Martin Hairer got the Fields Medal for his study of the KPZ equation

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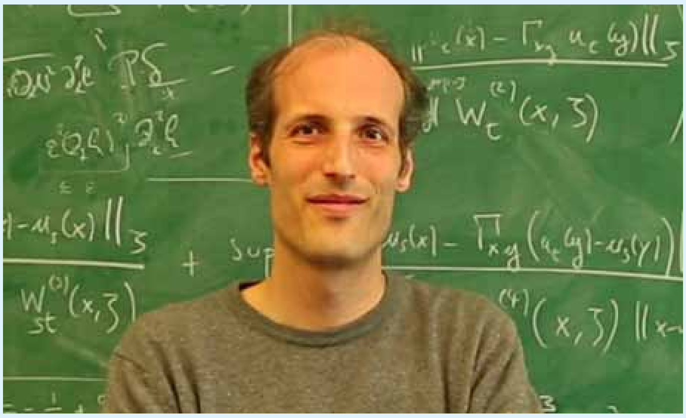
*"**Martin Hairer** is awarded a Fields Medal for his outstanding contributions to the theory of stochastic partial differential equations, and in particular for the creation of a theory of regularity structures for such equations."*

The statement by the International Mathematical Union (Seoul 2014):



A mathematical problem that is important throughout science is to understand the influence of noise on differential equations, and on the long time behavior of the solutions. This problem was solved for ordinary differential equations by Itō in the 1940s. For partial differential equations, a comprehensive theory has proved to be more elusive, and only particular cases (linear equations, tame nonlinearities, etc.) had been treated satisfactorily.

Hairers work addresses two central aspects of the theory. Together with Mattingly he employed the Malliavin calculus along with new methods to establish the ergodicity of the two-dimensional stochastic Navier-Stokes equation.

Building on the rough-path approach of Lyons for stochastic ordinary differential equations, Hairer then created an abstract theory of regularity structures for stochastic partial differential equations (SPDEs). This allows Taylor-like expansions around any point in space and time. The new theory allowed him to construct systematically solutions to singular non-linear SPDEs as fixed points of a renormalization procedure.

Hairer was thus able to give, for the first time, a rigorous intrinsic meaning to many SPDEs arising in physics.

The Swiss scientific community is very happy that Martin Hairer got the Fields Medal, the highest mathematical distinction at the International Congress of Mathematics 2014. Hairer's contact with Switzerland is is particularly strong, since he grew up in Geneva, where he visited all schools, and studied theoretical physics at the University of Geneva. He finished his PhD in December 2001, on "Comportement Asymptotique d'Équations à Dérivées Partielles Stochastiques."

This work deals with long time behavior of differential equations with some noise term ξ added. The novelty of the work leading to his PhD was to work in infinite dimension (for example the Ginzburg-Landau equation with periodic boundary conditions) for the function $u(x, t)$:

$$\dot{u} = \Delta u + u - u^3 + \xi\,, \qquad (0.1)$$

with $u(x, 0)$ given. Since the equation is on a finite domain with periodic boundary conditions, it can be rewritten in terms of Fourier coefficients $u_k(t)$:

$$\dot{u}_k = (1 - k^2)\,u_k - \sum_{k_1+k_2+k_3=k} u_{k_1} u_{k_2} u_{k_3} + \xi_k\,,$$

where now the terms ξ_k describe the stochastic driving. It was then shown that this system has a unique equilibrium state, *even if only the high frequency modes are forced*. So the "long wave- length," (small *k* modes) get to feel the noise only indirectly, through the nonlinear coupling in (0.1). To give a proof of this, a version of infinite dimensional Malliavin Calculus had to be developed, which is a method to show that, by taking sufficiently many commutators of the generator of the equation, all modes can be reached, [2].

After his PhD, Hairer went on to study the stochastically driven Navier-Stokes equation in 2 dimensions (again with periodic boundary conditions):

$$d_t u + (u \cdot \nabla)\,u = \nu \Delta u - \nabla p + \xi\,, \quad div\, u = 0\,, \qquad (0.2)$$

Together with Mattingly, he was able to show that it actually suffices to stochastically force only a *finite number of long range modes*, and still get a unique equilibrium measure [3]. This again needed the development of a new technique, called the *asymptotic strong Feller*[1] property. It means that the smoothing of the transition probabilities does not have to appear immediately, but only as time goes on, and thus the intrinsic smoothing properties of the system under study can be used.

The next achievement of Hairer was the study of the KPZ equation. This equation was discovered and discussed by Kardar, Parisi, and Zhang [1] (who is currently professor of theoretical physics in Fribourg, a second Swiss connection). It describes the dynamics of advancing fronts in 1-dimensional systems. For the discussion here, let me say that there are two archetypal such equations:

1 In [3] they describe this as follows: "In many dissipative systems, including the stochastic Navier-Stokes equations, only a finite number of modes are unstable. Conceivably, these systems are ergodic even if the noise is transmitted only to those unstable modes rather than to the whole system. The asymptotic strong Feller property captures this idea. It is sensitive to the regularization of the transition densities due to both probabilistic and dynamic mechanisms." Strong Feller means that the transition "matrix" P_t maps measurable function to continuous ones: $|\nabla P_t\varphi(x)| \le C(\|x\| \cdot \|\varphi\|_\infty)$ and asymptotically strong Feller requires less, namely only $|\nabla P_{t_n}\varphi(x)| \le C(\|x\|) \cdot (\|\varphi\|_\infty + \delta_n \|\nabla\varphi\|_\infty)$, with $\delta_n \to 0$ as $t_n \to \infty$ This captures *progressive smoothing*.

$$\partial_t h(t,x) = \partial_x^2 h(t,x) + \xi(t,x)\,, \quad EW \qquad (0.3)$$

and

$$\partial_t h(t,x) = (\partial_x h(t,x))^2 + \partial_x^2 h(t,x) + \xi(t,x)\,, \quad KPZ \qquad (0.4)$$

where the first equation is sometimes called the Edwards-Wilkinson equation and the second is the KPZ equation. They can be viewed as continuous versions of depositing of particles illustrated by the two drawings below.

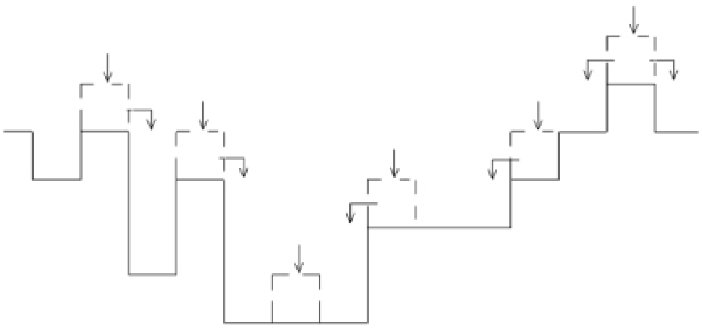


Figure 1: A "raining" mechanism which leads to the EW equation (0.3): Particles are rained randomly, but move over by 1 site if the valley there is deeper.

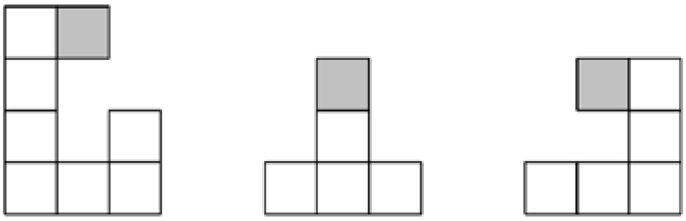


Figure 2: A "raining" mechanism which leads to the KPZ equation (0.4): Particles are rained randomly, but stick immediately to any wall.

Clearly, the first example is "smoother" than the second, because inequalities in height get softened.
In both cases, particles are rained randomly on the surface. The point here is that in the first case, if there is a deeper hole on the side, then the particle "corrects" its position to that lower site. In the second case, as illustrated, the particle sticks to the first neighbor it meets when it falls down. The two equations (0.3) and (0.4) are obtained by taking suitable limits as the particle sizes are scaled towards 0.
The paper [1] has, over the years, obtained over 3000 citations, so one might ask whether there is still anything unknown about it, and why the new result by Hairer is considered so good as to merit the most prestigious prize in mathematics.
Up to Hairer's work, most rigorous results on the KPZ equation were obtained by applying the so-called Cole-Hopf transformation to (0.4). It transforms the *non-linear* equation (0.4) by the substitution $h(x,t) = \log H(x,t)$ into the *linear* equation

$$\partial_t H = \partial_x^2 H + H\xi\,. \qquad (0.5)$$

Linear equations are much easier to analyze than non-linear ones, and, apart from the multiplicative noise term $H\xi$ it is really the same as (0.3). And thus, "everything" is known for (0.5), and beautiful connections have been discovered between this equation and many other problems, such as random matrix theory.
However, all this does not tell us that the KPZ equation itself is really well defined. Namely, the noise is non-smooth, and so even such things as the existence of the nonlinear gradient term $(h')^2$ are unclear. This is what Hairer was able to solve. Starting with a renormalization procedure[2] he first takes care of the ultraviolet divergencies of the problem[3]. The novel difficulty, which was brilliantly solved is the proof that the remainder is well-defined. If one naively tries to solve the equation for this remainder, one can show that no classical space of functions will do the job. But Hairer used [4] the theory of "rough paths" by Terry Lyons (a generalization of bounds on convolution integrals to fractional derivatives) to overcome these difficulties. With this method he was able to show [5]:

*With probability 1, the KPZ equation **has** a solution which is defined for some finite time, which can depend on the noise, and any (very general) initial condition.*

Combining this with the existence of (0.5) for all times, and using that very rough initial conditions are allowed in his theorem, one concludes (by restarting) that

*KPZ has solutions (with probability 1) for **all** times.*

The techniques he developed for this proof were then extended to a very general form of "Hairer-calculus" [6], which is called the theory of regularity structures. This allows him to treat, in the same vein, other singular evolution equations, such as the dynamical Φ_3^4 quantum field model (in finite volume).
My heartfelt congratulations to Martin! We are already looking forward to your coming successes.

References

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2 Using the Wild expansion, something like Feynman graphs

3 In quantum field theory jargon, the KPZ equation is superrenormalizable, needing only a finite number of counterterms