BIFURCATIONS FOR MAPS

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The study of complicated dynamical systems is, of course, intimately related to the differential equation governing the evolution of such a system, which we suppose of the form

$$\frac{dx}{dt} = F(x), \; x \in \mathbb{R}^{\nu} \; . \tag{1}$$

This system is too complicated for our purposes, and we consider a quite strong simplification

$$x_{n+1} = f(x_n) \qquad x_n \in \mathbb{R} \; , \; n = 1,2, \ldots \tag{2}$$

The equation (2) can be obtained from (1) in several more or less realistic ways which I shall briefly sketch.

1) Let $x(t)$ be a solution of (1). Then one defines $x_{n+1} = x(n+1)$, that is f is the time one map associated to the flow defined by F (for $\nu=1$).

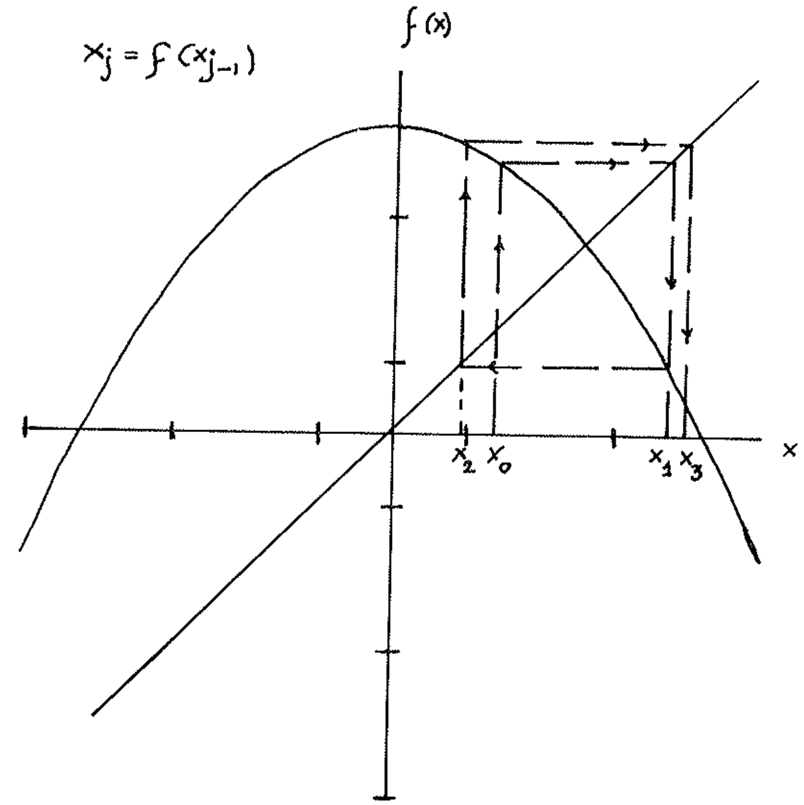
2) One approximates (for $\nu=1$), $\frac{dx}{dt} \sim x(t+1) - x(t)$, so that $x_{n+1} = F(x_n) + x_n = f(x_n)$ is the difference approximation of (1).

3) Even for $\nu>1$, f may be viewed as the Poincaré map of the flow $t \to x(t)$ with respect to some special surface.{This method is in fact often quite successful in concrete examples and was a motivation for Hénon to find his map on $\mathbb{R}^2$[H], or for Landford, who showed that the Lorenz model [Lo] can be studied as a map on $\mathbb{R}^1$[La]}. This approach is the most tedious, but also the most interesting, because it shows that the study of maps on $\mathbb{R}$ may be realistically related to flows in $\mathbb{R}^{\nu}$.

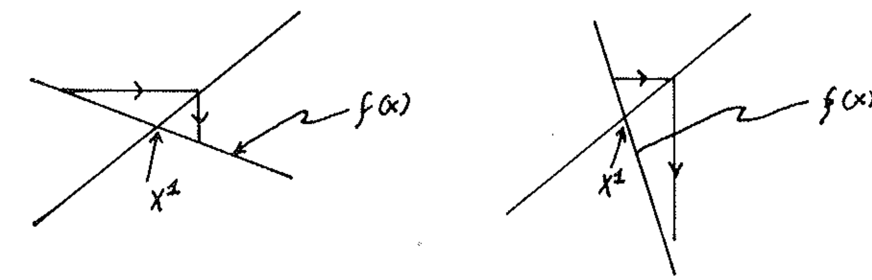
We shall now concentrate on maps on $\mathbb{R}$, which have the additional property that they map $[-1,1]$ into itself . Since

C. Bardos and D. Bessis (eds.), Bifurcation Phenomena in Mathematical Physics and Related Topics, 115–135.

we are interested in the long time behaviour of the dynamical systems whose evolution this map is supposed to mimic, we shall study the iterates of maps from [-1,1] to itself. Such iterates can be conveniently described graphically. The following figure shows the graph of f and the iterates x_1, x_2, x_3 of a point x_o.



Let us perform, as a first exercise, an analysis of this map near x^1, defined by $x^1 = f(x^1)$. It is straight-forward to see, that if $|f'(x^1)| < 1$ [' denotes derivative] then points near x^1 will eventually evolve towards x^1 while if $|f'(x^1)| > 1$ they will tend to evolve away from x^1.



We shall call x^1 a stable, respectively unstable fixed point. Globally, the situation may be a lot more complex. There may occur stable and unstable fixed points or periodic orbits of any order (= number of points on the orbit), but it is also possible that for almost all initial points x_o, the orbit never settles down on an orbit, but wanders truly ergodically through the interval. In addition two nearby points may evolue into totally different histories, giving rise to mixing [Ru].

We shall now concentrate on maps from [-1,1] to itself, <u>depending on a parameter</u> μ. One can think of this parameter as coming from a parameter in F (Eq. 1) where it has maybe the role of a Reynolds number of a temperature difference (in hydrodynamics) or a (re)production rate (in chemistry or biology). We may now ask how the number and position of periodic orbits evolves as one looks at $x_n, n \to \infty$ for different μ. There is the following experimental evidence.

1^o) For some "decent" functions f_μ there is a range $\mu \in (\mu_n, \mu_{n+1})$ of parameters where f_μ has exactly a stable period 2^n, and no other stable periods. In particular, as suggested by the notation, these ranges are contiguous.

2^o) In each interval (μ_n, μ_{n+1}), there is one point $\bar{\mu}_n$ for which $f_{\bar{\mu}_n}$ has the property

$$\frac{d}{dx} f_{\bar{\mu}_n}^{2^n}(x) = 0 \qquad (3)$$

for any point x on the period 2^n. By the chain rule

$$\frac{d}{dx} f^k(x) = f'(f^{k-1}(x))\, f'(f^{k-2}(x)) \ldots f'(x), \qquad (4)$$

so that (3) says that one of the points of the period is the x for which of takes its maximum, cf Fig. 1. Such periods are called <u>superstable</u>.

3^o) Feigenbaum has discovered the following regularity

1. $\lim_{n \to \infty} \bar{\mu}_n = \bar{\mu}_\infty$ exists.

2. For "all decent" families f_μ,

$$\lim_{n \to \infty} \frac{\log|\bar{\mu}_n - \bar{\mu}_\infty|}{n} = \log \delta^{-1}$$

where δ in <u>independent</u> of the family f_μ,

$\delta = 4.66920160903\ldots$.

3. There is for each period 2^n a point x_n^1 nearest to the $\bar{x}$ for which $f_{\bar{\mu}_n}$ takes its maximum. In fact $x_n^1 = f_{\bar{\mu}_n}^{2n-1}(\bar{x})$. For "all decent" families f_μ

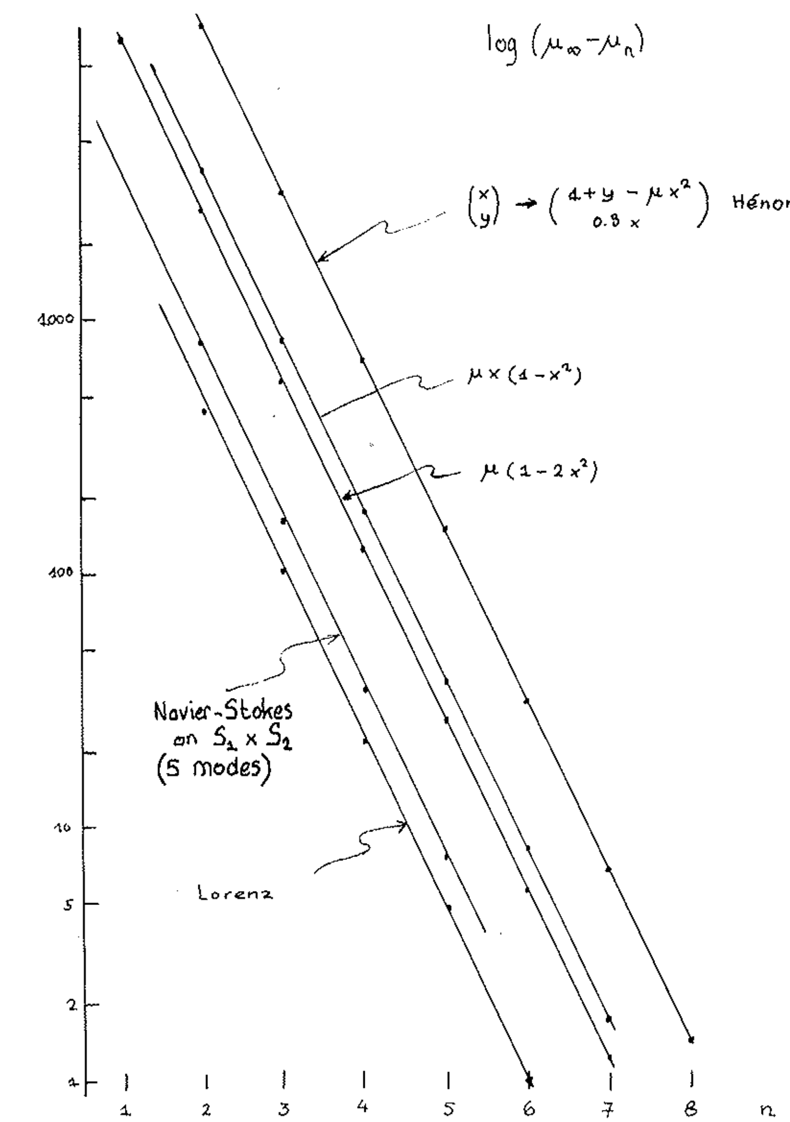
$$\lim_{n \to \infty} \frac{\log|x_n^1 - \bar{x}|}{n} = \log \lambda$$

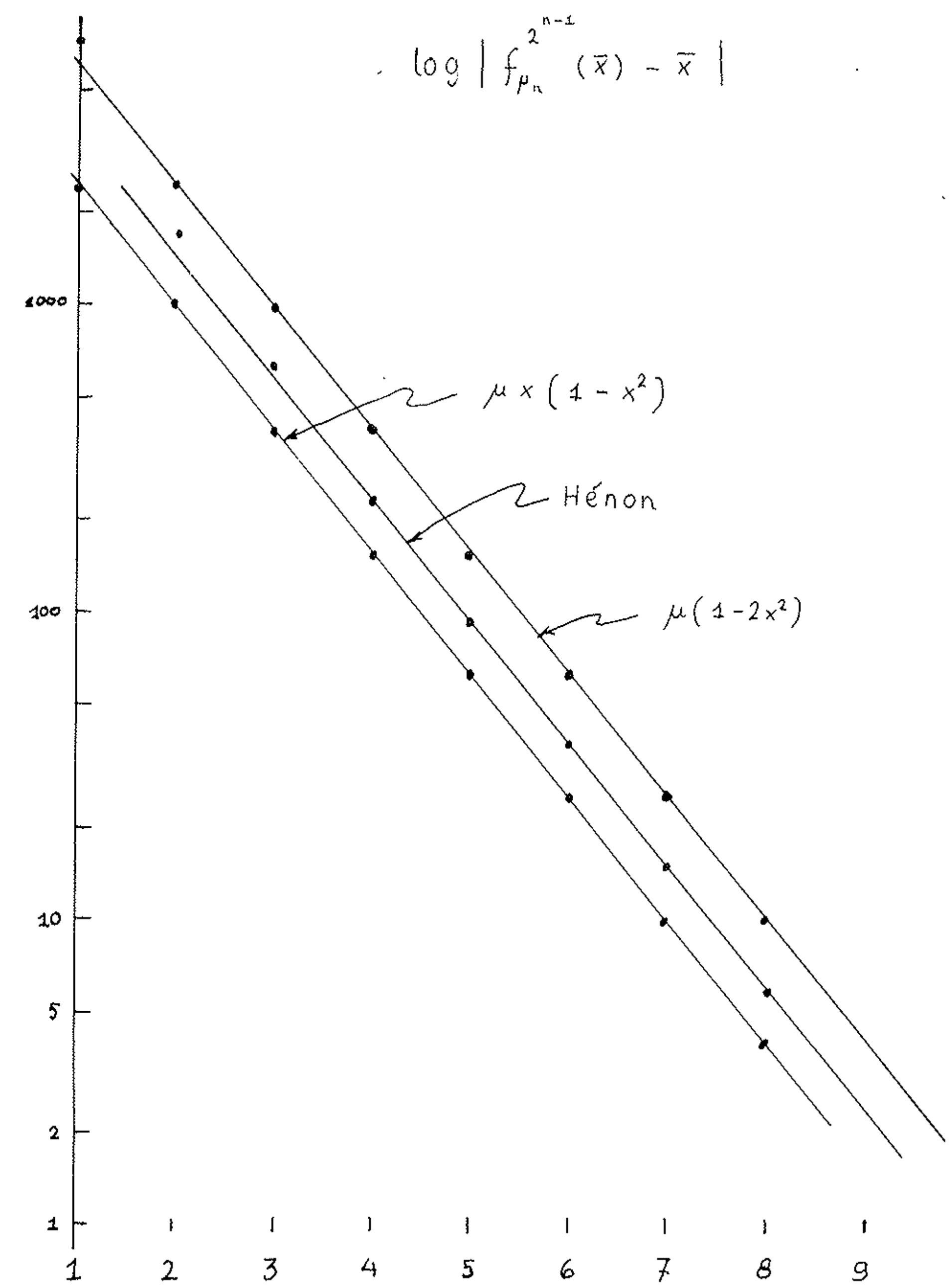
where λ is <u>independent</u> of the family f_μ,

$\lambda = 1/2.5029078750957\ldots$.

In addition, $(-1)^n[\hat{x}_n - \bar{x}]$ has constant sign as a function of n. I would like to illustrate this universality for the following examples :

- $f_\mu(x) = \mu x(1-x^2)$ Feigenbaum [F]; shown is μ_n
- $f_\mu(x) = (1-2x^2)$ Feigenbaum [F]; shown is $\underline{\mu_n}$





- $f_\mu\binom{x}{y} = \binom{1 + y - x^2}{0.3x}$ Derrida et al [D] shown is μ_n
- Navier Stokes on torus $S_1 \times S_1$ (5 Fourier components)[BF] shown is $\bar{\mu}_n$
- Lorenz System [L]. Calculation done by Franceschini [Fr]. Shown is $\bar{\mu}_n$

Note that the 3 latter examples are not maps on $\mathbb{R}$!

I shall now try to argue why the non-linear map

$f \to Nf$

$$(Nf)(x) = f(1)^{-1}\, f o f(f(1)x) \qquad (5)$$

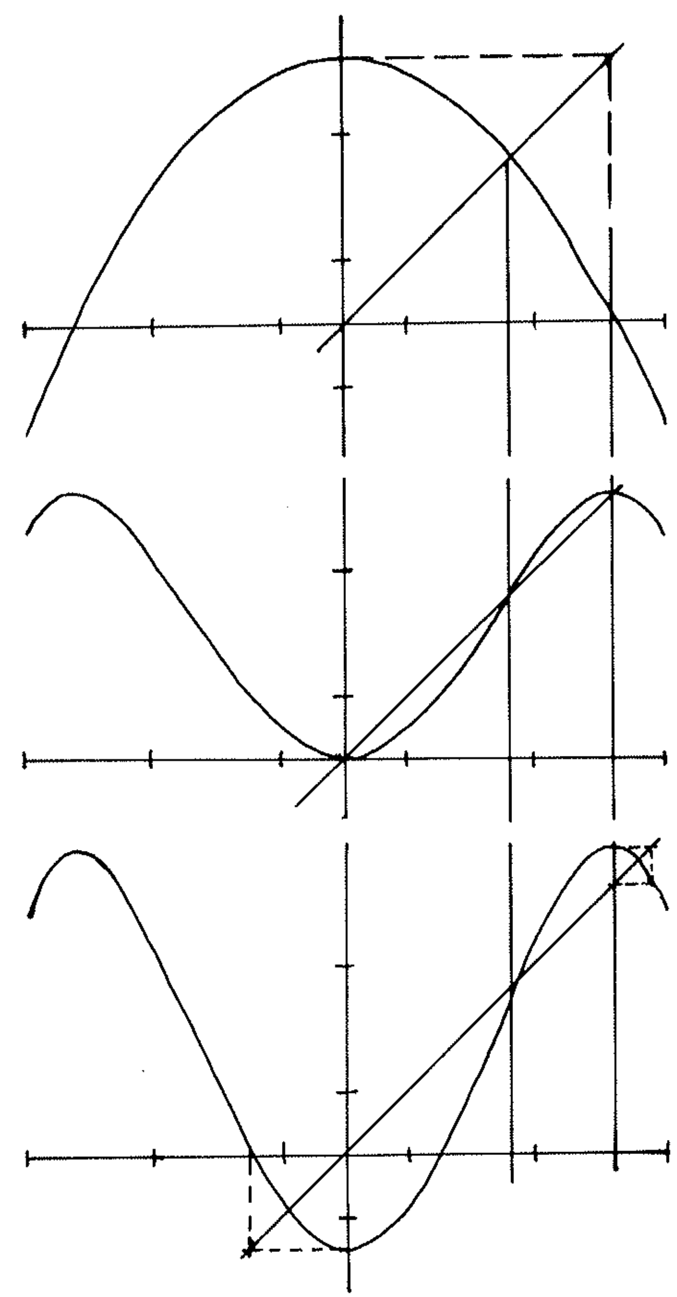
is important in studying the above facts. We view the facts discovered by Feigenbaum as expressing that there is an intimate connection between "any" map $x \to f^{2^n}(x)$ and $x \to f^{2^{n+1}}(x)$ provided n is very large [on purpose, we have omitted the μ-dependence]. In particular the most intimate connection possible would be $f^{2^n}(x) = f^{2^{n+1}}(x)$. This is however not possible except in the trivial case $f(x) = x$ but we could ask that the two functions are at least similar, i.e.

$$f^{2^n}(x) = -\lambda^{-1} f^{2^{n+1}}(\lambda x)$$

for some λ. Choosing $\lambda = f(1)$ amounts to break the dilatation symmetry of the problem, and we may fix $f(0) = 1$. Our "most intimate" connection has thus become the question of looking for a fixed point of the map N on a space of functions which take the value 1 for $x = 0$. (N preserves this property !)

The possibility that such a function may indeed exist is indicated through Fig. 3 below, where we plot $1 - x^2$ and $(1 - 1.411x^2)^{\circ 2}$.

Note the similarity of the two square pieces



Unfortunately, to date, existence of a solution to $Nf = f$ is not known in C^{2}*. We now generalize the problem and explain the mechanism in special cases. We look for solutions ϕ to $N\phi = \phi$ which are of the form

$$\phi_\varepsilon(x) = f_\varepsilon(|x|^{1+\varepsilon}) \qquad (6)$$

with $t \to f(t)$ being C^2. We claim that there is some sort of bifurcation associated to the problem

$$\phi(1)\,\phi(x) = \phi o \phi(\phi(1)x). \qquad (7)$$

Indeed, (7) has for all ε the trivial solution $\phi(x) = 1 - |x|^{1+\varepsilon}$, corresponding to $\phi(1) = 0$. At $\varepsilon = 0$ another, non-trivial solution bifurcates, as a formal power series in ε, $\log\varepsilon$ and t which is of the form

$$\begin{aligned} f_\varepsilon(t) &= 1 - (1+\lambda)t + \lambda^2 g(t) \\ \lambda &= \lambda(\varepsilon) = \varepsilon\,(-1 + \log 1/\varepsilon)\,(1 + O(\varepsilon)). \end{aligned} \qquad (8)$$

This branch does not come from the sort of singular eigenvalue described in the lectures of Crandall and Rabinowitz, as you may easily check.

Our first aim will then be to show that to (8) there exists a solution $\phi_\varepsilon(x) = f_\varepsilon(|x|^{1+\varepsilon})$ for small $\varepsilon > 0$, with $\lambda > 0$. From this we shall then derive the following theorem. Let $\mathcal{B}$ be the Banach space of functions analytic in $|t| < 2$, equipped with the sup norm.

* There is evidence of existence, O. Lanford, privat communication.

THEOREM : Fix $\varepsilon > 0$ sufficiently small, let $\psi(x) = h(|x|^{1+\varepsilon})$, with $h \in \mathcal{B}$ and $\|h - f_\varepsilon\|$ small. Let $\psi_\mu(x) = \psi(\mu x)$. For any such family $\mu \to \psi_\mu$ of functions, the μ_n (or $\bar{\mu}_n$) defined as before satisfy

i) $\lim_{n\to\infty} \mu_n = \mu_\infty$ exists

ii) $$\lim_{n\to\infty} \frac{\log|\mu_n - \mu_\infty|}{n} = \log \delta(\varepsilon)^{-1}$$

where $\delta(\varepsilon)$ depends on ε but not on ψ

iii) $$\lim_{n\to\infty} \frac{\log(-1)^{n+1}\psi_{\mu_n}^{2^n+1}(0)}{n} = \log \lambda(\varepsilon)$$

where $\lambda(\varepsilon)$ depends on ε but not on ψ

Note : the condition $h \in \mathcal{B}$ is probably too strong. On the other hand $h \in C^1$ is definitely too weak.

We now come to the existence proof for f_ε. For technical reasons, it is easier to consider f_ε as a function of λ and to write an equation for $\varepsilon(\lambda)$. Since $\varepsilon(\lambda)$ will turn out to be monotone, this is certainly a legitimate procedure. The result is the following

THEOREM : There is a constant K such that for sufficiently small $\lambda > 0$, the Equation (7) has a unique solution $\phi = \phi_\lambda$ of the form $\phi_\lambda(x) = f_\lambda(|x|^{1+\varepsilon(\lambda)})$, satisfying

i) $t \to f_\lambda(t)$ is C^2 in $0 < t < 1$,

ii) $f_\lambda(t) = 1 - (1+\lambda)t + \lambda^2 g_\lambda(t)$, $\|g_\lambda\|_{C^2} < K$,

iii) $\varepsilon(\lambda) = -2/(1+\log\lambda) + \lambda^2 E(\lambda)$, $|E(\lambda)| < K$.

For sufficiently small $\lambda > 0$, f_λ is analytic in $|t| < 2$, and both f_λ and $\varepsilon(\lambda)$ are analytic in $\{\lambda | \mathrm{Re}|\lambda| > 0, |\mathrm{Arg}\lambda| < \Theta_o\}$ for some small $\lambda_o, \Theta_o > 0$.

Once we have found out how to set up the problem, the proof is really not that hard. I will only sketch it. Because λ is small, $f_\lambda(\lambda^{1+\varepsilon}t)$ will be positive, and it suffices to show existence of a solution of

$$\lambda f(t) + f(f(\lambda^{1+\varepsilon}t)^{1+\varepsilon}) = 0, \tag{9}$$

where $f(t) = 1 - (1+\lambda)t + \lambda^2 g(\lambda,t)$ and ε, g are unknowns, $g(\lambda,0) = g(\lambda,1) = 0$.

Differentiating Eq. (9) twice with respect to t and solving for ε and g one gets equations of the following form :

$$E = N_\lambda(g,\varepsilon)$$

and (10)

$$g = L\,(1 - \lambda K_\lambda(g,\varepsilon))^{-1} M_\lambda(g,\varepsilon).$$

The quantities K, L, M, N are defined as follows :

Let $\mathcal{D} = \{(g,\varepsilon), g \in C^2[0,1], g(0) = g(1) = 0, \varepsilon \in C\}$ equipped with norm

$$\|(g,\varepsilon)\| = \sup_{x\in 0,1} |g''(x)| + |\varepsilon|\,[\log^1/\lambda - 1]/2\lambda,$$

let $\mathcal{D}_1$ be the unit ball in $\mathcal{D}$.

$$N_\lambda(g,\varepsilon) = \lambda/(-1 + \log 1/\lambda) + \text{remainder}$$

and in fact $N_\lambda(g,\varepsilon)$ is a bounded, contracting map from $\mathcal{D}_1$ to $\mathbb{C}$.

$$[M_\lambda(g,\varepsilon)](t) = -\lambda^{2\varepsilon}(\varepsilon/\lambda)\, f'(f(\lambda^{1+\varepsilon}t))^{\varepsilon-1}(1+\varepsilon)\, f'(\lambda^{1+\varepsilon}t)^2$$

is a bounded, contracting map from $\mathcal{D}_1$ to $C^1[0,1]$.

$K_\lambda(g,\varepsilon)$ is a bounded linear operator from $C^0[0,1]$ to $C^0[0,1]$ and from $C^1[0,1]$ to $C^1[0,1]$ and

$$\|[K_\lambda(g_1,\varepsilon_1)-K_\lambda(g_2,\varepsilon_2)]h\|_{C^0} < O(1)\,\|(g_1,\varepsilon_1)-(g_2,\varepsilon_2)\|_{\mathcal{D}}|h|_{C^1}$$

provided $g_i,\varepsilon_i \in \mathcal{D}_1$, $i = 1,2$.

The explicit definition is

$$(K_\lambda(g,\varepsilon)h)(t) = -\lambda^{2\varepsilon}\{h(f(\lambda^{1+\varepsilon}t)^{1+\varepsilon})f(\lambda^{1+\varepsilon}t)^{2\varepsilon}(1+\varepsilon)^2 f'(\lambda^{1+\varepsilon}t)$$
$$+ h(\lambda^{1+\varepsilon}t)f'(f(\lambda^{1+\varepsilon}t)^{1+\varepsilon})f(\lambda^{1+\varepsilon}t)^{\varepsilon}(1+\varepsilon)\}$$

Finally L is bounded linear from $C^0[0,1]$ to $C^2_0[0,1]$;

$$(Lh)(t) = \int_0^t (t-\tau)h(\tau)d\tau - t\int_0^1 (1-\tau)h(\tau)d\tau.$$

The above properties are easily verified, and by using the identity

$$[1 - \lambda K(X_1)]^{-1} - [1 - \lambda K(X_2)]^{-1}$$
$$= [1 - \lambda K(X_1)]^{-1}[K(X_1) - K(X_2)][1 - \lambda K(X_2)]^{-1}$$

where $X_i = (g_i,\varepsilon_i)$ $i = 1,2$,

one verifies that the equations (10) have a unique solution, and that this solution solves (9), and hence it solves (7).

Existence (and, of cours uniqueness) of an analytic solution can be shown in a similar way. I refer to the original paper.

Let us now come back to the map

$$f \to Nf$$
$$(Nf)(x) = f(1)^{-1}\, f\circ f(f(1)x).$$

It should be obvious that a good understanding of N will bring

us nearer to an understanding of f^{2^n} since, up to a change of scale, f^{2^n} equals $N^n f$. The importance of having found a fixed point of the map $f \to Nf$ lies then in the fact that in a neighborhood of the fixed point f_λ, the nonlinear map N can be essentially understood in terms of the (linear) tangent map $\mathcal{D}N$ at f_λ.

Note that if $f(0) = 1$ then $(Nf)(0) = 1$. Therefore, since $f_\lambda(0) = 1$, $\mathcal{D}N$ at f_λ will have the property that if $g(0) = 0$ then $(\mathcal{D}Ng)(0) = 0$.

A natural space of functions to work on is therefore the space E of functions (of t) analytic in $|t| < 2$ and vanishing at $t = 0$.

We now <u>fix</u> $\varepsilon > 0$ sufficiently small and we study $\mathcal{D}N$. Viewed as a map on $\mathcal{B}$, the Banach space of functions analytic in $|t| < 2$, (recall $t = |x|^{1+\varepsilon}$) the formula for $A = \mathcal{D}N$ at $f = f_\lambda$ is

$$(Ah)(t) = -h(y)/\lambda - f'(y)(1+\varepsilon)f^{\varepsilon}(z)h(z)/\lambda$$
$$+ h(1)(f(t) - tf'(t)(1+\varepsilon)/\lambda,$$

where $\lambda = -f(1)$, $z = \lambda^{1+\varepsilon}t$, $y = f(z)^{1+\varepsilon}$.

In order to study A we shall first analyze its first order approximation (in λ), A_0, which is given by

$$(A_0 h)(t) = -h(1-\lambda t)/\lambda + (1+\varepsilon)(1+\lambda)h(\lambda t)/\lambda$$
$$+ h(1)[1 + (1+\lambda)\varepsilon t]/\lambda.$$

<u>LEMMA</u> : <u>For small</u> $\lambda > 0$, <u>the operator</u> A_0 <u>is compact on the space</u> E. <u>Its spectrum is the closure of the set</u> $\{\delta_i, i = 1,2,..\}$, <u>where</u> $\delta_1 = 1 + (1+\lambda)(1+\varepsilon/\lambda+\varepsilon)$, $\delta_n = \lambda^{n-1}((-1)^{n-1} + (1+\lambda)(1+\varepsilon))$, $n = 2,3\ldots$. <u>The corresponding eigenfunctions are polynomials of increasing degree</u>. <u>In particular the eigenfunction corresponding to</u> δ_1 <u>is the function</u> t.

Since, in these lectures, we insist more about the kind of theorems which it is necessary to prove in a "renormalization group" situation, rather than to elaborate on the technical details, we again only give some clues concerning the proof of the lemma. The details can be found again in [CEL].

The operator A_o is a sum of three pieces, of which the third is manifestly a rank one operator. On the other hand, the maps $h(.) \to h(\lambda .)$ and $h(.) \to h(1 - \lambda .)$ are both compact, as can be seen by using Montel's theorem. A combinatorial argument shows that for every n, there is (up to a factor) exactly one polynomial of degree n in E which is an eigenvector of A_o.

The following calculation shows that A_o is uniformly bounded for small λ, i.e. the factor $^1/\lambda$ does not matter :

$$- h(1 - \lambda t)/\lambda + h(1)/\lambda + (1 + \lambda + \varepsilon)h(\lambda t)/\lambda$$

$$= -\frac{1}{\lambda}\int_{1-\lambda t}^{1} d\tau h'(\tau) + \frac{1 + \lambda + \varepsilon}{\lambda}\int_{o}^{\lambda t} d\tau h'(\tau) \quad (\text{since } h(0) = 0)$$

$$= (2\pi i\lambda)^{-1} \int_{1-\lambda t}^{1} d\tau \int_{\Gamma} dz_o\, h(z_o)\, (\tau - z_o)^{-2}$$

$$- (1 + \lambda + \varepsilon)\int_{o}^{\lambda t} d\tau \int_{\Gamma} dz_o\, h(z_o)(\tau - z_o)^{-2}$$

$$= (2\pi i\lambda)^{-1}\int_{\Gamma} dz_o h(z_o)$$

$$\left[\frac{1}{1 - \lambda t - z_o} - \frac{1}{1 - z_o} + (1 + \lambda + \varepsilon)\left[\frac{1}{\lambda t - z_o} - \frac{1}{z_o}\right]\right]$$

(using Fubini's theorem)

$$= (2\pi i)^{-1}\int_{\Gamma} dz_o h(z_o)\left[\frac{t}{(1 - z_o)(1 - \lambda t - z_o)} + \frac{t(1 + \varepsilon + \lambda)}{z_o(\lambda t - z_o)}\right] .$$

A similar calculation shows $\|A - A_o\| = O(\lambda)$ and from this we conclude that A is bounded, has one eigenvalue of modulus greater than 1 and the remainder of the spectrum in $|z| < O(\lambda)$. Let us call $\delta = \delta(\varepsilon)$ the largest eigenvalue. We have

$$\delta = 2 + (\varepsilon/\lambda) + O(\lambda),$$

by standard methods [K] and the corresponding eigenfunction e_1 satisfies

$$e_1(t) = t + O(\lambda), \qquad \text{in } E.$$

We now have, so to speak, an infinitesimal picture of what happens in a neighborhood of the fixed point. What we want to know next, is, whether to that infinitesimal picture there corresponds a global picture in a neighborhood of the fixed point. We use here the machinery from [HPS].

The following estimate is crucial in establishing the connection between the infinitesimal and the local picture. Note that we only get a local picture and not a global one at this stage, since N is not defined on functions f with $f(1) = 0$.

<u>LEMMA</u> : <u>Let</u> $\sigma < \lambda$. <u>For</u> $\|h_1\|_E, \|h_2\|_E, \|h_1 - h_2\|_E < \sigma$, <u>one has</u> $\|(N - A)(h_1) - (N - A)(h_2)\| = O(\sigma/\lambda)\,\|h_1 - h_2\|_E$.

The proof of this lemma is just done by brute force by writing the difference as an integral of the second derivative.

Let $(Th) = N(\phi + h) - \phi$, let E_s and E_u be the stable and unstable spectral subspaces of $DT = A$.

Define now $E(\sigma)$ to be the ball of radius σ in E, centered at zero, and let $E_u(\sigma) = E(\sigma)\ E_u$ and $E_s(\sigma) = E(\sigma)\ E_s$. By multiplying T by a cutoff function χ which is equal to one to $E(\sigma)$ we have thus

LEMMA : For a fixed ratio $\sigma/\lambda < 1$ the map $\chi T - \mathcal{D}\chi T_{g=0}$ is for sufficiently small $\lambda > 0$ a Lipshitz continuous map from E to itself with Lipshitz constant $\mathcal{O}(\sigma/\lambda)$. In addition, $\chi T - \mathcal{D}\chi T_{g=0} = \mathcal{O}(\sigma)$.

The preceding lemma allows us to apply Corollary 5.4 of [HPS] and we get the

THEOREM : The stable and unstable manifolds W_s, W_u through 0, tangent to E_s, E_u exist, are unique and are locally the graphs of two C^∞ functions $f_s : E_u(\sigma) \to E_s(\sigma)$, $f_u : E_s(\sigma) \to E_u(\sigma)$.

The next operation is to find a C^1 map U of E to E which diagonalizes $T : E \to E$, in the following sense. Write an arbitrary vector in E as $x \oplus y$ with x in the subspace of e_1, y in its spectral complement. Then we want to find U such that

$$UTU^{-1} \binom{x}{y} = \begin{bmatrix} \delta x \\ Ay + M(x,y) \end{bmatrix}$$

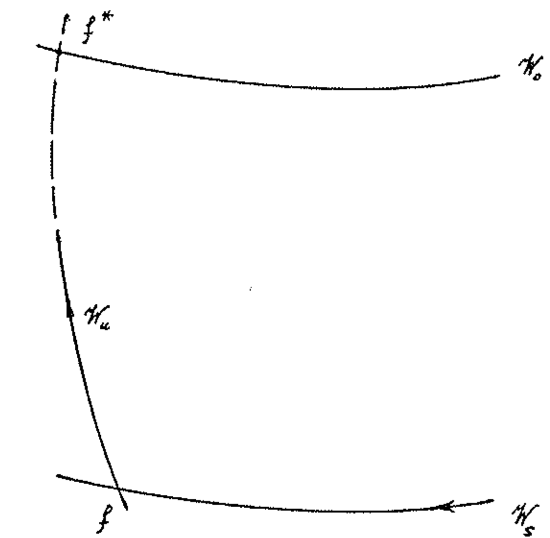
with $M(x,0) = 0$. Note that in general it is not possible to find a U which is C^1 and which would completely linearize T, since there are an infinity of eigenvalues of $DT = A$.

However, for the purpose of renormalization group theory, it is totally sufficient to diagonalize T in the unstable direction only. On the other hand it is crucial to have this diagonalization at least once differentiable, because we want to take the distance on the unstable manifold to be used as a coordinate, after linearization.

The proof of existence of U is through a sequence of transformations using the implicit function theorem; in fact U is near to the identity [CE pp 131 - 134].

We can now harvest the fruits of our efforts. The geometical situation is as depicted in the figure below. Note that we draw not E but $f_\varepsilon + E$, so that the relevant map is N. The

manifold W_o is defined to be those functions $f \in E + f_\varepsilon$ for which $f(1) = 0$ (and any way, $f(0) = 1$). That its position is as indicated in the figure follows at once from our knowledge of the approximate direction of W_u.



To this picture, we want to add one more curve, namely the objects which have motivated this study. This curve is obtained as follows.

1) Choose any point in $E + f_\varepsilon$ in a neighborhood (sufficiently small) of f_ε call it Ψ.

2) Define $\psi(x) = \Psi(|x|^{1+\varepsilon})$.

3) Define $\psi_\mu(x) = \psi(\mu x)$. It is for the family of functions $\mu \to \psi_\mu$ that we shall prove the universality properties.

4) We next analyze the behaviour of the curve $\mu \to \psi_\mu$ in a neighborhood of f_ε and in particular we claim it is transversal to W_u. To see this we just have to check its derivative with respect to μ. Since Ψ_μ is very near to $f_{\varepsilon,\mu}$ it obviously suffices to verify this property for the

curve $\mu \to f_{E,\mu}$, where

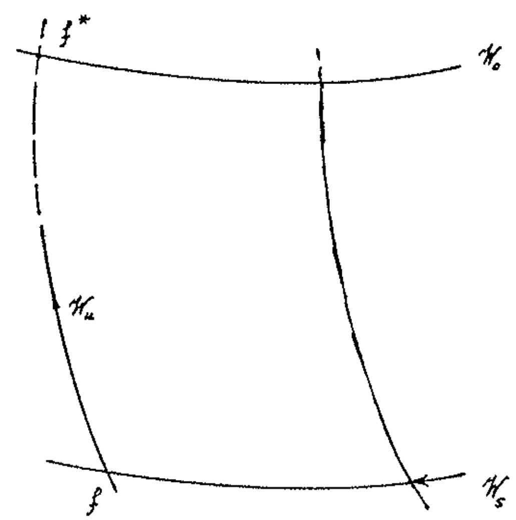
$$f_{\varepsilon,\mu}(t) = f_{\varepsilon}(\mu^{1 + \varepsilon}t).$$

Now $d_{\mu} f_{\varepsilon}(\lambda^{1 + \varepsilon}t)\big|_{\mu = 1} = f_{\varepsilon}'(t)(1 + \varepsilon)t$

$$= -(1 + \lambda)(1 + \varepsilon)t + O(\lambda^2)$$

and this is certainly not in the complement of E_u for sufficiently small ε. Hence we have transversality in this case and so the same by continuity if Ψ is near f_{ε}.

5) For simplicity of the argument, we assume now that the curve $\mu \to \Psi_{\mu}$ extends as much as shown in the next figure.



At the point μ_o, Ψ_{μ_o} lies in W_o and thus has the property $\Psi_{\mu_o}(0) = 1, \Psi_{\mu_o}(1) = 0$, i.e. Ψ_{μ_o}, and hence ψ_{μ_o} has a superstable period of length two.

6) By using the implicit function theorem, we can show that for every $n > n_o$, n_o sufficiently large, such that there is a unique μ_n such that

$$N^n(\psi_{\mu_n}) \in W_o .$$

This means

$$[N^n(\psi_{\mu_n})]^2(0) = 0,$$

i.e.

$$\prod_{j=1}^{n-1} \psi_{\mu_n}^{2^j}(1)^{-1} \psi_{\mu_n}^{[2^n]}\Big(x \prod_{j=1}^{n-1} \psi_{\mu_n}^{2^j}(1)\Big)\Big|_{x=0} = 0$$

so that

$$\psi_{\mu_n}^{2^n}(0) = 0;$$

ψ_{μ_n} is superstable of period 2^n !

Let $h_{n,\mu} = N^n(\psi_{\mu}) - f$

We define an approximate μ_n as follows. According to the transversality statement, and by virtue of the fact that W_u is tangent to $e_1(t) = t + O(\lambda)$, we can write, for some constant $a > 0$:

$$h_{o,\mu}(t) = (1 - \mu)\, a\, e_1(t) + r_{\mu}$$

where $r_{\mu} = O(1 - \mu)$ is in W_s. Having linearized T, we find

$$h_{n,\mu}(t) = \delta^n(1 - \mu)\, a\, e_1(t) + r_{n,\mu},$$

where $r_{n,\mu} = O((2\lambda)^n(1 - \mu))$. We now define $\tilde{\mu}_{n + 1} = -\lambda\delta^{-n}(1/a) + 1$, so that $h_{n,\mu}(1) + f_{\lambda}(1)$ is approximately zero. The value of μ_n can now be found in principle by solving a fixed point problem for a_n in $\mu_{n+1} = 1 - \lambda\delta^{-n}(1/a_n)$, where one will find $a_n \to a$ as $n \to \infty$, by continuity.

To analyze the second problem, which is the scaling of the solutions, it is more convenient to consider the functions

$$\psi_{m,\mu}(x) = f_{\lambda}(|x|^{1 + \varepsilon(\lambda)}) + h_{m,\mu}(|x|^{1 + \varepsilon(\lambda)}).$$

We intend to compare $\psi_{n,\mu_n}(x)$ to $\psi_{n-1,\mu_{n-1}}(x)$. By the definition of T, we find

$$\psi_{m,\mu}(x) = \frac{1}{\psi_{m-1,\mu}(1)}\, \psi^{(2)}_{m-1,\mu}(\psi_{m-1,\mu}(1)x),$$

and interating this,

$$\psi_{m,\mu}(x) = \prod_{j=0}^{m-1} \psi_{j,\mu}(1)^{-1}\psi^{(2^m)}_{o,\mu}\Big(x.\prod_{j=0}^{m-1}\psi_{j,\mu}(1)\Big).$$

By construction, the sequence $\psi_{n,\mu_n}(x)$ converges to $f^*(|x|^{1+(\lambda)})$, and we may therefore compare the relative total scales of ψ_{n,μ_n} and $\psi_{n-1,\mu_{n-1}}$. The scale

$\rho_m = \prod_{j=0}^{m-1}\psi_{j,\mu_m}(1)$ satisfies, due to the linearization,

$$\rho_m = (-\lambda)^m \prod_{j=0}^{m-1}\Big(1 - \delta^{j-m}\frac{a + o(j^{-1})}{a + o(n^{-1})} + (Q\lambda)^j\delta^{-m}O(1)\Big),$$

where we have used

$$\psi_{j,\mu_m}(1) = -\lambda(1 - \delta^{j-m}a_j/a_m + (Q\lambda)^{j-1}(1-\mu_m)O(1)).$$

Therefore, we find $\log|\rho_m|/m \to \log\lambda$ as $m \to \infty$.
We consider now,

$$\begin{aligned} x^1_n &= \psi^{(2^n-1)}(\mu_n 0) \\ &= \psi^{(2^n-1)}_{o,\mu_n}(0) \\ &= \Big[\prod_{j=0}^{n-2}\mu_{j,\mu_n}(0)\Big]\,\mu_{n-1,\mu_n}(0) \\ &= \rho_n, \end{aligned}$$

so for the problem scaled by $\psi(\mu j)$, the assertion is proved.

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