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Are there static textures?

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We consider harmonic maps from Minkowski space into the three-sphere. We are especially interested in solutions which are asymptotically constant, i.e., converge to the same value in all directions of spatial infinity. Physical three-space can then be compactified and topologically (but not metrically) identified with a three-sphere. Therefore for fixed time, the winding of the map is defined. We investigate whether static solutions with a nontrivial winding number exist. The answer which we can prove here is only partial: We show that within a certain family of maps no static solutions with a nonzero winding number exist. We discuss the existing static solutions in our family of maps. An extension to other maps or a proof that our family of maps is sufficiently general remains an open problem. [S0556-2821(99)04612-3]

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I. INTRODUCTION

Many physical problems can be described by scalar fields φ with topologically nontrivial target spaces. The equation of motion for φ often requires φ to represent a harmonic map from spacetime into the target space (in the physics literature such maps are better known under the name ''nonlinear σ model'').

The question arises as to whether or not a given field configuration is topologically trivial (continuously deformable to the constant map). Topological defects are topologically nontrivial field configurations. If we consider a topologically trivial four dimensional space-time manifold $\mathcal{M}$, nontrivial field configurations are in general singular on a certain submanifold $\mathcal{S}\subset\mathcal{M}$. The dimension of $\mathcal{S}$ depends on the first nontrivial homotopy of $\varphi(\mathcal{M}\setminus\mathcal{S})\equiv\mathrm{Im}\varphi$: If $\pi_0(\mathrm{Im}\varphi)$ is nontrivial, the submanifold $\mathcal{S}$ forms a network of ''domain walls'' of space-time dimension 3. If $\pi_1(\mathrm{Im}\varphi)$ is nontrivial, a network of ''strings'' of space-time dimension 2 is formed. If $\pi_2(\mathrm{Im}\varphi)$ is nontrivial ''monopoles'' of space-time dimension 1 appear. And if $\pi_3(\mathrm{Im}\varphi)$ is nontrivial ''textures,'' singular events of space-time dimension 0 appear.

Higher homotopy groups do not lead to topological defects in four space-time dimensions. The simplest and most common examples of topological defects are the cases $\mathrm{Im}\varphi=S^n$, where S^n denotes the sphere of dimension n and $S^0=\{-1,1\}$. But also other examples play an important role in solid state physics (helium [1], liquid crystals [2]) and cosmology [3,4].

In the case of a field living on S^n with $n\leq 2$ simple static domain wall $(n=0)$, string $(n=1)$ and monopole $(n=2)$ solutions are known (see, e.g. [4] or [5]). The question which we want to address here is whether there also exist static texture solutions $(n=3)$. A static texture solution is different from monopoles, strings and domain walls in that it is nonsingular: A map from $\mathbf{R}^3$ to S^3 which is asymptotically constant, $\lim_{|x|\to\infty}\varphi(x,t)=\varphi_0(t)$, can wind around S^3 without being singular anywhere. Derrick's theorem [6] then already implies that there is no static texture solution with finite energy. However, the simple static domain wall, string and monopole solutions we are alluding to (which contain a singular sheet, line and point, respectively), have infinite total energy and we thus want to allow also for infinite energy solutions. We therefore cannot apply Derrick's theorem. Nevertheless, numerical simulations [7,8] indicate, that winding texture configurations always shrink, leading to a singularity, the unwinding event, at a finite time t_c, after which the configuration becomes topologically trivial and approaches the constant solution, as predicted by Derrick's theorem. The total energy of the initial configuration is, however, in general infinite so that Derrick's theorem cannot be applied.

This numerical finding prompted us to search for a proof for the nonexistence of static texture on flat physical space. Clearly, the result depends on the geometry of physical space. If space is a three-sphere, the identity map represents a well defined static texture solution. We want to investigate whether such solutions are excluded, for example in Minkowski space.

We do not quite succeed in this task. First, we shall assume that the searched for static winding solution has a spherically symmetric Lagrangian density. This assumption does not bother us too much. It seems physically well motivated (we can, however not use any rigorous energy arguments to justify it, since the total energy of our solution must be infinite). Also within the class of solutions with spherically symmetric Lagrangian densities we have a proof only for a special ansatz for the field configuration and it remains an open problem how general our ansatz is.

Our paper is organized as follows: In the next section we write down the equations of motion for the scalar field and specify our ansatz. In Sec. III we then show that within this ansatz no static solution with nontrivial winding number can exist and discuss the nature of the globally existing (nonwinding) static solutions. In Sec. IV we present the conclusions and an outlook.

II. SPHERICALLY SYMMETRIC "TEXTURE" FIELDS

We consider a scalar field (order parameter) $\varphi:\mathcal{M}\rightarrow S^3$ and we ask φ to be harmonic, a nonlinear σ model. A harmonic map satisfies the Euler-Lagrange equations for the action

$$\mathcal{S}(\varphi)=\frac{1}{2}\int_{\mathcal{M}}|d\varphi|^2_{S^3}dx. \tag{1}$$

We consider the situation where $\mathcal{M}$ is four-dimensional Minkowski spacetime with the flat Lorentzian metric g and S^3 the unit three-sphere with the standard metric which we denote by G.

Here dx denotes the volume element of the metric g on $\mathcal{M}$ and

$$\begin{aligned}\mathcal{L}(\varphi)&=|d\varphi|^2_{S^3}=\mathrm{Trace}_g(\varphi^*G)\\&=g^{\mu\nu}(x)G_{ij}(\varphi(x))\frac{\partial\xi^i}{\partial x_\mu}(x)\frac{\partial\xi^j}{\partial x_\nu}(x),\end{aligned} \tag{2}$$

for some (local) coordinates $x=(x_0,\ldots,x_3)$ on $\mathcal{M}$ and $\varphi=(\xi_1,\xi_2,\xi_3)$ on S^3 (we always assume summation over repeated indices).

We only consider regular maps φ, i.e., maps that have finite energy density $|d\varphi|^2_{S^3}$ everywhere. In addition to being stationary points of Eq. (1) we also demand our maps to be asymptotically constant,

$$\lim_{|x|\rightarrow\infty}\varphi(x,t)=\varphi_0(t).$$

At fixed time we then can consider them as maps $\overline{\varphi_t}$ from compactified $\mathbf{R}^3,\overline{\mathbf{R}^3}=\mathbf{R}^3\cup\{\infty\}\equiv S^3$ to S^3, assigning $\overline{\varphi_t}(x)=\varphi(x,t)$ and $\overline{\varphi_t}(\infty)=\varphi_0(t)$.

The winding number of this extended map $\overline{\varphi_t}:S^3\rightarrow S^3$ is a topological invariant and counts how many times $\overline{\varphi_t}(S^3)$ winds around the target S^3. This number cannot change under continuous time evolution. We would like to show that there are no static solutions φ with nonzero winding number.

Unfortunately, we are not able to solve the problem in this generality. We thus impose some restrictions on the maps φ. One way of doing this is to demand φ to obey certain symmetry properties. We want to impose spherical symmetry, i.e., invariance under $SO(3)$, the group of rotations of physical space.

The action of an element of the rotation group, $g\in SO(3)$ on the maps $\varphi:M\rightarrow S^3$ is given by

$$(g\cdot\varphi)(x)=\varphi(g^{-1}\cdot x) \tag{3}$$

(scalar field). The fixed points of the action (3) are the spherically symmetric fields of the form $\varphi=\varphi(r,t),\ \ r=|\vec{x}|$. We might want to require spherical symmetry of the field φ itself. For our purposes however, this restriction is too severe: Since the image of a smooth map can never have dimension greater than the dimension of the domain, this would limit us to only two-dimensional ranges (one-dimensional in the static case) which are topologically not interesting. Instead we will only demand that the Lagrangian density

$$\mathcal{L}=|d\varphi|^2_{S^3} \tag{4}$$

be $SO(3)$ invariant, $\mathcal{L}(g\varphi)=\mathcal{L}(\varphi)$.

We proceed as follows: We first derive the full Euler-Lagrange equations and then make an ansatz for φ (which is inspired by the symmetry requirements). We then insert our ansatz into the Euler-Lagrange equations. Since the equations remain self-consistent, we can try and solve them. The solutions we find are then always solutions of the full Euler-Lagrange equations. It remains to investigate whether they can be topologically nontrivial, i.e., whether there exist solutions with nonzero winding number.

We use standard spherical coordinates (r,θ,ϕ) for the spatial part and write the standard metric on flat Minkowski spacetime M as

$$g=-dt^2+dr^2+r^2(d\theta^2+\sin^2(\theta)d\phi^2). \tag{5}$$

For $y\in S^3$ we use the standard spherical coordinates

$$\begin{aligned}y=(&\sin\xi_3\sin\xi_2\sin\xi_1,\sin\xi_3\sin\xi_2\cos\xi_1,\\&\sin\xi_3\cos\xi_2,\cos\xi_3).\end{aligned}$$

In our case the ξ_i are functions living on spacetime M,

$$\xi_i:M\rightarrow\mathbf{R},\quad \xi_i=\xi_i(t,r,\theta,\phi). \tag{6}$$

The standard ranges for the angles ξ_i are $\xi_1\in[0,2\pi]$, $\xi_2\in[0,\pi]$ and $\xi_3\in[0,\pi]$. It is important to note that for a map to cover all of S^3, ξ_2 and ξ_3 have to assume both boundary values, 0 and π, and ξ_1 must assume both 0 and 2π.

The standard metric on S^3 expressed in terms of (ξ_i) is

$$G=d\xi_3^2+d\xi_2^2\sin^2(\xi_3)+d\xi_1^2\sin^2(\xi_3)\sin^2(\xi_2). \tag{7}$$

The equations of motion corresponding to the Lagrangian (1) are

$$\nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\xi_i)}\right)=\frac{\partial\mathcal{L}}{\partial\xi_i},\quad 1\leqslant i\leqslant 3. \tag{8}$$

With the standard metric (7) on S^3 they become

$$\begin{aligned}0=&\nabla^\mu\nabla_\mu\xi_3-\sin(\xi_3)\cos(\xi_3)[(\nabla_\mu\xi_2)(\nabla^\mu\xi_2)\\&+\sin^2(\xi_2)(\nabla_\mu\xi_1)(\nabla^\mu\xi_1)],\\0=&\nabla^\mu\nabla_\mu\xi_2+2\cot(\xi_3)(\nabla_\mu\xi_3)(\nabla^\mu\xi_2)\\&-\sin(\xi_2)\cos(\xi_2)(\nabla_\mu\xi_1)(\nabla^\mu\xi_1),\\0=&\nabla^\mu\nabla_\mu\xi_1+2\cot(\xi_2)(\nabla_\mu\xi_2)(\nabla^\mu\xi_1)\\&+2\cot(\xi_3)(\nabla_\mu\xi_3)(\nabla^\mu\xi_1).\end{aligned} \tag{9}$$

To obtain a spherically symmetric Lagrangian density we use a generalized hedgehog ansatz:

$$\xi_i = \xi_i(\phi,\theta), \quad i=1,2 \quad \text{and} \quad \xi_3 = \xi_3(r,t). \tag{10}$$

The idea behind this ansatz is that we want to make use of the vast knowledge on harmonic maps on two-dimensional spaces, which will help us to first solve the two-dimensional problem for ξ_1 and ξ_2 in the coordinates θ and ϕ and then afterwards solve the equation for the remaining function ξ_3. The crucial point in order for this to work is that the equations of motion must respect this ''decomposition'' of φ in a (r,t)-dependent and a (θ,ϕ)-dependent part. This is the subject of the theory of (ρ,σ)-equivariant maps which is explained in great detail in [11].

With the ansatz (10) we may introduce a map

$$\Omega: S^2 \to S^2, \quad \Omega(\phi,\theta) = (\xi_1,\xi_2) \tag{11}$$

with Lagrangian density

$$|d\Omega|^2 = (\nabla_\mu \xi_2)(\nabla^\mu \xi_2) + \sin^2(\xi_2)(\nabla_\mu \xi_1)(\nabla^\mu \xi_1). \tag{12}$$

The total Lagrangian density of the map φ is then

$$|d\varphi|^2 = (\nabla_\mu \xi_3)(\nabla^\mu \xi_3) + \sin^2(\xi_3)|d\Omega|^2. \tag{13}$$

Our ansatz (10) yields

$$|d\Omega|^2 = \frac{\lambda(\phi,\theta)}{r^2}. \tag{14}$$

Spherical symmetry of the Lagrangian density then requires $\lambda = \text{const}(\geqslant 0)$. Inserting our ansatz (10) into the Euler-Lagrange equations (10), we find after multiplying with r^2 that for the components ξ_1 and ξ_2 these equations are just the Euler-Lagrange equations of the map $\Omega: S^2 \to S^2$. Thus Ω has to be harmonic (on S^2) with constant Lagrangian density $|d\Omega|^2_{S^2} = \lambda$, where $|.|^2_{S^2}$ now denotes the Lagrangian density on S^2. But this means that Ω has to be an eigenmap of the Laplacian on S^2 with eigenvalue λ (in the sense of [11]). Therefore the components Ω_i have to be given by linear combinations of spherical harmonics of a fixed degree k, Y_{km},

$$\Omega_i = \sum_m a_{im} Y_{km}.$$

If we apply this to our map $\Omega: S^2 \to S^2$, we obtain for $|d\Omega|^2$ (with respect to the metric g on spacetime)

$$|d\Omega|^2 = \frac{k(k+1)}{r^2}, \quad k \in \mathbf{N}. \tag{15}$$

We will always assume $\lambda = k(k+1) > 0$ since we are only interested in spherically nontrivial maps. Note that $\lambda = 2$ just corresponds to $\Omega = id$, the identity map, used for example in the ''hedgehog'' monopole.

The remaining Euler-Lagrange equation for the last component ξ_3 is now

$$0 = \nabla^\mu \nabla_\mu \xi_3 - \frac{k(k+1)}{2r^2}\sin(2\xi_3), \quad k \in \mathbf{N}. \tag{16}$$

It is this equation that we would like to analyze in this paper. In the next section we will prove the nonexistence of *static* solutions with nonzero winding number, whereas an infinite family of *time-dependent* solutions of Eq. (16) in Minkowski space can be found for any winding number $n \in \mathbf{N}$ [13].

In [10] solutions from a geometrical S^3 to S^3 are studied with an ansatz that is less general than our ansatz (10). Here we consider compactified $\mathbf{R}^3$, which is topologically equivalent to S^3, but with flat geometry. Using our ansatz (10) on the geometrical S^3, we can also generalize the results of [10] by showing that there are actually two countable families of such maps for every $k > 0$, $k \in \mathbf{N}$, where $k(k+1)$ is the eigenvalue of the map Ω defined in Eq. (11). This will be done in a subsequent paper [13].

All of these results can also be found in [12].

III. NONEXISTENCE OF STATIC WINDING SOLUTIONS

We consider maps to the standard three sphere where spacetime is parametrized by standard Cartesian coordinates r and t. In what follows we will use the notation

$$\dot{} = \frac{\partial}{\partial t}, \quad \text{and} \quad {}' = \frac{\partial}{\partial r}. \tag{17}$$

Then Eq. (16) becomes

$$\xi_3'' - \ddot{\xi}_3 + \frac{2}{r}\xi_3' - \frac{k(k+1)}{2r^2}\sin(2\xi_3) = 0, \quad k \in \mathbf{N}. \tag{18}$$

An exact solution to Eq. (18) which describes a winding time dependent texture for the case $k=1$ has been found in [14]. [Time-dependent solutions to Eq. (18) can in fact be found for any $k \in \mathbf{N}$ and for any winding number $n \in \mathbf{N}$ [13]]. The r-dependence of the last term in Eq. (18) shows that nontrivial solutions of the form $\xi_3 = \xi_3(t)$ are impossible for $k \neq 0$. However, for static maps the ansatz (10) becomes $\xi_3 = \xi_3(r)$ and Eq. (18) reduces to $(\xi \equiv \xi_3)$

$$\xi'' + \frac{2}{r}\xi' - \frac{k(k+1)}{2r^2}\sin(2\xi) = 0, \tag{19}$$

the equation we will discuss in the following.

A. Local properties

Equation (19) has a singular point at $r=0$. Since we require solutions to be regular for all r and t, we assume that $\xi(r)$ is described in a neighborhood of $r=0$ by some power series

$$\xi(0+\epsilon) = \sum_{j=0}^{\infty} a_j \epsilon^j. \tag{20}$$

If we multiply Eq. (19) by r^2 and insert Eq. (20) at $r=0+\epsilon$ we obtain an equation in powers of ϵ:

$$0=\sum_{j=0}^{\infty}((j+2)(j+1)a_{j+2}\epsilon^{j+2}+2(j+1)a_{j+1}\epsilon^{j+1})-\frac{k(k+1)}{2}\left[\cos(2a_0)\sin\left(2\sum_{j=1}^{\infty}a_j\epsilon^j\right)+\sin(2a_0)\cos\left(2\sum_{j=1}^{\infty}a_j\epsilon^j\right)\right]. \tag{21}$$

From comparison of the lowest order terms ϵ^0 we obtain immediately

$$\sin(2a_0)=0\Rightarrow a_0=m\pi/2, \quad m\in\mathbf{Z}. \tag{22}$$

Let $l\geqslant 1$ be the smallest integer for which $a_l\neq 0$. Then the lowest order terms ϵ^l yield

$$l(l-1)a_l+2la_l-\frac{k(k+1)}{2}\cos(2a_0)\cdot 2a_l=0. \tag{23}$$

Therefore

$$a_l(l(l+1)-k(k+1)\cos(2a_0))=0. \tag{24}$$

Combining this with Eq. (22) leads to

$$a_l(l(l+1)-(-1)^m k(k+1))=0, \tag{25}$$

thus for m odd, $a_l=0$ for all $l>0$ and thus ξ is constant, but for m even, $\xi(r)$ has a nontrivial power series expansion at $r=0$ for every eigenmap $\Omega:S^2\rightarrow S^2$ with eigenvalue $k(k+1)$, where the first nonvanishing expansion coefficient is just a_k. For the next higher orders ϵ^{k+1} and ϵ^{k+2} we get in a similar way

$$a_{k+1}=0 \quad \text{and} \quad a_{k+2}=-\frac{k(k+1)}{3(2k+3)}a_k^3. \tag{26}$$

Table I summarizes this information.

B. Global Properties

For a global analysis of the behavior of solutions of Eq. (19) it is convenient to transform the equation into an autonomous one by the transformation

$$s=\frac{1}{\beta}\ln r, \quad \text{where} \quad \beta=\sqrt{\frac{2}{k(k+1)}}. \tag{27}$$

TABLE I. Expansion coefficients for ξ_3 at $r=0$ where $k(k+1)$ is the eigenvalue of the map $\Omega:S^2\rightarrow S^2$ and $m\in\mathbf{Z}$.

$a_0=$	$m\pi$	$(2m+1)\pi/2$
$a_j=\quad(0<j<k)$	0	0
$a_k=$	free	0
$a_{k+1}=$	0	0
$a_{k+2}=$	$-\frac{k(k+1)}{3(2k+3)}a_k^3$	0

The new variable s runs from $-\infty$ to $+\infty$. Remember that we always require $k>0$, $k\in\mathbf{N}$, so that $0<\beta\leqslant 1$. Denoting the derivative with respect to s again by a prime, Eq. (19) transforms into

$$\xi''+\beta\xi'-\sin(2\xi)=0. \tag{28}$$

This differential equation describes the motion of a particle with constant damping β and potential $\sin(2\xi)$. If we switch off the damping, we obtain the conservative system

$$\xi''=-\operatorname{grad}U(\xi), \tag{29}$$

where the potential $U(\xi)$ is given by

$$U(\xi)=\int_{\xi_0}^{\xi}-\sin(2\xi)d\xi=-\sin^2\xi-U_0. \tag{30}$$

The ''energy'' of this system

$$E(\xi,\xi')=\frac{1}{2}\xi'^2-\sin^2\xi-U_0 \tag{31}$$

is conserved, and all solutions are periodic, lying on the sets $E=$const. When we switch on the damping, the solutions no longer remain on levels with $E=$const, but ''fall down into the potential wells'' at $[(2m+1)\pi/2,0]$ (see Fig. 1).

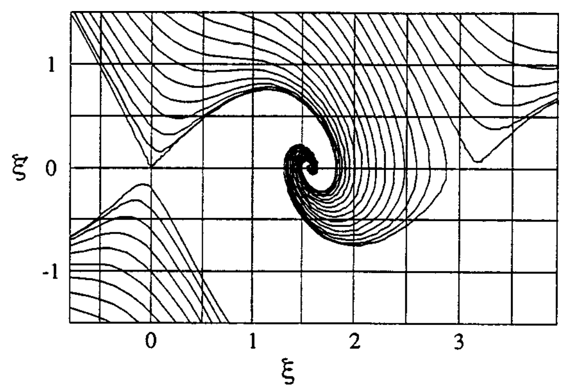

FIG. 1. Phase diagram for the solutions $\xi(s)$ and $\xi'=d\xi/ds$ of the damped system $(k=1,\beta=1)$. Because of the energy loss the solutions ''fall down'' into the potential wells at $[(2m+1)\pi/2,0]$.

If the damping is weak ($\beta \ll 1$) they spiral several times around those points, gradually loosing energy and moving towards the center (underdamped motion), whereas if the damping is strong ($\beta \gg 1$) they move towards the center quickly (overdamped motion). In any case, since the energy is no longer conserved, nontrivial periodic solutions are not possible.

We can formulate this more precisely: If we differentiate the ''energy'' (31) with respect to s and insert Eq. (28) we obtain

$$\frac{dE}{ds} = \xi' \xi'' - \sin(2\xi)\xi' = -\beta \xi'^2. \tag{32}$$

Thus the energy is always decreasing with growing s (for nonconstant ξ). With this we can show the following Lemma.

Lemma 3.1. If $\xi(s_0) = m\pi$, $m \in \mathbf{Z}$, for some $s_0 > -\infty$, and if ξ is not constant, it tends monotonically to $\pm\infty$ for $s \to -\infty$.

Proof. (The idea of the proof is taken from [10]). Let $\xi(s_0) = m\pi$ with $\xi'(s_0) = a$. For $s < s_0$ we have

$$0 < E(s) - E(s_0) = \frac{1}{2}\xi'^2 - \sin^2\xi - \frac{1}{2}a^2 \leqslant \frac{1}{2}\xi'^2 - \frac{1}{2}a^2 \tag{33}$$

and thus $\xi'^2 > a^2, \forall s < s_0$. Therefore E is monotonically increasing for $s \to -\infty$ and $\xi'^2 \to \infty$ with $\xi'^2 > 0, \forall s < s_0$. Correspondingly, ξ tends monotonically to $\pm\infty$, the sign depending on the sign of ξ'.

Lemma 3.2. If $\xi(s_0) = m\pi$, $m \in \mathbf{Z}$ for some s_0, and $\xi'(s_0) = 0$, then if ξ is not constant either $m\pi < \xi(s) < (m+1)\pi$ or $(m-1)\pi < \xi(s) < m\pi \quad \forall s > s_0$.

Proof. Let $\xi(s_0) = m\pi$ with $\xi'(s_0) = 0$. If ξ is not constant then for $s > s_0$ we have

$$0 > E(s) - E(s_0) = \frac{1}{2}\xi'^2 - \sin^2\xi \tag{34}$$

and thus $\sin^2\xi > \frac{1}{2}\xi'^2 \geqslant 0$, $\forall s > s_0$. Especially, $\sin^2\xi(s) \neq 0 \quad \forall s > s_0$ which implies our statement.

Corollary 3.1. There is no static texture solution which satisfies the ansatz (10) *(a texture solution being a solution with homotopy degree $\neq 0$, i.e., one that really winds).*

Proof. From Lemma 3.1: The only nonconstant regular solutions through a point $m\pi$ are the ones with

$$\xi(-\infty) = \lim_{s \to -\infty} \xi(s) = m\pi. \tag{35}$$

Furthermore, for a regular solution

$$\xi'(-\infty) = \lim_{s \to -\infty} \xi'(s) = \lim_{r \to 0} \beta r \frac{d\xi}{dr} = 0. \tag{36}$$

To fully wind around S^3, $\xi = \xi_3$ with $\xi = m\pi$ at $r = 0$ ($s = -\infty$) would have to assume either the value $(m-1)\pi$ or $(m+1)\pi$ which, according to Lemma 3.2 is not possible.

C. Existence and stability of static solutions

Finally, we would like to briefly verify that nonconstant solutions to Eq. (28) do indeed exist globally, and we want to review their properties. With the substitution

$$x = \xi, \tag{37}$$

$$y = \xi', \tag{38}$$

Eq. (28) is equivalent to the autonomous system of first order differential equations

$$x' = y, \tag{39}$$

$$y' = \sin(2x) - \beta y. \tag{40}$$

Together with some initial conditions $x(0) = u_0$, $y(0) = v_0$, this is an initial value problem (IVP) of the form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = f(x,y),$$

$$(x(0), y(0)) = (u_0, v_0). \tag{41}$$

The local existence and uniqueness of a solution to the IVP (41) follow from standard theorems on ordinary differential equations.

All solutions $[x(s), y(s)]$ with

$$\lim_{s \to -\infty} x(s) = m\pi, \quad m \in \mathbf{Z} \ \left[\text{and thus } \lim_{s \to -\infty} y(s) = 0\right] \tag{42}$$

are bounded for every $s \in \mathbf{R}$ as follows directly from Lemma 3.2 and its proof. Therefore these solutions exist globally (cf. [15], Corollary 3.2). [The local existence of solutions satisfying Eq. (42) follows from our series expansion in Sec. III B.]

Now let us discuss the stability of the critical points (x_0, y_0) for this system which are given by

$$(x_0, y_0) = (m\pi/2, 0), \quad m \in \mathbf{Z}. \tag{43}$$

In some neighborhood of a critical point we can approximate Eq. (41) by the linearized system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = (Df)(x_0, y_0) \begin{pmatrix} x \\ y \end{pmatrix}, \tag{44}$$

where Df is the first derivative of f (with respect to x and y). If we calculate the eigenvalues of $Df(x_0, y_0)$ and use the principle of linearized stability [9] we find that the critical points $(x_0,y_0)=(m\pi,0)$, $m\in\mathbf{Z}$ are unstable, while the critical points $(x_0,y_0)=[(2m+1)\pi/2,0]$, are stable, and attractive [in the sense, that if we start at any point close enough to (x_0,y_0) then we will always end up at the critical point itself for $s\to\infty$]. The solutions thus spiral into a focus at $[(2m+1)\pi/2,0]$. This behavior is clearly seen in Fig. 1: All trajectories that come close to the points $(\xi,\xi')=(0,0)$ or $(\xi,\xi')=(\pi,0)$ are repelled and spiral into $\xi=\pm\pi/2$ and $\xi=3\pi/2$ respectively, depending on the sign of ξ'.

In Fig. 1 we show the phase diagram for $\xi(s)$ and $\xi'=d\xi/ds$ for the case $k=1(\beta=1)$. Only the solutions with

$$\lim_{s\to-\infty}\xi(s)=\xi(r=0)=m\pi/2 \quad\text{and}\quad \lim_{s\to-\infty}\frac{d\xi}{ds}=0$$

yield regular solutions in physical space. Remember that from our power series expansion we need $\xi(r=0)=m\pi$, $m\in\mathbf{Z}$ for a nonconstant solution (regular at $r=0$). Furthermore, since the ‘‘energy’’ (31) is always decreasing there are no periodic solutions, and thus all non-constant regular solutions must end in one of the two focal points $((2m\pm1)\pi/2,0)$. We can therefore conclude:

Proposition 3.1. The only nonconstant solutions to Eq. (19) *with $k>0$ that are regular for all r are the ones starting at $\xi(r=0)=m\pi$ and ending in a focus at $\xi(r\to\infty)=(m\pi\pm\pi/2)$ without ever leaving the strip $[m\pi,m\pi+c]$ respectively $[m\pi-c,m\pi]$ for $m\in\mathbf{Z}$ and some $0<c<\pi$.*

This result is also visible in Fig. 1.

IV. CONCLUSIONS

We have found that static ‘‘spherically symmetric’’ harmonic maps (solutions of the nonlinear σ model) from compactified $\mathbf{R}^3$ into S^3 which satisfy our ‘‘ansatz’’ cannot have nontrivial topology. Therefore, if static winding solutions exist, this map cannot be represented as the tensor product of a map of the spherical angles (our Ω) and a radial function. It is not clear to us whether such a decomposition should always exist globally.

We also could not show that each static solution should be homeotopic to a spherically symmetric static solution and thus spherical symmetry remains a nontrivial condition which we have to pose.

In this sense, our partial result only hints to the following which still remains to be fully proven (if true):

Conjecture 4.1. There exist no static textures with everywhere finite energy density.

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