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Hydrodynamic Lyapunov Modes in Translation-Invariant Systems

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We study the implications of translation invariance on the tangent dynamics of extended dynamical systems, within a random matrix approximation. In a model system, we show the existence of hydrodynamic modes in the slowly growing part of the Lyapunov spectrum, which are analogous to the hydrodynamic modes discovered numerically by Dellago, Posch, and Hoover. The hydrodynamic Lyapunov vectors lose the typical random structure and exhibit instead the structure of weakly perturbed coherent long-wavelength waves. We show further that the amplitude of the perturbations vanishes in the thermodynamic limit, and that the associated Lyapunov exponents are universal.

KEY WORDS: Nonlinear dynamics; Hamiltonian dynamics; extended systems; random matrices; Lyapunov spectrum; hydrodynamic modes.

1. INTRODUCTION

The Lyapunov spectrum is recognized as an important diagnostic of chaotic dynamical systems. As such, it has been studied intensely in the context of extended systems.[1, 2] It has been observed that in the thermodynamic limit the spectrum seems to approach a continuous density, and some theoretical studies have focused on this phenomenon.[3] However, despite the large amount of available data, there is an unsatisfactory degree of understanding of the relation between the Lyapunov spectrum of extended systems, and their global or collective properties.

In connection with these problems, the recent study of ref. 4 presents an interesting development (see also ref. 5 for further results and references). In the context of molecular dynamics simulations, they find hydrodynamic,

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i.e., slow, long-wavelength behavior in the *tangent space* dynamics. Namely, they observe that the Lyapunov vectors associated with the Lyapunov exponents of small absolute value have *ordered*, *wave-like structure*, and that the exponents themselves follow an ordered pattern. Hydrodynamic behavior in *phase space* is of course present in every extended system with a continuous symmetry. In the models considered in ref. 4 the symmetries in question are translation and Galilei invariance, precisely those which enable the hydrodynamic description of a fluid in terms of the Navier–Stokes equations.[6] However, it is for the first time that a similar phenomenon is observed in tangent space.

In this paper, we study theoretically the slow Lyapunov modes (vectors and exponents) of extended systems with translation invariance. We focus attention on a simplified model which shares the essential features with the more elaborate model of ref. 4. This simplified model is constructed only in tangent space without an accompanying real space dynamics, and is based on a random matrix approximation. As has been often found before, in systems with strong chaos, qualitative features of the Lyapunov spectrum are well reproduced by approximating the tangent matrices by independent random matrices with appropriately chosen distributions.[2, 3] We prove several statements on the slow Lyapunov modes of this model in the thermodynamic limit, which show that in this limit the Lyapunov vectors and exponents are indeed well described as being hydrodynamic.

The basic reason for the existence of these hydrodynamic modes is evidently the translation invariance. Its presence dictates that the dynamics are indifferent to a uniform shift of all the particles (or their momenta), so that the associated Lyapunov vectors are decoupled from the rest of the dynamics, and the associated Lyapunov exponents vanish. We show that slowly growing large wavelength disturbances are nearly decoupled for the same reason, and use this property to show how the clean wave structure is obtained as a result of the orthogonalization procedure which involves all the faster growing Lyapunov vectors. It should be emphasized that the wave-like structure characterizes the Lyapunov vector at any given instant and is not an average property. Our arguments depend essentially on the local and hyperbolic character of the interactions, in addition to translation invariance. The absence of translation invariance has been recognized to ruin the hydrodynamic modes.[7] In translation invariant anharmonic chains, the absence of short time hyperbolicity seems to ruin the hydrodynamic modes.[8] We present theoretical arguments for the existence of hydrodynamic modes in the simplified model, which are complemented by numerical verifications. The outline of the paper is as follows. In Section 2 the hydrodynamic phenomenology is described in some more detail, the definition of the random matrix model is presented and motivated, and the results are stated. They are derived in Sections 3–5. Section 6 is devoted to numerical studies of the hydrodynamic properties of Lyapunov vectors.

2. HYDRODYNAMIC BEHAVIOR IN TANGENT SPACE

The systems studied in ref. 4 consist (among others) of a large set of disks moving in a two dimensional box Ω with periodic boundary conditions (torus geometry), with elastic scattering. In this case, the phase space is $4N$-dimensional where N is the number of disks. The Lyapunov vectors have $4N$ components which we label as

$$(\delta x_n, \delta y_n, \delta p_{x,n}, \delta p_{y,n}) \qquad 1 \leqslant n \leqslant N$$

with evident notation. To give them a geometrical meaning the components of the Lyapunov vectors are drawn in ref. 4 at the instantaneous position of the particles which carry a given specific index. That is, one constructs a vector field $\vec{v}(t, \vec{x})$ with values in $\mathbf{R}^4$, which are defined only at the instantaneous positions $\vec{x}_n(t)$ of the particles, for example

$$v_x(t, \vec{x}_n(t)) = \delta x_n(t)$$

and similarly for the other components.

The vector fields of the slow Lyapunov vectors, defined as the Lyapunov vectors with small corresponding Lyapunov exponents, are very well approximated by the long wavelength eigenmodes of a "reverse wave equation" in the domain Ω:

$$\partial_t^2 \vec{v}(t, \vec{x}) = -\frac{1}{N^2} \nabla^2 \vec{v}(t, \vec{x}) \tag{1}$$

(note the unusual sign in front of ∇^2). That is, the vectors look like long wavelength waves with, say, n nodes in the x direction and m nodes in the y direction, and the corresponding Lyapunov exponent is proportional to

$$\pm \frac{1}{N} \sqrt{\left(\frac{m}{L_x}\right)^2 + \left(\frac{n}{L_y}\right)^2}$$

Note that the translation modes—constant Lyapunov vectors with zero exponents, which are trivially present in any system with translation invariance, correspond to the special case $m = n = 0$. This phenomenology was observed in simulations with widely varying parameters, such as aspect ratio, density, and the shape of the particles.[4, 5]

Before we go on, it is necessary to make precise what we mean by Lyapunov vectors. As is well-known, the standard numerical method for calculating the tangent space dynamics of the the Lyapunov vectors v_n is defined as follows: One starts with an orthogonal matrix Q, multiplies it from the left with the tangent matrix (in the present case S) and decomposes the result as $SQ = Q'R$, where Q' is orthogonal and R is upper triangular. This procedure is iterated to yield a sequence of Q_t. The columns of the orthogonal matrices Q_t are what we will call the Lyapunov vectors. One can attach an intuitive meaning to the Lyapunov vectors by noticing that the subspace spanned by a set of any initial p tangent vectors will be mapped after sufficiently long time to a subspace exponentially close to that spanned by the first p Lyapunov vectors, with probability one. The reader should note that the Lyapunov vectors as defined here are not the ones whose existence is proved in the multiplicative ergodic theorem.

The tangent flow of the molecular dynamics system can be written as

$$\partial_t \begin{pmatrix} \delta\vec{x} \\ \delta\vec{p} \end{pmatrix} = G(t) \begin{pmatrix} \delta\vec{x} \\ \delta\vec{p} \end{pmatrix} \tag{2}$$

where the components $\delta\vec{x}, \delta\vec{p}$ are column vectors with N entries each of which is a vector in $\mathbf{R}^2$. The quantity $G(t)$ is the action on the tangent space induced by the flow $\Phi(t)$ of the dynamical system: If ψ_0 is the instantaneous state of the system then $G(t)$ is given by $G(t)\, f = D\Phi(t)_{\psi_0} f$, where $\Phi(t)(\psi_0 + \varepsilon f) = \Phi(t)\, \psi_0 + \varepsilon D\Phi(t)_{\psi_0} f + O(\varepsilon^2)$. The evolution operator of Eq. (2) may be written formally as

$$U = \top \exp \int G(t)\, dt \tag{3}$$

The purpose of this section is, starting from the picture just described, to arrive at a simplified model of the tangent dynamics, which contains both the essential ingredients of the original molecular dynamics model of ref. 4, and displays the hydrodynamic properties which are the subject of our study. This procedure is then by nature heuristic, and it yields a precisely defined model, whose properties are studied in a precise manner in the following sections.

As a first step toward the construction of our model, we replace the hard-core interaction with a short range "soft" potential. In that case G will have a block structure of the from

$$G(t) = \begin{pmatrix} 0 & 1 \\ A(t) & 0 \end{pmatrix} \tag{4}$$

where the symmetric $N \times N$ matrix A depends on the instantaneous positions of the particles and couples only nearest neighbors. Since the interactions of the original hard-core model are purely repulsive, the corresponding flow is hyperbolic in the sense that each collision increases the distance between any two given close trajectories. In order that the flow generated by the matrices of the form (4) share this property, the matrices $A(t)$ are chosen to be positive.

At this stage one may note that if the matrix $A(t)$ in Eq. (4) were replaced by the negative of the discrete Laplacian, the Lyapunov spectrum of G would be precisely that described in Eq. (1). However, the matrices $A(t)$ are in fact generated by chaotic dynamics, and therefore fluctuate rapidly. Furthermore, the particles in the gas rearrange in time, so that the positions of the non-zero elements in the matrix also evolve. In our study we concentrate on the first feature. That is, we show how tangent dynamics of the type (4) result in slow hydrodynamic modes in spite of the fluctuations in $A(t)$; the effects of particle rearrangement may in principle dealt with similarly, but need to be studied further.

The above discussion allows us to conclude that the matrices $A(t)$ should have non-zero elements only at those positions which are nonzero in the discrete Laplacian. Furthermore, momentum conservation implies that the sum of elements in any row and column of A must vanish. This specifies completely the matrix structure of A, and it remains to model the time dependence of the off-diagonal non-zero elements of A. For this we invoke the hypothesis of strong chaos:[1, 3] The elements of A may be treated as independent random processes, with a correlation time τ which is short with respect to other time scales of the system. It is commonly found that this approximation yields results which are in good qualitative agreement with those of the actual tangent flow.

With this in mind we model the evolution operator U by a product of independent random matrices S_n

$$U = \prod_n S_n \tag{5}$$

where

$$S_n \sim \top \exp \int_{(n-1)\tau}^{n\tau} G(t)\, dt \tag{6}$$

During the time interval of length τ, A, and therefore G, may be considered constant, so the simplest model for S would be $S = \begin{pmatrix} 1 & \tau \\ \tau A & 1 \end{pmatrix}$. However it is more convenient to correct this form by a second order term in τ in order to preserve the symplectic property which holds for U. We thus arrive at our model:[2, 3]

$$S=\begin{pmatrix} 1 & \tau \\ \tau A & 1+\tau^2 A \end{pmatrix} \tag{7}$$

The matrices A are independent, and their off-diagonal elements are independent and identically distributed. The actual probability distribution of the off-diagonal elements can be chosen arbitrarily, subject to the constraint of uniform hyperbolicity, namely that the support of the distribution is strictly negative, and bounded away from zero.

The model as defined above makes sense in any space dimension, but for the sake of simplicity we study it in one dimension. There it bears similarity to the tangent dynamics of an anharmonic chain. However, in the latter case the matrices S would be elliptic rather than hyperbolic. As we show below this is an essential ingredient in the mechanism for hydrodynamic modes, which are not present in the Lyapunov spectra of anharmonic chains.[8] Unfortunately, we have not been able to find a model dynamics in real space whose tangent space dynamics would resemble that generated by the matrices of type (7). On the other hand our results do not use explicitly the dimensionality of the system and seem to be generalizable to higher dimensions.

Since the individual matrices are symplectic, the Lyapunov exponents of (5) come in pairs of equal absolute value and opposite signs. Translation and Galilei invariance imply the existence of two vanishing exponents. We concentrate our attention on the Lyapunov exponents λ_{N-1} and λ_{N-2} of smallest positive value, and the corresponding Lyapunov vectors v_{N-1} and v_{N-2}.

We can now state the main result of this paper.

Existence of Hydrodynamic Lyapunov Modes. As the size N of the matrices tends to infinity the exponents λ_{N-1} and λ_{N-2} as well as the vectors v_{N-1} and v_{N-2} are asymptotic to the exponents and vectors that would be obtained if the matrices A where replaced everywhere by the negative of the discrete Laplacian (properly rescaled). The statement holds for the Lyapunov vectors, which are random objects, in probability.

The statement is spelled out only for two Lyapunov modes, which have nearly equal exponents, and where the deviation from hydrodynamic behavior is the smallest. However, as will become apparent from the arguments below, the result can be extended to a number of Lyapunov modes near the middle of the spectrum which is proportional to $\sqrt{N}$. Our numerical studies also indicate that this is in fact true.

As already explained, the basic reason for the existence of the hydrodynamic modes is translation invariance. However, this general observation is not sufficient, and the actual proof is not trivial. It depends on the random nature of successive matrices, i.e., on strong chaos. Our strategy will be to show the existence of hydrodynamic modes first in the spectrum of a *single* matrix of type A, i.e., a negative random Laplacian (Section 3); then this will be used to show that such modes exist in the Lyapunov spectrum of non-symplectic products of type $\Pi(1+A_n)$ in Section 4, which in turn will be used to show the same property for symplectic products in Section 5.

3. SPECTRAL PROPERTIES OF A SINGLE MATRIX

The matrix A defined in Section 2 takes in one dimension the explicit form

$$A = \begin{pmatrix} a_1+a_2 & -a_2 & 0 & 0 & \cdots & 0 & -a_1 \\ -a_2 & a_2+a_3 & -a_3 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & & \vdots \\ -a_1 & 0 & 0 & 0 & \cdots & -a_N & a_N+a_1 \end{pmatrix} \tag{8}$$

where the a_n are positive identically distributed independent random numbers. The distribution of the a_n is arbitrary, subject to the condition $0 < a_{\min} < a < a_{\max}$ with $a_{\min} < a_{\max}$, and normalized such that $\langle a^{-1} \rangle = 1$, for later convenience. The $\langle \cdot \rangle$ always denote expectation with respect to the probability distribution of the a. We are not going to assume that the width of the distribution is small.

The matrix A may be written as a product

$$A = -\underline{\partial} \mathscr{A} \, \bar{\partial} \tag{9}$$

where $\underline{\partial}$ and $\bar{\partial}$ are the discrete derivatives whose action on a vector $v \in \mathbf{R}_N$ is

$$(\underline{\partial} v)_n = v_n - v_{n-1}, \qquad (\bar{\partial} v)_n = v_{n+1} - v_n \tag{10}$$

and $\mathscr{A}$ is a diagonal matrix with diagonal elements a_n. (The indices are extended periodically so that $a_{N+1} \equiv a_1$). If all the a_n were equal to one, $-A$ would reduce to the discrete Laplacian matrix $\partial^2 \equiv \underline{\partial} \bar{\partial}$.

We define the Fourier transform matrix F with elements

$$F_{kn} = \frac{1}{\sqrt{N}} e^{i(2\pi/N)\, kn} \tag{11}$$

which is a unitary transformation taking $\mathbf{R}^N$ to $\tilde{\mathbf{C}}_N$, the subset of $\mathbf{C}_N$ (with standard basis vectors e_n) consisting of vectors $\tilde{v}$ for which $v_{-k} = v_k^*$, which is an N dimensional vector space over $\mathbf{R}$. The components of A in the new basis are

$$\tilde{A}_{kl} \equiv (FAF^\dagger)_{kl} = \mu_k^* (\langle a \rangle \, \delta_{k,l} + \tilde{a}_{k-l}) \, \mu_l \tag{12}$$

where $\mu_k = 1 - \exp(-(2\pi i/N)k)$, and $\tilde{a}$ is related to the Fourier transform of a considered as a vector in $\mathbf{R}^N$ by

$$\tilde{a} = N^{-1/2} F(\vec{a} - \langle \vec{a} \rangle) \tag{13}$$

The random variables $\tilde{a}_k$ are centered, and as sums of independent random numbers their "single-point" distribution is nearly Gaussian with variance

$$\langle |\tilde{a}|^2 \rangle = \frac{\langle a^2 \rangle - \langle a \rangle^2}{N} \tag{14}$$

so that they are typically small, of order $\mathcal{O}(N^{-1/2})$. The joint distribution is not Gaussian.

Note that $\mu_0 = 0$, so that row 0 and column 0 of $\tilde{A}$ are zero, with the translation vector e_0 being trivially a zero eigenvector. We define the slow subspace

$$V_{\mathrm{s}} = \mathrm{Span}(\{e_0, e_1, e_{-1}\}) \cap \tilde{\mathbf{C}}^N \tag{15}$$

and its orthogonal complement, the fast subspace V_{f}. We will consider often below the block decomposition of $\tilde{A}$ and other matrices into the fast and slow subspaces, e.g.,

$$\tilde{A} = \begin{pmatrix} A_{\mathrm{ff}} & A_{\mathrm{fs}} \\ A_{\mathrm{sf}} & A_{\mathrm{ss}} \end{pmatrix} \tag{16}$$

Note that V_{f} contains slow as well as fast modes.

The block A_{ss} has small norm of order $\mathcal{O}(N^{-2})$, and the off-diagonal blocks have norm of order $\mathcal{O}(N^{-1})$. However, there are more specific properties of A which are needed to establish the existence of hydrodynamic eigenmodes. Consider the eigenvalue problem $Av = \lambda v$. Letting $v = \underline{\partial} u$, and using the representation (9) gives an equation for u

$$-\partial^2 u = \lambda \mathscr{A}^{-1} u \tag{17}$$

It is convenient to proceed by writing Eq. (17) in Fourier component form

$$(|\mu_k|^2 - \lambda)\, \tilde{u}_k = \lambda \sum_q \tilde{b}_{k-q} \tilde{u}_q \tag{18}$$

with the $\tilde{b}_k$ bearing a relation to a_n^{-1} analogous to that between $\tilde{a}_k$ and a_n, namely

$$\tilde{b}_k = N^{-1/2} F_{kn} \left(\frac{1}{a_n} - 1 \right) \tag{19}$$

Since $\langle 1/a^2 \rangle < 1/a_{\min}^2$ is $\mathcal{O}(1)$ we find that the $\tilde{b}_k$ are $\mathcal{O}(N^{-1/2})$ for the same reason that the $\tilde{a}_k$ are.

We claim that *given a fixed* m, *and for* $N \to \infty$ *the system* (18) *has two linearly independent solutions* $u^{(\pm m)}$, $\lambda_{\pm m}$ such that

$$\frac{1}{|u_m^{(\pm m)}|} \sum_{|k| \neq m} |u_k^{(\pm m)}| = \mathcal{O}(N^{-1/2}) \tag{20}$$

and

$$\left| \frac{\lambda_{\pm m}}{|\mu_m|^2} - 1 \right| = \mathcal{O}(N^{-1/2}) \tag{21}$$

We justify the claim by showing that Eqs. (20) and (21) are consistent with the eigenvalue Equation (18). For this we rewrite (18) as

$$\tilde{u}_k^{(m)} = \frac{\lambda_m}{|\mu_k|^2 - \lambda_m} \sum_q \tilde{b}_{k-q} \tilde{u}_q^{(m)} \tag{22}$$

We assume that (20–21) hold; this implies that the sum over q in (22) is dominated by the two terms with $q = \pm m$, that is,

$$\tilde{u}_k^{(m)} = \frac{|\mu_m|^2}{|\mu_k|^2 - |\mu_m|^2} (b_{k-m} \tilde{u}_m^{(m)} + b_{k+m} \tilde{u}_{-m}^{(m)}), \qquad \text{for} \quad |k| \neq m \tag{23}$$

On substituting this expression in the left-hand-side of (20) the sum over k is observed to be local, in the sense that it is dominated by terms with $|k| \sim m$, where $|\mu_k|^2 \sim (2\pi k/N)^2$. Since $\tilde{b}_k$ is $\mathcal{O}(N^{-1/2})$, assumption (20) is verified. On the other hand, using (20) in (22) for $k = m$ gives

$$\tilde{u}_m^{(m)} = \frac{\lambda_m}{|\mu_m|^2 - \lambda_m} (b_0 \tilde{u}_m^{(m)} + b_{2m} \tilde{u}_{-m}^{(m)}) \tag{24}$$

Since b_0 and $\tilde{b}_{2m}$ are $O(N^{-1/2})$ it follows that $\lambda_m/(|\mu_m|^2-\lambda_m)=O(N^{1/2})$ verifying (21), which shows that (20)–(21) are indeed consistent with (18).

In terms of the original variables v, Eq. (23) reads

$$\tilde{v}_k^{(m)}=\frac{\mu_k^*\mu_m}{|\mu_k|^2-|\mu_m|^2}(b_{k-m}\tilde{v}_m^{(m)}+b_{k+m}\tilde{v}_{-m}^{(m)}),\qquad \text{for}\quad |k|\neq m \tag{25}$$

so that the norm of $v_\perp^{(m)}$, the component of $v^{(m)}$ orthogonal to $e_{\pm m}$ is small,

$$\|v_\perp^{(m)}\|^2\sim\frac{1}{N}\sum_{|k|\neq m}\frac{k^2m^2}{(k^2-m^2)^2}=O\left(\frac{1}{N}\right)$$

In words, these eigenvectors are almost pure Fourier modes, i.e., eigenvectors of the discrete Laplacian.

For further developments we also need to show these modes are the only ones with eigenvalues of order $\mathcal{O}(N^{-2})$.

This is established easily by noting that the sharp cutoff on the probability distribution of the a implies that every realization A satisfies the bounds

$$-a_{\min}\,\partial^2<A<-a_{\max}\,\partial^2 \tag{26}$$

and then by the minimax principle it follows that the pth eigenvalue of A is larger than $a_{\min}$ times the pth eigenvalue of $-\partial^2$ (sorting the eigenvalues of both matrices in increasing order).

The results of this section can be summarized using the decomposition of $\tilde{\mathbf{C}}^N$ into slow and fast subspaces defined above. We have shown there exist small numbers ε and λ, and a number $0<\alpha<1$, such that the matrix $\tilde{A}$ can be block diagonalized,

$$\tilde{A}=RDR^T,\qquad RR^T=1,\qquad D=\begin{pmatrix}D_{\mathrm{f}} & 0\\ 0 & D_{\mathrm{s}}\end{pmatrix} \tag{27}$$

with the off-diagonal blocks bounded by $\|R_{\mathrm{sf}}\|,\ \|R_{\mathrm{fs}}\|<\varepsilon$, and the diagonal blocks obeying

$$D_{\mathrm{f}}>\lambda>\alpha\lambda>D_{\mathrm{s}}\geqslant 0 \tag{28}$$

and furthermore

$$\|A_{\mathrm{ss}}\|<\alpha\lambda \tag{29}$$

The orders of magnitude for λ and ε are $\varepsilon = \mathcal{O}(N^{-1/2})$ and $\lambda = \mathcal{O}(N^{-2})$, whereas $\alpha \sim 1/4$. However, to keep the discussion reasonably general we are not going to use these specific values in our arguments below. Rather, we will make statements regarding arbitrary matrices which satisfy the conditions (27)–(29).

Although this will not be used below, it is relevant to note that if we let m vary, a slight generalization of the arguments given above shows that (20) and (21) remain valid provided that $m/N^{1/2} \ll 1$ and that the right-hand-side is replaced by $\mathcal{O}(m/N^{1/2})$. This means that we can expect a number of hydrodynamic eigenmodes which is proportional to $\sqrt{N}$. Another way to see this is related to the study of the vibrations of one-dimensional disordered lattices which are modeled precisely by the eigenmodes of matrices of type (8). There it is known that the localization length ξ is proportional to λ_m^{-1}. Since $\lambda_m \sim (2\pi/N)^2$ the localization length will reach N when $m = \mathcal{O}(N^{1/2})$. Thus, again, we only expect wave-like modes when $m < \mathcal{O}(N^{1/2})$.

4. PRODUCTS OF MATRICES OF THE FORM $1 + \tau A$

In this section we use the properties derived in Section 3 to derive the existence of hydrodynamic modes in the Lyapunov spectrum of the product $\prod_n (1 + \tau A_n)$, where the matrices A_n are independent realizations of the random matrix defined in Eq. (8). Beside providing a step towards proving the existence of hydrodynamic modes in symplectic products, such a product may be regarded as the discrete approximation to a continuous tangent flow given by

$$U = \top \exp \int A(t)\, dt \tag{30}$$

[Compare Eqs. (3) and (4).] Although this does not correspond to the tangent flow of a mechanical system, it is nonetheless the simplest example where hydrodynamic Lyapunov modes can be expected. For convenience of further analysis we absorb τ into the definition of A and change to Fourier basis once and for all, so the problem becomes that of a product

$$\prod_n (1 + \tilde{A}_n) \tag{31}$$

Since the Lyapunov exponents of the slow part are expected to be smaller than the rest we aim at showing that the first $N - 3$ Lyapunov vectors span a subspace L_{f} (of $\tilde{\mathbf{C}}_N$) which is almost orthogonal to V_{s} in the sense that for any two unit vectors $u_{\mathrm{f}} \in L_{\mathrm{f}}$ and $v_{\mathrm{s}} \in V_{\mathrm{s}}$ one has $|u_{\mathrm{f}} \cdot v_{\mathrm{s}}| \ll 1$. Although the subspace L_{f} changes after each step, we show below that the "almost orthogonality" is propagated from step to step.

To show this, we propose the following scheme. Take an arbitrary vector $u \in L_{\mathrm{f}}$, whose components in V_{f} and V_{s} are u_{f} and u_{s} respectively, normalized so that $\|u_{\mathrm{f}}\| = 1$, and assume that u_{s} is small. The action of $1 + A$ generates a new normalized vector u' by

$$u' = \frac{(1+A)\,u}{\|[(1+A)\,u]_{\mathrm{f}}\|} \tag{32}$$

where $[\,\cdot\,]_{\mathrm{f}}$ is the projection onto the f-component. We would like to show that u_{s} remains small after repeated iteration of this process.

The block diagonalization (27) shows that the subspaces V_{f} and V_{s} are indeed almost invariant under the transformation $\tilde{A}$. However, in trying to apply this fact to the Lyapunov vectors of the product (31) we immediately encounter the danger that the small perturbations may accumulate. The basic problem is that a vector in V_{s} is contracted with respect to the "slowest" direction in V_{f} by a factor of only $1 - \mathcal{O}(\lambda)$ (as can be seen from the bounds on D_{f} and D_{s}), whereas the perturbations which tilt a vector in L_{f} with respect to V_{f} are of order ε, which is the typical size of the off-diagonal blocks [cf. Eqs. (27)–(29)], and since we are interested in the case $\lambda \ll \varepsilon$ this contraction is not strong enough to overcome the perturbation.

This order of magnitude argument can be made explicit by constructing a series of matrices with the properties given by Eqs. (27)–(29), which take a vector in V_{f} and rotate it such that the outcome is a vector which has an angle with V_{f} of order 1. This counter-example is given in Appendix A.

An essential ingredient in the construction of this counter-example is that the matrix $\tilde{A}$ has to be chosen specifically given u which in turn depends on former realizations, in violation of the independence assumption. In other words, although such a "bad" sequence is possible one naturally expects that this is an event with very low probability. Typically the perturbations to u_{s} generated by the off-diagonal part of the matrices R do not have the same direction, and should serve to cancel one another. Therefore the statement one can hope to show is that *in the sequence generated by iteration of Eq. (32), the probability that $\|u_{\mathrm{s}}\| > C\varepsilon$ for some fixed C is very small*, as was shown for a similar example in ref. 9. Here we will only prove the weaker statement that *the variance* $\langle \|u_{\mathrm{s}}\|^2 \rangle$ is $\mathcal{O}(\varepsilon^2)$, and take that as an indication that the probabilistic statement is correct, since the behavior of higher moments may be treated in an analogous manner.

To prove this statement we look at the s-component of Eq. (32),

$$u_s' = \frac{A_{sf} u_f + (1 + A_{ss})\, u_s}{\|(1 + A_{ff})\, u_f + A_{fs} u_s\|} \tag{33}$$

The quantity $\langle \|u_s'\|^2 \rangle$ is a sum of three terms $E_1 + E_2 + E_3$:

$$\begin{aligned} E_1 &= \left\langle \frac{\|A_{sf} u_f\|^2}{\ell^2} \right\rangle \\ E_2 &= 2 \left\langle \frac{A_{sf} u_f \cdot (1 + A_{ss})\, u_s}{\ell^2} \right\rangle \\ E_3 &= \left\langle \frac{\|(1 + A_{ss})\, u_s\|^2}{\ell^2} \right\rangle \end{aligned} \tag{34}$$

where $\ell = \|(1 + A_{ff})\, u_f + A_{fs} u_s\|$.

To bound these terms we first need a lower bound on the denominator ℓ. Let $v_f = R_{ff}^T u_f + R_{fs}^T u_s$, and define d by

$$\|D_f v_f\| \equiv d\, \|v_f\| \tag{35}$$

Note that d can vary widely between $\mathcal{O}(1)$ values and $\mathcal{O}(\lambda)$. But, using the lower bound on D_f of (28), we see that

$$\|(1 + D_f)\, v_f\|^2 = \|v_f\|^2 + 2 v_f \cdot D_f v_f + \|D_f v_f\|^2 \geqslant (1 + 2\lambda + d^2)\, \|v_f\|^2 \tag{36}$$

Expanding ℓ as

$$\ell = \|R_{ff}(1 + D_f)\, v_f + R_{fs} D_s v_s\| \tag{37}$$

and using the estimates $\|R_{fs} D_s\| = \mathcal{O}(\varepsilon\lambda)$ and $\|1 - R_{ff}\| = \mathcal{O}(\varepsilon^2)$ (cf. Eq. (28)), we get from (36) the desired lower bound on ℓ:

$$\ell^2 > (1 + 2\lambda + d^2)\, \|v_f\|^2\, (1 - \mathcal{O}(\varepsilon)) \tag{38}$$

We can now bound E_1, E_2 and E_3. First, we have

$$\|A_{sf} u_f\| = \|R_{sf} D_f v_f\| + \mathcal{O}(\lambda\varepsilon) < \varepsilon d\, \|v_f\| + \mathcal{O}(\lambda\varepsilon) \tag{39}$$

Thus, neglecting higher order corrections in ε, we get

$$E_1 < \frac{\varepsilon^2 d^2}{1 + 2\lambda + d^2} \tag{40}$$

The bound on the term E_2 makes essential use of the translation invariance.

For this, we note that

$$\frac{A_{\mathrm{fs}} u_{\mathrm{s}}}{\|(1+A_{\mathrm{ff}})\, u_{\mathrm{f}}\|^2} \tag{41}$$

transforms as a vector, that is, its kth component is multiplied by $\exp(i(2\pi/N)\, kx)$ under a relabeling of the coordinates $n \to n+x$. Therefore, because of translation invariance, the expectation value of (41) must remain invariant under such transformation, which means it must vanish. Since the denominator in E_2 is ℓ^2 (which also depends on u_{s}) and not $\|(1+A_{\mathrm{ff}})\, u_{\mathrm{f}}\|^2$, we need some gymnastics to exhibit the vanishing term. In order to see this we write

$$\begin{aligned} E_2 = 2 \Bigg\langle & \frac{A_{\mathrm{sf}} u_{\mathrm{f}} \cdot u_{\mathrm{s}}}{\|(1+A_{\mathrm{ff}})\, u_{\mathrm{f}}\|^2} + \frac{A_{\mathrm{sf}} u_{\mathrm{f}} \cdot A_{\mathrm{ss}} u_{\mathrm{s}}}{\|(1+A_{\mathrm{ff}})\, u_{\mathrm{f}}\|^2} \\ & - \frac{A_{\mathrm{sf}} u_{\mathrm{f}} \cdot (1+A_{\mathrm{ss}})\, u_{\mathrm{s}} [2(1+A_{\mathrm{ff}})\, u_{\mathrm{f}} \cdot A_{\mathrm{fs}} u_{\mathrm{s}} + \|A_{\mathrm{fs}} u_{\mathrm{s}}\|^2]}{\|(1+A_{\mathrm{ff}})\, u_{\mathrm{f}}\|^2 \ell^2} \Bigg\rangle \end{aligned} \tag{42}$$

The first term in (42) vanishes because of translation invariance, as explained before. The second term is bounded by

$$2\alpha\lambda\varepsilon \langle \|u_{\mathrm{s}}\| \rangle \tag{43}$$

and the dominant part of the third is

$$4\, \frac{(A_{\mathrm{sf}} u_{\mathrm{f}} \cdot u_{\mathrm{s}})((1+A_{\mathrm{ff}})\, u_{\mathrm{f}} \cdot A_{\mathrm{fs}} u_{\mathrm{s}})}{\|(1+A_{\mathrm{ff}})\, u_{\mathrm{f}}\|^2 \ell^2} < \frac{4\, d\varepsilon^2 \langle \|u_{\mathrm{s}}\|^2 \rangle}{1+2\lambda+d^2} \tag{44}$$

The last term is bounded by

$$E_3 < \frac{1+2\alpha\lambda}{1+2\lambda+d^2} \langle \|u_{\mathrm{s}}\|^2 \rangle$$

Collecting the bounds yields

$$\langle \|u_{\mathrm{s}}'\|^2 \rangle < \frac{d^2\varepsilon^2 + (1+2\alpha\lambda+4\, d\varepsilon^2) \langle \|u_{\mathrm{s}}\|^2 \rangle}{1+2\lambda+d^2} + 2\alpha\lambda\varepsilon \tag{45}$$

It appears from (45) that although large perturbations are possible when d is $\mathcal{O}(1)$, the contraction rate increases precisely enough to compensate this contribution. Thus if $\langle \|u_s\|^2 \rangle$ is $\mathcal{O}(\varepsilon^2)$ to start with, it will stay so indefinitely.

In summary, assuming that the variance is indeed a measure of typical fluctuations, we have shown that, for $N \gg 1$, the subspace L_f spanned by the first $N-3$ Lyapunov vectors of the product (31) is, with very high probability, almost orthogonal to V_s. This implies that the last three Lyapunov vectors (including the translation) remain approximately in V_s. This means by definition that they are hydrodynamic, in the sense that they are well approximated by eigenvectors of the discrete Laplacian, as has been defined precisely in Section 2. Since the action of $\tilde{A}$ on V_s has two eigenvalues close to $(2\pi/N)^2$ as shown in the previous section, it follows as a corollary that the two smallest non-trivial Lyapunov exponents have approximately this value, so that they are also hydrodynamic. This completes the demonstration.

5. PRODUCTS OF SYMPLECTIC MATRICES

We now turn to products of matrices of the form

$$S = \begin{pmatrix} 1 & \tau \\ \tau A & 1 + \tau^2 A \end{pmatrix} \tag{46}$$

We disregard the two translation modes in S for convenience and view the matrices S as $(2N-2) \times (2N-2)$ matrices. Let us recall that since the matrices S are symplectic and hyperbolic, the Lyapunov exponents are non-zero and come in pairs of opposite signs. We concentrate on modes number $N-2$ and $N-1$ which are the smallest positive ones.

We reduce the problem to an equivalent one to which the results of Section 4 can be applied directly. We denote by L_+ the subspace spanned by the first $N-1$ Lyapunov vectors. It is spanned by a set of $N-1$ independent vectors, which we display in the form of a $(N-1) \times (2N-2)$ matrix $\mathscr{V}$. The $N-1$ vectors can always be chosen it such a way that $\mathscr{V}$ is of the normal form

$$\mathscr{V} = \begin{pmatrix} \mathbf{1} \\ V \end{pmatrix} \tag{47}$$

where both blocks are $(N-1) \times (N-1)$. Acting on $\mathscr{V}$ with S gives a spanning set of the image subspace L'_+,

$$S\mathscr{V} = \begin{pmatrix} 1 + \tau V \\ \tau A + (1 + \tau^2 A)\, V \end{pmatrix} \tag{48}$$

and changing basis to normal form gives $\mathscr{V}'=\binom{\mathbf{1}}{V'}$ where

$$V'=\tau A+\frac{V}{1+\tau V} \tag{49}$$

A convenient property of this matrix dynamical system is that if V is symmetric to begin with, it stays so as a consequence of the symplectic property of S.[10]

By definition, any vector $v\in L_+$ has a block representation $v=(u, Vu)$. From Eq. (48) it follows that its image is $(u', V'u')$, where

$$u'=(1+\tau V)\,u \tag{50}$$

Hence, the first $N-1$ Lyapunov modes of the products of the S are the same as those of the product $\prod_n (1+\tau V_n)$ where the matrices V_n are evolving according to Eq. (49): $V_{n+1}=\tau A_n+V_n/(1+\tau Vn)$.

In view of this equivalence, it suffices to show that the matrix V has the properties formulated in Eqs. (27)–(29) and to apply the results of Section 4. First note if $A_n=-\partial^2$ (minus the discrete Laplacian) for all n, then the V_n converge to $f(-\partial^2)$, where

$$f(x)=\frac{\tau x}{2}+\sqrt{x+\left(\frac{\tau x}{2}\right)^2}$$

is the larger root of the quadratic equation $f(x)=\tau x+f(x)/(1+\tau f(x))$. For small $x>0$ this is close to $x^{1/2}$, and therefore we assume that V has a representation of the type given by Eqs. (27)–(29), with $\varepsilon=\mathcal{O}(N^{-1/2})$ as before and λ is now $f(4\pi/N^2)=\mathcal{O}(N^{-1})$. The aim is to show that this property is carried on to V'.

In order to avoid the necessity of presenting even more technical details, we present the argument for the case where the slow subspace contains a single mode, rather than a pair of nearly degenerate modes. Since V' is symmetric its smallest eigenvalue is given by

$$\lambda_{V'}=\min_{\|u\|=1} u\cdot\left(\tau A+\frac{V}{1+\tau V}\right)u \tag{51}$$

It follows from our assumptions that there exist (normalized) eigenvectors of V and A:

$$Ae_A=\lambda_1(1+c_A\varepsilon)\,e_A,\qquad Ve_V=f(\lambda_1)(1+c_V\varepsilon)\,e_V \tag{52}$$

where λ_1 is the smallest positive eigenvalue of $-\partial^2$ and e_A and e_V are close to e_1, the corresponding eigenvector. The variational principle then gives immediately a lower bound on $\lambda_{V'}$,

$$\lambda_{V'} > f(\lambda_1)(1+\varepsilon c_{V'}) \tag{53}$$

for some $c_{V'}$, between c_A and c_V. To get an upper bound on $\lambda_{V'}$ recall that it was shown in Section 3 Eq. (25) that the k component of e_A is of order εk^{-1}, and note that the bound (26) implies

$$f(-a_{\min}\,\partial^2) < V < f(-a_{\max}\,\partial^2) \tag{54}$$

We now use $u = e_A$ in Eq. (51) and get

$$\lambda_{V'} < \lambda_1(1+c_A\varepsilon) + e_A \cdot f(-a_{\max}\,\partial^2)\, e_A < f(\lambda_1)(1+\bar{c}\varepsilon) \tag{55}$$

for some constant $\bar{c}$. Equations (53) and (55) establish the desired property of the eigenvalues.

The corresponding eigenvector $e_{V'}$ is the one which minimizes Eq. (51). Because of the minimax principle applied to A and V, letting $u = e_1 + w$ with $w \cdot e_1 = 0$ and w small, the quadratic form $u \cdot Au$ may be approximated by

$$u \cdot Au \sim (w - w_A) \cdot A(w - w_A) + \lambda_A$$

and similarly

$$u \cdot Vu \sim (w - w_V) \cdot V(w - w_V) + \lambda_V$$

Therefore, in order to find $w_{V'}$ we need to minimize the quadratic form

$$\tau(w - w_A) \cdot A(w - w_A) + (w - w_V) \cdot \frac{V}{1+\tau V}(w - w_V)$$

The minimum occurs at

$$w_{V'} = (1+B)^{-1}(Bw_A + w_V) \tag{56}$$

where

$$B = \tau \frac{1+\tau V}{V} A$$

We can use again the bounds (26) and (54) to show that

$$b_{\min} g(-\partial^2) < B < b_{\max} g(-\partial^2),$$

for some positive numbers $b_{\min}, b_{\max}$ and a positive function g, and thus bound the components of $w_{V'}$,

$$|(w_{V'})_k| < \frac{(w_V)_k + b_{\max} g(|\mu_k|^2)(w_A)_k}{1 + b_{\min} g(|\mu_k|^2)}$$

where $|\mu_k|^2$ is the kth eigenvalue of $-\partial^2$ (see Section 3). This shows that $(w_{V'})_k \sim \varepsilon k^{-1}$. This completes the demonstration of the desired properties of V', and, on applying the results of Section 4, the existence of hydrodynamic Lyapunov modes in the symplectic case.

6. NUMERICAL TESTS

The purpose of this Section is to verify numerically some of the statements given above, and on to further study numerically the dependence of hydrodynamic behavior of several Lyapunov modes as a function of noise level as well as system size.

The simplest system we discuss is a product $\prod_n A_n$ of independent matrices of the form (8). Since the relative gap between the first two nonzero eigenvalues is $\mathcal{O}(1)$ [see Eqs. (27)–(29)], the contraction in this case is strong, and the potential problems of the accumulation of errors as discussed in Section 4 and Appendix A are absent. Nevertheless, even in this case, there are some qualitative differences between the behavior of the Lyapunov modes, and the corresponding eigenmodes of a single matrix.

We quantify the degree of hydrodynamic behavior in the Lyapunov modes as follows. For the Lyapunov vectors v_i we computed the *residuals* r_i, that is, the norm of the orthogonal complement

$$r_i = \| v_i - (v_i \cdot e_k)\, e_k - (v_i \cdot e_{-k})\, e_{-k} \|$$

where $k = k(i)$ is the wave vector associated with vector i. (For example, $k = 1$ for the vectors v_{N-1} and v_{N-2} discussed above.) In fact, to get more precise results we subtracted from v_i all the components with lower-lying k:

$$r_i = \| v_i - \sum_{k \leqslant k(i)} ((v_i \cdot e_k)\, e_k - (v_i \cdot e_{-k}) e_{-k}) \|$$

(The results are not very different for the two definitions of r_i.)

Figure 1 presents these residuals for systems with different sizes of the matrices and different values of the noise variance σ. The vertical axis

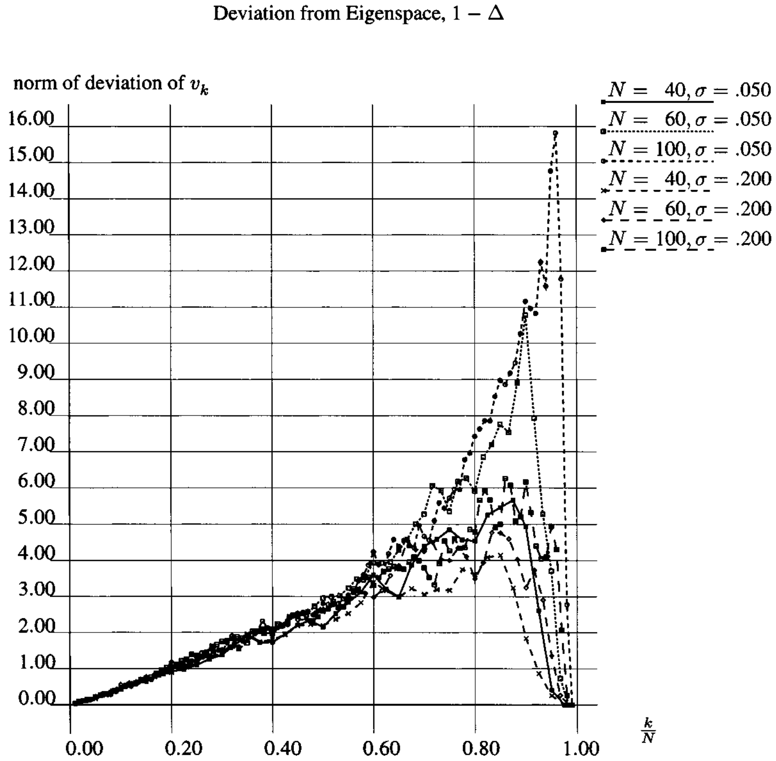

Fig. 1. Data collapse of the residuals of the Lyapunov vectors of a product of random Laplacians. The curves show $(r_k/\sigma)^2$ as a function of the wave number k/N for several values of N and σ. The data points are averages of 10 realizations, averaged also within each pair which corresponds to the same $|k|$. As expected, the range of collapse increases with the system size. The random variables are chosen as $a = 0.5 \pm \sigma$ with probability 1/2.

measures $(r_k/\sigma)^2$, and the horizontal axis gives k/N. The approximate collapse of the graphs for small k implies that the dependence of the residuals on system size N and noise strength (variance) σ) is given by the scaling form

$$r_k = \sigma f_1\left(\frac{k}{N}\right)$$

The behavior of f_1 for small x is approximately $f_1(x) = \mathcal{O}(\sqrt{x})$, which implies that for a fixed k

$$r_k \sim \frac{\sigma}{\sqrt{N}}$$

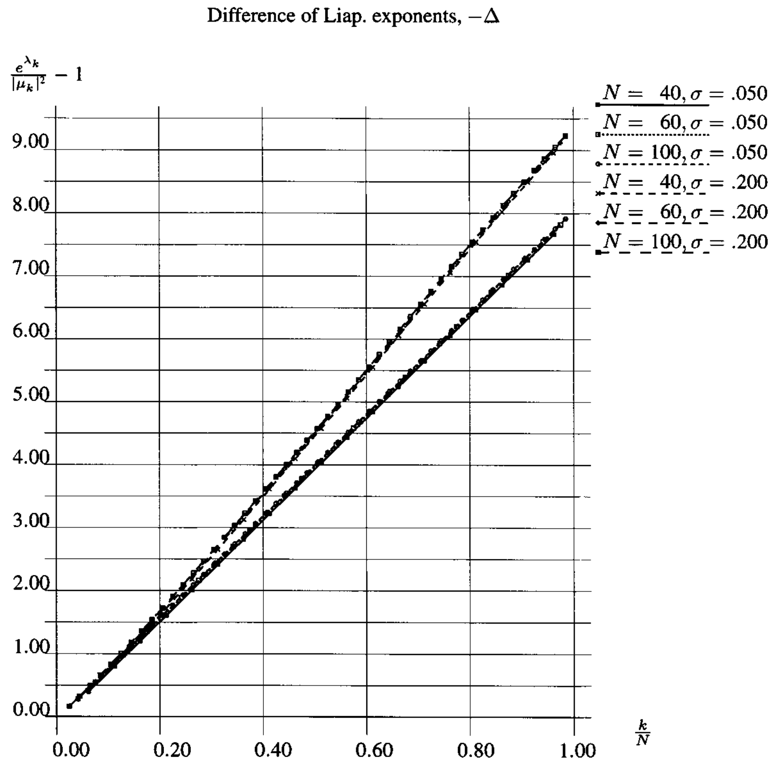

Fig. 2. Data collapse of the relative deviations of the Lyapunov exponents of a product of random Laplacians. The curves show δ_k/σ^2 as a function of the wave number k for several values of N and σ. In fact the date collapse is perfect for fixed σ.

the *same* dependence as in the residuals of the eigenvectors of a single matrix. However, the dependence of the residuals as a function of k is $r_k \sim \sqrt{k}$, slower the linear dependence on k in the case of a single matrix.

For the Lyapunov exponents λ_i we measure the relative deviation δ_k from the respective eigenvalues $|\mu_k|^2$ of the discrete Laplacians (see Section 3),

$$\delta_k = \frac{\exp(\lambda_k)}{|\mu_k|^2} - 1$$

The results for the deviations δ_i of the Lyapunov exponents are displayed in Fig. 2, where δ_k/σ^2 is plotted against k/N. The data collapse implies that

$$\delta_k = \sigma^2 f_2\left(\frac{k}{N}\right)$$

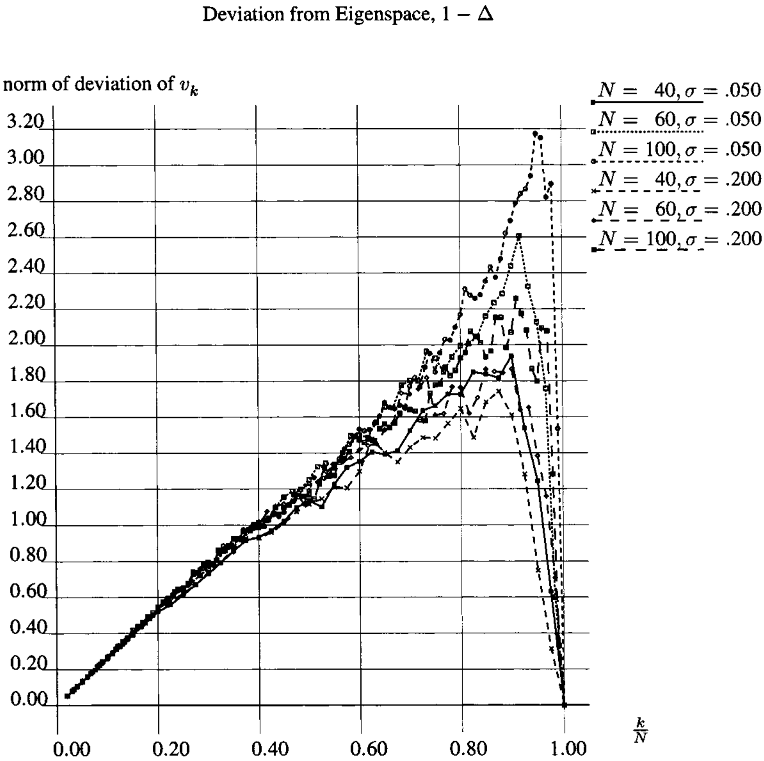

Fig. 3. Same as Fig. 1 for the product $\prod_n (1 + A_n)$, except that the vertical axis measures r_k/σ.

The function $f_2(x)$ is approximately linear for small x which implies for $k \ll N$:

$$\delta_k \sim r_k^2$$

This is not unreasonable, since the Lyapunov exponents, unlike the eigenvalues of a single matrix, are given as a result of an averaging process.

We next present a similar analysis for the product $\prod_n (1 + A_n)$ which was considered in Section 4. The results for the residuals of the Lyapunov vectors and the deviations of the exponents are presented in Figs. 3 and 4 respectively. The scaling form for the residuals is in this case

$$r_k = \sigma f_3\left(\frac{k}{N}\right)$$

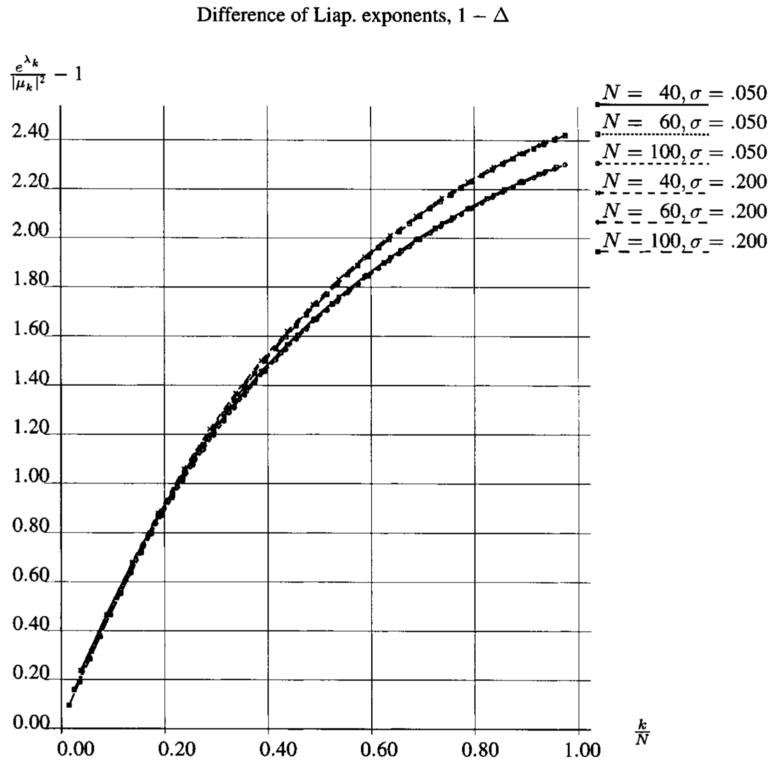

Fig. 4. Same as Fig. 2 for the product $\prod_n (1 + A_n)$, except that the vertical axis measures $\sqrt{\delta_k}/\sigma$.

where $f_3(x) \sim x$ for small x. Thus, in this case the residual for a fixed k decreases as

$$r_k \sim \frac{\sigma}{N}$$

that is, *faster* than the $N^{-1/2}$ decrease in the residuals of the eigenvectors of a single matrix. The analysis presented in Section 4 is too general to capture this behavior.

The relative deviations of the exponents scale in this case as

$$\delta_k = \sigma^2 f_4\left(\frac{k}{N}\right)$$

and $f_4(x) \sim x^2$ for small x, so that as in the product of random Laplacians, the relative size of r_k and δ_k is $\delta_k \sim r_k^2$.

APPENDIX A. COUNTER-EXAMPLE

We want to show here that an "unfortunate" choice of rotations can move the system out of the region where the Lyapunov vectors remain essentially aligned with the eigendirections of the Laplacian. The issue here is that, on one hand, the cones in which these vectors lie are slightly contracted and on the other hand slightly turned. The "counter-example" shows that the turning wins over the contraction.

Let $V_{\mathrm{f}} = \mathrm{Span}\{e_1, e_2\}$ and $V_{\mathrm{s}} = \mathrm{Span}\{e_3\}$. Suppose that L_{f} contains a vector with block representation $u = (u_{\mathrm{f}}, u_{\mathrm{s}})$, with $u_{\mathrm{s}} > 0$, normalized such that $\|u_{\mathrm{f}}\| = 1$, and let v_{f} span the orthogonal complement to u_{f} in V_{f}. We construct the matrix $\tilde{A}$ by giving the components in the representation (27),

$$R \sim \begin{pmatrix} \mathbf{1} & -\varepsilon v_{\mathrm{f}} \\ \varepsilon v_{\mathrm{f}}^T & 1 \end{pmatrix} \tag{57}$$

$D_{\mathrm{s}} = \alpha\lambda$, and D_{f} is such that

$$\begin{aligned} D_{\mathrm{f}} u_{\mathrm{f}} &= (\lambda + \varepsilon^2)\, u_{\mathrm{f}} + \varepsilon v_{\mathrm{f}} \\ D_{\mathrm{f}} v_{\mathrm{f}} &= \varepsilon u_{\mathrm{f}} + v_{\mathrm{f}} \end{aligned} \tag{58}$$

The image of u is

$$(1 + \tilde{A}) \begin{pmatrix} u_{\mathrm{f}} \\ u_{\mathrm{s}} \end{pmatrix} = \begin{pmatrix} (1 + \lambda + \varepsilon^2(1 - u_{\mathrm{s}}))\, u_{\mathrm{f}} + \varepsilon v_{\mathrm{f}} \\ (1 + \alpha\lambda - \varepsilon^2)\, u_{\mathrm{s}} + \varepsilon^2 \end{pmatrix} \tag{59}$$

After normalizing the f component to 1, the s component becomes

$$u'_{\mathrm{s}} \sim [1 + (\alpha - 1)\,\lambda + \varepsilon^2 (3/2 - u_{\mathrm{s}})]\, u_{\mathrm{s}} + \varepsilon^2 \tag{60}$$

Evidently, even if $u_{\mathrm{s}} = 0$ initially, by choosing $\tilde{A}$ as above, u_{s} can be increased to an $\mathcal{O}(1)$ value (as $\varepsilon, \lambda \to 0$) if $\lambda = \mathcal{O}(\varepsilon^2)$.

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