Dynamical Instabilities of the Randall–Sundrum Model

Timon Boehm[1], Ruth Durrer[1] and Carsten van de Bruck[2]
[1] *Département de Physique Théorique, Université de Genève, 24 quai E. Ansermet, CH–1211 Geneva 4 (Switzerland).*
[2] *Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK*
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We derive dynamical equations describing a single 3-brane containing fluid matter and a scalar field coupling to the dilaton and the gravitational field in a five dimensional bulk. First, we show that a scalar field or an arbitrary fluid on the brane cannot evolve to cancel the cosmological constant in the bulk. Then we show that the Randall–Sundrum model is unstable under small deviations from the fine–tuning between the brane tension and the bulk cosmological constant and even under homogeneous gravitational perturbations. Implications for brane world cosmologies are discussed.

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I. INTRODUCTION

Up today, string theories are the most promising fundamental quantum theories at hand which include gravity. Open strings carry gauge charges and end on so-called Dp–branes, $p+1$ dimensional hyper-surfaces of the full spacetime. Correspondingly, gauge fields may propagate only on the $p+1$ dimensional brane, and only modes associated with closed strings, like the graviton, the dilaton and the axion, live in the full spacetime [1]. Super–string theories and especially M–theory suggest that the observable universe is a $3+1$ dimensional hyper-surface, a 3–brane, of a 10 or 11 dimensional spacetime. This fundamental spacetime could be a product of a four dimensional Lorentz manifold with an n dimensional compact space of volume V_n (n is the number of extra dimensions). Then, the relation between the $4+n$ dimensional fundamental Planck mass, $M_{\rm f}$, and the effective four dimensional Planck mass, $M_{\rm eff} \equiv \sqrt{1/(8\pi G_N)} \simeq 2.4 \times 10^{18}$ GeV, is

$$M_{\rm eff}^2 = M_{\rm f}^{n+2} V_n. \tag{1}$$

If some of the extra dimensions are much larger than the fundamental Planck scale, $M_{\rm f}$ is much smaller than $M_{\rm eff}$ and may even be close to the electro-weak scale, thereby relieving the long-standing hierarchy problem [2]. For example, if one allows for two 'large' extra dimensions of the order of 1 mm, one obtains a fundamental Planck mass of 1 TeV. However, a new hierarchy between the electro-weak scale and the mass-scale associated with the compactification volume, $\frac{1}{{V_n}^{1/n}}$, is introduced.

Clearly, this idea is very interesting from the point of view of bringing together fundamental theoretical high energy physics and experiments, which have been diverging more and more since the advent of string theory. While the four dimensionality of gauge interactions has been tested down to scales of about $1/200\,\mathrm{GeV}^{-1} \simeq 10^{-15}$ mm, Newton's law is experimentally confirmed only above 1 mm. Therefore, 'large' extra dimensions are not excluded and should be tested in the near future by refined micro gravity experiments [3]. The fundamental string scale might in principle be accessible to LHC [4].

In the past, it was commonly assumed that the fundamental spacetime is factorizable, and that the extra dimensional space is compact. Recently, Randall and Sundrum [5] proposed a five dimensional model, in which the metric on the 3–brane is multiplied by an exponentially decreasing 'warp' factor such that transverse lengths become small already at short distances along the fifth dimension. This idea allows for a non–compact extra dimension without getting in conflict with observational facts. In this scenario the brane is embedded in an Anti–de Sitter space, and a fine–tuning relation,

$$\Lambda = -\frac{{\kappa_5}^2}{6} V^2, \tag{2}$$

between the brane tension, V, and the negative cosmological constant in the bulk, Λ, has to be satisfied. Here, κ_5 is related to the five dimensional Newton's constant by ${\kappa_5}^2 = 8\pi G_5 = {M_5}^{-3}$. Randall–Sundrum also proposed a model with two branes of opposite tension which provides an elegant way to relieve both hierarchy problems mentioned above [6]. However, also this model requires the fine–tuning (2). Here, we will only consider the case of a single brane.

The main unattractive feature of the Randall–Sundrum (RS) model is the fine-tuning condition (2). Both from the particle physics and the cosmological point of view this relation between two a priori independent quantities appears unlikely. One would like to put it on a physical basis, such as a fundamental principle, or explain it due to some dynamical process.

This paper has the purpose to point out the cosmological problems associated with the fine–tuning condition (2). The outline of the paper is as follows: In section II, we derive dynamical equations describing the gravitational field and the dilaton in the bulk coupling to fluid matter and a scalar field on the brane. These equations allow for a dynamical generalization of the RS model, which is a special static solution of our equations with vanishing dilaton. Our equations also provide a starting point for further studies of various issues in cosmology, for example inflation. In Section III we discuss a cosmological version of the RS model and show that the fine–tuning condition (2) cannot be stabilized by an arbitrary scalar field or fluid on the brane. In section IV we discuss linear perturbations of the static RS model and derive gauge invariant perturbation equations from our general setup. We prove that the full RS spacetime is unstable against homogeneous processes on the brane such as cosmological phase transitions: The solutions run quadratically fast away from the static RS spacetime. This instability reminds that of the static homogeneous and isotropic Einstein universe [7]. In linear perturbation theory the RS spacetime is unstable even against purely gravitational perturbations. In the last section we present our results and the conclusions.

II. EQUATIONS OF MOTION

In this section we provide the equations of motion. For generality and for future work we have included the dilaton, although its does not play a role in the present discussion of RS stability. Works on dilaton gravity and the brane world have also been done by [8] and [9].

A. General case

We consider a five dimensional spacetime with a metric g_{MN} parameterized by coordinates $(x^M) = (x^\mu, y)$, where $M = 0,1,2,3,5$ and $\mu = 0,1,2,3$, with a 3–brane fixed at $y = 0$. We shall use units in which $2{\kappa_5}^2 = 1$. In the string frame our action is

$$\begin{aligned} S_{\rm string} &= \int d^5x\sqrt{-g}e^{-2\phi}\left(R + 4(\nabla_M\phi)(\nabla_N\phi)g^{MN} - \Lambda(\phi)\right) \\ &- \int d^4x\sqrt{-\overline{g}}e^{-2\phi}\left(\frac{1}{2}(\overline{\nabla}_\mu\varphi)(\overline{\nabla}_\nu\varphi)\overline{g}^{\mu\nu} + V(\varphi) + \mathcal{L}_{\rm fluid}\right), \end{aligned} \tag{3}$$

which describes the dilaton, ϕ, coupling to gravity as well as to a scalar field, φ, with a potential $V(\varphi)$ and to a fluid. The graviton, the dilaton and the 'bulk potential' $\Lambda(\phi)$ live in the five dimensional (bulk) spacetime, whereas the fluid and the scalar field are confined to the brane. The induced four dimensional metric is[1]

$$\overline{g}_{\mu\nu} = \delta_\mu^M\delta_\nu^N g_{MN}(y=0). \tag{4}$$

The action in the Einstein–frame is obtained by the conformal transformation

$$g_{MN} \to e^{-\frac{4\phi}{D-2}}g_{MN} \tag{5}$$

with $D = 5$. We find

$$\begin{aligned} S_{\rm Einstein} &= \int d^5x\sqrt{-g}\left(R - \frac{4}{3}(\nabla_M\phi)(\nabla_N\phi)g^{MN} - e^{(4/3)\phi}\Lambda(\phi)\right) \\ &- \int d^4x\sqrt{-\overline{g}}\left(\frac{1}{2}e^{-(2/3)\phi}(\overline{\nabla}_\mu\varphi)(\overline{\nabla}_\nu\varphi)\overline{g}^{\mu\nu} + e^{(2/3)\phi}\left(V(\varphi) + \mathcal{L}_{\rm fluid}\right)\right), \end{aligned} \tag{6}$$

[1] Where confusion could arise, we over-line four dimensional quantities.

where g now denotes the metric tensor in the Einstein–frame, and $R, \nabla, \overline{\nabla}$ are constructed from g. From this action we derive the equations of motion by varying with respect to the dilaton, the brane scalar field and the metric:

$$\frac{8}{3}\nabla^2\phi - \frac{4}{3}e^{(4/3)\phi}\Lambda(\phi) - e^{(4/3)\phi}\frac{\partial\Lambda(\phi)}{\partial\phi} + \frac{\sqrt{-\overline{g}}}{\sqrt{-g}}\delta(y)\left(\frac{1}{3}e^{-(2/3)\phi}(\overline{\nabla}\varphi)^2 - \frac{2}{3}e^{(2/3)\phi}(V(\varphi)+\mathcal{L}_{\text{fluid}})\right) = 0, \tag{7}$$

$$e^{-(2/3)\phi}(\overline{\nabla}^2\varphi) - e^{(2/3)\phi}\frac{\partial V(\varphi)}{\partial\varphi} = 0, \tag{8}$$

$$G_{MN} = \frac{4}{3}\left((\nabla_M\phi)(\nabla_N\phi) - \frac{1}{2}g_{MN}(\nabla\phi)^2\right) - \frac{1}{2}g_{MN}e^{(4/3)\phi}\Lambda(\phi) \tag{9}$$

$$- \frac{\sqrt{-\overline{g}}}{\sqrt{-g}}\delta(y)\delta^\mu_M\delta^\nu_N\left(-e^{-(2/3)\phi}\frac{1}{2}\left((\overline{\nabla}_\mu\varphi)(\overline{\nabla}_\nu\varphi) - \frac{1}{2}\overline{g}_{\mu\nu}(\overline{\nabla}\varphi)^2\right) + \frac{1}{2}\overline{g}_{\mu\nu}e^{(2/3)\phi}V(\varphi) - \frac{1}{2}e^{(2/3)\phi}(T_{\text{fluid}})_{\mu\nu}\right),$$

where G_{MN} is the five dimensional Einstein tensor of the metric g_{MN}.

As we are interested in cosmological solutions, we require the 3–brane to be homogeneous and isotropic and make the ansatz

$$ds^2 = -e^{2N(t,y)}dt^2 + e^{2R(t,y)}d\vec{x}^2 + e^{2B(t,y)}dy^2\ , \tag{10}$$

where we have assumed the ordinary spatial dimensions to be flat. Note that this metric is not factorizable as the scale factor on the brane, $e^{R(t,y)}$, and the lapse function, $e^{N(t,y)}$, depend on time as well as on the fifth dimension. The factor $e^{B(t,y)}$ is a modulus field. The energy–momentum tensor of a homogeneous and isotropic fluid, representing matter in the universe, is

$$({T_{\text{fluid}}}^\mu{}_\nu(t)) = diag(-\rho(t), p(t), p(t), p(t)), \tag{11}$$

and for the dilaton and the brane scalar field we shall assume $\phi = \phi(t,y), \varphi = \varphi(t)$. Finally, the Lagrangian density of the fluid, $\mathcal{L}_{\text{fluid}}$, is given by its free energy density F (see [10]).

With these assumptions the equations of motion take the following form: (A dot and a prime refer to the derivatives with respect to t and y, and quantities on the brane carry a subscript zero, for example $N_0 \equiv N(t, y=0)$.)

$$\phi: \quad \frac{8}{3}e^{-2N}(\ddot\phi - \dot\phi\dot N + 3\dot\phi\dot R + \dot\phi\dot B) - \frac{8}{3}e^{-2B}(\phi'' + \phi'N' + 3\phi'R' - \phi'B') + \frac{4}{3}e^{(4/3)\phi}\Lambda(\phi)$$
$$+ e^{(4/3)\phi}\frac{\partial\Lambda(\phi)}{\partial\phi} + \delta(y)e^{-B}\left(\frac{1}{3}e^{-(2/3)\phi}e^{-2N}\dot\varphi^2 + \frac{2}{3}e^{(2/3)\phi}(V(\varphi)+F)\right) = 0\ , \tag{12}$$

$$\varphi: \quad e^{-2N_0}(\ddot\varphi - \dot\varphi\dot N_0 + 3\dot\varphi\dot R_0) + e^{(4/3)\phi_0}\frac{\partial V(\varphi)}{\partial\varphi} = 0\ , \tag{13}$$

$$\textit{fluid}: \quad p = p(\rho)\ , \tag{14}$$

$$00: \quad 3e^{-2N}(\dot R^2 + \dot R\dot B - \frac{2}{9}\dot\phi^2) + 3e^{-2B}(-R'' - 2R'^2 + R'B' - \frac{2}{9}\phi'^2) - \frac{1}{2}e^{(4/3)\phi}\Lambda(\phi)$$
$$- \delta(y)e^{-B}\left(\frac{1}{4}e^{-(2/3)\phi}e^{-2N}\dot\varphi^2 + \frac{1}{2}e^{(2/3)\phi}(V(\varphi)+\rho)\right) = 0\ , \tag{15}$$

$$11: \quad e^{-2N}(-2\ddot R - \ddot B - 3\dot R^2 - \dot B^2 + 2\dot N\dot R + \dot N\dot B - 2\dot R\dot B - \frac{2}{3}\dot\phi^2)$$
$$+ e^{-2B}(N'' + 2R'' + N'^2 + 3R'^2 + 2N'R' - N'B' - 2R'B' + \frac{2}{3}\phi'^2) + \frac{1}{2}e^{(4/3)\phi}\Lambda(\phi)$$
$$- \delta(y)e^{-B}\left(\frac{1}{4}e^{-(2/3)\phi}e^{-2N}\dot\varphi^2 + \frac{1}{2}e^{(2/3)\phi}(p - V(\varphi))\right) = 0\ , \tag{16}$$

$$05: \quad \dot R' + \dot R R' - N'\dot R - R'\dot B + \frac{4}{9}\dot\phi\phi' = 0\ , \tag{17}$$

$$55: \qquad 3e^{-2N}(-\ddot{R}-2\dot{R}^2+\dot{N}\dot{R}-\frac{2}{9}\dot{\phi}^2)+3e^{-2B}(N'R'+R'^2-\frac{2}{9}\phi'^2)+\frac{1}{2}e^{(4/3)\phi}\Lambda(\phi)=0 \ . \tag{18}$$

In order to have a well defined geometry, the metric has to be continuous across $y=0$. However, first derivatives with respect to y may not be continuous at $y=0$, and second derivatives may contain delta–functions. Such distributional parts can be treated separately by writing

$$f''=f''_{reg}+\delta(y)[f'] \ , \tag{19}$$

where

$$[f']\equiv\lim_{y\to 0}[f'(y)-f'(-y)] \tag{20}$$

is the jump of f' across $y=0$, and f''_{reg} is the part which is regular at $y=0$. By matching the delta–functions from the second derivatives of ϕ, N and R with those in equations (12), (15) and (16), one obtains the junction conditions

$$[\phi']=\frac{1}{8}e^{-(2/3)\phi_0}e^{B_0-2N_0}\dot{\varphi}^2+\frac{1}{4}e^{(2/3)\phi_0}e^{B_0}(V(\varphi)+F) \ , \tag{21}$$

$$[N']=\frac{5}{12}e^{-(2/3)\phi_0}e^{B_0-2N_0}\dot{\varphi}^2+\frac{1}{6}e^{(2/3)\phi_0}e^{B_0}(3p+2\rho-V(\varphi)) \ , \tag{22}$$

$$[R']=-\frac{1}{12}e^{-(2/3)\phi_0}e^{B_0-2N_0}\dot{\varphi}^2-\frac{1}{6}e^{(2/3)\phi_0}e^{B_0}(V(\varphi)+\rho) \ . \tag{23}$$

Eqs.(22) and (23) are equivalent to Israel's junction conditions [11]. Our equations agree with those found by other authors in special cases, see *e.g.* [12] and [13].

To ensure that our brane does stay in place, we shall assume Z_2 symmetry in the remainder of this paper. Furthermore, we neglect the dilaton and consider a simple cosmological constant in the bulk. For the sake of generality we have not used these assumptions so far.

B. Special case: The Randall–Sundrum model

The RS model is a special static solution of the equations derived in the previous section with $N(y)=R(y), B=0$, when Λ is taken to be a pure cosmological constant, and V represents a constant brane tension. All other fields are set to zero. The RS metric is

$$ds^2=e^{2\alpha|y|}(-dt^2+d\vec{x}^2)+dy^2 \ . \tag{24}$$

Our equations of motion then reduce to

$$6R'^2=-\frac{\Lambda}{2} \ , \tag{25}$$

$$3R''=-\delta(y)\frac{V}{2} \ . \tag{26}$$

Eq. (25) can now be solved by

$$R(y)=-\sqrt{\frac{-\Lambda}{12}}|y|\equiv\alpha|y| \ , \tag{27}$$

which respects Z_2 symmetry and leads to an exponentially decreasing 'warp-factor'. To satisfy simultaneously equation (26), one must fine-tune the brane tension and the (negative) bulk cosmological constant:

$$\Lambda+\frac{V^2}{12}=0 \ . \tag{28}$$

This is the RS solution. A priori, Λ and V are independent constants, and there is no reason for such a relation. However, in a realistic time-dependent cosmological model this relation must be satisfied in order to recover the usual Friedmann equation for a fluid with $\rho\ll V$ see [14]. In the next section we study, whether Eq. (28) can be obtained by some dynamics on the brane.

III. A DYNAMICAL BRANE

We first consider a dynamical scalar field on the brane. The fine-tuning condition (28) corresponds to the requirement that the negative bulk cosmological constant, Λ, can be canceled by the brane tension, V, which we try to identify with the potential energy of the scalar field, φ. If, starting with some initial conditions on φ and $\dot{\varphi}$, the evolution of the system would stabilize at $\Lambda + \frac{V^2}{12} = 0$, the cancelation could be accomplished dynamically. If this would be the case for a 'large class' of initial conditions, the RS solution (27) would be an attractor of the system.

We start from Eqs. (13-18) for the case of a vanishing dilaton. Taking the 'mean value' of the 55 equation across $y = 0$, inserting the junction conditions (22,23) and taking into account Z_2 symmetry, one obtains (see [12])

$$\ddot{R}_0 + 2\dot{R}_0^2 = -\frac{1}{144}\rho_b(\rho_b + 3p_b) + \frac{\Lambda}{6}, \tag{29}$$

where $\rho_b = \rho + \rho_\varphi$ and $p_b = p + p_\varphi$ are the total energy density and the total pressure on the brane due to the fluid and the scalar field. In this section the dot denotes the derivative with respect to the time coordinate τ given by $d\tau = e^{N_0(t)}dt$. Using the energy conservation equation on the brane,

$$\dot{\rho}_b = -3\dot{R}_0(\rho_b + p_b), \tag{30}$$

one can eliminate the pressure and integrate Eq. (29) to obtain a 'Friedmann' equation for the expansion of the brane (see [14])

$$H^2 = \frac{1}{12}\Lambda + \frac{1}{144}{\rho_b}^2 + \frac{\mathcal{C}}{a_0^4}, \tag{31}$$

where $a_0(t) \equiv e^{R(t,y=0)}$ denotes the scale factor on the brane, $H = \dot{a}_0/a_0 = \dot{R}_0$, and $\mathcal{C}$ is an integration constant. If the dilaton vanishes, Eq. (13) becomes the ordinary equation of motion for a scalar field

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{\partial V}{\partial \varphi} = 0 \tag{32}$$

with an energy density and a pressure

$$\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V(\varphi), \tag{33}$$

$$p_\varphi = \frac{1}{2}\dot{\varphi}^2 - V(\varphi). \tag{34}$$

We now assume that the energy density of the scalar field dominates any other component on the brane, that is $\rho_\varphi >> \rho$. This may be the case in the early universe. Later in this section we will see that this assumption does not affect our result. In the same sense we neglect the radiation term, so that equation (31) reduces to

$$H = +\sqrt{\frac{1}{12}\left(\Lambda + \frac{{\rho_\varphi}^2}{12}\right)}. \tag{35}$$

The positive sign corresponds to an expanding brane. The question whether the system evolves towards $\Lambda + \frac{V^2}{12} = 0$ is now translated into the question whether the Hubble–parameter vanishes at some time τ_1. From Eqs. (32) and (35) together with Eq. (33) one finds

$$\dot{H} = -\frac{1}{48}\rho_\varphi\dot{\varphi}^2, \tag{36}$$

which is always negative. (The case $\dot{\varphi}(\tau_1) = 0$ simultaneously with $H(\tau_1) = 0$ will be treated separately.) Starting with an expanding universe, $H > 0$, this implies that, indeed, H is decreasing and $H = 0$ may well be obtained within finite or infinite time depending on the details of the potential $V(\varphi)$. However, at τ_1 the scale factor has reached a maximum ($\ddot{a}_0(\tau_1) = a_0(\tau_1)\dot{H}(\tau_1) < 0$) and, after a momentary cancelation of Λ with ${\rho_\varphi}^2$, H changes sign and the brane begins to contract with

$$H = -\sqrt{\frac{1}{12}\left(\Lambda + \frac{{\rho_\varphi}^2}{12}\right)}\ . \tag{37}$$

In order for H to stop evolving at τ_1 when the RS condition, $\Lambda + \rho_\varphi{}^2/12 = 0$ is satisfied, we need $\frac{d^n}{dt^n}H(\tau_1) = 0$ for all $n \geq 0$, which implies $\frac{d^n}{d\tau^n}\rho_\varphi = 0$ and also $\frac{d^n}{d\tau^n}\varphi(\tau_1) = 0$ for all $n \geq 1$. Therefore, the scalar field has to be constant with value $\varphi_1 \equiv \varphi(\tau_1)$ and $V(\varphi_1) = \sqrt{-12\Lambda}$. But this is only possible if V_1 is a minimum of the potential, and we have to put φ into this minimum with zero initial velocity from the start. This of course corresponds to the trivial static fine-tuned RS solution.

We conclude that the fine-tuning condition (28) can not be obtained by such a mechanism. Note that our arguments have been entirely general and we have thus shown that the fine-tuning problem cannot be resolved by an arbitrary brane scalar field.

To illustrate the dynamics, we consider the potential $V(\varphi) = \frac{1}{2}m^2\varphi^2$. Eq. (35) then takes the form

$$H^2 = \frac{1}{12}\Lambda + \frac{1}{144}\left(\frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}m^2\varphi^2\right)^2 \ . \tag{38}$$

It is convenient to use dimensionless variables x, y, z and η related to φ, $\dot{\varphi}$, H and τ by

$$\varphi \equiv \sqrt{\frac{24}{m}}x, \quad \dot{\varphi} \equiv \sqrt{24m}y, \quad H \equiv mz, \quad \tau \equiv \frac{1}{m}\eta. \tag{39}$$

Eqs. (32) and (38) are equivalent to a two dimensional dynamical system in the phase space (x, y) with

$$x' = y, \quad y' = -x - 3zy, \tag{40}$$

with the constraint equation

$$z^2 = -K + (x^2 + y^2)^2. \tag{41}$$

The prime denotes the derivative with respect to the 'time parameter' η and $K \equiv -\frac{1}{12m^2}\Lambda$. In Fig. 1 two typical trajectories found by numerical solution of the system (40-41) are shown in the phase space (x, y). For large initial y, the damping term first dominates and lowers y until the potential term becomes comparable. Then, the system evolves towards the minimum of the potential until the curve hits the circle $x^2 + y^2 = \sqrt{K}$, where the damping term changes sign, and the trajectories move away nearly in y–direction.

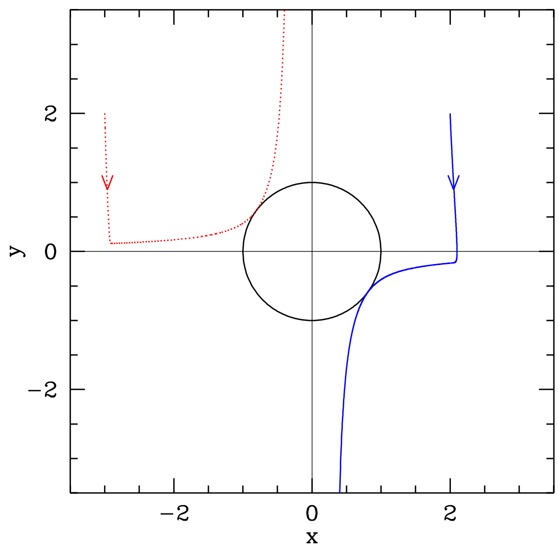

FIG. 1. Two trajectories in the phase space (x, y) which represent typical solutions of the system (40-41) for $K = 1$. The trajectory on the left (dotted, red), starting with an initial condition $x_{in} = -3, y_{in} = 2$, winds towards the circle $x^2 + y^2 = \sqrt{K}$, which corresponds to the condition $z = H = 0$. After reaching the circle, the solution moves away showing that it is not an attractor. The trajectory on the right (solid) with initial conditions $x_{in} = 2, y_{in} = 2$ shows a similar behaviour. It takes much longer to pass the region around the kinks than to trace out the remaining parts of the trajectories.

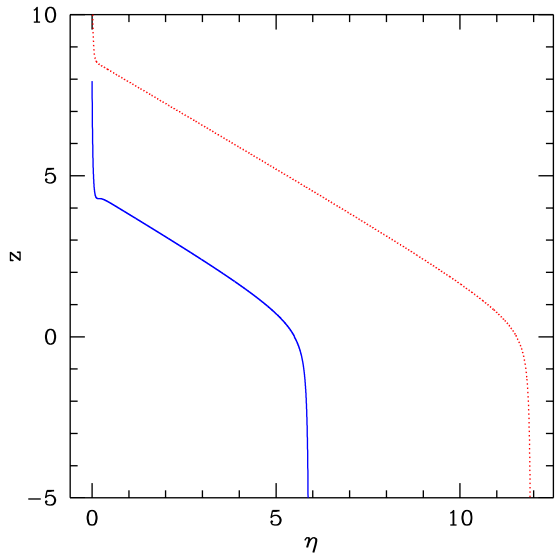

FIG. 2. The time evolution of the dimensionless Hubble parameter z for the two trajectories shown in Fig. 1.

In ordinary four dimensional cosmology there exists a 'no–go theorem' due to Weinberg [15], which states that the cosmological constant cannot be canceled by a scalar field. The argument is based on symmetries of the Lagrangian. In brane cosmology it is known [16] that for a 3–brane, embedded in a five dimensional spacetime, Einstein equations on the brane are the same as the usual four dimensional Einstein equations, apart from two additional terms: A term $S_{\mu\nu}$, which is quadratic in the energy–momentum tensor, $T_{\mu\nu}$, on the brane,

$$S_{\mu\nu} = -\frac{1}{4}T_{\mu\alpha}T_{\nu}{}^{\alpha} + \frac{1}{12}TT_{\mu\nu} + \frac{1}{8}\overline{g}_{\mu\nu}T_{\alpha\beta}T^{\alpha\beta} - \frac{1}{24}\overline{g}_{\mu\nu}T^2, \tag{42}$$

and a term $E_{\mu\nu}$, which is the projection of the five dimensional Weyl–tensor, $C^{A}{}_{BCD}$, onto the brane:

$$E_{\mu\nu} \equiv C^{A}{}_{BCD}n_A n^C \overline{g}_{\mu}{}^{B}\overline{g}_{\nu}{}^{D}, \tag{43}$$

where n_A is the normal vector to the brane and T the trace of the energy momentum tensor. Being purely constructed from $T_{\mu\nu}$, $S_{\mu\nu}$ does not introduce additional dynamical degrees of freedom. It just contributes the term ${\rho_\varphi}^2$ to the 'Friedmann' equation (31). It is also clear that $E_{\mu\nu}$, which is traceless, cannot cancel the cosmological constant on the brane. However, since the effective Einstein equations on the brane cannot be derived from a Lagrangian, and since $E_{\mu\nu}$ contains additional information from the bulk, it is not evident that Weinberg's theorem holds in our case.

More generally, our 'no-go' result also holds for any matter obeying an equation of state $p = \omega\rho$ when $\omega > -1$. This can be seen in a similar way: Initially the Hubble–Parameter is

$$H = +\sqrt{\frac{1}{12}\left(\Lambda + \frac{\rho^2}{12}\right)}. \tag{44}$$

Using the energy conservation equation,

$$\dot{\rho} = -3H(1+\omega)\rho, \tag{45}$$

one finds

$$\dot{H} = -\frac{1}{48}(1+\omega)\rho^2, \tag{46}$$

and hence $\dot{H} < 0$ as long as the weak energy condition, $\omega > -1$ (or $p > -\rho$) is satisfied. To relate our finding to previous results [14,17], let us note that Eqs. (44) and (46) imply the following condition for inflation on the brane:

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = \frac{\Lambda}{12} - \frac{2+3\omega}{144}\rho^2 > 0 \ , \tag{47}$$

For a brane energy density given by the brane RS tension $V = \sqrt{-12\Lambda}$ and an additional component indicated by a subscript $_f$, so that $\rho = V + \rho_f$ and $p = -V + p_f$ this gives

$$\frac{\ddot{a}}{a} = -[V(1+3\omega_f) + \rho_f(2+3\omega_f)]\frac{\rho_f}{144} > 0 \ , \tag{48}$$

which coincides with Eq. (8) of Ref [17]. If the RS term dominates, $V \gg \rho_f$ we obtain the usual strong energy condition for inflation, $1 + 3\omega_f < 0$, but if $V \ll \rho_f$ the condition is stronger, namely $2 + 3\omega_f < 0$.

Like for the scalar field, the brane starts to contract as soon as $H = 0$ is reached. We have thus shown that a relation like (28) cannot be realized in a cosmological setting which does not violate the weak energy condition.

After this section, in which we adopted the viewpoint of the brane, we now come back to the full five dimensional spacetime to investigate the stability of the RS model.

IV. GAUGE INVARIANT PERTURBATION EQUATIONS

We formally prove that the five dimensional RS spacetime is unstable under small perturbations of the brane tension.

A. Perturbations of the Randall–Sundrum model

The equations of motion derived in section II provide with $N(t,y), R(t,y)$, and $B(t,y)$ a dynamical generalization of the RS model. We consider Λ and V to be constant and set the dilaton, the scalar field on the brane and the energy-momentum tensor of the fluid to zero. Eqs. (15-18) now reduce to

$$\textit{00}: \quad 3e^{-2N}(\dot{R}^2 + \dot{R}\dot{B}) + 3e^{-2B}(-R'' - 2R'^2 + R'B') - \frac{1}{2}\Lambda - \delta(y)\frac{1}{2}e^{-B}V = 0 \ , \tag{49}$$

$$\begin{aligned} \textit{11}: \quad & e^{-2N}(-2\ddot{R} - \ddot{B} - 3\dot{R}^2 - \dot{B}^2 + 2\dot{N}\dot{R} + \dot{N}\dot{B} - 2\dot{R}\dot{B}) \\ & + e^{-2B}(N'' + 2R'' + N'^2 + 3R'^2 + 2N'R' - N'B' - 2R'B') + \frac{1}{2}\Lambda + \delta(y)\frac{1}{2}e^{-B}V = 0 \ , \end{aligned} \tag{50}$$

$$\textit{05}: \quad \dot{R}' + \dot{R}R' - N'\dot{R} - R'\dot{B} = 0 \ , \tag{51}$$

$$\textit{55}: \quad 3e^{-2N}(-\ddot{R} - 2\dot{R}^2 + \dot{N}\dot{R}) + 3e^{-2B}(N'R' + R'^2) + \frac{1}{2}\Lambda = 0 \ . \tag{52}$$

The RS solution (27) is a static solution of these equations, provided that condition (28) holds. We now derive linear perturbation equations from (49–52) which describe the time evolution of small deviations from RS. To this goal we set

$$N(t,y) = \alpha|y| + n(t,y), \tag{53}$$

$$R(t,y) = \alpha|y| + r(t,y), \tag{54}$$

$$B(t,y) = b(t,y), \tag{55}$$

where $\alpha = -\sqrt{\frac{-\Lambda}{12}}$ and $n(t,y), r(t,y), b(t,y)$ are small (with respect to 1) at $t = 0$. The perturbed metric is

$$ds^2 = -e^{2\alpha|y|+2n}dt^2 + e^{2\alpha|y|+2r}d\vec{x}^2 + e^{2b}dy^2. \tag{56}$$

We consider an energy-momentum tensor deviating from RS only by a slight mismatch of the brane tension,

$$T_{MN} = -\Lambda g_{MN} - \delta(y)\delta^\mu_M\delta^\nu_N e^{-b}V\overline{g}_{\mu\nu}, \tag{57}$$

with

$$V = \sqrt{-12\Lambda}(1+\Omega), \tag{58}$$

where $|\Omega| \ll 1$ parameterizes the perturbation of the brane tension, g_{MN} is the perturbed metric (56) and $\overline{g}_{\mu\nu}$ is its projection onto the brane. Clearly, if already this restricted set of perturbation variables contains an instability, the

RS solution is unstable under homogeneous and isotropic perturbations. Inserting this ansatz into equations (49- 52) and keeping only first order terms, we find

$$r'' - 4\alpha^2 b + \theta(y)\alpha(4r' - b') - \delta(y)2\alpha(b + \Omega) = 0 \ , \tag{59}$$
$$e^{-2\alpha|y|}(2\ddot{r} + \ddot{b}) - n'' - 2r'' + 12\alpha^2 b - \theta(y)\alpha(4n' + 8r' - 3b') + \delta(y)6\alpha(b + \Omega) = 0 \ , \tag{60}$$
$$\dot{r}' - \theta(y)\alpha\dot{b} = 0 \ , \tag{61}$$
$$e^{-2\alpha|y|}\ddot{r} + 4\alpha^2 b - \theta(y)\alpha(n' + 3r') = 0 \ , \tag{62}$$

where

$$\theta(y) = \begin{cases} +1 & \text{for } y > 0 \\ -1 & \text{for } y < 0 \ . \end{cases} \tag{63}$$

The junction conditions are

$$[r'] = [n'] = 2\alpha(b_0 + \Omega) \ . \tag{64}$$

Since we want to consider Z_2-symmetric perturbations, we require the functions n, r and b to be symmetric in y. In order to make coordinate-independent statements, we rewrite these equations in a gauge invariant way.

B. Gauge invariant perturbation equations

Under an infinitesimal coordinate transformation induced by the vector field

$$X = T(t,y)\partial_t + L(t,y)\partial_y, \tag{65}$$

the metric perturbations $g^{(1)}$ transform according to

$$g^{(1)} \to g^{(1)} + \mathcal{L}_X g^{(0)}, \tag{66}$$

where $g^{(1)}$ corresponds to the first order terms in the metric (56), and $\mathcal{L}_X g^{(0)}$ is the Lie derivative of the static background metric (24). One obtains the following transformation laws for the variables n, r and b:

$$n \to n + \theta(y)\alpha L + \dot{T}, \tag{67}$$
$$r \to r + \theta(y)\alpha L, \tag{68}$$
$$b \to b + L'. \tag{69}$$

Since we require the 05–component of the metric to vanish, it must remain zero under the coordinate transformation. This implies

$$\dot{L} = e^{2\alpha|y|}T'. \tag{70}$$

From Eq. (69), together with Z_2 symmetry, one finds that L' must be continuous and symmetric in y. Therefore L must be continuously differentiable and odd in y, which implies $L(t, y = 0) = 0$. Hence, the perturbation r restricted to the brane, r_0, is gauge invariant. Note that $L(t, y = 0) = 0$ also follows from Eq. (68) and $L \in C^1$. Hence the gauge invariance of r_0 is not a consequence of Z_2 symmetry, but is also preserved for non Z_2 symmetric perturbations. By computing the Lie derivative of the background energy-momentum tensor from Eq. (57) one finds that the perturbation of the brane tension, Ω, is gauge invariant. Condition (70) and the symmetry property of L' ensure that there is no energy flow onto or off the brane. With the following set of gauge invariant quantities

$$\Phi \equiv r' - \theta(y)\alpha b \ , \tag{71}$$
$$\Psi \equiv n' - \theta(y)\alpha b - \theta(y)\frac{1}{\alpha}e^{-2\alpha|y|}\ddot{r} \ , \tag{72}$$
$$r_0 \equiv r(t, y = 0) \ , \tag{73}$$
$$\Omega, \tag{74}$$

we can rewrite the perturbation equations (59-62) in terms of these variables

$$\Phi' + \theta(y)4\alpha\Phi - \delta(y)2\alpha\Omega = 0 \ , \tag{75}$$

$$\Psi' + 2\Phi' + \theta(y)4\alpha(\Psi + 2\Phi) + \delta(y)\left(\frac{2}{\alpha}\ddot{r}_0 - 6\alpha\Omega\right) = 0 \ , \tag{76}$$

$$\dot{\Phi} = 0 \ , \tag{77}$$

$$\Psi + 3\Phi = 0 \ . \tag{78}$$

The junction conditions are

$$[\Phi] = 2\alpha\Omega \ , \qquad [\Psi] = 2\alpha\Omega - \frac{2}{\alpha}\ddot{r}_0 \ . \tag{79}$$

The solutions of equations (75) with (77) and (78) are given by

$$\Phi(y) = -\frac{1}{3}\Psi(y) = \theta(y)\alpha\Omega e^{-4\alpha|y|} + \Phi_0 e^{-4\alpha|y|} \ . \tag{80}$$

Z_2 symmetry requires Φ to be odd in y and thus $\Phi_0 = 0$. Inserting equation (78) in (76) one obtains

$$\ddot{r}_0 = 4\alpha^2\Omega \tag{81}$$

and after integration

$$r_0(t) = 2\alpha^2\Omega t^2 + \mathcal{Q}t, \tag{82}$$

where $\mathcal{Q}$ is a small but arbitrary integration constant determined by the initial conditions. (An additive constant to r_0 can be absorbed in a redefinition of the spatial coordinates on the brane.) The scale factor on the brane is

$$e^{2r_0(t)} \simeq 1 + 2r_0(t) = 1 + 4\alpha^2\Omega t^2 + 2\mathcal{Q}t. \tag{83}$$

We have thus found a dynamical instability, which is quadratic in time, when the brane tension and the bulk cosmological constant are not fine–tuned. Our statement is valid in every coordinate system as r_0 is gauge invariant. In addition, more surprisingly, in linear perturbation theory there is no constraint on $\mathcal{Q}$, and it cannot be gauged away. This linear instability remains even for $\Omega = 0$, that is if the brane tension is not perturbed at all. We conclude that the RS model is unstable even under purely gravitational modes!

Let us finally discuss our solutions in two particular gauges. As a first gauge condition we set $r' = 0$, which fixes $L' = -\theta(y)\frac{1}{\alpha}r'$. The integration constant on L is determined by the condition $L(t, y = 0) = 0$. (Note that r' contains a θ-function and therefore L' is continuous.) For all values of y we have

$$r(t) = 2\alpha^2\Omega t^2 + \mathcal{Q}t. \tag{84}$$

Since $b(y) = -\theta(y)\frac{1}{\alpha}\Phi(y)$,

$$b(y) = -\Omega e^{-4\alpha|y|}. \tag{85}$$

From the definition of Ψ it follows

$$n' = -3\Phi + \theta(y)\alpha b + \theta(y)\frac{1}{\alpha}e^{-2\alpha|y|}\ddot{r}_0 \ , \tag{86}$$

which can be integrated to give

$$n(t, y) = \Omega e^{-4\alpha|y|} - 2\Omega e^{-2\alpha|y|} + \mathcal{N}(t) \ . \tag{87}$$

These n, r, and b solve equations (59-62). The integration constant $\mathcal{N}(t)$ can be absorbed in the gauge transformation T. Together with the choice of L', this fixes the gauge, and the solutions are therefore unique up to an additive purely time dependent function to T.

Another possible gauge is $b = 0$. Then, from $r' = \Phi$,

$$r(t,y) = -\frac{\Omega}{4}e^{-4\alpha|y|} + \mathcal{R}(t) \tag{88}$$

with

$$\mathcal{R}(t) = r_0(t) + \frac{\Omega}{4} = 2\alpha^2\Omega t^2 + \mathcal{Q}t + \frac{\Omega}{4} \tag{89}$$

and

$$n(t,y) = \frac{3}{4}\Omega e^{-4\alpha|y|} - 2\Omega e^{-2\alpha|y|} + \mathcal{N}(t) \ . \tag{90}$$

Again, the integration constant $\mathcal{N}(t)$ can be gauged away by choosing an appropriate T, and the solutions are uniquely determined by the gauge fixing.

Inserting these solutions in the perturbed metric

$$ds^2 = -e^{2\alpha|y|+2n}dt^2 + e^{2\alpha|y|+2r}d\vec{x}^2 + e^{2b}dy^2, \tag{91}$$

we find that the full RS spacetime, not only the brane, is unstable against homogeneous perturbations of the brane tension.

We must require the initial perturbations to be small, that is at some initial time, $t = 0$, the deviation from RS has to be small for all values of y. In the case of a compact spacetime, $|y| \leq y_{\max}$, this just requires $|\Omega| \ll e^{-4\alpha y_{\max}}$ (remember that α is a negative constant). For a non-compact spacetime, $-\infty < y < \infty$, we have to require $\Omega = 0$. In other words, for $V \not\equiv \sqrt{-12\Lambda}$ there exists no solution which is 'close' to RS in the sense of L^2 or $\sup_y$ at any given initial time!

Finally, we present a geometrical interpretation of the gauge invariant quantities Φ and Ψ. Since the five dimensional Weyl tensor of the RS solution vanishes, the perturbed Weyl tensor is gauge invariant according to the Steward-Walker lemma [18]. The 0505–component of the Weyl tensor of the perturbed metric (56) is up to first order

$$C_{0505} = \frac{1}{2}e^{2\alpha|y|}(n'' - r'' + \theta(y)\alpha(n' - r')) + \frac{1}{2}(\ddot{r} - \ddot{b}), \tag{92}$$

which can be expressed in terms of gauge invariant quantities

$$C_{0505} = -\frac{1}{2}e^{\alpha|y|}\left(e^{\alpha|y|}(\Phi - \Psi)\right)' + \delta(y)\frac{1}{\alpha}\ddot{r}_0. \tag{93}$$

All other non vanishing Weyl components are multiples of C_{0505}:

$$C_{0101} = C_{0202} = C_{0303} = C_{1212} = C_{1313} = C_{2323} = -\frac{1}{3}e^{2\alpha|y|}C_{0505} \quad \text{and} \tag{94}$$

$$C_{1515} = C_{2525} = C_{3535} = \frac{1}{3}C_{0505} \ . \tag{95}$$

In first order the projected Weyl tensor (defined in [16]) is $E_{11} = E_{22} = E_{33} = \frac{1}{3}E_{00} = 2\alpha^2\Omega$ with $E^{\mu}{}_{\mu} = 0$. The Weyl–tensor completely vanishes for $\Omega = 0$.

V. RESULTS AND CONCLUSIONS

In this paper we have addressed two main questions: First, we investigated whether the RS fine–tuning condition can be obtained dynamically by some matter component on the brane. As a concrete example, we studied a scalar field on the brane and found that a bulk cosmological constant cannot be canceled by the potential of the scalar field in a non-trivial way. This result can be generalized for any matter satisfying the weak energy condition.

Second, we studied the stability of the RS model in five dimensions. We have found that the RS solution is unstable under homogeneous and isotropic, but time dependent perturbations. For a small deviation of the fine–tuning condition parameterized by $\Omega \neq 0$, this instability was expected, it reminds strongly of the instability of the static Einstein universe, where the fluid energy density and the cosmological constant have to satisfy a delicate balance in order to keep the universe static. But even if $\Omega = 0$, there exists a purely gravitational mode, which represents an instability in first order perturbation theory. The physical interpretation of this mode is not yet clear to us. In a cosmological setting, our result means that a possible change in the brane tension, *e.g.* during a phase transition, or also quantum corrections to the bulk energy density (see [19]) give rise to instabilities of the full five dimensional spacetime.

Even if one would consider a dynamical scalar field in the bulk (which does not couple to brane fields), which settles into a vacuum state such that its energy density is constant along the fifth dimension, one would not be able to solve the cosmological constant problem without falling back to some fine–tuning mechanism. From our results we can conclude that in order to have a chance to solve the RS fine tuning problem dynamically, we have to consider fully dynamical *bulk* fields. This can in principle be done with the system of equations, which we have presented in section II, and which also applies to the effective five dimensional low energy theory suggested by heterotic M–theory [20].

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