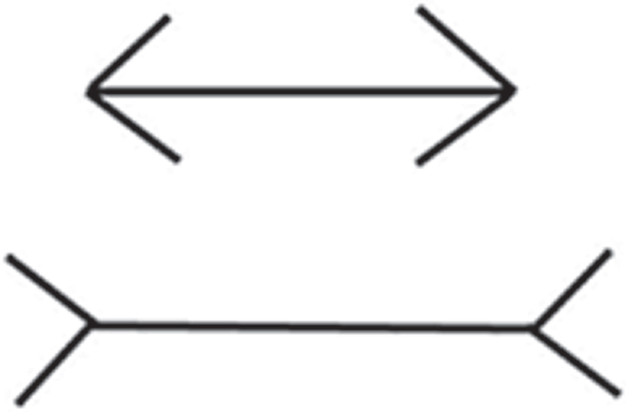

FIG. 17. The well-known optical illusion that makes the lower line look longer than the upper one. Apparently invented by Franz Carl Müller-Lyer (image in the public domain) (Müller-Lyer, 1889).

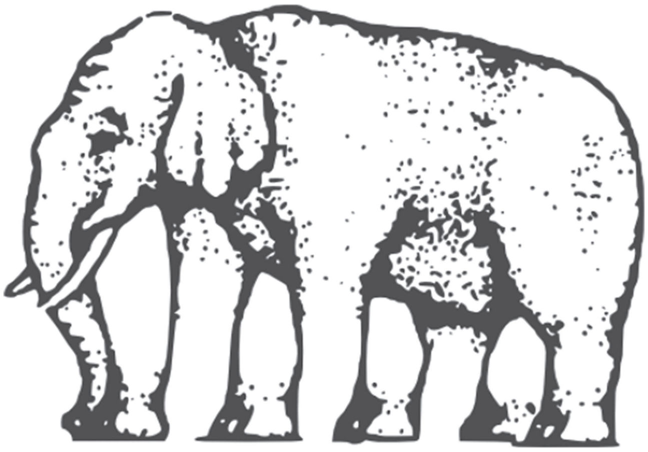

FIG. 19. The famous Shepard elephant riddle, which confuses about the number of legs. Reproduced open access modified version from https://static.parade.com/wp-content/uploads/2016/02/elephant_legs_illusion.png (Shepard, 1990).

Many people are familiar with Feigenbaum's discovery about the universality of period doubling and the constant 4.6692 . . . , which now takes his name Feigenbaum (1978; 1979).

Obviously, one may think that this book is an addition to his very successful findings of the 1970s. However, he developed here a very different subject, namely, optics, and, in particular, its relation to vision, the construction of the eye, and the interpretation by the brain of what is being seen.

Feigenbaum had, for many years before he started to write this book, been interested in optics and vision. This began with work on the problem of why the moon seems to be larger when it is near the horizon than when it is high in the sky. This appears to be unpublished. His work with the Hammond company, Hammond, 1997, in making a digital atlas that scales properly, is another example of a visual problem (the atlas also deals with the problem of non-overlapping labels, which is more a problem of statistical mechanics). His favorite example was a river that meanders around a city. On which side of the river will the city be shown when you scale down the picture? Another is his work on making bank notes that you cannot photocopy because they contain so many scales. At least one of the many scales will be badly sampled to the fixed pixel size of any scanner. Another project was the question of how to guarantee that the photograph of, say, a painting has the same colors as the painting itself. Strictly speaking, this is not possible since the material is not the same. However, Feigenbaum's idea was to actually re-photograph a first printout and then find the correct color mapping by comparing the original photograph with the second one. (For experts, this problem does not have a unique solution, and one gets better results when there are many colors in the picture.) Finally, he had a keen interest in photography, and clearly, his love for what a camera (and the eye) can do infuses his text. Petzval's study (and design) of optics is another example showing his fascination with the subject. I mention all this because it shows a logical evolution of his thinking about optics, vision, and optical resolution, which finally led to his manuscript.

Feigenbaum worked for a very long time on this project, and, from the oldest files he shared with me, I found a program, called anamorph.exe, dated July 12, 2006 (written in Pascal), in which he already programmed what he calls 3D-anamorphs (those are the "correct" anamorphs.) For a modern version, see Sec. V.

X. EVOLUTION AND DESIGN

The book had, in principle, an interesting section "Evolution and Design," which had no text at Feigenbaum's death. However, we corresponded (and discussed) extensively about this subject in the summer of 2009. Feigenbaum insisted that the design of the fish eye is excellent and that the design of the land eye is awful. He also insisted that this means that evolution does not always find the best solution, even if there is much time (since animals left water about 500×10^6 years ago). He also complained about the lack of good experimental measurements. The following paragraphs are taken from a letter of June 18, 2009. I did not edit the text because it shows a good example of Feigenbaum's thinking.

"This matters. The point is that I say I am showing that the simplest well-designed optics for the task of air to water imaging already bests evolution by almost an order of magnitude. I have not gone here further in actually optimizing, but evidently, it is hard to do worse than biology for land eyes with any theoretical knowledge. Now, this isn't quite true. I've explored replacing my one uniform spherical lens by a variable density one. I can numerically assert that the uniform version is the optimal among them. (I can't figure out how to prove this.) In some ways, my simple design appears to be the end of the line for this genre of optics with the strongest refractor in front. It is important then, without much more elaborate designing, to be satisfied that we have already bested evolution. This is what has bothered me, but that I now feel reasonably confident about. Too bad that someone hasn't built such a simple eye and experimentally checked it out.

Anyway, the story for the fish eye is totally different. Here the design of evolution is almost perfect: It has optimized the design of optics within its diffraction limits. A careful fit to the crummy plot you've seen, determines two radii at which they determined variations that they claim to be optically significant. However the better to fit to my analytic three parameter family plus Gaussian bumps, shows the bumps are also where the authors claim, and precisely where the 5th order caustic of the analytic family has reached its final tangency to be within the self-determined aperture (in the book, he analyzes the experiments of Kröger et al., 2001; they show that the density in the fish eye is not constant; Feigenbaum then deduces what he alludes to in his letter). Indeed the bumps are with the correct sign and size to now induce a new cusp, turning it into a 7th order caustic. The spill-over is within the faster than exponential short diffractive tail of the caustic diffraction, and so improves the lens from f/1.6 to at least f/1.5. This is impressive to discover merely by fitting. Things are as right as could have been ordered.

The reason is that the fish eye is better than a usual optimization problem. By studying 5th order isotropic optics, it turns out that any mutation that has the lens grow first with one index, and then uniformly with another is win-win: It either simultaneously improves both the brightness and resolution, or simultaneously worsens both. In the second case the fish is dead in the water against predators and finding food, while the former is highly favored. Each further striation in radial density works the same way. This is why evolution made extravagantly good eyes for fish. It matters everything what the environment is, and what its ambient physics can provide gratis. Where physics is less forth-coming in its abundance, as in the land eye case, evolution falls flat on its face, and simply constructs engineering kluges.

This is a précis of what I intend to say about what we have learned about evolution from the comparative study of water and land eyes. This is why I need my land eye analysis to be impeccable."

XI. USING A HOLOGRAM INSTEAD OF THE CYLINDRICAL TUBE

I devised and have made a hologram—made from a 3D anamorph—which shows the reflections from a tube in a quality similar to viewing an actual cylinder. The point is that holograms reproduce the interferences of light waves, and therefore, they are as good as seeing a scene. Thus, they can capture what a normal camera cannot distinguish (as in Fig. 7) other than by changing focus.

The idea of making a hologram came already up in early 2015, in discussions with Karl Knop, a specialist in optics (among many other things). Feigenbaum was immediately interested in the idea, and Knop was looking for somebody to make such a hologram, but without success. I made a second attempt while editing the book.

Hologram making was very fashionable in the 1980s, but currently, it is difficult to find professional hologram makers. The hologram of the empire state building was made by Walter Spierings, who is one of the leaders in this trade. Making the hologram is extremely delicate, as the objects in question should not move by more that 1/20th of the wavelength of light, about 50 nm (0.5×10^{-7} m). Hair has a diameter of about 50 000 nm. This needs precise mounting of whatever is photographed, but even the fluctuations of the density of air during exposition time matter. Making a good hologram needs, among other parameters, a careful control of the technical details of the optical setup. In the case at hand, laser speckle was an issue since the laser beam is reflected from the anamorph to the mirror. The hologram is made onto photoresist (photolithography), whose surface consists of optical ridges at a submicrometer scale that will manipulate incoming light waves into reconstructing the 3D scene. From this master hologram, multiple copies can be made by embossing plastic with a metal shim made by galvanic means from the photoresist.

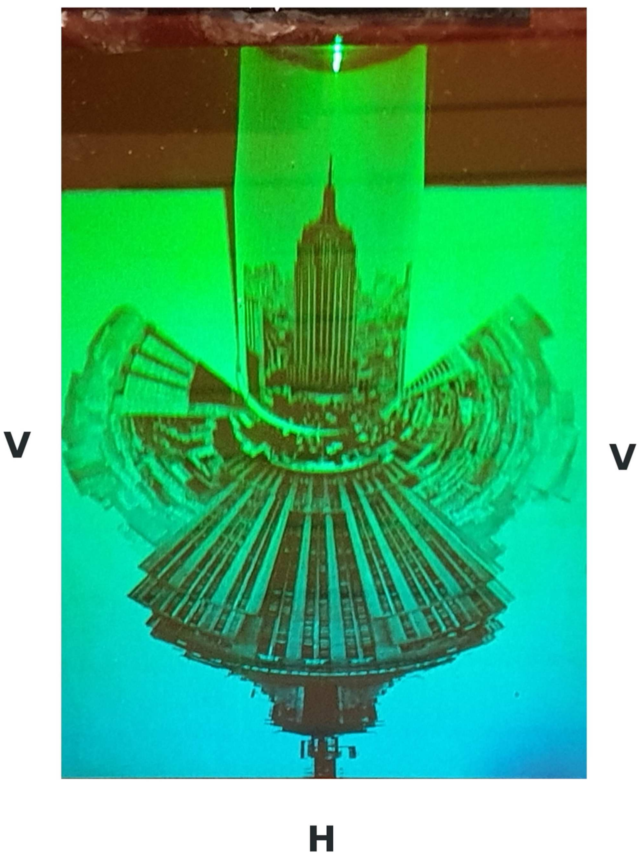

FIG. 20. Approximate rendering of the scene in the hologram. For best results, the observer should position the eye above the center of the hologram, looking down perpendicularly. If you stand on the **H** side, you should see the empire state building in the tube, but standing on either of the **V** sides, you should see it in the same plane as the circular figure, as illustrated in Fig. 5. Note that the position of the image in the tube does not change.

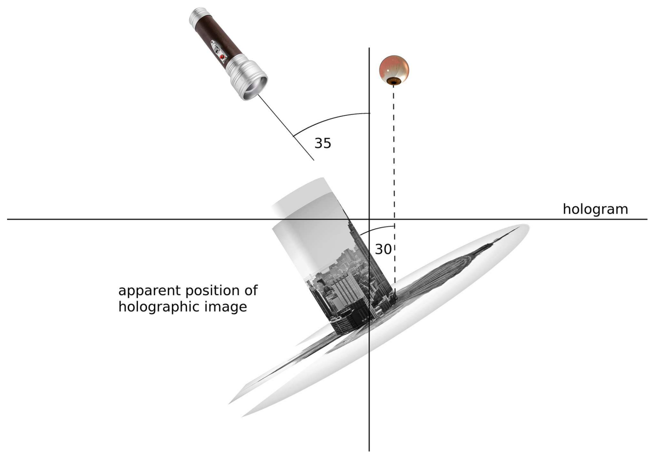

FIG. 21. A sketch of how the hologram will appear if the sheet is laid flat on a table. The illumination with a phone will work well. Results will be better if there is not too much stray light from other sources. Holding the light source at a larger distance is preferable. The angle should be 35° from the vertical (away from the top edge of the cardboard). Keep the light source fixed when viewing the hologram from the side. It is important that the light source is aligned with the axis of the hologram and that the eye is really vertically above the center of the hologram. When you have the correct illumination, the hologram will appear uniformly green.

The hologram has some advantages over viewing the scene with an actual tube, apart from not needing to make a tube. In particular, instead of turning the head, one can just turn the hologram by 90°. This avoids the neuro-ophthalmological signals mentioned in Sec. V. The best results are obtained if the hologram is illuminated from a fixed source, and then the viewer walks around the table, as shown in Fig. 20. The hologram is made in such a way that the best view is obtained when the eye is perpendicularly above the center of the hologram as in Fig. 21.

XII. MISSING FIGURES

There were some pictures that had not made it into the manuscript. I mention here again the one, Fig. 7, which "proves" the **H** vs **V** difference and which seems to me the most important: It is a high quality photograph of a few of the dots in Fig. 7.

Another image that should have appeared inside Feigenbaum's text is the reflection of the rays in Fig. 22. Such drawings in two dimensions have a long history, but somehow, Feigenbaum's study in three dimensions seems to be absent in the literature.

XIII. ABOUT REFERENCES

Feigenbaum's manuscript did not contain references. Fortunately, I could find from my exchanges with him several files with

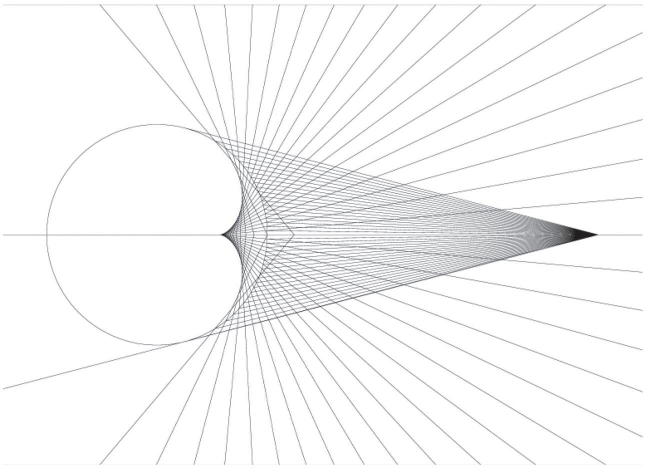

FIG. 22. The reflection of the rays emitted from a point as reflected from the circular cylinder. The caustic is inside the circle and is the place where the rays are denser.

relevant literature. I decided to put them all as references at the end of this article whenever I could make out the source. The reader will find that some of these references clearly address issues described in the planned book. This collection also shows the eclectic interests of Feigenbaum. He certainly was inspired by these references, but, knowing his way of re-deriving everything by himself, we can assume that he did not copy the results of the books and papers.

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First of all, I am grateful to Mitchell Feigenbaum to have shared, over the many years, his ideas, worries, and questions about the manuscript with me. I hope this outline gives justice to his work. I also thank my many colleagues who commented on my foreword, in particular, Michael Berry, David Campbell, Neil Dobbs, Jérémie Francfort, Gemunu Gunaratne, Karsten Kruse, Alberto Morpurgo, Jacques Rougemont, and David Ruelle. Some of their wishes could not be followed, because I wanted to keep this summary of Feigenbaum's treatise as close as possible to the spirit of the text. Noé Cuneo was instrumental in translating Feigenbaum's "Pascal" program to HTML/JavaScript so that readers can make their own anamorphs. I am grateful to Walter Spierings for his dedicated effort to make the best possible hologram. I thank Léo Lévy for help with making the index. The cost of producing the holograms was kindly supported by the Fondation Schmidheiny, Geneva, and by the overhead of an ERC advanced grant, 290843 Bridges.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Jean-Pierre Eckmann: Writing – original draft (lead).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

RELATED RESOURCES

Abbe, E., "Über die Bbedingungen des Aplanatismus der Linsensysteme," Jena. Z. Naturwiss. **13 (Neue Folge 6ter Band)**, 129–142 (1879).

Adams, C. K., Perez, J. M., and Hawthorne, M. N., "Rod and cone densities in the rhesus," Invest. Ophthalmol. Vis. Sci. **13**, 885–888 (1974).

Bonds, A. B., "Optical quality of the living cat eye," J. Physiol. **243**, 777–795 (1974).

Borghero, F. and Bozis, G., "A two-dimensional inverse problem of geometrical optics," J. Phys. A: Math. Gen. **38**, 175–184 (2004).

Born, M. and Wolf, E., *Principles of Optics: Electromagnetic Theory of Propagation, Interference and Diffraction of Light*, 2nd ed. (Pergamon, Oxford, 1964).

Browne, R. G., Kelly, T. C., Culloty, S., Davenport, J., and Deeny, T., "The schematic eye of the rook, *Corvus frugilegus*," Folia Zool. **56**, 399–404 (2007).

Campbell, F. W. and Gubisch, R. W., "Optical quality of the human eye," J. Physiol. **186**, 558–578 (1966).

Carkeet, A., Leo, S. W., Khoo, B. K., and Au Eong, K. G., "Modulation transfer functions in children: Pupil size dependence and meridional anisotropy," Invest. Ophthalmol. Vis. Sci. **44**, 3248–3256 (2003).

Collewijn, H., Steinman, R., Erkelens, C., and Regan, D., "Binocular fusion, stereopsis and stereoacuity with a moving head," in *Vision and Visual Dysfunction: Binocular Vision and Psychophysics* (Macmillan Press, London, 1991), pp. 121–136.

Fernald, R., "Evolution of eyes," in *Encyclopedia of Neuroscience* (Springer, Berlin, 2009).

Gårding, J., Porrill, J., Mayhew, J. E., and Frisby, J. P., "Stereopsis, vertical disparity and relief transformations," Vision Res. **35**, 703–722 (1995).

Glaeser, G. and Schröcker, H.-P., "Reflections on refractions," J. Geom. Graph. **4**, 1–18 (2000).

Gordon, J. M., "Spherical gradient-index lenses as perfect imaging and maximum power transfer devices," Appl. Opt. **39**, 3825–3832 (2000).

Guirao, A., Gonzalez, C., Redondo, M., Geraghty, E., Norrby, S., and Artal, P., "Average optical performance of the human eye as a function of age in a normal population," Invest. Ophthalmol. Vis. Sci. **40**, 203–213 (1999).

Hamilton, W. R., "Theory of systems of rays," Trans. R. Ir. Acad. **15**, 69–174 (1828).

Hamilton, W. R., "Supplement to an essay on the theory of systems of rays," Trans. R. Ir. Acad. **16**, 4–62 (1830).

Hamilton, W. R., "Second supplement to an essay on the theory of systems of rays," Trans. R. Ir. Acad. **16**, 93–125 (1831).

Hamilton, W. R., "On a general method of expressing the paths of light and of the planets by the coefficients of a characteristic function," Dubl. Univ. Rev. Quart. Mag. **1**, 795–826 (1833).

Hamilton, W. R., "Third supplement to an essay on the theory of systems of rays," Trans. R. Ir. Acad. **17**, 1–144 (1837).

Helmholtz, H. V., *Treatise on Physiological Optics* (English translation, 1924) (Dover, New York, 1962/Menasha, WI, 1866).

Jagger, W. S., "The optics of the spherical fish lens," Vision Res. **32**, 1271–1284 (1992).

Koretz, J. F., Cook, C. A., and Kaufman, P. L., "Accommodation and presbyopia in the human eye. Changes in the anterior segment and crystalline lens with focus," Invest. Ophthalmol. Vis. Sci. **38**, 569–578 (1997).

Land, M. F., "The optical structures of animal eyes," Curr. Biol. **15**, R319–R323 (2005).

Land, M. F. and Burton, F. A., "The refractive index gradient in the crystalline cones of the eyes of a Euphausiid crustacean," J. Exp. Biol. **82**, 395–399 (1979).

Malmstrom, T., "Pupil shapes and lens optics in the eyes of terrestrial vertebrates," J. Exp. Biol. **209**, 18–25 (2006).

Matthews, N., Meng, X., Xu, P., and Qian, N., "A physiological theory of depth perception from vertical disparity," Vision Res. **43**, 85–99 (2003).

Morlet, N., Minassian, D., and Dart, J., "Astigmatism and the analysis of its surgical correction," Br. J. Ophthalmol. **86**, 1458–1459 (2002).
Nasse, H., "How to read MTFcurves," Zeiss, Internal Report, 2008.
Nilsson, D.-E. and Pelger, S., "A pessimistic estimate of the time required for an eye to evolve," Proc. R. Soc. London, Ser. B **256**, 53–58 (1994).
O'Kane, L. M. and Hibbard, P. B., "Vertical disparity affects shape and size judgments across surfaces separated in depth," Perception **36**, 696–702 (2007).
Qian, N., "Binocular disparity and the perception of depth," Neuron **18**, 359–368 (1997).
Read, J. C. A. and Cumming, B. G., "Does depth perception require vertical-disparity detectors?," J. Vis. **6**, 1323–1355 (2006).
Schwarzschild, K., "Untersuchungen zur geometrischen Optik I," Astron. Mitt. Univ. Goettingen **9**, 1–31 (1905).
Schwarzschild, K., "Untersuchungen zur geometrischen Optik II," Astron. Mitt. Univ. Goettingen **10**, 1–28 (1905).
Schwarzschild, K., "Untersuchungen zur geometrischen Optik III," Astron. Mitt. Univ. Goettingen **11**, 1–54 (1905).
Sereno, M. and De Luca, F., "Primary caustics and critical points behind a Kerr black hole," Phys. Rev. D **78**, 023008 (2008).
Shealy, D. L., "Analytical illuminance and caustic surface calculations in geometrical optics," Appl. Opt. **15**, 2588–2596 (1976).
Smith, G. H., *Practical Computer-Aided Lens Design* (Willmann-Bell, 1998).
Southall, J. P. C., *Mirrors, Prisms and Lenses: A Text-Book of Geometrical Optics* (BiblioLife, 2010).
Southall, J. P. C., *The Principles and Methods of Geometrical Optics* (BiblioLife, 2011).
Stavroudis, O. N., *The Optics of Rays, Wavefronts, and Caustics* (New York, Academic Press, 1972).
Synge, J. L., *Geometrical Optics: An Introduction to Hamilton's Method*, Cambridge Tracts in Mathematics and Mathematical Physics (Cambridge University Press, 1937).
Thorne, K. S. and Blandford, R. D., *Modern Classical Physics Optics, Fluids, Plasmas, Elasticity, Relativity, and Statistical Physics* (Princeton University Press, 2017).
Veltman, K. H., "Perspective, anamorphosis and vision," Marburger Jahrb. **21**, 93–117 (1986).
Westheimer, G., "Vertical disparity detection: Is there an induced size effect?," Invest. Ophthalmol. Vis. Sci. **17**, 545–551 (1978).
Westheimer, G., "The resolving power of the eye," Vision Res. **45**, 945–947 (2005).

REFERENCES

Arnol'd, V. I., "Normal forms of functions in the neighborhood of degenerate critical points," Usp. Mat. Nauk **29**, 11–49 (1974), Collection of articles dedicated to the memory of Ivan Georgievič Petrovskiĭ (1901–1973), I.
Arnol'd, V. I., "Critical points of smooth functions and their normal forms," Russ. Math. Surv. **30**, 1–75 (1975).
Bartlett, A. A., Lucero, R., and Johnson, G. O., "Note on a common virtual image," Am. J. Phys. **52**, 640–643 (1984).
Berry, M. and Upstill, C., *IV Catastrophe Optics: Morphologies of Caustics and Their Diffraction Patterns* (Elsevier, 1980), pp. 257–346.
Eckmann, J.-P., "Broken pencils and moving rulers: After an unpublished book by Mitchell Feigenbaum," Am. J. Phys. **89**, 955–962 (2021).
Eckmann, J.-P., Gunaratne, G. H., Shulman, J., and Wood, L. T., "Imaging in reflecting spheres," Chaos **32**, 093136 (2022).
Feigenbaum, M. J., "Quantitative universality for a class of nonlinear transformations," J. Stat. Phys. **19**, 25–52 (1978).
Feigenbaum, M. J., "The universal metric properties of nonlinear transformations," J. Stat. Phys. **21**, 669–706 (1979).
Hammond, *Hammond Atlas of the World (1995)* (Hammond World Atlas Corporation, 1997).
Horgan, J., *The End of Science: Facing the Limits of Knowledge in the Twilight of the Scientific Age* (Broadway Books, 1997).
Horváth, G., Buchta, K., and Varjú, D., "Looking into the water with oblique head tilting: Revision of the aerial binocular imaging of underwater objects," J. Opt. Soc. Am. A: Opt. Image Sci. Vis. **20**, 1120–1131 (2003).
Kinsler, L. E., "Imaging of underwater objects," Am. J. Phys. **13**, 255–257 (1945).
Kröger, R. H., Campbell, M. C., and Fernald, R. D., "The development of the crystalline lens is sensitive to visual input in the African cichlid fish, *Haplochromis burtoni*," Vision Res. **41**, 549–559 (2001).
Minnaert, M., *Light and Colour in the Outdoors* (Springer-Verlag GmbH & Co. KG, Berlin, 1993).
Müller-Lyer, F. C., "Optische Urtheilstäuschungen," Arch. Physiol. Suppl. Bd. **1889**, 263–270.
Niceron, J.-F., *La Perspective Curieuse OV Magie Artificiele Des Effets Merveillevx* (Billaine, Paris, 1638).
Nussenzweig, H. M., "The theory of the rainbow," Sci. Am. **1977**, 116–127
Shepard, R. N., *Mind Sights: Original Visual Illusions, Ambiguities, and Other Anomalies, With a Commentary on the Play of Mind in Perception and Art* (W.H. Freeman & Co, 1990).
Thom, R., *Stabilité Structurelle et Morphogénèse* (W. A. Benjamin, Inc., Reading, MA, 1972), p. 362, pp. (loose errata), *essai d'une théorie générale des modèles*, Mathematical Physics Monograph Series.
Thom, R., *Structural Stability and Morphogenesis* (W. A. Benjamin, Inc., Reading, MA, 1976), pp. xxv+348, an outline of a general theory of models, Translated from the French by D. H. Fowler, With a foreword by C. H. Waddington, second printing.
Watson, W., *A Text-Book of Physics*, 4th ed. (Longmans, Green, and Co., New York, 1907).