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Gauge-Invariant Cosmological Perturbation Theory with Seeds

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Abstract

Gauge-invariant cosmological perturbation theory is extended to handle perturbations induced by seeds. A calculation of the Sachs–Wolfe effect is presented. A second order differential equation for the growth of density perturbations is derived and the perturbation of Liouville's equation for collisionless particles is also given.
The results are illustrated by a simple analytic example of a single texture knot, where we calculate the induced perturbations of the energy of microwave photons, of baryonic matter and of collisionless particles.

1 Introduction

Within standard cosmology the formation of large scale structure is probably the biggest unsolved problem.

The most extensively worked out scenarios like isocurvature baryons or baryons and collisionless matter (CDM or HDM) face severe problems (for a short review see [1]).

On the other hand there are lots of interesting possibilities which induce perturbations in the baryonic and dark matter by **seeds**, e.g. cosmic strings, primordial black holes, a first generation of stars or texture [2].

By seeds we mean a inhomogenously distributed form of energy which contributes only a small fraction of the total energy density of the universe. Linear perturbation theory is thus justified. Gauge-invariant linear perturbation theory [3] is superior to any arbitrary choice of gauge since it is not plagued by gauge modes and it leads in all known cases to the simplest systems of equations.

The goal of this paper is to expand gauge-invariant linear perturbation theory to include seed perturbations. In Section 2 the terminology is laid down and the gauge-invariant forms of the perturbation of Einstein's equations, the conservation equations and Liouville's equation are given. A new formula for the Sachs-Wolfe effect is also presented.

In Section 3 the introduction of seeds is discussed and a gauge-invariant equation for the growth of density perturbations induced by seeds is derived.

In Section 4 a simple example of a single texture solution [4] is discussed.

2 Gauge-invariant perturbation theory and the Sachs–Wolfe effect

2.1 Basic equations

A gauge transformation in the context of linear perturbation theory of gravity is a linearized coordinate transformation. It is thus given by a vector field X. Any tensor field Q changes under a gauge transformation by the linearized flux in direction of X which is given by the Lie derivative

$$Q \to Q + \epsilon L_X Q \ . \tag{2.1}$$

Separating Q into a background component and a small perturbation, $Q = Q^{(0)} + \epsilon Q^{(1)}$, we find the following transformation law for $Q^{(1)}$:

$$Q^{(1)} \to Q^{(1)} + L_X Q^{(0)} \ . \tag{2.2}$$

Thus tensors with vanishing background component are gauge-invariant. Since all the relativistic equations are covariant, it is always possible to express the corresponding perturbation equations in terms of gauge-invariant variables.

In this subsection we define the well-known gauge-invariant variables which describe the perturbations of the metric, the energy momentum tensor and the one particle distribution function in a Friedman background (see also [5] and [6]). We then write down the perturbation of Einstein's equation, energy momentum conservation and Liouville's equation in a form which will be convenient later. All these equations are most easily derived using the 3+1 formalism of gravity (see [5]).

Using conformal time we define the perturbation in the lapse function α, the shift vector $\boldsymbol{\beta}$ and the 3-metric $\boldsymbol{g}$ of the slices of constant time by

$$\alpha \ = \ a(1 + A) \ , \tag{2.3}$$

$$\boldsymbol{\beta} \;=\; alB^{|i}\partial_i \;, \tag{2.4}$$

$$\boldsymbol{g} \;=\; a^2[(1+2H_L-(2/3)l^2\Box H_T)\gamma_{ij}+2l^2H_{T|ij}]dx^idx^j \;. \tag{2.5}$$

(3–dimensional vector and tensor fields are denoted by bold face letters, | and $\Box$ denote the covariant derivative and Laplacian with respect to the metric $\boldsymbol{\gamma}$, which is the metric of a three space with constant curvature K.)
A, B, H_L, and H_T are arbitrary functions of space and time. To keep them dimensionless we have introduced the length l which in applications will be chosen to be a characteristic scale of the problem.

By defining the metric perturbations according to (2.3), (2.4) and (2.5), we restrict ourselves to **scalar** type perturbations, but we shall not perform the harmonic analysis any further. There are of course also vector and tensor type perturbations but we ignore them in what follows since they don't give rise to density perturbations.

Writing the 4–dimensional metric in the form

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + a^2 h_{\mu\nu} \;, \tag{2.6}$$

the above definitions of the perturbation variables yield

$$h = -2A(dt)^2 - 2lB_{;i}\,dtdx^i + 2(H_L-(l^2/3)\triangle H_T)\gamma_{ij}dx^idx^j + l^2H_{T|ij}dx^idx^j \;. \tag{2.7}$$

From (2.3), (2.4) and (2.5) one can calculate the 3–dimensional Riemann scalar and the extrinsic curvature with the result

$$\delta\boldsymbol{R} = a^{-2}(\Box - 3K)\mathcal{R} \;, \qquad \mathcal{R} = H_L - (l^2/3)\Box H_T \;, \tag{2.8}$$

$$K^{(aniso)}_{ij} = -al(\sigma_{|ij} - 1/3\Box\sigma) \;, \qquad l\sigma = l^2\dot{H}_T - lB \;. \tag{2.9}$$

$K^{(aniso)}$ is the traceless contribution to the extrinsic curvature of the slices of constant time or, what amounts to the same thing, the shear of the normal to the slices.

Deriving the gauge transformation properties of A, $\mathcal{R}$, and σ, one easily finds that the following variables, the so called Bardeen potentials are gauge-invariant (see [5] and [6]):

$$\Phi = \mathcal{R} - (\dot{a}/a)l\sigma \tag{2.10}$$

$$\Psi = A - (\dot{a}/a)l\sigma - l\dot{\sigma} \;. \tag{2.11}$$

Now we proceed to the perturbations of the energy momentum tensor. We define the perturbed energy density, ρ and velocity field u as the timelike eigenvalue and eigenvector of the energy momentum tensor (note that apart from symmetry we do not make any assumptions on the nature of ${T_\mu}^\nu$):

$${T_\mu}^\nu u^\mu = -\rho u^\nu \;\;,\;\; u^2 = -1 \;.$$

We then define the perturbations in the density and velocity field by

$$\rho = \rho^{(0)}(1+\delta) \;\;, \tag{2.12}$$

$$u = u^0\partial_t + u^i\partial_i \;,\; \text{with}\;\; u^0 = (1-A)\;,\; u^i = -lv^{,i} \;. \tag{2.13}$$

u^0 is already fixed by the normalization condition. In the 3–space orthogonal to u we define the perturbations of the stress tensor by

$$\tau_i^{\ j} = p[(1 + \pi_L - (l^2/3)\Box\pi_T)\delta_i^{\ j} + l^2 \pi_{T|i}^{\ \ |j}] \ . \tag{2.14}$$

Calculating the behavior of the quantities δ , v , π_L and π_T under gauge transformations, one finds the following gauge-invariant variables:

$$\begin{aligned}
\Pi &= \pi_T \text{ , anisotropic stress,} \\
\Gamma &= \pi_L - (c_s^2/w)\delta \text{ , entropy perturbation,} \\
lV &= lv - l^2\dot{H}_T \text{ , peculiar velocity,} \\
D_s &= \delta + 3(1+w)(\dot{a}/a)l\sigma \\
D_g &= \delta + 3(1+w)\mathcal{R} \\
D &= D_s + 3(1+w)(\dot{a}/a)lV \ .
\end{aligned}$$

D_g, D_s and D are different choices for a gauge-invariant density perturbation variable. For a physical interpretation of these variables see [5] and [6]. Here we just want to show that for perturbations which are small compared to the horizon distance, l_H all the gauge-invariant combinations V and $D_{(.)}$ approach the usual v and δ. Let us now choose our free length scale l to be the typical size of a given perturbation. From the above equation it is then clear that for $l \ll l_H = \tau$, $D \approx D_s$.

Noting that perturbations to the Einstein tensor are given by second derivatives of the metric perturbations (Palatini's identity, see e.g. [7]), we obtain the following order of magnitude equation:

$$\mathcal{O}(\frac{\delta T}{T})\mathcal{O}(8\pi G T_{\mu\nu}) = \mathcal{O}(\tau^{-2}\frac{\delta g}{g} + (l\tau)^{-1}\frac{\delta g}{g} + l^{-2}\frac{\delta g}{g}) \ . \tag{2.15}$$

Using Friedman's equation

$$\mathcal{O}(8\pi G T_{\mu\nu}) = \mathcal{O}(\dot{a}/a)^2 = \mathcal{O}(1/\tau^2)$$

this yields

$$\mathcal{O}(\frac{\delta T}{T}) = \mathcal{O}(\frac{\delta g}{g} + (l_H/l)\frac{\delta g}{g} + (l_H/l)^2\frac{\delta g}{g}) \ . \tag{2.16}$$

On subhorizon scales the metric perturbations are thus always much smaller than the matter perturbations and the difference between the gauge-invariant quantities V, $D_{(.)}$ and v, δ becomes negligible for $l/l_H \ll 1$.

The perturbations of Einstein's equations and energy momentum conservation can be expressed in terms of these gauge-invariant variables (A simple derivation is given in [5]):

Constraint equations

$$4\pi G a^2 \rho D = -(\Box + 3K)\Phi \tag{2.17}$$

$$4\pi G a^2 (\rho + p) lV = (\dot{a}/a)\Psi - \dot{\Phi} \ . \tag{2.18}$$

Dynamical equations

$$-8\pi G a^2 p\Pi = \Box(\Phi+\Psi) \ . \tag{2.19}$$

$$8\pi G a^2 p(\Gamma + (c_s^2/w)D_g - (2/3)l^2\Box\Pi) = (\dot a/a)\{\dot\Psi - [(1/a)(\frac{a^2\Phi}{\dot a})^\cdot]^\cdot\} +$$
$$\{2a(\dot a/a^2)^\cdot + 3(\dot a/a^2)^2\}[\Psi - 1/a(\frac{a^2\Phi}{\dot a})^\cdot] \ . \tag{2.20}$$

Conservation equations

$$\dot D_\alpha - 3w_\alpha(\dot a/a)D_\alpha = (\Box + 3K)[(1+w_\alpha)lV_\alpha + 2(\dot a/a)w_\alpha l^2\Pi_\alpha] + 3(1+w_\alpha)4\pi G a^2(\rho+p)(lV - lV_\alpha) \ , \tag{2.21}$$

$$l\dot V_\alpha + (\dot a/a)lV_\alpha = \frac{c_\alpha^2}{1+w_\alpha}D_\alpha + \frac{w_\alpha}{1+w_\alpha}\Gamma_\alpha + \Psi + 2/3(\Box+3K)\frac{w_\alpha}{1+w_\alpha}l^2\Pi_\alpha \ . \tag{2.22}$$

Equations (2.21, 2.22) are the conservation equations for a matter component α, which does not interact other than by gravity with the rest of the content of the universe. The total perturbations are defined as the sums:

$$\rho D = \sum_\alpha \rho_\alpha D_\alpha \ , \quad (\rho+p)V = \sum_\alpha(\rho_\alpha + p_\alpha)V_\alpha \quad ... \tag{2.23}$$

The adiabatic sound speed, c_α and enthalpy, w_α are

$$c_\alpha^2 = \dot{p_\alpha}/\dot{\rho_\alpha} \quad , \quad w_\alpha = p_\alpha/\rho_\alpha \ \text{ and } \ w = \frac{\sum_\alpha p_\alpha}{\sum_\alpha \rho_\alpha} \ .$$

The corresponding equations for interacting matter components are given in [6]. In order to complete the above analysis one also needs to include matter equations[1].

Collisionless matter

As one example of a matter equation let us now briefly discuss the case of collisionless particles. They are described by their one particle distribution function

$$f = f^{(0)} + f^{(1)}$$

which lives on the mass bundle, $P_m = \{(p,x)|g(x)_{\mu\nu}p^\mu p^\nu = -m^2\}$. The matter equation is Liouville's equation [7]. From the background Liouville equation it is easy to see that the unperturbed distribution function, $f^{(0)}$ is a function of the redshift corrected momentum, $v := \frac{a}{m}\sqrt{g_{ij}^{(0)}p^ip^j}$ alone.

Studying the somewhat complicated gauge transformation properties of $f^{(1)}$ one finds the following gauge-invariant combination (see [5] and [9]):

$$\mathcal{F} = f^{(1)} - \{v\mathcal{R} + (q/v)lv^i\partial_i\sigma\}\frac{df^{(0)}}{dv} \ , \tag{2.24}$$

[1]To calculate the time evolution of the perturbations in the background matter components, we need not to make explicit use of the somewhat unwieldy equation (2.20), but we can use (2.19) and one of the conservation equations (2.21) and (2.22) which are of course equivalent to (2.19, 2.20).

with $q := (v^2 + a^2)^{1/2}$.
Note that in an orthonormal frame, $p = p^\mu e_\mu$, (for $K = 0$ we have $e_\mu = a^{-1}\partial_\mu$) v and q are just given by

$$v = \frac{a}{m}|\boldsymbol{p}| \text{ and } q = \frac{a}{m}p^0 .$$

The perturbation of Liouville's equation can now be expressed in terms of $\mathcal{F}$:

$$\{q\partial_\tau + v^k\partial_k + K/2[x^iv_iv^j - v^2x^j]\frac{\partial}{\partial v^j}\}\mathcal{F} = \frac{df}{dv}^{(0)}[(q/v)v^k\partial_k\Psi - (v/q)v^k\partial_k\Phi] , \tag{2.25}$$

where we treat $\mathcal{F}$ as a function of $(\tau, \boldsymbol{x}, \boldsymbol{v})$. A derivation of this equation is given in [5] and [9] for the case $K = 0$. The generalization to $K \neq 0$ is discussed in [10].

To connect (2.25) to Einstein's equation, we have to calculate the energy momentum tensor from f. The gauge-invariant perturbation variables are then found to be the following momentum integrals of $\mathcal{F}$ (see [5] and [9]):

$$D_g = \frac{m^4}{a^4\rho^{(0)}}\int v^2 q\mathcal{F}dvd\Omega \tag{2.26}$$

$$l\Box V = \frac{-m^4}{a^4(\rho^{(0)} + p^{(0)})}\int v^2v^i\partial_i\mathcal{F}dvd\Omega \tag{2.27}$$

$$l^2\Pi_{|ij} = \frac{m^4}{a^4p^{(0)}}\int \frac{v^2}{q}(v^iv^j - \frac{v^2}{3}\delta_{ij})\mathcal{F}dvd\Omega \tag{2.28}$$

$$\Gamma = \frac{m^4}{a^4p^{(0)}}\int (v^4/3q - c_s^2v^2q)\mathcal{F}dvd\Omega . \tag{2.29}$$

$$\tag{2.30}$$

These matter variables inserted in Einstein's equations (2.17) and (2.19) yield the geometrical perturbations Ψ and Φ which enter in (2.25). In Section 3 we shall discuss how this closed system is altered in the presence of seeds.

2.2 Sachs–Wolfe effect

On their way from the last scattering surface into our antennas, the microwave photons travel through a perturbed Friedman geometry. Thus, even if the photon energy density was completely homogeneous at the last scattering surface, we would receive it slightly perturbed. — This is the Sachs–Wolfe effect [11]. we will now calculate it to first order perturbation theory.

For sake of simplicity we shall restrict ourselves to $K = 0$ in this subsection. Two metrics which are conformally equivalent,

$$d\bar{s}^2 = a^2ds^2 ,$$

have the same lightlike geodesics, only the corresponding affine parameters are different. Let us denote the affine parameters by $\bar{\lambda}$ and λ respectively and the tangent vectors to the geodesic by

$$n = \frac{dx}{d\lambda} \text{ and } \bar{n} = \frac{dx}{d\bar{\lambda}} \ , \quad n^2 = \bar{n}^2 = 0 \ , \quad n^0 = 1 \ , \quad \boldsymbol{n}^2 = 1$$

Setting $n^0 = 1 + \delta n^0$ the geodesic equation for

$$ds^2 = (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu$$

yields to first order

$$\delta n^0|_i^f = [h_{00} + h_{0i}n^i]_i^f - 1/2\int_i^f \dot{h}_{\mu\nu}n^\mu n^\nu d\lambda \ . \tag{2.31}$$

The ratio of the energy of a photon measured by some observer at t_f to the energy emitted at t_i is given by

$$E_f/E_i = \frac{(\bar{n}\cdot u)_f}{(\bar{n}\cdot u)_i} = (T_f/T_i)\frac{(n\cdot u)_f}{(n\cdot u)_i} \ , \tag{2.32}$$

where u_f and u_i are the four velocities of the observer and the emitter respectively and the factor T_f/T_i is the usual redshift which relates n and $\bar{n}$. We write T_f/T_i and not a_f/a_i here since also this redshift is slightly perturbed in general and we want to reserve a to denote the unperturbed background expansion factor.

Since this is a physical, in principle measurable quantity it is independent of coordinates. It must thus be possible to write it in terms of gauge-invariant variables. The gauge-invariant expression for E_f/E_i can be obtained in the following way: Let us assume the observer and emitter are moving with the cosmic fluid. We thus have

$$u = (1-A)\partial_t - lv^{|i}\partial_i \ .$$

Further, since the photon density may itself be perturbed

$$T_f/T_i = (a_i/a_f)(1 + \frac{\delta T_f}{T_f} - \frac{\delta T_i}{T_i}) = a_i/a_f(1 + (1/4)\delta^{(\gamma)}|_i^f) \ ,$$

where $\delta^{(\gamma)}$ is the intrinsic density perturbation in the radiation. This term was neglected in the original analysis of Sachs and Wolfe, but since it is gauge dependent doing so violates gauge invariance. We therefore keep it for the time being . Inserting all this and (2.31) into (2.32) yields

$$E_f/E_i = (a_i/a_f)[1 - n^j v_{,j}|_i^f + A|_i^f - (1/4)\delta^{(\gamma)}|_i^f - 1/2\int_i^f \dot{h}_{\mu\nu}n^\mu n^\mu d\lambda] \ .$$

With the help of equation (2.7) for the definition of $h_{\mu\nu}$ one finds after several integrations by part

$$E_f/E_i = (a_i/a_f)\{1 - [(1/4)D_s^{(\gamma)} + lV_{|j}^{(m)}n^j - \Psi]|_i^f - \int_i^f (\dot{\Phi} - \dot{\Psi})d\lambda\} \ . \tag{2.33}$$

Here $D_s^{(\gamma)}$ denotes the density perturbation in the radiation and $V^{(m)}$ is the peculiar velocity of the matter component (the emitter and observer of radiation).

For a discussion of the Sachs–Wolfe effect we neglect the intrinsic density perturbation of the radiation, i.e., we set $D_s^{(\gamma)} = 0$, which now is a gauge-invariant statement. $V^{(m)}$ is a Doppler term due to the relative motion of the emitter and receiver. The Ψ – term accounts for the redshift due to the gravitational field and the integral is a path dependent contribution to the redshift. (A similar equation was obtained in [12].)

3 Gauge-invariant perturbation theory in the presence of seeds

By seeds we mean density perturbations originating from an inhomogenously distributed form of energy whose mean density is much smaller than the density of the Friedman background, e.g. a first generation of stars, primordial blackholes, cosmic strings, texture, ... We assume that these seeds do not interact with the rest of the matter other than gravitationally.

Since the energy momentum tensor of the seeds, $T^{\mu\nu}_{(s)}$ has no homogeneous background contribution, it is gauge-invariant by itself according to (2.2).

$T^{\mu\nu}_{(s)}$ can be calculated by solving the matter equations for the seeds in the Friedman *background* geometry. (Since $T^{\mu\nu}_{(s)}$ has no background component it satisfies the unperturbed matter and conservation equations.) Let us assume that we can express the solution $T^{\mu\nu}_{(s)}$ in terms of scalar functions: If not we just neglect vectorial and tensorial contributions. Since they do not give rise to density perturbations and since they decouple within linear perturbation theory, this will not affect our results.

$$T^{(s)}_{00} = a^2\rho^{(s)} = (M^2/l^2)f_\rho \; , \tag{3.34}$$

$$T^{(s)}_{i0} = -a^2 l v^{(s)}_{|i} = -(M^2/l)f_{v\;|i} \; , \tag{3.35}$$

$$\begin{aligned} T^{(s)}_{ij} &= a^2[(p^{(s)} - (l^2/3)\Box\Pi^{(s)}) + l^2\Pi^{(s)}_{|ij}] \\ &= M^2[f_p/l^2 - (1/3)\Box f_\pi)\gamma_{ij} + f_{\pi\;|ij}] \; . \end{aligned} \tag{3.36}$$

Here l is introduced merely to keep the functions $f.$ dimensionless. It may be chosen to denote a typical size of the seeds. M denotes a typical mass of the seeds. (It is of course possible to choose $l = M^{-1}$.)

If we are given the energy momentum tensor $T^{(s)}_{\mu\nu}$ which may still contain vectorial and tensorial contributions, the scalar parts f_v and f_π are in general determined by the identities

$$T^{(s)\;|j}_{0j} = (M^2/l)\Box f_v$$

$$(T^{(s)}_{ij} - 1/3\gamma_{ij}\gamma^{kl}T_{kl})^{|ij} = \frac{2}{3}M^2(\Box + 3K)\Box f_\pi \; .$$

On the other hand $\Box f_v$ and $\Box(\Box+3K)f_\pi$ are also determined in terms of f_ρ and f_p by the conservation equations:

$$\dot{f}_\rho - l\Box f_v + (\dot{a}/a)(f_\rho + 3f_p) = 0 \tag{3.37}$$

$$l\dot{f}_v + 2(\dot{a}/a)lf_v + f_p + (2/3)l^2(\Box + 3K)f_\pi = 0 \tag{3.38}$$

For seeds the energy momentum tensor is determined by background variables alone. Interactions with the perturbations of the other components do not contribute to first order.The geometrical perturbations can then be separated into a part induced by the seeds and a part caused by the perturbations in the remaining matter components:

$$\Psi = \Psi_s + \Psi_m \quad \text{and} \quad \Phi = \Phi_s + \Phi_m \; .$$

By Einstein's equations we can directly calculate the geometry perturbations induced by the seeds:

$$-(\Box + 3K)\Phi_s = \epsilon(f_\rho/l^2 + 3(\dot a/a)f_v/l) \;, \tag{3.39}$$

$$\Box(\Phi_s + \Psi_s) = -2\epsilon\Box f_\pi \;, \tag{3.40}$$

$$\tag{3.41}$$

where $\epsilon = 4\pi GM^2$ is assumed to be much smaller than 1, to justify linear perturbation analysis.
The geometry perturbations induced by the matter, Ψ_m and Φ_m are determined by equations (2.17) to (2.19) as before. But in the conservation equations and in any matter equations the full geometry perturbations, Ψ and Φ have to be inserted.

We now have a closer look at the example of a single fluid where Π and Γ are given in terms of D and V. We assume that in addition to the seeds we have one perturbed matter component which we indicate by a subscript m. Other components which contribute to the background but whose perturbations can be neglected may also be present. The conservation equation (2.21) then reads

$$\dot D_m - 3w_m(\dot a/a)D_m = (\Box + 3K)[(1+w_m)lV_m + 2(\dot a/a)w_m l^2\Pi_m] + 3(1+w_m)\epsilon f_v/l \;. \tag{3.42}$$

The last term describes the additional work done upon the spacetime due to the perturbation of the expansion rate by the seeds.

Solving this equation for $(\Box + 3K)lV_m$ and inserting the result and its time derivative into (2.22) yields a second order equation for D_m. Using

$$(\Box + 3K)\Psi = 4\pi G\rho a^2(D_m - 2w_m l^2(\Box + 3K)\Pi_m) + \epsilon(f_\rho/l^2 - 2(\Box + 3K)f_\pi)$$

and the conservation equation (3.38) we find

$$\begin{aligned}
&\ddot D - (\Box + 3K)c_s^2 D + (1 + 3c_s^2 - 6w)(\dot a/a)\dot D - 3(w(\ddot a/a) - 3(\dot a/a)^2(c_s^2 - w) + \\
&+(1+w)(4\pi/3)G\rho a^2)D \;\; = \\
&(\Box + 3K)w\Gamma + 2(\dot a/a)wl^2(\Box + 3K)\dot\Pi \\
&+\{2(\ddot a/a)w - 6(\dot a/a)^2(c_s^2 - w) + (1+w)8\pi Ga^2p + 2/3(\Box + 3K)w\}l^2(\Box + 3K)\Pi \\
&+(1+w)\epsilon(f_\rho + 3f_p)/l^2 \;,
\end{aligned} \tag{3.43}$$

where we have dropped the subscript m.
This equation describes the behavior of perturbations in the presence of seeds in an arbitrary Friedman background. We have not used Friedman's equations to express $\ddot a/a$ in terms of w and $\dot a/a$, or ρ in terms of $(\dot a/a)^2$ so that (3.43) is valid also if there are unperturbed components which contribute to the expansion but not to the perturbation. Note that within this gauge-invariant treatment the source term is up to a factor $(1+w)$ just the naively expected term $4\pi Ga^2(\rho^{(s)} + 3p^{(s)})$ for all types of fluids.

Let us now simplify equation (3.43) in the case where $\Pi = \Gamma = 0$ (adiabatic perturbations and no anisotropic stresses) and $K = 0$:

$$\ddot D - \Box c_s^2 D + (1 + 3c_s^2 - 6w)(\dot a/a)\dot D - 3[w(\ddot a/a) - 3(\dot a/a)^2(c_s^2 - w) + (1+w)(4\pi/3)G\rho a^2]D = S \;, \tag{3.44}$$

where $S = (1+w)\epsilon(f_\rho + 3f_p)/l^2$.

If we Fourier transform (3.44) we find

$$\ddot{D}+k^2c_s^2D+(1+3c_s^2-6w)(\dot{a}/a)\dot{D}-3[w(\ddot{a}/a)-3(\dot{a}/a)^2(c_s^2-w)+(1+w)(4\pi/3)G\rho a^2]D = \tilde{S} \ . \quad (3.45)$$

$\tilde{S} = (1+w)\epsilon(\tilde{f}_\rho + \tilde{f}_p)/l^2$ is the Fourier transform of S and we denote the Fourier transform of D again with D.

If we know the homogeneous solutions D_1 and D_2 of (3.45), we can find the perturbation induced by S by the Wronskian method:

$$D = c_1D_1 + c_2D_2 \ \text{ with} \quad (3.46)$$

$$c_1 = -\int(\tilde{S}D_2/W)d\tau \quad , \quad c_2 = \int(\tilde{S}D_1/W)d\tau \ , \quad (3.47)$$

where $W = D_1\dot{D}_2 - \dot{D}_1D_2$ is the Wronskian determinant of the homogenous solution. This leads to the following general behavior: If the time dependence of D_1, D_2 and $\tilde{S}$ can be approximated by power laws, the contribution to D with maximum growth behaves like $D \propto \tilde{S}\tau^2$. If D_1 and D_2 are waves with approximately constant amplitude and frequency ω, D can be approximated by a wave with amplitude proportional to $\omega^{-1}\int e^{i\omega\tau}\tilde{S}d\tau$. Thus after a long time only typical frequencies of the source survive.

As a second example, we look at collisionless particles. The source term on the r.h.s. of Liouville's equation, (2.25) can be separated as above into a part due to the collisionless component and a part induced by the seeds. Equation (2.25) then becomes

$$\{q\partial_\tau + v^k\partial_k + K/2[x^iv_iv^j - v^2x^j]\frac{\partial}{\partial v^j}\}\mathcal{F} = \frac{df}{dv}^{(0)}[(q/v)v^k\partial_k\Psi_m - (v/q)v^k\partial_k\Phi_m] + \mathcal{S} \ , \quad (3.48)$$

with

$$\mathcal{S} = \frac{df}{dv}^{(0)}[(q/v)v^k\partial_k\Psi_s - (v/q)v^k\partial_k\Phi_s] \ . \quad (3.49)$$

If one chooses a density parameter $0.2 \le \Omega \le 1$, which one might do in a realistic calculation, the curvature term in (3.48) can always be neglected at early times, e.g., for redshifts $z \ge 5$. It is of the order $(l/l_K)^2$ as compared to the other contributions. (l and $l_K = K^{-1/2}$ denote the typical size of the perturbation and the radius of curvature respectively.)

With the integrals for the fluid variables D_g, V, Γ and Π as given in Section 2 and Einstein's equations (2.17) to (2.19) for the geometrical perturbations Ψ_m and Φ_m induced by the collisionless component this forms a closed system.

4 The texture knot

As an analytic application we now discuss the perturbations induced in the microwave background, in a cosmic dust component (baryons) and in collisionless particles by a single texture knot [2] on subhorizon scales.

Turok & Spergel [4] have obtained (to first order perturbation theory) the energy momentum tensor of a spherical texture knot in flat spacetime:

$$\begin{aligned} \delta T_{00} &= 2\eta^2\tfrac{r^2+3t^2}{(r^2+t^2)^2} \\ \delta T_{0i} &= \eta^2(\tfrac{2t}{r^2+t^2})_{,i} \\ \delta T_{ij} &= 2\eta^2\tfrac{r^2-t^2}{(r^2+t^2)^2}\delta_{ij} \ , \end{aligned} \quad (4.50)$$

where η is given by the symmetry breaking scale.
We now set $M^2 = 4\eta^2$ and $l = r_c$, the core radius of a texture knot, i.e., the scale where the σ-model treatment of the field equations for texture and therefore the result (4.50) break down (see [4]). Defining $x = r/r_c$, $y = t/r_c$ we find with the terminology of the previous section

$$f_\rho = (1/2)\frac{x^2+3y^2}{(x^2+y^2)^2} , \qquad (4.51)$$

$$f_v = (1/2)\frac{y}{(x^2+y^2)} , \qquad (4.52)$$

$$f_p = (1/2)\frac{x^2-y^2}{(x^2+y^2)^2} , \qquad (4.53)$$

$$f_\pi = 0 \qquad (4.54)$$

The variable t in (4.50) is defined to vanish when the knot collapses. It is not the age of the universe. From equations (3.39) to (3.40) in Minkowski space, i.e., $\dot{a} = 0$ we find the following perturbation of the geometry:

$$\Phi_s = -1/4\epsilon\log(\frac{r^2+t^2}{r_c^2}) \qquad (4.55)$$

$$\Psi_s = -\Phi_s + f(t) , \qquad (4.56)$$

with $\epsilon = 16\pi G\eta^2$. Ψ_s is only determined up to a function of time, which we choose to insure $\Psi_s \to 0$, for $t \to \pm\infty$. This initial condition yields

$$\Psi_s = 1/4\epsilon\log(\frac{r^2+t^2}{t^2}) . \qquad (4.57)$$

Of course, physical observables shall not depend on this choice.

The 3–dimensional Riemann scalar on the surfaces of constant time is then

$$\delta^3 R = a^{-2}\Box\mathcal{R} \approx a^{-2}\Box\Phi = -\frac{\epsilon}{2a^2}\frac{r^2+3t^2}{(r^2+t^2)^2} , \qquad (4.58)$$

where we again have neglected the expansion of the background. So, far away from the collapsing knot and at early and late times we approach flat space.

4.1 Distortion of the microwave sky

This result can be used to calculate the energy shift which a photon experiences by passing a texture knot. If we neglect the distinctive dipole term and intrinsic density perturbations, equation (2.33) leads to

$$(\delta E/E)_i^f = (a_i/a_f)[\int_i^f(\dot{\Phi}-\dot{\Psi})d\lambda + \Psi|_i^f] . \qquad (4.59)$$

Denoting the impact parameter of the photon trajectory by R and the time when the photon passes the texture knot by t_*, we get $r^2 = R^2 + (t-t_*)^2$. (4.59) then yields

$$(\delta E/E)_i^f = (a_i/a_f)\frac{-\epsilon t_*}{(t_*^2+2R^2)^{1/2}}[\mathrm{arctg}(\frac{2t-t_*}{(t_*^2+2R^2)^{1/2}})]_i^f . \qquad (4.60)$$

For $t_f,\ -t_i \gg t_*,\ R$, we obtain

$$(\delta E/E)_i^f \approx -(\epsilon\pi/2)\frac{t_*}{(t_*^2+2R^2)^{1/2}}\cdot\frac{a_i}{a_f}\ . \tag{4.61}$$

Another derivation of this result and its interpretation is presented in [4]. We just notice that those photons which pass the texture knot before it collapses, $t_* < 0$ are blueshifted and those passing the knot after collapse $t_* > 0$ are redshifted. This results in a very distinctive hot spot — cold spot signal in the microwave sky wherever a texture has collapsed.

Of course our result is not strictly correct in the expanding universe since we neglected expansion in the calculation of Ψ_s and Φ_s. But since the main contribution to the energy shift comes from times $|t| \le |t_*| + R$, our approximation is reasonable also for the expanding case, if $|t_*| \le R_H$ and $R \le R_H$, where R_H denotes the horizon distance at the time of collapse, $t = 0$. On the other hand, by causality arguments the texture cannot have a big effect on photon trajectories with $|t_*| > R_H$ or $R > R_H$. A good approximation to the situation in the expanding universe is thus

$$(\frac{\delta E}{E})_i^f = \begin{cases} -\epsilon\pi/2\frac{t_*}{(t_*^2+2R^2)^{1/2}}(a_i/a_f) & ,\ \text{for } |t_*| < R_H \text{ and } R < R_H \\ 0 & ,\ \text{for } |t_*| > R_H \text{ or } R > R_H\ . \end{cases} \tag{4.62}$$

4.2 Baryons around a texture knot

Let us now briefly discuss the behavior of cosmic dust (baryons) in the field of a single texture knot. Equation (3.43) for a knot in a flat dust universe ($c_s^2 = w = 0$, $\dot{a} = 0$) yields

$$\frac{d^2D}{dy^2} + 4\pi G\rho^{(0)}r_c^2 D = S\ , \tag{4.63}$$

with

$$S = 2\epsilon\frac{x^2}{(x^2+y^2)^2}\ .$$

The term $4\pi G\rho^{(0)}r_c^2 D$ leads to exponential growth of perturbations which is a feature of the non expanding universe only. But our approximation, neglecting expansion, means that all times involved are much smaller than Hubble time. This coincides with $4\pi G\rho^{(0)}r_c^2 \ll 1/x$. Within our approximation it is thus consistent to neglect this self gravitating term in (4.63). Direct integration then yields the solution

$$\begin{aligned} D &= \epsilon[(y/x)\mathrm{arctg}(y/x) + c_1(x)y + c_2(x)] \\ &= \epsilon(t/r)[\mathrm{arctg}(t/r) + \pi/2] + \epsilon\ , \end{aligned} \tag{4.64}$$

where we have chosen the integration constants c_1 and c_2 such that D converges to 0 for large negative times, $D(t = -\infty) = 0$ and D converges to a constant for large radii, $D(t, r = \infty) = \epsilon$. Since D is only a function of the self similarity variable t/r we cannot consistently choose both boundary conditions to be 0. For late times, $t/r \gg 1$ D grows linearly with time:

$$D = \epsilon\pi(t/r)\ ,$$

Near the time of collapse, $|t/r| \ll 1$, D is of the order of ϵ, $D(t = 0, r) = \epsilon$. At a given time t_* after the knot has collapsed which is small compared to the Hubble time, D has the following profile: For large radii $D \propto 0.5\epsilon\pi t_*/r + \epsilon$ and roughly at $r = t_*$ bends into $D \propto \epsilon\pi t_*/r$ and diverges for $r \to 0$. This divergence leads to early formation of nonlinear structure on small scales. At time t_* perturbations on scales of the order of $r \leq r_{nl} = \epsilon\pi t_*$ have become nonlinear.

The total mass accumulated around a texture diverges like the mass of the texture itself (see [4]). But in the real, expanding universe one has to cut it of at roughly the Hubble radius at the time when the texture collapses, R_H.

In this simple approximation we end up with the following picture: Due to the textures forming at a time t in the universe, objects of mass $M \approx 2\epsilon M_H(t)$, form at separations on the order of $p^{-1}R_H(t)$. Where M_H denotes the horizon mass at the time when the texture collapses and p is the probability that a four component vector field which is distributed in a completely uncorrelated manner over a 2-sphere winds around a 3-sphere (i.e. the probability of texture formation at the horizon). This probability has been found to be substantially less than 1 [13].

A thorough discussion of the linear perturbation spectrum which is expected from this simple picture will be presented elsewhere [14].

From matter conservation, (3.42) for $w = 0$ we obtain

$$\Box V = \epsilon/r[\text{arctg}(t/r) + \pi/2] - (\epsilon/2)\frac{t}{r^2 + t^2}$$

and therefore

$$v_i = -\partial_i V = -(\epsilon/2)n_i[\text{arctg}(t/r) + \pi/2]. \tag{4.65}$$

The total change in a particle's velocity as the knot collapses is thus independent of the particles distance from the knot and is given by

$$\Delta v_j = v_j(\infty) - v_j(-\infty) = -(\epsilon/2)\pi n_j \tag{4.66}$$

in agreement with [4].

A numerical calculation for the distribution of texture in an expanding Friedman universe, where the growth of density perturbations is given according to (3.43) with $w = c_s^2 = 0$, is underway [13].

4.3 Collisionless particles around a texture knot

Let us calculate the perturbations in the distribution function of collisionless particles induced by a texture knot. As for dust, we neglect self gravity. Setting $K = 0$ in (3.48) we obtain

$$q\partial_t\mathcal{F} + v^k\partial_k\mathcal{F} = \frac{df}{dv}^{(0)}[(q/v)v^k\partial_k\Psi_s - (v/q)v^k\partial_k\Phi_s] \equiv \mathcal{S}\ , \tag{4.67}$$

where Ψ_s and Φ_s are the metric perturbations due to the texture. Inserting the results (4.57) and (4.55) yields

$$\mathcal{S} = (\epsilon/2)\frac{df}{dv}^{(0)}(q/v + v/q)\frac{\boldsymbol{v}\cdot\boldsymbol{x}}{r^2 + t^2}\ . \tag{4.68}$$

The general solution of (4.67) is

$$\mathcal{F}(t, \boldsymbol{x}, \boldsymbol{v}) = \mathcal{F}(t_0, \boldsymbol{x} - \boldsymbol{w}(t - t_0), \boldsymbol{v}) + \int_{t_0}^{t}\mathcal{S}(t', \boldsymbol{x} - \boldsymbol{w}(t - t'), \boldsymbol{v})dt'\ , \tag{4.69}$$

with $\boldsymbol{w} = \boldsymbol{v}/q$.
In the case of the texture knot the integral in (70) can be solved analytically. Let us assume that at some initial time t_0, long before the knot collapses, we can neglect the perturbation of the distribution function, $\mathcal{F}(t_0, ...) = 0$. Then $\mathcal{F}$ is given by the integral above:

$$\mathcal{F}(t,\boldsymbol{x},\boldsymbol{v}) \quad = (\epsilon/2)\tfrac{df}{dv}^{(0)}(q/w) \quad \{\frac{\boldsymbol{w}\cdot(\boldsymbol{x}-\boldsymbol{w}t)}{\Delta}[\mathrm{arctg}(\frac{\boldsymbol{x}\cdot\boldsymbol{w}+t}{\Delta}) - \mathrm{arctg}(\frac{\boldsymbol{x}\cdot\boldsymbol{w}-w^2(t-t_0)+t_0}{\Delta})] + \frac{w^2}{2}\log(\frac{r^2+t^2}{(\boldsymbol{x}-(t-t_0)\boldsymbol{w})^2+t_0^2})\} \; . \qquad (4.70)$$

Here Δ is given by

$$\Delta^2 = w^2r^2 - (\boldsymbol{w}\cdot\boldsymbol{x})^2 + (\boldsymbol{x}-\boldsymbol{w}t)^2 \; .$$

This result has to be inserted in equations (2.26) to (2.29) to obtain the induced perturbations of the energy momentum tensor. Since we neglected expansion of space, it is not worth doing this with the full result (4.70). Instead we take the nonrelativistic limit, $w \ll 1$. Then $\Delta \approx |\boldsymbol{x}-\boldsymbol{w}t|$ and the logarithmic term in (4.70) can be neglected.

Nonrelativistic limit:

$$\mathcal{F}(t,\boldsymbol{x},\boldsymbol{v}) = (\epsilon/2)q\frac{df}{dv}^{(0)}\frac{\boldsymbol{w}\cdot(\boldsymbol{x}-\boldsymbol{w}t)}{w|\boldsymbol{x}-\boldsymbol{w}t|}[\mathrm{arctg}(\frac{\boldsymbol{x}\cdot\boldsymbol{w}+t}{|\boldsymbol{x}-\boldsymbol{w}t|}) + \pi/2] \; , \qquad (4.71)$$

where we have taken the limit $t_0 \to -\infty$.
We want to use this result to calculate the induced velocity perturbations. In the baryonic case we saw that most of the velocity perturbations at a given distance r are induced at times $t \le r$. We thus make the additional assumption $\mathcal{O}(t) \le \mathcal{O}(r)$ so that terms wt are much smaller than t and r. With this additional approximation we find

$$\mathcal{F} = (\epsilon/2)\frac{df}{dv}^{(0)} q\frac{\boldsymbol{x}\cdot\boldsymbol{w}}{wr}[\mathrm{arctg}(t/r) + \pi/2] \; . \qquad (4.72)$$

Setting $\mu = \boldsymbol{x}\cdot\boldsymbol{w}/(wr)$ we then obtain

$$v^i\partial_i\mathcal{F} = (\epsilon/2)\frac{df}{dv}^{(0)}\{(1-\mu^2)[\mathrm{arctg}(t/r)+\pi/2] - \mu^2\frac{t}{r^2+t^2}\}vq \; . \qquad (4.73)$$

Using now $\int(1-\mu^2)d\Omega = 8\pi/3$ and $\int\mu^2 d\Omega = 4\pi/3$ equation (2.27) yields

$$l\Box V = -\epsilon[\mathrm{arctg}(t/r) - \frac{t}{2(t^2+r^2)}]\frac{m^4 4\pi}{3a^4(\rho^{(0)}+p^{(0)})}\int v^3 q\frac{df^{(0)}}{dv}dv \; . \qquad (4.74)$$

Reinserting the definitions $|\boldsymbol{p}| = vm/a$, $p^0 = qm/a$ and integrating by parts finally leads to

$$l\Box V = \epsilon\;\mathrm{arctg}(t/r) - (\epsilon/2)\frac{t}{r^2+t^2} \; . \qquad (4.75)$$

As expected, we obtain the same result as for cosmic dust, i.e. baryons (4.65).

Of somewhat more interest is of course the corresponding calculation in the expanding universe, for which we leave for a future project.

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