

CMB anisotropies from vector perturbations in the bulk

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The vector perturbations induced on the brane by gravitational waves propagating in the bulk are studied in a cosmological framework. Cosmic expansion arises from the brane motion in a non-compact Z_2 symmetric five-dimensional anti-de Sitter space-time. By solving the vector perturbation equations in the bulk, for generic initial conditions, we find that they give rise to growing modes on the brane in the Friedmann-Lemaître era. Among these modes, we exhibit a class of normalizable perturbations, which are exponentially growing with respect to conformal time on the brane. The presence of these modes is strongly constrained by the current observations of the cosmic microwave background (CMB). We estimate the anisotropies they induce in the CMB, and derive quantitative constraints on the allowed amplitude of their primordial spectrum.

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I. INTRODUCTION

The idea that our universe may have more than three spatial dimensions has been originally introduced by Nordström [1], Kaluza [2] and Klein [3]. The fact that super string theory, the most promising candidate for a theory of quantum gravity, is consistent only in ten space-time dimensions (11 dimensions for M-theory) has led to a revival of these ideas [4–6]. It has also been found that string theories naturally predict lower dimensional “branes” to which fermions and gauge particles are confined, while gravitons (and the dilaton) propagate in the bulk [7–9]. Such “braneworlds” have been studied in a phenomenological way already before the discovery that they are actually realized in string theory [10, 11].

Recently it has been emphasized that relatively large extra-dimensions (with typical length $L \simeq \mu\text{m}$) can “solve” the hierarchy problem: The effective four-dimensional Newton constant given by $G_4 \propto G/L^N$ can become very small even if the fundamental gravitational constant $G \simeq m_{\text{Pl}}^{-(2+N)}$ is of the order of the electro-weak scale. Here N denotes the number of extra-dimensions [12–15]. It has also been shown that extra-dimensions may even be infinite if the geometry contains a so-called “warp factor” [16].

The size of the extra-dimensions is constrained by the requirement of recovering usual four-dimensional Einstein gravity on the brane, at least on scales tested by experiments [17–19]. Models with either a small Planck mass in the bulk [12–14], or with non-compact warped extra-dimensions [15, 16], have been shown to lead to an acceptable cosmological phenomenology on the brane [20–26], with or without Z_2 symmetry in the bulk [27–29]. Explicit cosmological scenarii leading to a

nearly Friedmann-Lemaître universe at late time can be realized on a 3-brane at rest in a dynamical bulk [30, 31] or, alternatively, on a brane moving in an anti-de Sitter bulk [32, 33]. It has been shown that both approaches are actually equivalent [34].

One can also describe braneworlds as topological defects in the bulk [35–39]. This is equivalent to the geometrical approach in the gravity sector [40], while it admits an explicit mechanism to confine matter and gauge fields on the brane [40–51]. Depending on the underlying theory, the stability studies of these defects have shown that dynamical instabilities may appear on the brane when there are more than one non-compact extra-dimensions [52–54], whereas this is not the case for a five-dimensional bulk [55], provided that a fine-tuning between the model parameters is fixed [56].

The next step is now to derive observational consequences of braneworld cosmological models, *e.g.* the anisotropies of the cosmic microwave background (CMB). To that end, a lot of work has recently been invested to derive gauge invariant perturbation theory in braneworlds with one co-dimension [57–61]. Again, the perturbation equations can be derived when the brane is at rest [62], or when it is moving in a perturbed anti-de Sitter space-time [34, 63–67]. Whatever the approach chosen, the perturbation equations are quite cumbersome and it is difficult to extract interesting physical consequences analytically. Also the numerical treatment is much harder than in usual four-dimensional perturbation theory, since it involves partial differential equations.

Nevertheless, it is useful to derive some simple physical consequences of perturbation theory for brane worlds before performing intensive numerical studies. This has been done for tensor perturbations on the brane in a very phenomenological way in Ref. [68] or on a more fundamental level in Ref. [69]. Tensor modes in the bulk which induce scalar perturbations on the brane have been studied in Ref. [70] and it was found that they lead to important constraints for braneworlds.

In this article we consider a braneworld in a five-

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dimensional bulk where cosmology is induced by the motion of a “3-brane” in AdS_5 . The bulk perturbation equations are considered without bulk sources and describe gravity waves in the bulk. The present work concentrates on the part of these gravity waves which results in vector perturbations on the brane.

For the sake of clarity, we first recall how cosmology on the brane can be obtained via the junction conditions, particularly emphasizing how Z_2 symmetry is implemented [20–26]. After re-deriving the bulk perturbation equations for the vector components in terms of gauge invariant variables [34, 63–66], we analytically find the most general solutions for arbitrary initial conditions. The time evolution of the induced vector perturbations on the brane is then derived by means of the perturbed junction conditions. The main result of the paper is that vector perturbations in the bulk generically give rise to vector perturbations on the brane which grow either as a power law or even exponentially with respect to conformal time. This behavior essentially differs from the usual decay of vector modes in standard four-dimensional cosmology, and may lead to observable effects of extra-dimensions in the CMB.

The outline of the paper is as follow: in the next section, the cosmological braneworld model obtained by the moving brane in an anti-de Sitter bulk is briefly recalled. In Sect. III we set up the vector perturbation equations and solve them in the bulk. In Sect. IV the induced perturbations on the brane are derived and compared to those in four-dimensional cosmology, while Sect. V deals with the consequences of these new results on CMB anisotropies. The resulting new constraints for viable braneworlds are discussed in the conclusion.

II. BACKGROUND

As mentioned in the introduction, our universe is considered to be a 3-brane embedded in five-dimensional anti-de Sitter space-time

$$ds^2 = g_{AB} dx^A dx^B = \frac{r^2}{L^2} (-dt^2 + \delta_{ij} dx^i dx^j) + \frac{L^2}{r^2} dr^2. \quad (1)$$

The capital Latin indices A, B run from 0 to 4 and the flat spatial indexes i, j from 1 to 3. Anti-de Sitter space-time is a solution of Einstein’s equations with a negative cosmological constant Λ

$$G_{AB} + \Lambda g_{AB} = 0, \quad (2)$$

provided that the curvature radius L satisfies

$$L^2 = -\frac{6}{\Lambda}. \quad (3)$$

Another coordinate system for anti-de Sitter space can be defined by the coordinates transformation $r^2/L^2 = \exp(-2\varrho/L)$. Then, the metric takes the form

$$ds^2 = g_{AB} dx^A dx^B = e^{-2\varrho/L} (-dt^2 + \delta_{ij} dx^i dx^j) + d\varrho^2, \quad (4)$$

which is often used in braneworld models.

A. Embedding and motion of the brane

The position of the brane in the AdS_5 bulk is given by

$$x^M = X^M(y^\mu), \quad (5)$$

where X^M are embedding functions depending on the internal brane coordinates y^μ ($\mu = 0, \dots, 3$). Using the reparametrization invariance on the brane, we choose $x^i = X^i = y^i$. The other embedding functions are written

$$X^0 = t_b(\tau), \quad X^4 = r_b(\tau), \quad (6)$$

where $\tau \equiv y^0$ denotes cosmic time on the brane. Since we want to describe a homogeneous and isotropic brane, X^0 as well as X^4 are required to be independent of the spatial coordinates y^i . The four tangent vectors to the brane are given by

$$e_\mu^M \partial_M = \frac{\partial X^M}{\partial y^\mu} \partial_M, \quad (7)$$

and the unit space-like normal 1-form n_M is defined (up to a sign) by the orthogonality and normalization conditions

$$n_M e_\mu^M = 0, \quad g^{AB} n_A n_B = 1. \quad (8)$$

Adopting the sign convention that n points in the direction in which the brane is moving (growing r_b for an expanding universe), one finds using

$$e_\tau^0 = \dot{t}_b, \quad e_\tau^4 = \dot{r}_b, \quad e_j^i = \delta_j^i, \quad (9)$$

the components of the normal

$$n_0 = -\dot{r}_b, \quad n_4 = \dot{t}_b, \quad n_i = 0. \quad (10)$$

The other components are vanishing, and the dot denotes differentiation with respect to the brane time τ .

This embedding ensures that the induced metric on the brane describes a spatially flat homogeneous and isotropic universe,

$$ds_b^2 = q_{\mu\nu} dy^\mu dy^\nu = -d\tau^2 + a^2(\tau) \delta_{ij} dy^i dy^j, \quad (11)$$

where $a(\tau)$ is the usual scale factor, and $q_{\mu\nu}$ is the pull-back of the bulk metric onto the brane

$$q_{\mu\nu} = g_{AB} e_\mu^A e_\nu^B, \quad (12)$$

(see e.g. [71, 72]). The first fundamental form q_{AB} is now defined by

$$q^{AB} = q^{\mu\nu} e_\mu^A e_\nu^B, \quad (13)$$

i.e. the push-forward of the inverse of the induced metric tensor [71, 73]. One can also define an orthogonal projector onto the brane which can be expressed in terms of the normal 1-form

$$\perp_{AB} = n_A n_B = g_{AB} - q_{AB}, \quad (14)$$

in the case of only one codimension.

Upon inserting the equations (1), (10) and (13) into the above equation, one finds a parametric form for the brane trajectory [32, 33, 65, 66]

$$\begin{aligned} r_b(\tau) &= a(\tau)L, \\ \dot{t}_b(\tau) &= \frac{1}{a} \sqrt{1 + L^2 H^2}, \end{aligned} \quad (15)$$

where $H = \dot{a}/a$ denotes the Hubble parameter on the brane. Alternatively, this result can be obtained by comparing expression (12) with the Friedmann metric (11).

Therefore, the unperturbed motion induces a cosmological expansion on the 3-brane if r_b is growing with t_b .

B. Extrinsic curvature and unperturbed junction conditions

The cosmological evolution on the brane is found by the Lanczos–Sen–Darmois–Israel junction conditions¹. They relate the jump of the extrinsic curvature across the brane to its surface energy-momentum content [74–77]. The extrinsic curvature tensor projected on the brane can be expressed in terms of the tangent and normal vectors as

$$K_{\mu\nu} = -e_\mu^A e_\nu^B \nabla_A n_B = -\frac{1}{2} e_\mu^A e_\nu^B \mathcal{L}_n g_{AB}. \quad (16)$$

Here ∇ denotes the covariant derivative with respect to the bulk metric, and \mathcal{L}_n is the five-dimensional Lie-derivative in the direction of the unit normal on the brane. With the sign choice in Eq. (16), the junction conditions read [78]

$$K_{\mu\nu}^> - K_{\mu\nu}^< = \kappa_5^2 \left(S_{\mu\nu} - \frac{1}{3} S q_{\mu\nu} \right) \equiv \kappa_5^2 \hat{S}_{\mu\nu}, \quad (17)$$

where $S_{\mu\nu}$ is the energy momentum tensor on the brane with trace S , and

$$\kappa_5^2 \equiv 6\pi^2 G_5 = \frac{1}{M_5^3}, \quad (18)$$

where M_5 and G_5 are the five-dimensional (fundamental) Planck mass and Newton constant, respectively. The superscripts “>” and “<” stand for the bulk sides with

$r > r_b$ and $r < r_b$. As already noticed, the brane normal vector n^M points into the direction of increasing r [see Eq. (10)]. Eq. (17) is usually referred to as second junction condition. The first junction condition simply states that the first fundamental form (13) is continuous across the brane.

In general, there is a force acting on the brane which is due to its curvature in the higher dimensional geometry. It is given by the contraction of the brane energy momentum tensor with the average of the extrinsic curvature on both sides of the brane [28]

$$S^{\mu\nu} (K_{\mu\nu}^> + K_{\mu\nu}^<) = 2f. \quad (19)$$

This force f , normal to the brane, is exerted by the asymmetry of the bulk with respect to the brane [28, 71]. In this paper, we consider only the case in which the bulk is Z_2 symmetric across the brane, hence $f = 0$. In this case the motion of the brane is caused by the stress energy tensor of the brane itself which is exactly the cosmological situation we have in mind.

From Eqs. (10), (11), (15) and (16), noting that the extrinsic curvature can be expressed purely in terms of the internal brane coordinates [65, 66], one has

$$K_{\mu\nu} = -\frac{1}{2} [g_{AB} (e_\mu^A \partial_\nu n^B + e_\nu^A \partial_\mu n^B) + e_\mu^A e_\nu^B n^C g_{AB,C}]. \quad (20)$$

A short computation shows that the non-vanishing components of the extrinsic curvature tensor are

$$\begin{aligned} K_{\tau\tau} &= \frac{1 + L^2 H^2 + L^2 \dot{H}}{L \sqrt{1 + L^2 H^2}}, \\ K_{ij} &= -\frac{\dot{a}^2}{L} \sqrt{1 + L^2 H^2} \delta_{ij}. \end{aligned} \quad (21)$$

It is clear, that the extrinsic curvature evaluated at some brane position r_b does not jump if the presence of the brane does not modify anti-de Sitter space. Like in the Randall–Sundrum (RS) model [16], in order to accommodate cosmology, the bulk space-time structure is modified by gluing the mirror symmetric of anti-de Sitter space on one side of the brane onto the other [34]. There are two possibilities: one can keep the “ $r > r_b$ ” side and replace the “ $r < r_b$ ” side to get

$$K_{\mu\nu}^> = K_{\mu\nu}, \quad K_{\mu\nu}^< = -K_{\mu\nu}, \quad (22)$$

where $K_{\mu\nu}$ is given by Eq. (21). Conversely, keeping the $r < r_b$ side leads to

$$K_{\mu\nu}^> = -K_{\mu\nu}, \quad K_{\mu\nu}^< = K_{\mu\nu}. \quad (23)$$

Note that both cases verify the force equation (19). From the time and space components of the junction conditions (17) one obtains, respectively

$$\pm \frac{1 + L^2 H^2 + L^2 \dot{H}}{L \sqrt{1 + L^2 H^2}} = \frac{1}{2} \kappa_5^2 (P + \rho) - \frac{1}{6} \kappa_5^2 (\rho + \mathcal{T}), \quad (24)$$

$$\pm \frac{\sqrt{1 + L^2 H^2}}{L} = -\frac{1}{6} \kappa_5^2 (\rho + \mathcal{T}). \quad (25)$$

¹ In the following, they will be simply referred to as “junction conditions”.

Here the brane stress tensor is assumed to be that of a cosmological fluid plus a pure tension \mathcal{T} , i.e.

$$S_{\mu\nu} = (P + \rho) u_\mu u_\nu + P q_{\mu\nu} - \mathcal{T} q_{\mu\nu}, \quad (26)$$

ρ and P being the usual energy density and pressure on the brane, and u^μ the comoving four-velocity. The “ \pm ” signs in Eqs. (24) and (25) are obtained by keeping, respectively, the $r > r_b$, or $r < r_b$, side of the bulk. In order to allow for a positive total brane energy density, $\rho + \mathcal{T}$, we have to keep the $r < r_b$ side and glue it symmetrically on the $r > r_b$ one². In the trivial static ($H = 0$) case this construction reproduces the Randall Sundrum II [16] solution with warp factor $\exp(-|\varrho|/L)$, for $-\infty < \varrho < \infty$ if we choose $r_b = L = \text{constant}$. In our coordinates, we just have $0 < r \leq r_b$ on either side of the brane, and the bulk is now described by two copies of the “bulk behind the brane”. Even if r only takes values inside a finite interval, and even though the volume of the extra dimension,

$$V = 2 \int_0^{r_b} \sqrt{|g|} dr = \frac{r_b}{2} \left(\frac{r_b}{L} \right)^3, \quad (27)$$

is finite, the bulk is semi-compact and its spectrum of perturbation modes has no gap (like in the RS model).

From Eqs. (24) and (25), one can check that energy conservation on the brane is verified

$$\dot{\rho} + 3H(P + \rho) = 0. \quad (28)$$

Solving Eq. (25) for the Hubble parameter yields

$$H^2 = \frac{\kappa_5^4 \mathcal{T}}{18} \rho \left(1 + \frac{\rho}{2\mathcal{T}} \right) + \frac{\kappa_5^4}{36} \mathcal{T}^2 - \frac{1}{L^2}. \quad (29)$$

At “low energies”, $|\rho/\mathcal{T}| \ll 1$, the usual Friedmann equation is recovered provided the fine-tuning condition

$$\frac{\kappa_5^4}{36} \mathcal{T}^2 = \frac{1}{L^2}, \quad (30)$$

is satisfied. The four-dimensional Newton constant is then given by

$$\kappa_4^2 \equiv 8\pi G_4 = \frac{\kappa_5^4 \mathcal{T}}{6}, \quad (31)$$

Thus a positive tension is required to get a positive effective four-dimensional Newton constant. Note also that low energy means $\tau^2 \sim H^{-2} \gg L^2$. In the Friedmann-Lemaître era, the solution of Eq. (29) reads

$$\begin{aligned} H &\simeq H_0 \left(\frac{a}{a_0} \right)^{-3(1+w)/2}, \\ \dot{H} &\simeq -\frac{3}{2}(1+w)H_0^2 \left(\frac{a}{a_0} \right)^{-3(1+w)}, \end{aligned} \quad (32)$$

² Note that we obtain the same result as in Ref. [66]: a positive brane tension for an expanding universe is obtained by keeping the anti-de Sitter side which is “behind the expanding brane with respect to its motion”.

for a cosmological equation of state $P = w\rho$ with constant w . The parameters H_0 and a_0 refer, respectively, to the Hubble parameter and the scale factor today. For the matter era we have $w = 0$, and during the radiation era $w = 1/3$.

III. GAUGE INVARIANT PERTURBATION EQUATIONS IN THE BULK

A general perturbation in the bulk can be decomposed into “3-scalar”, “3-vector” and “3-tensor” parts which are irreducible components under the group of isometries (of the unperturbed space time) $SO(3) \times E_3$, the group of three dimensional rotations and translations. In this paper we restrict ourselves to 3-vector perturbations³ and consider an “empty bulk”, i.e. the case where there are no sources in the bulk except a negative cosmological constant. With respect to the bulk, and its four spatial dimensions, only bulk gravity waves are therefore considered since they are the only modes present when the energy momentum tensor is not perturbed. It is well known (see e.g. Ref. [62]) that gravity waves in 4 + 1 dimensions have five degrees of freedom which can be decomposed with respect to their spin in 3 + 1 dimensions into a spin 2 field, the ordinary graviton, a spin 1 field, often called the gravi-photon and into a spin 0 field, the gravi-scalar. In this work we study the evolution of the gravi-photon in the background described in the previous section.

After setting up our notations, we find the gauge invariant vector perturbation variables in the bulk and write down the perturbed Einstein equations. We derived analytic solutions for all vector modes in the bulk.

A. Bulk perturbation variables

Considering only vector perturbations in the bulk, the five dimensional perturbed metric can be parameterized as

$$\begin{aligned} d\tilde{s}^2 &= -\frac{r^2}{L^2} dt^2 + \frac{r^2}{L^2} (\delta_{ij} + \nabla_i E_j + \nabla_j E_i) dx^i dx^j \\ &+ \frac{L^2}{r^2} dr^2 + 2B_i \frac{r^2}{L^2} dt dx^i + 2C_i dx^i dr, \end{aligned} \quad (33)$$

where ∇_i denotes the connection in the three dimensional subspace of constant t and constant r . Assuming this space to be flat one has $\nabla_i = \partial_i$. The quantities E^i , B^i , and C^i are divergenceless vectors i.e. $\partial_i E^i = \partial_i B^i = \partial_i C^i = 0$.

As long as we want to solve for the vector perturbations in the bulk only, the presence of the brane is not yet

³ The prefix “3-” will be dropped in what follows, and the term “vector” will be always applied here for spin 1 with respect to the surfaces of constant t and r .

relevant. Later it will appear as a boundary condition for the bulk perturbations via the junction conditions as will be discussed in Sect. IV C.

Under a linearized vector type coordinate transformation in the bulk, $x^M \rightarrow x^M + \varepsilon^M$, with $\varepsilon_M = (0, \varepsilon_i, 0)$, the perturbation variables defined above transform as

$$\begin{aligned} E_i &\rightarrow E_i + \frac{L^2}{r^2} \varepsilon_i, \\ B_i &\rightarrow B_i + \frac{L^2}{r^2} \partial_t \varepsilon_i, \\ C_i &\rightarrow C_i + \partial_r \varepsilon_i - \frac{2}{r} \varepsilon_i. \end{aligned} \quad (34)$$

As expected for three divergenceless vector variables and one divergenceless vector type gauge transformation, there remain four degrees of freedom which are described by the two gauge invariant vectors

$$\Sigma_i = B_i - \partial_t E_i, \quad (35)$$

$$\Xi_i = C_i - \frac{r^2}{L^2} \partial_r E_i. \quad (36)$$

Note that in the gauge $E_i = 0$ these gauge-invariant variables simply become B_i and C_i respectively.

B. Bulk perturbation equations and solutions

A somewhat cumbersome derivation of the Einstein tensor from the metric (33) to first order in the perturbations leads to the following vector perturbation equations,

$$\partial_t \Sigma - \frac{L}{r} \partial_r \left(\frac{r^3}{L^3} \Xi \right) = 0, \quad (37)$$

$$\frac{r^4}{L^2} \partial_r^2 \Sigma + 5 \frac{r^3}{L^2} \partial_r \Sigma - L^2 \partial_t^2 \Sigma + L^2 \Delta \Sigma = 0, \quad (38)$$

$$\begin{aligned} \frac{r^4}{L^2} \partial_r^2 \left(\frac{r^3}{L^3} \Xi \right) - \frac{r^3}{L^2} \partial_r \left(\frac{r^3}{L^3} \Xi \right) - L^2 \partial_t^2 \left(\frac{r^3}{L^3} \Xi \right) \\ + L^2 \Delta \left(\frac{r^3}{L^3} \Xi \right) = 0, \end{aligned} \quad (39)$$

where Δ denotes the spatial Laplacian, i.e.

$$\Delta = \delta^{ij} \partial_i \partial_j, \quad (40)$$

and the spatial index on Σ and Ξ has been omitted. One can check that these equations are consistent, e.g. with the master function approach of Ref. [64].

A complete set of solutions for these equations can easily be found by Fourier transforming with respect to x^i , and making the separation ansatz:

$$\Sigma(t, r, \mathbf{k}) = \Sigma_{\text{T}}(t, \mathbf{k}) \Sigma_{\text{R}}(r, \mathbf{k}), \quad (41)$$

$$\Xi(t, r, \mathbf{k}) = \Xi_{\text{T}}(t, \mathbf{k}) \Xi_{\text{R}}(r, \mathbf{k}). \quad (42)$$

The most general solution is then a linear combination of such elementary modes. Eq. (38) splits into two ordinary differential equations for Σ_{T} and Σ_{R} ,

$$r^4 \frac{\partial_r^2 \Sigma_{\text{R}}}{\Sigma_{\text{R}}} + 5r^3 \frac{\partial_r \Sigma_{\text{R}}}{\Sigma_{\text{R}}} = \pm L^4 \Omega^2, \quad (43)$$

$$\frac{\partial_t^2 \Sigma_{\text{T}}}{\Sigma_{\text{T}}} + k^2 = \pm \Omega^2, \quad (44)$$

where k is the spatial wave number, and $\pm \Omega^2$ the separation constant having the dimension of an inverse length squared. The frequency Ω represents the rate of change of Σ_{R} at $r \sim L$, while the rate of change of Σ_{T} is $\sqrt{|\Omega^2 \mp k^2|}$. From the four-dimensional point of view, $-\Omega^2$ can also be interpreted as the mass m^2 of the mode so that $\pm \Omega^2 = -m^2$. The signs in Eqs. (43) and (44) come from the choice $\Omega^2 \geq 0$. Eq. (43) is a Bessel differential equation of order two for the “-” sign and a modified Bessel equation of order two for the “+” sign [79], while Eq. (44) exhibits oscillatory or exponential behavior in bulk time. From Eq. (39), similar equations are derived for $\Xi_{\text{T}}(t, \mathbf{k})$ and $\Xi_{\text{R}}(r, \mathbf{k})$. This time, the radial function is given by Bessel functions of order one. The constraint equation (37) ensures that the separation constant $\pm \Omega^2$ is the same for both vectors and it also determines their relative amplitude. The general solution of Eqs. (37) to (39) is a superposition of modes Ω, \mathbf{k} which are given by

$$\begin{aligned} \Sigma \propto & \begin{cases} \frac{L^2}{r^2} \text{K}_2 \left(\frac{L^2 \Omega}{r} \right) e^{\pm t \sqrt{\Omega^2 - k^2}} \\ \frac{L^2}{r^2} \text{I}_2 \left(\frac{L^2 \Omega}{r} \right) e^{\pm t \sqrt{\Omega^2 - k^2}} \\ \frac{L^2}{r^2} \text{J}_2 \left(\frac{L^2 \Omega}{r} \right) e^{\pm i t \sqrt{\Omega^2 + k^2}} \\ \frac{L^2}{r^2} \text{Y}_2 \left(\frac{L^2 \Omega}{r} \right) e^{\pm i t \sqrt{\Omega^2 + k^2}}, \end{cases} \quad (45) \\ \Xi \propto & \begin{cases} \pm \sqrt{1 - k^2/\Omega^2} \frac{L^2}{r^2} \text{K}_1 \left(\frac{L^2 \Omega}{r} \right) e^{\pm t \sqrt{\Omega^2 - k^2}} \\ \pm \sqrt{1 - k^2/\Omega^2} \frac{L^2}{r^2} \text{I}_1 \left(\frac{L^2 \Omega}{r} \right) e^{\pm t \sqrt{\Omega^2 - k^2}} \\ \pm \sqrt{1 + k^2/\Omega^2} \frac{L^2}{r^2} \text{J}_1 \left(\frac{L^2 \Omega}{r} \right) i e^{\pm i t \sqrt{\Omega^2 + k^2}} \\ \pm \sqrt{1 + k^2/\Omega^2} \frac{L^2}{r^2} \text{Y}_1 \left(\frac{L^2 \Omega}{r} \right) i e^{\pm i t \sqrt{\Omega^2 + k^2}}. \end{cases} \quad (46) \end{aligned}$$

Here K_p and I_p are the modified Bessel functions of order p while J_p and Y_p are the ordinary ones. The \pm signs in Eqs. (45) and (46) correspond to the two linearly independent solutions of Eq. (44), whereas the sign of the separation constant determines the kind of Bessel functions: $+\Omega^2$ (or $m^2 < 0$) for the modified Bessel function K and I ; $-\Omega^2$ (or $m^2 > 0$) for the ordinary Bessel func-

tions J and Y. In general, each of these modes⁴ can be multiplied by a proportionality coefficient which depends on the wave vector \mathbf{k} and Ω . Eq. (37) ensures that this coefficient is the same for Σ and Ξ . Furthermore, notice that if $\Omega^2 > k^2$ the K- and I-modes can have an exponentially growing behaviour, whereas for $\Omega^2 < k^2$ one sets $\sqrt{\Omega^2 - k^2} = i\sqrt{|\Omega^2 - k^2|}$ such that the modes become oscillatory. The J- and Y-modes are always oscillating.

For a given perturbation mode to be physically acceptable one has to require that, at some initial time t_i , the perturbations are small for all values $0 < r \leq r_b(t_i)$, compared to the background. To check that, we use the limiting forms of the Bessel functions [79]. For large arguments, the ordinary Bessel functions behave as

$$\begin{aligned} J_p(x) &\underset{x \rightarrow \infty}{\sim} \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{2}p - \frac{\pi}{4}\right), \\ Y_p(x) &\underset{x \rightarrow \infty}{\sim} \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{2}p - \frac{\pi}{4}\right), \end{aligned} \quad (47)$$

while the modified Bessel functions grow or decrease exponentially

$$\begin{aligned} K_p(x) &\underset{x \rightarrow \infty}{\sim} \sqrt{\frac{\pi}{2x}} e^{-x}, \\ I_p(x) &\underset{x \rightarrow \infty}{\sim} \frac{1}{\sqrt{2\pi x}} e^x. \end{aligned} \quad (48)$$

Therefore, in Eqs. (45) and (46) all modes, except for the K-mode, diverge as $r \rightarrow 0$. Hence the only regular modes are

$$\begin{aligned} \Sigma &= A(\mathbf{k}, \Omega) \frac{L^2}{r^2} K_2\left(\frac{L^2}{r} \Omega\right) e^{\pm t \sqrt{\Omega^2 - k^2}}, \\ \Xi &= \pm A(\mathbf{k}, \Omega) \sqrt{1 - \frac{k^2}{\Omega^2}} \frac{L^2}{r^2} K_1\left(\frac{L^2}{r} \Omega\right) e^{\pm t \sqrt{\Omega^2 - k^2}}, \end{aligned} \quad (49)$$

$$(50)$$

where the amplitude $A(\mathbf{k}, \Omega)$ is determined by the initial conditions and carries an implicit spatial index. For small wave numbers $k^2 < \Omega^2$ the growing solution rapidly dominates, whereas for large wave numbers $k^2 > \Omega^2$ both solutions are comparable and oscillating in time. It is easy to see that the K-mode is also normalizable in the sense that

$$\begin{aligned} \int_0^{r_b} \sqrt{|g|} |\Sigma|^2 dr &\propto \int_0^{r_b} \frac{1}{r} \left[K_2\left(\frac{L^2}{r} \Omega\right) \right]^2 dr < \infty, \\ \int_0^{r_b} \sqrt{|g|} |\Xi|^2 dr &\propto \int_0^{r_b} \frac{1}{r} \left[K_1\left(\frac{L^2}{r} \Omega\right) \right]^2 dr < \infty. \end{aligned} \quad (51)$$

Note that also the J-modes and Y-modes are normalizable. One might view this integrability condition as

a requirement to insure finiteness of the energy of these modes. This suggests that the J- and Y-modes could also be excited by some physical process. Indeed, from Eq. (4), their divergence for $r \rightarrow 0$ can be recast in terms of the ϱ coordinate, with $\varrho \rightarrow \infty$. Expressed in terms of ϱ , the integrability condition (51) ensures that the J- and Y-modes are well defined in the Dirac sense, and thus that a superposition of them may represent physical perturbations⁵ [80].

Let us briefly discuss also the zero-mode $\Omega = 0$. The solution of Eqs. (37), (38) and (39) are then

$$\Sigma = A e^{\pm i k t} \quad (52)$$

$$\Xi = \pm i k L e^{\pm i k t} \left[\frac{A L}{2 r} + B \left(\frac{L}{r} \right)^3 \right] \quad (53)$$

These solutions (which can also be obtained from Eqs. (45) and (46) in the limit $\Omega \rightarrow 0$) diverge for $r \rightarrow 0$ but the A-mode is normalizable in the sense that the integrals defined in Eq. (51) converge.

Clearly the most intriguing solutions are the K-modes, especially for values of the separation constant verifying $\Omega^2 > k^2$. Then, if present, these fluctuations soon dominate the others in the bulk. The fact that $m^2 = -\Omega^2 < 0$ in this case implies that the K-modes are tachyonic modes, and it is thus not surprising that they may generate instabilities.

Before we go on, let us just note, that all these solutions are also valid solutions of the bulk vector perturbation equations in the RS model. In their original work [16], Randall and Sundrum have obtained very similar equations (we used somewhat different variables). However, they considered only the solutions with $m^2 = +\Omega^2 > 0$ and therefore did not find the growing K-modes. As we shall see in the next section, this choice is justified when one considers boundary conditions which do not allow for any anisotropic stresses on the brane. This is indeed well motivated as far as cosmology is not concerned. A more detailed discussion of the relevance of these modes for the RS model is given in Appendix A.

In our cosmological framework however, if there is no physical argument which forbids these modes, they have to be taken seriously since they represent solutions of the perturbations equations which are small at very early times and grow exponentially with respect to bulk time. Note that this instability is linked to the particular bulk structure considered here where the brane lies at one boundary of the space-time. In the full AdS_5 , $0 < r < \infty$, the K-modes are clearly not normalizable since $K_p(L^2\Omega/r)$ diverges for $r \rightarrow \infty$.

At last, one may hope that the K-modes are never generated. However, during any bulk inflationary phase which leads to the production of 4 + 1 dimensional gravity waves, as we shall see now, if the anisotropic stresses

⁴ In the following, they will be labeled by the kind of Bessel function they involve, e.g. ‘‘K-mode’’, ‘‘I-mode’’ etc.

⁵ This is of course not the case for the I-modes.

on the brane do not vanish identically the K-modes are perfectly admissible solutions. For a given inflationary model, it should be also possible to calculate the spectrum of fluctuations, $|A(\mathbf{k}, \Omega)|^2$.

At this stage, the perturbation modes have been only derived in the bulk. In the next section, we shall determine the induced perturbations on the brane using the perturbed junction conditions.

IV. THE INDUCED PERTURBATIONS ON THE BRANE

A. Brane perturbation variables

Since we are interested in vector perturbations on the brane induced by those in the bulk, we parameterize the perturbed induced metric as

$$\begin{aligned} d\tilde{s}_b^2 &= \tilde{q}_{\mu\nu} dy^\mu dy^\nu \\ &= -d\tau^2 + 2ab_i d\tau dy^i \\ &\quad + a^2 (\delta_{ij} + \nabla_i e_j + \nabla_j e_i) dy^i dy^j, \end{aligned} \quad (54)$$

where e^i and b^i are divergence free vectors. The junction conditions which relate the bulk perturbation variables to the perturbations of the brane can be written in terms of gauge invariant variables. Under an infinitesimal transformation $y^\mu \rightarrow y^\mu + \xi^\mu$, where $\xi_\mu = (0, a^2 \xi_i)$, we have

$$\begin{aligned} e_i &\rightarrow e_i + \xi_i, \\ b_i &\rightarrow b_i + a\dot{\xi}_i. \end{aligned} \quad (55)$$

Here the dot is the derivative with respect to the brane time τ and ξ_i is a divergence free vector field. Hence the gauge invariant vector perturbation is [81]

$$\sigma_i = b_i - a\dot{e}_i. \quad (56)$$

This variable fully describes the vector metric perturbations on the brane.

The brane energy momentum tensor $S_{\mu\nu}$ given in Eq. (26) has also to be perturbed. As we shall see, the junction conditions (together with Z_2 symmetry) do in general require a perturbed energy momentum tensor on the brane. Since we only consider vector perturbations $\delta\rho = \delta P = 0$. However, the perturbed four-velocity of the perfect fluid does contain a vector part $\tilde{u}^\mu = u^\mu + \delta u^\mu$, with

$$\delta u^\mu = \begin{pmatrix} 0 \\ v^i \\ a \end{pmatrix} \quad (57)$$

and where v^i is divergenceless. Under $y^\mu \rightarrow y^\mu + \xi^\mu$,

$$v_i \rightarrow v_i - a\dot{\xi}_i, \quad (58)$$

where $v_i \equiv \delta_{ij} v^j$. A gauge invariant perturbed velocity can therefore be defined as

$$\vartheta_i = v_i + a\dot{e}_i. \quad (59)$$

In addition, the anisotropic stresses contain a vector component denoted π_i . Since the corresponding background quantity vanishes, this variable is gauge invariant according to the Stewart–Walker lemma [82].

In summary, there are three gauge invariant brane perturbation variables. We shall use the combinations

$$\begin{aligned} \sigma_i &= b_i - a\dot{e}_i, \\ \vartheta_i &= v_i + a\dot{e}_i, \\ \pi_i &. \end{aligned} \quad (60)$$

To apply the junction conditions we need to determine the perturbations of the reduced energy momentum tensor defined in Eq. (17). In terms of our gauge invariant quantities they read

$$\delta\hat{S}_{\tau\tau} = 0, \quad (61)$$

$$\delta\hat{S}_{\tau i} = -a \left(P + \frac{2}{3}\rho - \frac{1}{3}\mathcal{T} \right) \sigma_i - a(P + \rho) \vartheta_i, \quad (62)$$

$$\delta\hat{S}_{ij} = a^2 P (\partial_i \pi_j + \partial_j \pi_i). \quad (63)$$

B. Perturbed induced metric and extrinsic curvature

We now express the perturbed induced metric, and the perturbed extrinsic curvature in terms of the bulk perturbation variables [66]. In principle there are two contributions to the brane perturbations: perturbations of the bulk geometry as well as perturbations of the brane position. A bulk perturbed quantity has then to be evaluated at the perturbed brane position [see Eq. (6)]. Using reparametrization invariance on the brane [66], the perturbed embedding can be described in terms of a single variable Υ ,

$$\tilde{X}^M = X^M + \Upsilon n^M, \quad (64)$$

where all quantities are functions of the brane coordinates y^μ . Since Υ is a scalar perturbation it does not play a role in our treatment, and we can consider only the perturbations δg_{AB} due to the perturbed bulk geometry evaluated at the unperturbed brane position. The induced metric perturbation is then given by

$$\delta q_{\mu\nu} = \tilde{q}_{\mu\nu} - q_{\mu\nu} = e_\mu^A e_\nu^B \delta g_{AB}. \quad (65)$$

From Eqs. (33), (35), (36) and (65) one finds in the gauge $E_i = 0$

$$\begin{aligned} \delta q_{\tau\tau} &= 0, \\ \delta q_{\tau i} &= a\sqrt{1 + L^2 H^2} \Sigma_i + aLH\Xi_i, \\ \delta q_{ij} &= \delta g_{ij} = 0. \end{aligned} \quad (66)$$

The time component vanishes as it is a pure scalar, and the purely spatial components can be set to zero without loss of generality by gauge fixing ($E_i = e_i = 0$).

In the same way, perturbing Eq. (20), and making use of Eqs. (7), (8) in order to derive the perturbed normal vector, leads to (again we use the gauge $E_i = 0$)

$$\delta K_{\tau\tau} = 0 \quad (67)$$

$$\delta K_{\tau i} = \frac{1}{2}\partial_t \Xi_i - \frac{1}{2}a^2\partial_r \Sigma_i - aH\sqrt{1+L^2H^2}\Xi_i - \frac{a}{L}(1+L^2H^2)\Sigma_i, \quad (68)$$

$$\delta K_{ij} = \frac{1}{2}aLH(\partial_i \Sigma_j + \partial_j \Sigma_i) + \frac{1}{2}a\sqrt{1+L^2H^2}(\partial_i \Xi_j + \partial_j \Xi_i), \quad (69)$$

where $\delta K_{\mu\nu} = \tilde{K}_{\mu\nu} - K_{\mu\nu}$, and all bulk quantities have to be evaluated at the brane position. In the derivation we have also used that on the brane $\partial_\mu = e_\mu^A \partial_A$.

C. Perturbed junction conditions and solutions

The first junction condition requires the first fundamental form q_{AB} to be continuous across the brane. Therefore, the components of the induced metric (54) are given by the explicit expressions (66). This leads to the following relations

$$\begin{aligned} e_i &= E_i, \\ b_i &= \sqrt{1+L^2H^2}B_i + LHC_i, \end{aligned} \quad (70)$$

where the bulk quantities have to be evaluated at the brane position (t_b, r_b). For $\sigma_i = b_i - a\dot{e}_i$ we use

$$\begin{aligned} a\dot{e}_i &= a(\dot{t}_b\partial_t E_i + r_b\partial_r E_i) \\ &= \sqrt{1+L^2H^2}\partial_t E_i + a^2LH\partial_r E_i. \end{aligned} \quad (71)$$

Together with Eqs. (35) and (36) this gives

$$\sigma_i = \sqrt{1+L^2H^2}\Sigma_i + LH\Xi_i. \quad (72)$$

The equations corresponding to the second junction condition are obtained by perturbing Eq. (17) (using Z_2 symmetry, $K_{\mu\nu}^> = -K_{\mu\nu}^< = -K_{\mu\nu}$) and inserting the expressions (68), (69) for the perturbed extrinsic curvature tensor, with Eqs. (62), (63) for the perturbed energy-momentum tensor on the brane. After some algebra one obtains for the $(0i)$ and the (ij) components, respectively

$$\frac{2L\dot{H}}{\sqrt{1+L^2H^2}}a(\sigma_i + \vartheta_i) = a^2\partial_r \Sigma_i - \partial_t \Xi_i, \quad (73)$$

$$\kappa_s^2 a P \pi_i = -LH\Sigma_i - \sqrt{1+L^2H^2}\Xi_i, \quad (74)$$

where we have used the unperturbed junction conditions, Eqs. (24) and (25), and the fact that on the brane $\partial_\mu = e_\mu^A \partial_A$.

In the RS model one has $H = \dot{H} = 0$ and the requirement that the anisotropic stresses vanish identically.

We show in Appendix A that the well-known results of Refs. [16] and [80] are recovered in this limit.

Hence, if by some mechanism, like e.g. bulk inflation, gravity waves are produced in the bulk, their vector parts Σ_i and Ξ_i will induce vector metric perturbations σ_i on the brane according to Eq. (72). The vorticity $\sigma_i + \vartheta_i$ and anisotropic stresses π_i on the brane define boundary conditions for the bulk variables according to Eqs. (73) and (74). In general, the time evolution of π_i may be given by some additional matter equation, like e.g. the Boltzmann equation or some dissipation equation which usually depends also on the metric perturbations. It is interesting to note that for generic initial conditions in the bulk, the amplitude of the K-mode does not vanish, which means that the anisotropic stresses on the brane may grow exponentially. At late time (for $LH \ll 1$) Eq. (74) reduces to $\kappa_s^2 a P \pi_i = -\Xi_i$. A generally covariant equation of motion for π_i must be compatible with this behavior since it is a simple consequence of the 5-dimensional Einstein equations for a certain choice of initial conditions.

In the following we do not want to specify a particular mechanism which generates Σ_i and Ξ_i , and just assume they have been produced with some spectrum given by $A(\mathbf{k}, \Omega)$. In usual 4-dimensional cosmology it is well-known that vector perturbations decay. Therefore, in ordinary 4-dimensional inflationary scenarios they are not considered. Only if they are continuously re-generated like, e.g. in models with topological defects (see e.g. Ref. [83]), vector modes affect CMB anisotropies. Here the situation is different since the modes considered are either exponentially growing or oscillating with respect to bulk time. Therefore, we expect the behavior of vector perturbations to be very different from the usual 4-dimensional results even in the absence of K-modes.

In the following, we assume that the boundary and initial conditions are such the $\pi_i(t_i) \neq 0$. They therefore allow for K-mode contributions. Clearly, if this happens it leads to exponential growth of σ_i and π_i . However, before concluding about the viability of these modes, one has to check if they have observable consequences on the brane. Indeed, anisotropic stresses are often very small (e.g. of second order only) and one may therefore hope that the initial amplitudes of the K-modes are also very small and do not lead to destructive effects, at least on time scales equal to the age of the universe. By estimating the induced CMB anisotropies, we show in the next section that this is not the case.

V. CMB ANISOTROPIES

To calculate the CMB anisotropies from the vector perturbations induced by bulk gravity waves, the relevant quantities are σ and $\vartheta + \sigma$ given in terms of the bulk variables by Eqs. (72) and (73). Inserting the solutions (49) and (50) for the K-mode into (72) and (73) yields

$$\sigma(t_b, \mathbf{k}) = A(\mathbf{k}, \Omega) \left[\sqrt{1 + L^2 H^2} \frac{1}{a^2} K_2\left(\frac{L\Omega}{a}\right) \pm LH \sqrt{1 - \frac{k^2}{\Omega^2}} \frac{1}{a^2} K_1\left(\frac{L\Omega}{a}\right) \right] e^{\pm t_b \sqrt{\Omega^2 - k^2}}, \quad (75)$$

$$(\sigma + \vartheta)(t_b, \mathbf{k}) = A(\mathbf{k}, \Omega) \frac{k^2}{\Omega^2} \frac{\sqrt{1 + L^2 H^2}}{2L^2 \dot{H}} \frac{L\Omega}{a} \frac{1}{a^2} K_1\left(\frac{L\Omega}{a}\right) e^{\pm t_b \sqrt{\Omega^2 - k^2}}, \quad (76)$$

where again we have omitted the spatial index i on σ , ϑ and A . Similar equations can be obtained for the J- and Y-modes by replacing, in Eqs. (75) and (76), the modified Bessel functions by the ordinary ones, plus the transformations: $-k^2 \rightarrow k^2$ and $\pm \rightarrow \pm i$.

These equations are still written in bulk time t_b which is related to the conformal time η on the brane by

$$dt_b = \sqrt{1 + L^2 H^2} d\eta. \quad (77)$$

Therefore, at sufficiently late time $L^2 H^2 \ll 1$ such that $dt_b \simeq d\eta$. Note that L is the size of the extra-dimension which must be smaller than micrometers while H^{-1} is the Hubble scale which is larger than 10^5 light years at times later than recombination which are of interest for CMB anisotropies.

As a result, the growing or oscillating behavior in bulk time carries over to conformal time. Moreover there are additional time dependent terms in Eqs. (75) and (76) with respect to Eqs. (49) and (50) due to the motion of the brane. As can be seen from Eqs. (75) and (76), the modes evolve quite differently for different values of their physical bulk wave number Ω/a . In the limit $\Omega/a \ll 1/L$ and for $\Omega^2 > k^2$, the growing K-modes behave like

$$\begin{aligned} \sigma &\sim \frac{2A}{(\Omega L)^2} e^{\eta \sqrt{\Omega^2 - k^2}}, \\ \sigma + \vartheta &\sim \frac{A}{(\Omega L)^2} \frac{k^2}{2a^2 \dot{H}} e^{\eta \sqrt{\Omega^2 - k^2}}, \end{aligned} \quad (78)$$

where use has been made of $L^2 H^2 \ll 1$, and of the limiting forms of Bessel function for small arguments [79]

$$K_p(x) \underset{x \rightarrow 0}{\sim} \frac{1}{2} \Gamma(p) \left(\frac{2}{x}\right)^p. \quad (79)$$

In the same way, from Eq. (48), the K-modes verifying $\Omega/a \gg 1/L$ reduce to

$$\begin{aligned} \sigma &\sim \frac{A}{(\Omega L)^2} e^{\eta \sqrt{\Omega^2 - k^2}} \sqrt{\frac{\pi}{2}} \left(\frac{\Omega L}{a}\right)^{3/2} e^{-\Omega L/a}, \\ \sigma + \vartheta &\sim \frac{A}{(\Omega L)^2} \frac{k^2}{2a^2 \dot{H}} e^{\eta \sqrt{\Omega^2 - k^2}} \sqrt{\frac{\pi}{2}} \left(\frac{\Omega L}{a}\right)^{1/2} e^{-\Omega L/a}. \end{aligned} \quad (80)$$

They are exponentially damped compared to the former [see Eq. (78)]. As a result, the main contribution of the K-mode vector perturbations comes from the modes with a physical wave number Ω/a smaller than the energy

scale $1/L$ associated with the extra-dimension. As the universe expands, a mode with fixed value Ω remains relatively small as long as the exponents in Eq. (80) satisfy

$$\frac{\Omega}{a} L - \eta \sqrt{\Omega^2 - k^2} \simeq \frac{1}{a} \left(\Omega L - \tau \sqrt{\Omega^2 - k^2} \right) > 0. \quad (81)$$

When this inequality is violated, for $k \ll \Omega$ this is soon after $\tau \sim L$, the mode starts growing exponentially. The time $\tau \sim L$ also corresponds to the initial time at which the evolution of the universe starts to become Friedmannian.

In the same way, one can derive the behavior of the J- and Y-modes on the brane for physical bulk wave numbers greater or smaller than the size of the extra-dimension. This time, the exponentially growing terms are replaced by oscillatory ones, and the ordinary Bessel functions are approximated by (see Eq. (47) and Ref. [79])

$$\begin{aligned} J_p(x) &\underset{x \rightarrow 0}{\sim} \frac{1}{\Gamma(p+1)} \left(\frac{x}{2}\right)^p, \\ Y_p(x) &\underset{x \rightarrow 0}{\sim} -\frac{1}{\pi} \Gamma(p) \left(\frac{2}{x}\right)^p. \end{aligned} \quad (82)$$

From Eqs. (47) and (82), the equivalents of Eqs. (75) and (76) for J- and Y-modes can be shown to oscillate always. From Eq. (32), their amplitude is found to decay like $a^{-3/2}$ in the short wavelength limit $\Omega/a \gg 1/L$. In the long wavelength limit $\Omega/a \ll 1/L$, the amplitude of the Y-mode stays constant whereas the J-mode decreases as a^{-4} .

The vorticity is also found to oscillate in conformal time. This time, the amplitude of the long wavelength Y-modes always grows as a^{3w+1} while for the J-modes it behaves like a^{3w-1} . Finally, in the short wavelength limit, both Y and J vorticity modes grow like $a^{3w+1/2}$.

Whatever the kind of physical vector perturbation modes excited in the bulk, we have shown that there always exist bulk wave numbers Ω that give rise to growing vector perturbations on the brane. Although the J- and Y-modes generate vorticity growing like a power law of the scale factor, they can be, in a first approximation, neglected compared to the K-modes which grows like an exponential of the conformal time. We therefore now concentrate on the K-modes and derive constraints on their initial amplitude $A(\mathbf{k}, \Omega)$ by estimating the CMB anisotropies they induce.

In order to determine the temperature fluctuations in the CMB due to vector perturbations on the brane,

we have to calculate how a photon emitted on the last scattering surface travels through the perturbed geometry (54). A receiver today therefore measures different microwave background temperatures $T_{\text{R}}(n^i)$ for incident photons coming from different directions n^i . In terms of conformal time the vector-type temperature fluctuations are given by [81]

$$\begin{aligned} \frac{\delta T_{\text{R}}(n^i)}{T_{\text{R}}} &= n^i (\sigma_i + \vartheta_i)|_{\text{E}}^{\text{R}} + \int_{\text{E}}^{\text{R}} \frac{\partial \sigma_i}{\partial x^j} n^i n^j d\lambda, \\ &= -n^i \vartheta_i(\eta_{\text{E}}) + \int_{\text{E}}^{\text{R}} \sigma'_i n^i d\eta \end{aligned} \quad (83)$$

where λ denotes the affine parameter along the photon trajectory and the prime is a derivative with respect to conformal time η . The “R” and “E” index refer to the time of photon reception (today) and emission (recombination). For the second equality we have used

$$\frac{d\sigma_i}{d\lambda} = \sigma'_i - n^j \frac{\partial \sigma_i}{\partial x^j}, \quad (84)$$

where $-n^i$ is the direction of the photon momentum. We have also neglected the contribution from the upper boundary, “R”, in the first term since it simply gives rise to a dipole term. The first term in Eq. (83) is a Doppler shift, and the second is known as integrated Sachs-Wolfe effect. To determine the angular CMB perturbation spectrum C_ℓ , we apply the total angular momentum formalism developed by Hu and White [84]. According to this, a vector perturbation \mathbf{v} is decomposed as

$$\mathbf{v} = \mathbf{e}^+ v^+ + \mathbf{e}^- v^-, \quad (85)$$

where

$$\mathbf{e}^\pm = \frac{-i}{\sqrt{2}} \left(\mathbf{e}^{(1)} \pm i \mathbf{e}^{(2)} \right), \quad (86)$$

and $\mathbf{e}^{(1,2)}$ are defined so that $(\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \hat{\mathbf{k}} = \mathbf{k}/k)$ form a righthanded orthonormal system. Using this decomposition for ϑ_i and σ_i , one obtains the angular CMB perturbation spectrum C_ℓ via

$$C_\ell = \frac{2}{\pi} \ell(\ell+1) \int_0^\infty k^2 \langle |\Delta_\ell(k)|^2 \rangle dk \quad (87)$$

where

$$\begin{aligned} \Delta_\ell(k) &= -\vartheta^+(\eta_{\text{E}}, k) \frac{j_\ell(k\eta_0 - k\eta_{\text{E}})}{k\eta_0 - k\eta_{\text{E}}} \\ &+ \int_{\eta_{\text{E}}}^{\eta_0} \sigma^{+\prime}(\eta, k) \frac{j_\ell(k\eta_0 - k\eta)}{k\eta_0 - k\eta} d\eta. \end{aligned} \quad (88)$$

In Eq. (88) we have assumed that the process which generates the fluctuations has no preferred handedness so that $\langle |\sigma^+|^2 \rangle = \langle |\sigma^-|^2 \rangle$ as well as $\langle |\vartheta^+|^2 \rangle = \langle |\vartheta^-|^2 \rangle$. Omitting the “ \pm ” superscripts, we can take into account the negative helicity mode simply by a factor 2.

As shown in the previous section, the main contribution of the K-modes comes from those having long wavelengths $a/\Omega \gg L$, and $k < \Omega$. In the following, only these modes will be considered. Since they are growing exponentially in η , the integrated Sachs-Wolfe contribution will dominate and we concentrate on it in what follows. A more rigorous justification is given in Appendix B. Inserting the limiting form (78) for σ in Eq. (88) gives

$$\begin{aligned} \Delta_\ell(\bar{k}) &\simeq 2A_0 \Omega^n \bar{k}^n e^{\bar{\eta}_0} \sqrt{1-\bar{k}^2} \sqrt{\frac{1}{\bar{k}^2} - 1} \\ &\times \int_0^{x_{\text{E}}} \frac{j_\ell(x)}{x} e^{-x\sqrt{1/\bar{k}^2-1}} dx, \end{aligned} \quad (89)$$

where a simple power law ansatz has been chosen for the primordial amplitude

$$\sqrt{\langle |A(\mathbf{k}, \Omega)|^2 \rangle} = A_0(\Omega) \Omega^2 L^2 k^n. \quad (90)$$

A dimensionless wave number \bar{k} , and conformal time $\bar{\eta}$, have also been introduced as

$$\bar{k} = \frac{k}{\Omega}, \quad \bar{\eta} = \eta\Omega, \quad (91)$$

in order to measure their physical counterparts in units of the bulk wavelengths. The condition $k < \Omega$ now becomes $\bar{k} < 1$. The integration over η in the integrated Sachs-Wolfe term is transformed into an integration over the dimensionless variable x defined by

$$x = k(\eta_0 - \eta) = \bar{k}(\bar{\eta}_0 - \bar{\eta}), \quad (92)$$

the subscript “0” refers to the present time. Note that $x_{\text{E}} = k(\eta_0 - \eta_{\text{E}}) \simeq k\eta_0$.

By observing the CMB, one may naturally expect that the perturbations with physical wavelength greater than the horizon size today have almost no effect. In terms of our parameters, this means that the main contribution in the C_ℓ comes from the modes verifying $\Omega/a_0 > H_0$, hence $\bar{\eta}_0 > 1$.

In Appendix B, we derive a crude approximation for the angular power spectrum induced by the exponentially growing K-modes, in a range a little more constrained than the one previously motivated, namely

$$\ell_{\text{max}} H_0 < \frac{\Omega}{a_0} < \frac{L^{-1}}{1+z_{\text{E}}}, \quad (93)$$

where z_{E} is the redshift at photon emission which is taken to coincide with recombination, $z_{\text{E}} \simeq 10^3$. In order to simplify the calculation, we do not want the transition between the damped K-modes ($\Omega/a > L^{-1}$) and the exponentially growing ones ($\Omega/a < L^{-1}$) to occur between the last scattering surface and today. This requirement leads to the upper limit of Eq. (93). Moreover, in order to derive the C_ℓ , we perform an expansion with respect to a parameter $\ell/\bar{\eta}_0$ assumed small, and ℓ_{max} refers to the multipole at which this approximation breaks down.

The lower limit in Eq. (93) comes from this approximation. Using the values $L \simeq 10^{-3}$ mm, $H_0^{-1} \simeq 10^{29}$ mm, $\ell_{\max} \simeq 10^3$, and $z_E \simeq 10^3$, one finds

$$10^{-26} \text{ mm}^{-1} < \frac{\Omega}{a_0} < 1 \text{ mm}^{-1}. \quad (94)$$

The corresponding allowed range for the parameter $\bar{\eta}_0$ becomes [see Eq. (B11)]

$$10^3 < \bar{\eta}_0 < 10^{29}. \quad (95)$$

Clearly the detailed peak structure on the CMB anisotropy spectrum would have been different if we had taken into account the oscillatory parts ($k > \Omega$) of the K-modes, as well as the Y- and J-modes, but here we are only interested in estimating an order of magnitude bound. As detailed in Appendix B, for a scale invariant initial spectrum, *i.e.* $n = -3/2$, we obtain

$$\frac{\ell(\ell+1)}{2\pi} C_\ell \gtrsim (A_0 e^{\bar{\eta}_0})^2 \frac{e^{-\ell}}{\ell^{7/2}} \left(\frac{\ell}{\bar{\eta}_0}\right)^{\ell-1}. \quad (96)$$

From current observations of the CMB anisotropies, the left hand side of this expression is about 10^{-10} , and for $\ell \simeq 10$, one gets

$$A_0(\Omega) \lesssim \frac{e^{-[\bar{\eta}_0 - 5 \ln(\bar{\eta}_0)]}}{10^5}. \quad (97)$$

From Eq. (B11) and (95), one finds that the primordial amplitude of these modes must satisfy

$$A_0(\Omega) < e^{-10^3}, \text{ for } \Omega/a_0 \simeq 10^{-26} \text{ mm}^{-1} \quad (98)$$

and, more dramatically,

$$A_0(\Omega) < e^{-10^{29}}, \text{ for } \Omega/a_0 \simeq 1 \text{ mm}^{-1} \quad (99)$$

for the short wavelength modes. As expected, the perturbations with wavelength closer to the horizon today (smaller values of Ω) are less constrained than smaller wavelengths [see Eq. (98)]. Moreover, one may expect that the bound (99) is no longer valid for $\Omega/a_0 > L^{-1}/(1+z_E)$ since the modes in Eq. (80) start to contribute. However, the present results concern more than 20 orders of magnitude for the physical bulk wave numbers Ω/a_0 , and show that the exhibited modes are actually very dangerous for the braneworld model we are interested in.

It seems that the only way to avoid these constraints is to find a physical mechanism forbidding any excitation of these modes.

VI. CONCLUSION

In this paper we have shown that vector perturbations in the bulk generically lead to growing vector perturbations on the brane in the Friedmann-Lemaître era. This

behaviour radically differs from the usual one in four-dimensional cosmology, where vector modes decay like a^{-2} whatever the initial conditions.

Among the growing modes, we have identified so called K-modes which are perfectly normalizable and lead to exponentially growing vector perturbations on the brane with respect to conformal time. By means of a rough estimate of the CMB anisotropies induced by these perturbations, we have found that they are severely incompatible with a homogeneous and isotropic universe; they light up a fire in the microwave sky, unless their primordial amplitude is extremely small.

No particular mechanism for the generation of these modes has been specified. However, one expects that bulk inflation leads to gravitational waves in the bulk which do generically contain them. Even if they are not generated directly, they should be induced in the bulk by second order effects. Usually, these effects are too small to have any physical consequences, but here they would largely suffice due to the exponential growth of the K-modes [see Eqs. (98) and (99)]. This second order induction seems very difficult to prevent in the models discussed here.

It is interesting to note that this result is also linked to the presence of a non-compact extra-dimension which allows a continuum of bulk frequencies Ω . A closer examination of Eq. (44) shows that the mode $\Omega = 0$, admits only J- and Y-mode behaviours. In a compact space, provided the first quantized value of Ω is sufficiently large, one could expect the exponentially growing K-modes to be never excited by low energy physical processes. Another more speculative way to dispose of them could be to consider their causal structure: as we have noticed before, the modes with separation constant $+\Omega^2$ are tachyons of mass $-\Omega^2$ from the four-dimensional point of view. From the five-dimensional point of view, these are not “propagating modes”, but “brane-modes” which decay into the fifth dimension with penetration depth $d = L^2\Omega$.

In a more basic theory, which goes beyond our classical relativistic approach, these modes may thus not be allowed at all.

Finally, we want to retain that even if the K-modes can be eliminated in some way, the growing behavior of the Y- and J-modes remains. Although their power law growth is not as critical as the exponential growth of the K-modes, they should have significant effects on the CMB anisotropies. Indeed, they lead to amplified oscillating vector perturbations which are entirely absent in four-dimensional cosmology. This will be the object of a future study [85].

We therefore conclude that, if no physical mechanism forbids the generation of the discussed vector modes with time dependence $\propto \exp(\eta\sqrt{\Omega^2 - k^2})$, anti-de Sitter infinitely thin braneworlds, with non-compact extra-dimension, cannot reasonably lead to a homogeneous and isotropic expanding universe.

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APPENDIX A: COMPARISON WITH THE RANDALL-SUNDRUM MODEL

As already mentioned in the text, if the brane is at rest ($H = 0$) at $r_b = L$, our model reduces to the RS II model. One may ask therefore, quite naturally, why has our dangerous K-mode never been discussed in the context of RS II? In this appendix we address this question.

First of all, the bulk solutions Σ and Ξ for vector perturbations of AdS₅ with a brane, remain valid. The solutions with $m^2 = -\Omega^2 < 0$ have, however not been discussed in the literature so far. Also, when constructing the Green's function [16, 80, 86], these solutions have not been considered. As we shall see now, for most problems that is most probably very reasonable.

In the RS II model one considers perturbations which do not require anisotropic stresses on the brane, $\pi_i = 0$. Eq. (74) then reduces to

$$\Xi(r = L, t, \Omega, k) = 0, \quad (\text{A1})$$

such that Ξ has to vanish on the brane. We insert this into a general solution of the form

$$\begin{aligned} \Xi(r, t, \Omega, k) = & \pm i \sqrt{1 + k^2/\Omega^2} e^{\pm it \sqrt{\Omega^2 + k^2}} \frac{L^2}{r^2} \\ & \times \left[A J_1\left(\frac{L^2 \Omega}{r}\right) + B Y_1\left(\frac{L^2 \Omega}{r}\right) \right], \quad (\text{A2}) \\ & \text{for } m^2 = \Omega^2 > 0, \end{aligned}$$

$$\begin{aligned} \Xi(r, t, \Omega, k) = & \pm \sqrt{1 - k^2/\Omega^2} e^{\pm t \sqrt{\Omega^2 - k^2}} \frac{L^2}{r^2} \\ & \times \left[C K_1\left(\frac{L^2 \Omega}{r}\right) + D I_1\left(\frac{L^2 \Omega}{r}\right) \right], \quad (\text{A3}) \\ & \text{for } m^2 = -\Omega^2 < 0. \end{aligned} \quad (\text{A4})$$

The boundary condition (A1) then implies

$$B = -A \frac{J_1(L\Omega)}{Y_1(L\Omega)}, \quad \text{for } m^2 = \Omega^2 > 0, \quad (\text{A5})$$

$$D = -C \frac{K_1(L\Omega)}{I_1(L\Omega)}, \quad \text{for } m^2 = -\Omega^2 < 0. \quad (\text{A6})$$

Eq. (A5) is exactly the relation which has also been found in Ref. [16], while Eq. (A6) is new. However, if the solution is not allowed to grow exponentially when approaching the Cauchy horizon $r \rightarrow 0$, one has to require $D = 0$,

which implies $C = 0$ since K_1 has no zeros. With this physically sensible condition (see Ref. [80]), we can discard these solutions. Nevertheless, in cases where the I-modes can be regularized (e.g. by compactification, presence of a second brane *etc.*), the most general Green's function would include them. It is interesting to note that the calculation of the static potential of two masses M_1 and M_2 at distance x generated by the exchange of the zero-mode and the two continua of Kaluza-Klein modes with positive and with imaginary masses, simply leads to

$$\begin{aligned} V(x) & \sim G_4 \frac{M_1 M_2}{x} \left(1 + \int_0^\infty m L^2 e^{-mx} dm \right. \\ & \quad \left. - \int_0^\infty m L^2 e^{-imx} dm \right) \\ & = G_4 \frac{M_1 M_2}{x} \left(1 + \frac{2L^2}{x^2} \right). \quad (\text{A7}) \end{aligned}$$

The short distance modification hence deviates by a factor of 2 from the result of Ref. [16], if we include the tachyonic modes. One has to be aware of the fact that, like so often, the result is sensitive to the choice of the Green's functions.

Anyway, small initial perturbations of the RS solution which allow for small anisotropic stresses, so that the condition (A6) does not need to be imposed, will in general contain a small K-mode which grows exponentially and renders the cosmological model unstable. It seems to us that this possibility has been overlooked in the literature so far.

We end this appendix with a simple example which sketches the presence of this instability. We consider a 1+1 dimensional Minkowski space-time, with orbifold-like spatial sections which can be identified with two copies of $y \geq 0$. The "brane" is represented by the point $y = 0$ and the "bulk" by the two copies of $y > 0$. For an initially small perturbation $f(y, t)$ in the bulk, which satisfies a hyperbolic wave equation, we want to analyze whether an instability can build up. We are looking for solutions of

$$\partial_t^2 f - \partial_y^2 f = 0, \quad (\text{A8})$$

with small initial data, say $f(t = 0, y) \ll 1$ and $\partial_t f(t = 0, y) \ll 1$ for all $y \geq 0$. By separation of variables one can find a complete set of solutions, $f = f_\pm(k) \exp[\pm ik(y \pm t)]$. For a sufficiently small value of $f_\pm(k)$ these solutions satisfy the initial conditions. These solutions oscillate in time; they have constant amplitude. However, there are other solutions, $f = g_\pm(k) \exp[\pm k(y \pm t)]$. Since the initial data has to be small, the solutions $\propto \exp(+ky)$ are not allowed. But the solutions $f = g_- \exp[-k(y - t)]$ have perfectly small initial data and they represent an exponential instability. If we fix the boundary conditions, setting $f(t, y = 0) = 0$, or $\partial_t f(t, y = 0) = 0$, this instability disappears, but if $f(t, y = 0)$ is free, even a very small initial value $f(0, 0) \ll 1$ can induce an exponential instability. Clearly, this leads also to an exponential growth of the boundary value $f(t, y = 0)$.

If we give the initial conditions $f(0, y) = A \exp(-ky)$ and $\partial_t f(0, y) = kA \exp(-ky)$, the function $f(t, y) = A \exp[k(t - y)]$ solves the equation and generates the exponential growing. If we would require, as an additional boundary condition that, e.g. the solutions at $y = 0$ remain at least bounded, this mode would not be allowed and we would have to expand the initial data in terms of the oscillatory modes. However, it seems to us acausal to pose conditions of what is going to happen “on the brane” in the future. But mathematically, without any such “acausal” boundary conditions, the initial value prob-

lem is not well posed. This example is a simple analog of our instability. As long as anisotropic stresses vanish identically, only the J- and Y-modes are relevant. However, if the brane has arbitrarily small but non-vanishing anisotropic stresses on which we do not want to impose any constraints for their future behavior, an exponential instability can build up. This is a rather unnatural behaviour which may cast doubts on the RS realisation of braneworlds in the context of cosmological perturbation theory.

APPENDIX B: CMB ANGULAR POWER SPECTRUM

In this appendix we first present a crude and then a more sophisticated approximation for the C_ℓ -power spectrum from the exponentially growing K-modes. As we shall see, at moderate values of $\ell \sim 10-50$, both lead to roughly the same bounds for the amplitudes which are also presented in the text.

1. Crude approximation

Here we start from Eq. (89). In the integral

$$\int_0^{x_E} \frac{j_\ell(x)}{x} e^{-x\sqrt{1/\bar{k}^2-1}} dx, \quad (\text{B1})$$

we replace j_ℓ by its asymptotic expansion for small ℓ ,

$$j_\ell(x) \simeq \left(\frac{x}{2}\right)^\ell \frac{\sqrt{\pi}}{2\Gamma(\ell+3/2)}. \quad (\text{B2})$$

This is a good approximation if either $x_E \simeq k\eta_0 < \ell/2$ or $(\ell/2)(1/\bar{k}^2-1)^{1/2} > 1$. Since $k^2 < \Omega^2$, the first condition is always satisfied if the first of the two inequalities in (93) are fulfilled. The integral of x then gives

$$\langle |\Delta_\ell(\bar{k})|^2 \rangle \simeq \frac{\pi A_0^2 \Omega^{2n} \bar{k}^{2n}}{2^{2\ell} \ell^3} \left(\frac{1}{\bar{k}^2} - 1\right)^{1-\ell} e^{2\bar{\eta}_0 \sqrt{1-\bar{k}^2}}. \quad (\text{B3})$$

Integrating over k , we must take into account that our approximation is only valid for $k < k_{\text{max}} = (\Omega^2 - \eta_0^{-2})^{1/2}$. Since we integrate a positive quantity we certainly obtain a lower bound by integrating it only until k_{max} . To simplify the integral we also make the variable transform $y = \sqrt{1 - \bar{k}^2}$. With this and inserting our result (B3) in Eq. (87), we obtain

$$\ell^2 C_\ell \gtrsim \frac{2\ell}{2^{2\ell}} A_0^2 \Omega^{2n+3} \int_{1/\bar{\eta}_0}^1 (1-y^2)^{n+\ell-1/2} y^{3-2\ell} e^{2\bar{\eta}_0 y} dy. \quad (\text{B4})$$

For $\ell \geq 2$, $y^{3-2\ell} \geq 1$ on the entire range of integration. Hence we have

$$\ell^2 C_\ell \gtrsim \frac{2\ell}{2^{2\ell}} A_0^2 \Omega^{2n+3} \int_0^1 (1-y^2)^{n+\ell-1/2} e^{2\bar{\eta}_0 y} dy. \quad (\text{B5})$$

This integral can be expressed in terms of modified Struve functions [79]. In the interesting range, $\bar{\eta}_0 \gg 1$ we have

$$\int_0^1 (1-y^2)^{n+\ell-1/2} e^{2\bar{\eta}_0 y} dy \simeq \frac{\Gamma(n+\ell+1/2)}{4\bar{\eta}_0^{n+\ell+1/2}} e^{2\bar{\eta}_0}. \quad (\text{B6})$$

Inserting this result in Eq (B5) we then finally obtain

$$\ell^2 C_\ell \gtrsim \frac{\sqrt{2\pi}\sqrt{\ell} e^{-\ell}}{2^{2\ell+1}} \left(\frac{\ell}{\bar{\eta}_0}\right)^{n+\ell+1/2} A_0^2 \Omega^{2n+3} e^{2\bar{\eta}_0} \sim \frac{\sqrt{2\pi}\sqrt{\ell} e^{-\ell}}{2^{2\ell+1}} \left(\frac{\ell}{\bar{\eta}_0}\right)^{\ell-1} A_0^2 e^{2\bar{\eta}_0}, \quad (\text{B7})$$

where we have used Stirling's formula for $\Gamma(\ell + n + 1/2)$ and set $n = -3/2$ after the \sim sign.

In the next section we use a somewhat more sophisticated method which allows us to calculate also the Doppler contribution to the C_ℓ 's. For the ISW effect this method gives

$$\ell^2 C_\ell \simeq \sqrt{\frac{2}{\pi}} \frac{e^{-\ell}}{36\ell^{7/2}} \left(\frac{\ell}{\bar{\eta}_0}\right)^{\ell-1} A_0^2 e^{2\bar{\eta}_0} \quad (\text{B8})$$

for $n = -3/2$. Until $\ell \sim 15$ the two approximations are in reasonable agreement and lead to the same prohibitive bounds for $A_0(\Omega)$. For $\ell > 15$, Eq. (B8) becomes more stringent.

2. Sophisticated approximation

In Eq. (89) we have only considered the dominant contribution coming from the integrated Sachs-Wolfe effect. The general expression is obtained by inserting the solutions (78) for σ and ϑ in Eq. (88),

$$\Delta_\ell(\bar{k}) = 2A_0\Omega^n \bar{k}^n \left(1 - \frac{\bar{k}^2}{\bar{k}_E^2}\right) \frac{j_\ell(\bar{k}\bar{\eta}_0 - \bar{k}\bar{\eta}_E)}{\bar{k}\bar{\eta}_0 - \bar{k}\bar{\eta}_E} e^{\bar{\eta}_E \sqrt{1-\bar{k}^2}} + 2A_0\Omega^n \bar{k}^n e^{\bar{\eta}_0 \sqrt{1-\bar{k}^2}} \sqrt{\frac{1}{\bar{k}^2} - 1} \int_0^{x_E} \frac{j_\ell(x)}{x} e^{-x \sqrt{1/\bar{k}^2 - 1}} dx. \quad (\text{B9})$$

To derive the first term we have used Eq. (32) in the matter era. The parameter

$$\bar{k}_E^2 = 6(1 + z_E) \left(\frac{H_0 a_0}{\Omega}\right)^2, \quad (\text{B10})$$

reflects the change in behavior of the modes, redshifted by z_E to the emission time, which are either outside or inside the horizon today. It is important to note that the parameter $H_0 a_0 / \Omega$ completely determines the effect of the bulk vector perturbations on the CMB, together with the primordial amplitude A_0 . Indeed, solving Eq. (29) in terms of conformal time, and using Eqs. (28) and (32), yields $\eta_0 \simeq 2/(a_0 H_0)$ in the Friedmann-Lemaître era. Thus

$$\bar{\eta}_0 \simeq 2 \frac{\Omega/a_0}{H_0}, \quad \bar{\eta}_E \simeq \frac{1}{1+z_E} \frac{\Omega/a_0}{H_0}. \quad (\text{B11})$$

We now replace the spherical Bessel functions j_ℓ in the integrated Sachs-Wolfe term (ISW) using the relation [79]

$$j_\ell(x) = \sqrt{\frac{\pi}{2x}} J_{\ell+1/2}(x). \quad (\text{B12})$$

In the ISW term the upper integration limit can be taken to be infinity as the contribution from x_E to infinity can be neglected provided $x \sqrt{1/\bar{k}^2 - 1} > 1$. This restriction is equivalent to $\bar{k}^2 < 1 - 1/\bar{\eta}_0$ which is verified for almost all values of \bar{k} up to one, given that $\bar{\eta}_0$ varies in the assumed range (95). We remind that for the exponentially growing K-mode $k \leq \Omega$, and hence $0 \leq \bar{k} \leq 1$. This allows for the exact solution [87]

$$\int_0^\infty x^{-3/2} J_{\ell+1/2}(x) e^{-x \sqrt{1/\bar{k}^2 - 1}} dx = \frac{\bar{k}^\ell}{2^{\ell+1/2}} \frac{\Gamma(\ell)}{\Gamma(\ell+3/2)} F\left(\frac{\ell}{2}, \frac{\ell}{2} + 1; \ell + \frac{3}{2}; \bar{k}^2\right), \quad (\text{B13})$$

where F is the Gauss hypergeometric function. In regard to the subsequent integration over k we approximate F as follows. For small values of k , F is nearly constant with value 1, at $\bar{k} = 0$. As $\bar{k} \rightarrow 1$ the slope of F diverges and it cannot be Taylor expanded anymore. However, by means of the linear transformation formulas [79], F can be written as a combination of hypergeometric functions depending on $1 - \bar{k}^2$

$$F\left(\frac{\ell}{2}, \frac{\ell}{2} + 1; \ell + \frac{3}{2}; \bar{k}^2\right) = \frac{\Gamma\left(\ell + \frac{3}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\ell}{2} + \frac{3}{2}\right) \Gamma\left(\frac{\ell}{2} + \frac{1}{2}\right)} F\left(\frac{\ell}{2}, \frac{\ell}{2} + 1; \frac{1}{2}; 1 - \bar{k}^2\right) + \sqrt{1 - \bar{k}^2} \frac{\Gamma\left(\ell + \frac{3}{2}\right) \Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(\frac{\ell}{2}\right) \Gamma\left(\frac{\ell}{2} + 1\right)} F\left(\frac{\ell}{2} + \frac{3}{2}, \frac{\ell}{2} + \frac{1}{2}; \frac{3}{2}; 1 - \bar{k}^2\right). \quad (\text{B14})$$

These in turn can be expanded around $1 - \bar{k}^2 = 0$, and gives

$$F \underset{\bar{k} \rightarrow 1}{\sim} 2^{\ell+1/2} \left(1 - \ell \sqrt{1 - \bar{k}^2}\right). \quad (\text{B15})$$

These two approximations intersect at $\bar{k} = \sqrt{1 - 1/\ell^2}$. In this way, we can evaluate the mean value of F by integrating the two parts over the interval $[0, 1]$. Thus, the hypergeometric function is replaced by

$$F \left(\frac{\ell}{2}, \frac{\ell}{2} + 1; \ell + \frac{3}{2}; \bar{k}^2 \right) \simeq \frac{2^{\ell+1/2}}{6\ell^2}. \quad (\text{B16})$$

Furthermore, the Gamma functions in (B13) can be approximated using Stirling's formula [79]

$$\frac{\Gamma(\ell)}{\Gamma(\ell + 3/2)} \simeq \frac{1}{\ell^{3/2}}. \quad (\text{B17})$$

Putting everything together and squaring Eq. (B9) we obtain

$$\begin{aligned} |\Delta_\ell(\bar{k})|^2 = & 2\pi A_0^2 \bar{k}^{2n} \Omega^{2n} \left\{ \left(1 - \frac{\bar{k}^2}{k_E^2}\right)^2 \frac{e^{2\bar{\eta}_E \sqrt{1 - \bar{k}^2}}}{\bar{k}^3 (\bar{\eta}_0 - \bar{\eta}_E)^3} \left(J_{\ell+1/2} [\bar{k}(\bar{\eta}_0 - \bar{\eta}_E)] \right)^2 \right. \\ & + 2 \left(1 - \frac{\bar{k}^2}{k_E^2}\right) \frac{e^{(\bar{\eta}_0 + \bar{\eta}_E) \sqrt{1 - \bar{k}^2}}}{\bar{k}^{3/2} (\bar{\eta}_0 - \bar{\eta}_E)^{3/2}} J_{\ell+1/2} [\bar{k}(\bar{\eta}_0 - \bar{\eta}_E)] \frac{\bar{k}^{\ell-1} \sqrt{1 - \bar{k}^2}}{6\ell^{7/2}} \\ & \left. + e^{2\bar{\eta}_0 \sqrt{1 - \bar{k}^2}} \frac{\bar{k}^{2(\ell-1)} (1 - \bar{k}^2)}{36\ell^7} \right\} \end{aligned} \quad (\text{B18})$$

The C_ℓ 's are then found by integrating over all k-modes

$$\begin{aligned} C_\ell &= \frac{2}{\pi} \ell(\ell + 1) \Omega^3 \int_0^1 \bar{k}^2 |\Delta_\ell(\bar{k})|^2 d\bar{k} \\ &\equiv 4A_0^2 \ell(\ell + 1) \Omega^{2n+3} \left(C_\ell^{(1)} + C_\ell^{(2)} + C_\ell^{(3)} \right), \end{aligned} \quad (\text{B19})$$

where the $C_\ell^{(i)}$ correspond to the three terms in Eq. (B18). In the following we keep only the zero order terms in $\bar{\eta}_0/\bar{\eta}_E$. From Eqs. (B18), (B19) one finds

$$C_\ell^{(1)} = \frac{1}{\bar{\eta}_0^3} \int_0^1 \bar{k}^{2n-1} \left(1 - \frac{\bar{k}^2}{k_E^2}\right)^2 e^{2\bar{\eta}_E \sqrt{1 - \bar{k}^2}} \left[J_{\ell+1/2} (\bar{k}\bar{\eta}_0) \right]^2 d\bar{k} \quad (\text{B20})$$

First, notice that if the argument is larger or smaller than the index, the Bessel functions are well approximated by their asymptotic expansions (47) and (82), respectively. Therefore, we split the \bar{k} -integral into two integrals over the intervals $[0, \bar{k}_\ell]$ and $[\bar{k}_\ell, 1]$, in each of which the Bessel function is replaced by its limiting forms. The transition value \bar{k}_ℓ is given by $\bar{k}_\ell \simeq \ell/\bar{\eta}_0$. In the integral from \bar{k}_ℓ to 1, the $\sin^2(\bar{k}\bar{\eta}_0)$ is then replaced by its mean value $1/2$ which is justified if the multiplying function varies much slower in \bar{k} than the sine. To carry out the integration we make the substitution $y^2 = 1 - \bar{k}^2$, and in order to simplify the notation we define the integral

$$\mathcal{I}(a, b, \nu) = \int_a^b y(1 - y^2)^\nu e^{2\bar{\eta}_E y} dy \quad (\text{B21})$$

In this way we can write Eq. (B20) in the form

$$\begin{aligned} C_\ell^{(1)} &= \frac{1}{\pi \bar{\eta}_0^4} \left[\mathcal{I}(0, y_\ell, n - 3/2) - \frac{2}{k_E^2} \mathcal{I}(0, y_\ell, n - 1/2) + \frac{1}{k_E^4} \mathcal{I}(0, y_\ell, n + 1/2) \right] \\ &+ \frac{1}{\Gamma^2(\ell + 3/2)} \left(\frac{\bar{\eta}_0}{2} \right)^{2\ell+1} \left[\mathcal{I}(y_\ell, 1, \ell + n - 1/2) - \frac{2}{k_E^2} \mathcal{I}(y_\ell, 1, \ell + n + 1/2) + \frac{1}{k_E^4} \mathcal{I}(y_\ell, 1, \ell + n + 3/2) \right] \end{aligned} \quad (\text{B22})$$

Since $y_\ell = \sqrt{1 - \bar{k}_\ell^2}$ is very close to one, and the integrand is continuous in the interval $[0, 1]$, integrals of the form $\mathcal{I}(y_\ell, 1, \nu)$ can be well approximated by the mean formula

$$\mathcal{I}(y_\ell, 1, \nu) \simeq y(1 - y^2)^\nu e^{2\bar{\eta}_E y} \Big|_{y=y_\ell} (1 - y_\ell) \simeq \frac{e^{2\bar{\eta}_E}}{2} \left(\frac{\ell}{\bar{\eta}_0} \right)^{2(\nu+1)} \quad (\text{B23})$$

For the integrals of the type $\mathcal{I}(0, y_\ell, \nu)$ we distinguish between three cases:

Case a: $\nu > -1$. This case corresponds to a spectral index $n > 1/2$ in the first integral in Eq. (B20). We write $\mathcal{I}(0, y_\ell, \nu) = \mathcal{I}(0, 1, \nu) - \mathcal{I}(y_\ell, 1, \nu)$. The solution of the latter is given by Eq. (B23), whereas the former can be solved in terms of modified Bessel and Struve functions [87]

$$\mathcal{I}(0, 1, \nu) = \frac{1}{2(\nu+1)} + \frac{\sqrt{\pi}}{2} \bar{\eta}_E^{-1/2-\nu} \Gamma(\nu+1) [\text{I}_{\nu+3/2}(2\bar{\eta}_E) + \text{L}_{\nu+3/2}(2\bar{\eta}_E)]. \quad (\text{B24})$$

Since our derivation assumes $\bar{\eta}_E > \ell$, the large argument limit applies and we have

$$\text{I}_{\nu+3/2}(2\bar{\eta}_E) + \text{L}_{\nu+3/2}(2\bar{\eta}_E) \simeq \frac{e^{2\bar{\eta}_E}}{\sqrt{\pi\bar{\eta}_E}}, \quad (\text{B25})$$

independently of the index ν .

Case b: $\nu = -1$. Since the above expressions, Eq. (B24), diverge for $\nu = -1$, we approximate the integral by

$$\mathcal{I}(0, y_\ell, \nu) \simeq e^{2\bar{\eta}_E} \int_0^{y_\ell} y(1 - y^2)^{-1} dy = -e^{2\bar{\eta}_E} \ln \left(\frac{\ell}{\bar{\eta}_0} \right) \quad (\text{B26})$$

We have checked that the numerical solution of $\mathcal{I}(0, y_\ell, \nu)$ agrees well with the approximation, provided y_ℓ is close to 1.

Case c: $\nu < -1$. We use the same simplification as in Eq. (B26), and now the integral yields

$$\mathcal{I}(0, y_\ell, \nu) \simeq e^{2\bar{\eta}_E} \int_0^{y_\ell} y(1 - y^2)^\nu dy = -\frac{e^{2\bar{\eta}_E}}{2(\nu+1)} \left(\frac{\ell}{\bar{\eta}_0} \right)^{2(\nu+1)} \quad (\text{B27})$$

For the particular value $n = -3/2$, Eq. (B22) contains terms $\mathcal{I}(0, y_\ell, -3)$ and $\mathcal{I}(0, y_\ell, -2)$ which can be evaluated according to (B27), as well as a term $\mathcal{I}(0, y_\ell, -1)$ for which we use (B26). The remaining three integrals over the interval $[y_\ell, 1]$ are evaluated by (B23). The result is

$$C_\ell^{(1)} \simeq \frac{e^{\bar{\eta}_0/z_E}}{4\pi\ell^4} \left[1 - \frac{\ell^2}{6z_E} - \left(\frac{\ell^2}{12z_E} \right)^2 \ln \left(\frac{\ell}{\bar{\eta}_0} \right) + \frac{e^{2\beta\ell}}{2} \left(1 - \frac{\ell^2}{24z_E} \right)^2 \right] \quad (\text{B28})$$

The parameter β is a constant of order unity, within in our approximation it is $\beta = 1 - \ln 2 \sim 0.3$.

The second term $C_\ell^{(2)}$ in Eq. (B18) reads

$$C_\ell^{(2)} \simeq \frac{1}{3\ell^{7/2}\bar{\eta}_0^{3/2s}} \int_0^1 e^{(\bar{\eta}_0 + \bar{\eta}_E)\sqrt{1-\bar{k}^2}} \bar{k}^{2n+\ell-1/2} \sqrt{1-\bar{k}^2} \left(1 - \frac{\bar{k}^2}{\bar{k}_E^2} \right) \text{J}_{\ell+1/2}(\bar{k}\bar{\eta}_0) d\bar{k}, \quad (\text{B29})$$

where only the zero order terms in $\bar{\eta}_E/\bar{\eta}_0$ has been kept. Using the limiting forms for the Bessel function for arguments smaller and larger than the transition value \bar{k}_ℓ , yields

$$C_\ell^{(2)} \simeq \frac{1}{3\ell^{7/2}\bar{\eta}_0^{3/2}\Gamma(\ell+3/2)} \int_0^{\bar{k}_\ell} e^{(\bar{\eta}_0 + \bar{\eta}_E)\sqrt{1-\bar{k}^2}} \bar{k}^{2n+\ell-1/2} \sqrt{1-\bar{k}^2} \left(1 - \frac{\bar{k}^2}{\bar{k}_E^2} \right) \left(\frac{\bar{k}\bar{\eta}_0}{2} \right)^{\ell+1/2} d\bar{k} \\ + \frac{2^{1/2}}{3\pi^{1/2}\ell^{7/2}\bar{\eta}_0} \int_{\bar{k}_\ell}^1 e^{(\bar{\eta}_0 + \bar{\eta}_E)\sqrt{1-\bar{k}^2}} \bar{k}^{2n+\ell-1} \sqrt{1-\bar{k}^2} \left(1 - \frac{\bar{k}^2}{\bar{k}_E^2} \right) \sin \left(\bar{k}\bar{\eta}_0 - \frac{\pi}{2}\ell \right) d\bar{k}. \quad (\text{B30})$$

For consistency with the derivation of $C_\ell^{(1)}$, we have assumed that the main contribution comes from the first integral, while the second one is small due to the oscillating integrand. Since $\bar{k}_\ell \ll 1$, we can use again the mean formula to evaluate the first integral, and by the Stirling formula for $\Gamma(\ell + 3/2)$, Eq. (B30) becomes

$$C_\ell^{(2)} \simeq \frac{e^{\bar{\eta}_0}}{12\pi^{1/2}\ell^{11/2}} \left(\frac{\ell}{\bar{\eta}_0}\right)^{2n+\ell+2} e^{\beta\ell} \left(1 - \frac{\ell^2}{24z_E}\right). \quad (\text{B31})$$

Since $\ell < \bar{\eta}_0$, the spectrum is damped at large ℓ , while the other terms can lead to the appearance of a bump, depending on the value of $\bar{\eta}_0$ and n .

The last terms $C_\ell^{(3)}$ reads

$$C_\ell^{(3)} = \frac{1}{36\ell^7} \int_0^1 e^{2\bar{\eta}_0} \sqrt{1-\bar{k}^2} \bar{k}^{2n+2\ell} (1 - \bar{k}^2) d\bar{k}. \quad (\text{B32})$$

Splitting this expression in two terms over $1 - \bar{k}^2$, and using the substitution $y^2 = 1 - \bar{k}^2$ yields

$$C_\ell^{(3)} = \frac{1}{36\ell^7} \left[\mathcal{I} \left(0, 1, n + \ell - \frac{1}{2}\right) - \mathcal{I} \left(0, 1, n + \ell + \frac{1}{2}\right) \right], \quad (\text{B33})$$

where \mathcal{I} is given by Eq. (B21) with $\bar{\eta}_E \rightarrow \bar{\eta}_0$. As before, these two integrals can be expressed in terms of modified Bessel and Struve functions [87]. From Eq. (B24), taking their limiting forms at large argument, and expanding the Γ function by means of the Stirling formula gives

$$C_\ell^{(3)} \simeq \frac{1}{72\ell^7 (n + \ell + 3/2) (n + \ell + 1/2)} + \sqrt{\frac{\pi}{2}} \frac{e^{2\bar{\eta}_0}}{\ell^{15/2}} \frac{e^{-\ell}}{36} \left(\frac{\ell}{\bar{\eta}_0}\right)^{n+\ell+1/2}. \quad (\text{B34})$$

Clearly, $C_\ell^{(3)}$ dominates over the others since it involves $\exp(2\bar{\eta}_0)$ while $C_\ell^{(2)}$ and $C_\ell^{(1)}$ appear only with fractional power of this factor, namely $\exp(\bar{\eta}_0)$ and $\exp(\bar{\eta}_0/z_E)$. This is due to the fact that we are concerned with incessantly growing perturbations leading to the predominance of the integrated Sachs-Wolfe effect.

Inserting Eqs. (B28), (B31) and (B34) for the particular value $n = -3/2$ into Eq. (B19) gives the final CMB angular power spectrum

$$\begin{aligned} \frac{\ell(\ell+1)}{2\pi} C_\ell \simeq & \frac{2}{\pi} A_0^2 \left\{ \frac{e^{\bar{\eta}_0/z_E}}{4\pi} \left[1 - \frac{\ell^2}{6z_E} - \left(\frac{\ell^2}{12z_E}\right)^2 \ln\left(\frac{\ell}{\bar{\eta}_0}\right) + \frac{e^{2\beta\ell}}{2} \left(1 - \frac{\ell^2}{24z_E}\right)^2 \right] \right. \\ & \left. + \frac{e^{\bar{\eta}_0}}{12\pi^{1/2}\ell^{3/2}} \left(\frac{\ell}{\bar{\eta}_0}\right)^{\ell-1} e^{\beta\ell} \left(1 - \frac{\ell^2}{24z_E}\right) + \sqrt{\frac{\pi}{2}} \frac{e^{2\bar{\eta}_0}}{\ell^{7/2}} \frac{e^{-\ell}}{36} \left(\frac{\ell}{\bar{\eta}_0}\right)^{\ell-1} \right\}. \end{aligned} \quad (\text{B35})$$

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