Adiabatic perturbations in pre-big bang models: matching conditions and scale invariance

Ruth Durrer and Filippo Vernizzi
Département de Physique Théorique, Université de Genève, 24 quai E. Ansermet, CH-1211 Genève 4, Switzerland

At low energy, the four-dimensional effective action of the ekpyrotic model of the universe is equivalent to a slightly modified version of the pre-big bang model. We discuss cosmological perturbations in these models. In particular we address the issue of matching the perturbations from a collapsing to an expanding phase. We show that, under certain physically motivated and quite generic assumptions on the high energy corrections, one obtains $n = 0$ for the spectrum of scalar perturbations in the original pre-big bang model (with vanishing potential). With the same assumptions, when an exponential potential for the dilaton is included, a scale invariant spectrum ($n = 1$) of adiabatic scalar perturbations is produced under very generic matching conditions, both in a modified pre-big bang and ekpyrotic scenario. We also derive the resulting spectrum for arbitrary power law scale factors matched to a radiation dominated era.

I. INTRODUCTION

Observational cosmology has made enormous progress during the last couple of years. Most observations seem to agree with the fact that the total energy density of the universe ρ is very close to its critical value ρ_c, $\Omega \equiv \rho/\rho_c = 1$, and it is distributed in the form of pressureless dark matter ρ_m and dark energy with negative pressure, $P_\Lambda \lesssim -0.6\rho_\Lambda$, $\Omega = \Omega_\Lambda + \Omega_m = 1$ with $\Omega_\Lambda \simeq 0.7$ and $\Omega_m \simeq 0.3$. The clustering properties of the observed universe agree with a scale invariant spectrum of adiabatic scalar perturbations, $n \simeq 1$, with or without a tensor component. Many recent cosmological experiments measure one or several of these parameters. Most notably cosmic microwave background anisotropy experiments [1–3], supernovae type Ia measurements [4,5], cluster abundances [6], analysis of the observed galaxy distribution [7,8], and of peculiar velocities [9] (see also [10]).

Although the presence of dark energy, $\Omega_\Lambda \neq 0$, remains very mysterious, inflation explains why $\Omega = 1$ and $n \simeq 1$.

The basic idea of inflation is simple: If the energy density in a sufficiently smooth patch of space is dominated by the potential energy of some slowly varying scalar field, this patch will expand very rapidly and evolve into a large, very homogeneous, isotropic and flat universe. During this rapid expansion, the causal horizon becomes much larger than the Hubble horizon, alleviating the horizon problem. In addition, quantum fluctuations in the scalar field get amplified and grow larger than the Hubble scale, H^{-1}. They then 'freeze in' as classical fluctuations in the energy density or, equivalently, in the geometry, which obey a scale invariant spectrum.

This standard picture of inflation does not emerge in a direct way from any modern high energy physics model. This makes it very flexible which is probably one of the main reasons why the basic picture has survived for so long. If a given model does not work, one is free to slightly change the potential or other couplings of the scalar field. This has lead to many different models of inflation presented in the literature [11]. This flexibility may be considered either as a strong point or as a drawback. It is in any case certainly very important to investigate whether there are alternative explanations of the size and the flatness of the universe and of the observed scale invariant spectrum of adiabatic scalar fluctuations in the context of modern high energy physics.

In this paper we discuss two attempts in this direction which are both motivated by string theory: the pre-big bang model [12,13] and the ekpyrotic model [14–16]. Even though the high energy pictures of these models are very different, the four dimensional low energy effective actions agree and the models predict the same cosmology at low energy up to possible high energy 'relics'. In the following we call a model of the universe a 'pre-big bang model' if it contains a low curvature phase before the big bang. In this sense also the ekpyrotic scenario is a pre-big bang model.

The original pre-big bang model consists just of the dilaton and the metric, the two low energy degrees of freedom which are present in every string theory. The presence of the dilaton leads to a new symmetry called 'scale factor duality' of cosmological solutions: To each solution for the scale factor $a(t)$ corresponds a solution $a(t)^{-1}$, or $a(-t)^{-1}$ if combined with time reversal symmetry. If $a(t)$ is an expanding, decelerating solution, $a(-t)^{-1} \equiv \hat{a}(\hat{t})$ is an expanding accelerating solution, since

$$\frac{d\hat{a}}{d\hat{t}} = \frac{1}{a^2}\frac{da}{dt} > 0, \tag{1}$$

and

$$\frac{d^2\hat{a}}{d\hat{t}^2} = -\frac{1}{a^2}\frac{d^2a}{dt^2} + \frac{2}{a^3}\left(\frac{da}{dt}\right)^2 > 0. \tag{2}$$

The Hubble parameter $\hat{H}$ of this 'super-inflating' solution [12,13] grows as $\hat{t} = -t$ increases. The solution approaches trivial flat space and vanishing couplings in the past, $\hat{t} \to -\infty$, and a curvature singularity in the future, $\hat{t} \to 0^-$.

In this pre-big bang model, one supposes that curvature and strong coupling corrections of string theory 'bend' the evolution away from this singularity into an expanding, decelerating radiation dominated Friedmann model. Several studies of toy models where this can be achieved have been presented in the literature (see [17–20]), but they usually just represent second order corrections to the curvature and the coupling, and not full string theory solutions.

It has been shown [21] that a pure dilaton without potential cannot lead to a scale invariant spectrum of adiabatic scalar fluctuations. For this reason it has been proposed that fluctuations may be induced by axions via the so called seed mechanism [22]. Axions naturally display a scale invariant spectrum. However, the axion seed perturbations are of isocurvature nature, which is not in agreement with present observations. Mechanisms which may convert the axionic isocurvature fluctuations into adiabatic ones are currently under investigation [23].

In this paper, we will instead repeat the basic arguments of [21], but we will show that the spectrum of perturbations which one obtains in the radiation dominated post-big bang phase has the spectral index $n = 0$ and not $n = 4$ as claimed in [21]. We shall also show that when adding an exponential potential to this action, one obtains a scale invariant spectrum, $n = 1$.

The high energy picture behind the ekpyrotic scenario, the second pre-big bang model discussed in this paper, is quite different. There one starts with a five-dimensional universe containing two perfectly parallel 3-branes at rest [14,15], in a BPS state. One then supposes that the two branes approach each other with some very small initial velocity. It is argued that, from the four-dimensional point of view of an observer on one of the branes, this situation corresponds to a collapsing Friedmann universe with a scalar field, which is related to the distance between the two branes before the collision. After the collision the solution is supposed to turn into a radiation dominated Friedmann [14,15] (see [24–26] for critics).

It is assumed that the scalar field is minimally coupled and has a negative exponential potential V which describes the attraction of the two branes. The scalar field potential is due to non-perturbative string corrections but has not been derived from any string theory, so far. In Refs. [15,16] it has been argued that, if $V = -V_0 \exp(-c\varphi)$ at low curvature, with $c \gg 1$, a scale invariant spectrum of scalar perturbations develops. This result has been criticized in Refs. [27–32], where a spectral index $n = 3$ has been obtained. We shall show here that, even if the detailed arguments put forward in Refs. [15,16] might not be valid, under quite generic (although non trivial) assumptions one does obtain the spectral index $n = 1$.

Like the original pre-big bang, this model starts out at low curvature and develops a singularity in the future. Like there, the belief is that string theory corrections will change the behavior of the scale factor and the scalar field away from this singular evolution. In the five dimensional picture, this apparent 'singularity' corresponds to the collision of the two branes which then should result in the production of radiation leading to a thermal, radiation dominated Friedmann model. We call the phase before the high curvature regime the 'pre-big bang phase' and the regime after the big bang the 'post-big bang'.

Even if the string theory corrections, which must become important close to the singularity, are not fully understood, these models are promising candidates for alternatives to inflation: They certainly do not suffer from a horizon problem since their age can be arbitrarily large and is not related to the Hubble time. They do not dynamically imply flatness, but this comes from very natural vacuum (for the original pre-big bang) or BPS (for the ekpyrotic model) initial conditions which are posed at low curvature. Nevertheless, it is well known that these models are not very efficient in smoothing out classical inhomogeneities [33] and global anisotropies [34], and this may remain a problem. In the most recent version of the ekpyrotic model, a cyclic universe, flatness is also a consequence of a period of exponential expansion in the previous cycle [35]. A quite fair comparison of the ekpyrotic scenario and ordinary inflation is given in Ref. [36].

In this paper we do not address the important debate of the flatness problem, but we investigate the spectrum of perturbations generated during the pre-big bang phase. The aim of this paper is to learn as much as possible about such models without specifying the details of the high energy phase.

In the next section we write down the modified pre-big bang action and the action of the ekpyrotic model. We show that they are related by a conformal transformation and we solve the equations of motion in both Einstein and string frame. In Sections III and IV, which are the heart of this paper, we discuss scalar perturbations and the matching conditions between a contracting, scalar field dominated phase and an expanding, radiation dominated phase. In particular we show that, under certain well defined conditions, without knowing the details of the matching, one expects $n = 1$ for the modified pre-big bang and the ekpyrotic model. In Section V we generalize our results to arbitrary power law scale factors matched to a radiation dominated era. We end with our conclusions and an outlook.

II. THE BACKGROUND

The low energy effective action of the original pre-big bang model is simply gravity with a dilaton ϕ. Here we modify it by allowing for a dilaton potential. We assume that we have a four-dimensional effective theory, any extra dimensions being frozen at a very small scale. The low energy action for this theory is therefore [37]

$$\hat{S} = \frac{1}{2\kappa^2}\int dx^4\sqrt{-\hat{g}}e^{-\phi}\left[\hat{R} + (\hat{\nabla}\phi)^2 - 2\hat{V}(\phi)\right], \quad (3)$$

with $\kappa^2 = 8\pi G = 1/M_P^2$, where $M_P = 2.4\times 10^{18}$GeV is the reduced Planck mass. This action is written in the so-called string frame. The hat $\hat{}$ indicates that the corresponding quantities have to be computed using the metric in this frame. Therefore $\hat{g}$, $\hat{R}$, $\hat{\nabla}$, and $\hat{V}$ are the determinant of the metric, the Riemann scalar, the covariant derivative, and the dilaton potential, respectively, in the string frame. With this action ϕ is dimensionless and the usual scalar field with dimension of mass is simply $M_P\phi$. Correspondingly, the potential $\hat{V}$ has dimensions of (energy)^2 and the usual potential is $M_P^2\hat{V}$. We use the metric signature $-+++$.

It is possible to rewrite the action in Eq. (3) in a conformally related (and physically equivalent) frame. If we perform a conformal transformation $g_{\alpha\beta} = \Omega^2\hat{g}_{\alpha\beta}$ the action is modified to

$$S = \frac{1}{2\kappa^2}\int dx^4\sqrt{-g}\Omega^{-2}e^{-\phi}\left[R + (\nabla\phi)^2 + \right.$$
$$\left. +6(\nabla\ln\Omega)^2 + 6(\nabla\phi\cdot\nabla\ln\Omega) - 2\Omega^{-2}\hat{V}(\phi)\right]. \quad (4)$$

When choosing $\Omega = \exp(-\phi/2)$, we can obtain the Einstein frame action,

$$S_{\rm E} = \frac{1}{2\kappa^2}\int dx^4\sqrt{-g}\left[R - \frac{1}{2}(\nabla\phi)^2 - 2V(\phi)\right], \quad (5)$$

where

$$g_{\alpha\beta} = e^{-\phi}\hat{g}_{\alpha\beta} \quad \text{and} \quad V(\phi) = e^{\phi}\hat{V}(\phi) \quad (6)$$

are the metric and the scalar field potential, respectively, in the Einstein frame. Eq. (5) is the action for a minimally coupled scalar field. Notice that the dilaton has not been changed by the conformal transformation. We can also allow for a rescaling of the scalar field, $\varphi = \phi/\beta$, so that

$$S_{\rm E} = \frac{1}{2\kappa^2}\int dx^4\sqrt{-g}\left[R - \frac{1}{2}\beta^2(\nabla\varphi)^2 - 2V(\varphi)\right]. \quad (7)$$

String cosmology and, in particular, the original pre-big bang scenario, has been developed based on action (3) with the dilaton potential set to zero. In our modified pre-big bang model we will allow a non-zero potential. Since we want to obtain here the usual scalar field action presented in [14] starting from the string cosmology action (3), we have to require $\beta^2/2 = 1$. This fixes $\beta = \pm\sqrt{2}$. In terms of the new field φ the Einstein frame action now becomes

$$S_{\rm E} = \frac{1}{2\kappa^2}\int dx^4\sqrt{-g}\left[R - (\nabla\varphi)^2 - 2V(\varphi)\right]. \quad (8)$$

For an exponential potential

$$\hat{V}(\phi) = e^{-\phi}V(\phi) = -V_0e^{\lambda\phi}, \quad (9)$$

where $\lambda = -(1 + c/\beta)$ with $c \gg 1$, or equivalently for

$$V(\varphi) = -V_0e^{-c\varphi}, \quad (10)$$

we obtain precisely the low energy effective action of the ekpyrotic scenario [15,16]. The interpretation of the field φ is however quite different. There φ is related to the brane separation [15]. At early times when the two branes are separated by a large distance, the scalar field φ is very big and positive, $\varphi \to \infty$. Therefore the relation between the string cosmology dilaton ϕ which tends to $-\infty$ for very early times, $t \to -\infty$, and the field φ of the ekpyrotic scenario is $\phi = -\sqrt{2}\varphi$, $\beta = -\sqrt{2}$. Since $c \gg 1$ and β is negative, $\lambda > 0$ so that the potential (9) goes asymptotically to zero for very negative dilaton (at early time), and does not spoil the initial conditions of the pre-big bang.

Varying Eq. (8) with respect to φ we obtain the equation of motion

$$\Box\varphi - V(\varphi)_{,\varphi} = \Box\varphi - cV_0e^{-c\varphi} = 0, \quad (11)$$

where $\Box = \nabla_\alpha\nabla^\alpha$. Varying the action with respect to the metric yields the Einstein equations,

$$G_{\alpha\beta} = \kappa^2 T_{\alpha\beta}, \quad (12)$$

where $T_{\alpha\beta}$ is the energy-momentum tensor of the scalar field,

$$\kappa^2 T_{\alpha\beta} = \nabla_\alpha\varphi\nabla_\beta\varphi - \frac{1}{2}g_{\alpha\beta}\left[(\nabla\varphi)^2 + 2V(\varphi)\right]. \quad (13)$$

We want to consider a flat homogeneous and isotropic universe with metric $ds^2 = -dt^2 + a^2dx^2$. In this case Eq. (11) becomes

$$\ddot{\varphi} + 3H\dot{\varphi} + V_{,\varphi} = 0, \quad (14)$$

where the over-dot is a derivative with respect to the cosmic time t, and (12) turns into the Friedmann equation,

$$H^2 = \frac{\kappa^2}{3}\rho = \frac{1}{6}\dot{\varphi}^2 + \frac{1}{3}V(\varphi). \quad (15)$$

Eqs. (14,15) have the 'ekpyrotic solution' [15]

$$a(t) = (-t)^p, \qquad \varphi(t) = \frac{2}{c}\ln(-Mt), \quad (16)$$

with

$$p = \frac{2}{c^2}, \qquad M^2 = \frac{V_0}{p(1-3p)}. \quad (17)$$

At first it may seem strange that the enthalpy $w \equiv P/\rho$ and the sound speed $c_s^2 \equiv \dot{P}/\dot{\rho}$ are much larger than one, $c_s^2 = w \gg 1$, for small values of p (large c),

$$w = \frac{(1/2)\dot{\varphi}^2 - V}{(1/2)\dot{\varphi}^2 + V} = c_s^2 = \frac{2}{3p} - 1. \quad (18)$$

On the other hand, as long as we concentrate on a time interval bounded away from the singularity, we can always split the potential into $V = V_1(\varphi) + V_2$, where V_2 is a very negative constant and V_1 is always positive. Interpreting V_2 as a negative cosmological constant, we have

$$-1 < w_1 = \frac{(1/2)\dot{\varphi}^2 - V_1}{(1/2)\dot{\varphi}^2 + V_1} < 1, \tag{19}$$

as well as $-1 < c_1^2 < 1$ and $w_2 = c_2^2 = -1$. However, since $\Omega_1 = \rho_1/(\rho_1 + V_2) \gg 1$ and $\Omega_2 = V_2/(\rho_1 + V_2) \ll -1$, the 'effective' $w = w_1\Omega_1 - \Omega_2$ can become much larger than 1 without implying any pathological or even acausal behavior of the scalar field 'fluid'.

We shall see that the perturbations generated in this collapse phase acquire a scale invariant spectrum only if the collapse proceeds very slowly, *i.e.* when $0 < p \ll 1$. In the ekpyrotic scenario the collapse is followed by an expanding phase. Shortly before the bounce at $t \to 0^-$, when the scalar field, after having become negative, goes to minus infinity, $\varphi \to -\infty$, the shape of the potential has to change from the exponential expression, and turn upwards in such a way that $V \to 0$ for $\varphi \to -\infty$.

Let us give here, for completeness, the equations derived from the string frame action Eq. (3), where the potential $\hat{V}(\phi)$ is given by Eq. (9), and their solutions. By varying this action with respect to the field ϕ we obtain

$$2\hat{\nabla}_\alpha\hat{\nabla}^\alpha\phi - (\hat{\nabla}\phi)^2 + \hat{R} - 2\hat{V} + 2\hat{V}_{,\phi} = 0. \tag{20}$$

Varying the action with respect to $\hat{g}^{\alpha\beta}$ yields

$$\hat{G}_{\alpha\beta} = -\hat{\nabla}_\alpha\hat{\nabla}_\beta\phi - \frac{1}{2}\hat{g}_{\alpha\beta}\left[(\hat{\nabla}\phi)^2 - 2\hat{\nabla}_\alpha\hat{\nabla}^\alpha\phi + 2\hat{V}\right]. \tag{21}$$

For a homogeneous and isotropic universe with spatially flat sections, Eqs. (20) and (21) reduce to

$$\ddot{\phi} + 3\hat{H}\dot{\phi} - \dot{\phi}^2 + 2\hat{V} + 2\hat{V}_{,\phi} = 0, \tag{22}$$

$$\hat{H}^2 - \hat{H}\dot{\phi} + \frac{1}{6}\dot{\phi}^2 - \frac{1}{3}\hat{V} = 0, \tag{23}$$

where the over-dot here refers to cosmic time in the string frame, $\hat{t}$.

To find a solution to these equations we can simply transform the solution found in the Einstein frame using the relations

$$d\hat{t} = e^{\phi/2}dt = e^{-\varphi/\sqrt{2}}dt, \quad \hat{a} = e^{\phi/2}a = e^{-\varphi/\sqrt{2}}a. \tag{24}$$

The first relation gives

$$-\hat{M}\hat{t} = (-Mt)^{1-\sqrt{p}}, \tag{25}$$

where $\hat{M} = M(1 - \sqrt{p})$. For small p, $p \ll 1$, $\hat{t}$ is very close to t and, as long as $p < 1$, $\hat{t}$ grows from $-\infty$ to 0 with t. Inserting the ekpyrotic solutions in expressions (24) for $\hat{a}$ and ϕ, we obtain

$$\hat{a} = (-\hat{M}\hat{t})^{-\sqrt{p}}, \tag{26}$$

and

$$\phi = -\sqrt{2}\varphi = -\frac{2\sqrt{p}}{1-\sqrt{p}}\ln(-\hat{M}\hat{t}), \tag{27}$$

up to possible integration constants which we have fixed to obtain $\hat{a} = a$ and $\hat{t} = t$ in the limit $p \to 0$.

In this section we have first shown that, from a purely four-dimensional point of view the ekpyrotic scenario is equivalent to the pre-big bang scenario when the dilaton has an exponential potential that tends to zero at small coupling. In doing so we have presented the equations for these models, written in the string and Einstein frames, and we have written down the solutions that hold in either frames. These solutions are useful for discussing perturbations, which is the subject of the next section.

III. SCALAR PERTURBATIONS

We now want to study linear perturbations of a generic universe dominated by a minimally coupled scalar field with an exponential potential or an adiabatic fluid with $w = c_s^2 =$ constant. This last condition is automatically satisfied for a scalar field with exponential potential.

As discussed in the previous section, pre-big bang expansion in the string frame is equivalent to contraction in the Einstein frame, where the dilaton is minimally coupled. Therefore, pre-big bang with a dilaton corresponds to a collapsing universe dominated by a minimally coupled scalar field and is included in our study. It is important to note that physical quantities, like the spectral index or the perturbation amplitude are frame independent but they are more easily computed in the Einstein frame where linear perturbation theory is well established (see, *e.g.* the reviews [38,39]).

To discuss perturbations we work mainly in conformal time η, which is related to the physical time t by $ad\eta = dt$. The derivative with respect to conformal time is denoted by a prime, $'$. For the sake of simplicity we neglect a possible curvature of the spatial sections. In a flat universe dominated by a fluid or a scalar field with energy density ρ and pressure P the background Friedmann equations are

$$\mathcal{H}^2 = \frac{\kappa^2}{3}\rho a^2, \tag{28}$$

$$\mathcal{H}' = -\frac{\kappa^2}{6}(\rho + 3P)a^2 = -\mathcal{H}^2\frac{1+3w}{2}, \tag{29}$$

where $\mathcal{H} = a'/a$.

If the energy density is dominated by a scalar field, we have

$$\kappa^2\rho = \frac{1}{2a^2}\varphi'^2 + V(\varphi), \tag{30}$$

$$\kappa^2 P = \frac{1}{2a^2}\varphi'^2 - V(\varphi), \tag{31}$$

and

$$w+1=\frac{\varphi'^2}{3\mathcal{H}^2}. \tag{32}$$

When $w=c_s^2=$ constant, the solution to the Friedmann equation is a power law. In terms of conformal time η it is given by

$$a=\left|\frac{\eta}{\eta_1}\right|^q, \quad q=\frac{2}{1+3w}, \quad \mathcal{H}=\frac{q}{\eta}, \quad \mathcal{H}'=-\frac{q}{\eta^2}, \tag{33}$$

where we have chosen the normalization constant η_1 such that $-\eta_1<0$ is a very small negative time at which (higher order) corrections to the scalar field action become important. Since $a(\eta_1)=1$, $\eta_1=a(\eta_1)\eta_1\sim t_1$ corresponds to a physical quantity, *e.g.* the string scale in the pre-big bang model, $1/\eta_1\sim 10^{17}$ GeV. Comparing Eq. (33) with the ekpyrotic solutions in terms of physical time, we find $q=p/(1-p)$.

Let us now perturb the metric. In longitudinal gauge and in absence of anisotropic stresses, as it is the case for perfect fluids and for scalar fields, scalar metric perturbations are given by

$$ds^2=a^2(\eta)[-(1+2\Psi)d\eta^2+(1-2\Psi)\delta_{ij}dx^i dx^j]. \tag{34}$$

In this gauge, the metric perturbation Ψ corresponds to the gauge invariant Bardeen potential. Without gauge fixing the latter is given by a more complicated expressions of the metric perturbations [38–40]. The scalar field φ is also perturbed so that it can be divided into $\varphi(\eta)$ satisfying the background equation (14), and a perturbation $\delta\varphi(\eta,\mathbf{x})$.

We now want to compute the spectrum of metric perturbations generated from vacuum initial conditions. Generically, Ψ satisfies the equation [38,39]

$$\Psi''+3\mathcal{H}(1+c_s^2)\Psi'+ (2\mathcal{H}'+(1+3c_s^2)\mathcal{H}^2-\Upsilon\Delta)\Psi=0. \tag{35}$$

For adiabatic perturbations of a fluid, one finds $\Upsilon=c_s^2$, where c_s^2 is the adiabatic sound speed, while for a simple scalar field one finds $\Upsilon=1$ (see, *e.g.* Ref. [38]). Hence for a non-vanishing potential, $V\neq 0$ and hence $c_s^2\neq 1$, simple scalar field perturbations are not adiabatic in a thermodynamic sense.

If we restrict ourself to the case, $w=c_s^2=$ constant, the mass term in Eq. (35), $2\mathcal{H}'+(1+3c_s^2)\mathcal{H}^2$, vanishes by the use of the background Einstein equations, Eqs. (28) and (29). Thus, for scalar perturbations we obtain nearly the same equation as for tensor perturbations, which we can write in terms of Fourier modes as

$$\Psi''+3\mathcal{H}(1+w)\Psi'+\Upsilon k^2\Psi=0. \tag{36}$$

This equation is valid in both phases of the universe, before and after the big bang, depending on the corresponding value of w and Υ. We call Ψ_- the solutions obtained in the pre-big bang collapsing phase and Ψ_+ the one obtained in the radiation dominated phase. In the following we will work in Fourier space.

Let us now define the variable u in order to simplify Eq. (36) [38]. We set

$$u=\frac{M_P}{\mathcal{H}}a\Psi. \tag{37}$$

Eq. (36) can then be written in terms of u as

$$u''+\left(\Upsilon k^2-a(1/a)''\right)u=0. \tag{38}$$

Let us now suppose that the collapsing (or pre-big bang) phase $\eta<-\eta_1$ is dominated by the scalar field so that $\Upsilon=1$. Eq. (38) then has the general solution

$$u=(k|\eta|)^{\frac{1}{2}}[C(k)H_\mu^{(1)}(k\eta)+D(k)H_\mu^{(2)}(k\eta)], \tag{39}$$

with $\mu=q+1/2$. Here $H_\mu^{(i)}$ is the Hankel function of the i-th kind and of order μ. One can generalize this solution to the case of a fluid dominated universe simply by replacing $k\eta$ by $c_s k\eta$. This solution has to be generated from the incoming vacuum, so we assume that, for $k|\eta|\gg 1$,

$$\lim_{\eta\to-\infty}u=\frac{e^{-ik\eta}}{k^{3/2}}. \tag{40}$$

This assumption corresponds to normalizing the canonical variable which diagonalizes the perturbed second order action (called v in [38]) or equivalently the perturbation of the scalar field, $\delta\varphi$, to quantum vacuum fluctuations. With this normalization, the $H_\mu^{(1)}$ mode, which approaches $\exp(ik\eta)$ for $k|\eta|\gg 1$, has to be absent, $C(k)=0$, and the solution to Eq. (36) becomes

$$\Psi_-(k,\eta)=\frac{q}{M_P a\eta}D(k)(k|\eta|)^{1/2}H_\mu^{(2)}(k\eta), \tag{41}$$

where

$$D(k)=\sqrt{\pi/2}k^{-3/2}, \tag{42}$$

modulo some irrelevant phase.

At late time $k|\eta|\ll 1$, this solution approaches

$$\Psi_-(k,\eta)\simeq A_-(k)\frac{\mathcal{H}}{a^2}+B_-(k), \tag{43}$$

where A_- and B_- are determined by the exact solution (41) (up to logarithmic corrections),

$$A_-(k)\simeq\frac{2^\mu\Gamma(\mu)}{M_P\eta_1^q}k^{-\mu-1}, \tag{44}$$

$$B_-(k)\simeq\frac{\eta_1^q}{M_P 2^\mu\Gamma(\mu+1)}k^{\mu-1}\ . \tag{45}$$

The result (43) can be found directly by solving Eq. (36) neglecting the k^2-term. The full solution is however

needed to determine the pre-factors $A_-(k)$ and $B_-(k)$ from the vacuum initial condition. The A_--mode grows during the pre-big bang phase and becomes much larger than the constant B_--mode.

In the original pre-big bang, where the dilaton has no potential, *i.e.* $w = c_s^2 = 1$ and hence $q = 1/2$, we have $\mu = 1$. The A_--mode then has an $n = 0$ spectrum, $|A_-|^2 k^3 \propto k^{-1} \propto k^{n-1}$, while the B_--mode corresponds to $n = 4$, $|B_-|^2 k^3 \propto k^3 \propto k^{n-1}$. If we have an exponential potential as for the ekpyrotic model such that $p \ll 1$, and therefore $q \ll 1$, we have $\mu \simeq 1/2$ and hence $|A_-|^2 k^3$ is k-independent. The A_--mode has a scale invariant spectrum, $n = 1$, while $|B_-|^2 k^3 \propto k^2$, which corresponds to a blue spectrum, $n = 3$.

If the A_--mode has a red spectrum, as in the original pre-big bang scenario, we need to discuss its amplitude on large scales. It has been shown in [21] that a red ($n = 0$) A_--mode does not invalidate linear perturbation theory during the pre-big bang phase. Geometrically meaningful quantities like $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}/R^2 \equiv \Delta^2$, where $C_{\mu\nu\gamma\delta}$ is the Weyl tensor and R is the curvature scalar, remain small. In fact $\Delta^2 \propto |(k\eta)^2\Psi|^2 k^3$. We can therefore continue to use the Bardeen potential even though it may become large for certain k-modes. However, a red spectrum leads to serious problems in the subsequent radiation era where the Bardeen potential is constant on super horizon scales and Δ^2 grows larger than unity at horizon entry, $k\eta \sim 1$, for large scales.

In the modified pre-big bang models discussed here, this problem does not occur, since A_- has a scale invariant spectrum.

At very early time *after* the big bang, in the radiation dominated phase, we can neglect the term $\Upsilon k^2 = k^2/3$ in Eq. (38). We then have the same type of solution for super horizon modes,

$$\Psi_+(k,\eta) = A_+(k)\frac{\mathcal{H}}{a^2} + B_+(k). \tag{46}$$

In the next section we will work out the matching conditions between this solution and Eq. (43), in order to determine the coefficients A_+ and B_+.

IV. MATCHING CONDITIONS

We suppose that the solution given in Eq. (43) holds until $\eta = -\eta_1$, where higher order corrections begin to play a role. These corrections may be quite different for the modified pre-big bang model and for the ekpyrotic model, but in both cases they are supposed to lead over to a radiation dominated Friedmann model. Here we do not want to argue about the nature of the corrections and how to determine them from string theory (even if this probably has to be considered as the most difficult and the main problem of these models), but we study which statements can be made under certain assumptions on the transition. For this we neglect the details of the transition and match our pre-big bang solution at $\eta = -\eta_1$ to a radiation dominated universe at $\eta = +\eta_1$. In other words we suppose that the slice of spacetime 'squeezed' between $-\eta_1$ and η_1 is so thin compared to the scales we are interested in, that it can be replaced by a spacelike hypersurface. Therefore we can consistently use the thin shell formalism and apply the Israel junction conditions [41] for surface layers on the $\eta = \pm\eta_1$ hypersurface, in order to match the spacetime manifold $\mathcal{M}_-$ before the big bang to the spacetime manifold $\mathcal{M}_+$ after.

A. Matching the background

Before specifying the matching of the perturbations, we have to match the backgrounds, *i.e.* we have to impose the Israel junction conditions on the scale factor a and its first derivative. These conditions require the continuity of the induced metric,

$$q_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta, \tag{47}$$

where n_α is the normal to the $\eta =$ constant hypersurface, on the matching hypersurface $\eta = \pm\eta_1$. Thus we have

$$[q_{\alpha\beta}]_\pm = 0, \tag{48}$$

where we define

$$[h]_\pm \equiv \lim_{\eta \searrow \eta_1} (h(\eta) - h(-\eta)) \equiv h_+ - h_-, \tag{49}$$

for an arbitrary function $h(\eta)$. Here $\eta \searrow \eta_1$ indicates the right hand limit, *i.e.* η is decreasing towards η_1.

Our conformal time coordinate η itself jumps,

$$[\eta]_\pm = 2\eta_1. \tag{50}$$

This simply means that the coordinates of $\mathcal{M}_-$ and $\mathcal{M}_+$ are well defined only on the intervals $\eta \in (-\infty, -\eta_1]$ and $\eta \in [\eta_1, \infty)$, respectively. The limit (49) is well defined for every function which is continuous, monotonic and bounded in open intervals $(-\eta_2, -\eta_1)$ and (η_1, η_2), with $\eta_2 > \eta_1$, even if their value at $\pm\eta_1$ is not defined.

Eq. (48) implies $a_+ = a_- = a_\pm$. According to our normalization of the scale factor, Eq. (33), $a_\pm = 1$. We nevertheless prefer to leave $a_\pm$ in all the expressions where it appears, so that its normalization can be conveniently changed.

The second Israel junction condition concerns the extrinsic curvature K^ν_μ on the matching hypersurface with normal n^α,

$$K_{\alpha\beta} = \frac{1}{2}(q_\alpha{}^\rho \nabla_\rho n_\beta + q_\beta{}^\rho \nabla_\rho n_\alpha). \tag{51}$$

In a Friedmann universe this is

$$K^i_j = -\left(\frac{a'}{a^2}\right)\delta^i_j = -\frac{\mathcal{H}}{a}\delta^i_j. \tag{52}$$

The derivative a' changes sign in the transition from a contracting to an expanding phase. Hence, the extrinsic curvature is discontinuous in the four-dimensional, low energy picture if we simply 'glue' the contracting phase to the expanding phase with opposite sign for a' and conformal time $\eta = +\eta_1$. On the other hand, the Israel junction conditions allow for the existence of a surface stress tensor,

$$[K^i_j]_\pm = \kappa^2 S^i_j, \tag{53}$$

which in our case is non vanishing and diagonal, and it is characterized by a negative surface tension $P_s < 0$,

$$[K^i_j]_\pm = -\frac{\mathcal{H}_+ - \mathcal{H}_-}{a_\pm}\delta^i_j = \kappa^2 P_s \delta^i_j. \tag{54}$$

Within the four dimensional picture we have no explanation for this surface tension; it has to be introduced by hand in order for the extrinsic curvature to jump. Eq. (54) is a possibility to 'escape' the violation of the weak energy condition, $\rho + P < 0$, which is needed for a smooth transition from collapse to expansion. This has been one of the objections to the ekpyrotic scenario in Ref. [42]. Of course for $\eta = \eta_1$ the combination $\rho + P + P_s\delta(\eta - \eta_1)$ becomes negative, which, in the widest sense, can also be interpreted as an 'effective' violation of the weak energy condition. Clearly, this is the simplest way of connecting a contracting phase to an expanding phase, but it is relatively close to an approach motivated from the five-dimensional picture, where the singularity at $a = 0$ becomes a narrow 'throat' [15]. Here we replace this throat by a stiff 'collar' whose length we neglect (see also [16]).

B. Matching the perturbations

Let us now perturb the Israel junction conditions (48) and (53). Instead of considering the $\eta = \eta_1$ hypersurface we want, in general, to consider a hypersurface which is linearly perturbed from it, defined by $\tilde\eta = \eta + T = \eta_1$, where T is a small perturbation. The jump is now realized on the perturbed hypersurface $\tilde\eta = \eta_1$,

$$[h]_\pm \equiv \lim_{\bar\eta \searrow \eta_1} (h(\tilde\eta) - h(-\tilde\eta)) \equiv h_+ - h_-, \tag{55}$$

and in principle we cannot say anything about the continuity of T, which is also allowed to jump,

$$[T]_\pm = [\tilde\eta - \eta]_\pm = 2\eta_1 - [\eta]_\pm. \tag{56}$$

Nonetheless, this jump should be always small as it will become clear below.

We assume that the old coordinates (η, x^i) are those of longitudinal gauge, so that the metric perturbations are given by Eq. (34), but we want to determine the perturbation of the Israel junction conditions in the coordinate system $(\tilde\eta, x^i)$ on the surfaces $\tilde\eta =$ constant. The metric in this coordinate system is given by (see *e.g.* [39])

$$\begin{aligned} d\tilde s^2 = a^2(\tilde\eta)\{&-(1 + 2\Psi - 2(\mathcal{H}T + T'))d\tilde\eta^2 + 2T_{,i}\, d\tilde\eta dx^i \\ &+(1 - 2\Psi - 2\mathcal{H}T)\delta_{ij}dx^i dx^j\}. \end{aligned} \tag{57}$$

Hence the perturbation of the normal to the $\tilde\eta =$ constant slices is

$$\tilde\delta n = \frac{1}{a}\{(-\Psi + \mathcal{H}T + T')\partial_{\bar\eta} - T_{,i}\,\partial_i\}, \tag{58}$$

and the extrinsic curvature is given by [43]

$$\tilde\delta K^i_j = \frac{1}{a}\left\{\Psi' + \mathcal{H}\Psi + (\mathcal{H}' - \mathcal{H}^2)T\right\}\delta^i_j + T^{,i}_{,j}. \tag{59}$$

The matching conditions for the perturbations are obtained by perturbing Eqs. (48) and (53) on the $\tilde\eta = \eta_1$ hypersurface. They become

$$[\tilde\delta q^i_j]_\pm = 0, \qquad [\tilde\delta K^i_j]_\pm = \kappa^2 \tilde\delta S^i_j. \tag{60}$$

From the above expressions for $\tilde\delta g_{\mu\nu}$ and $\tilde\delta n^\mu$, the continuity of the perturbation of the induced metric δq_{ij} on the $\tilde\eta = \eta_1$ hypersurface leads to

$$[\Psi + \mathcal{H}T]_\pm = 0. \tag{61}$$

For reasons that become clear below, we assume in the following that $T = \tilde\eta - \eta$, the lapse of time between the background value η and the perturbed value $\tilde\eta$, remains a small perturbation on large scales. This implies that also $[T]_\pm$ has to remain small. What is the meaning of 'small perturbation' in this context? Once a gauge is fixed, the Bardeen potential Ψ is the only degree of freedom characterizing the perturbations. For dimensional reasons, it is natural to expect T to be given as a linear combination of Ψ and Ψ', in terms of

$$T = \eta P(k\eta)\Psi + \eta^2 Q(k\eta)\Psi', \tag{62}$$

where P and Q are polynomials of $k\eta$, which may have η/η_1 dependent coefficients. Here we assume that these polynomials do not contain any negative power of $k\eta$, *i.e.* that

$$|T/\eta\Psi| \sim |T/\eta^2\Psi'| \sim |P(k\eta)| + |Q(k\eta)| \stackrel{k\to 0}{\longrightarrow} \text{finite}. \tag{63}$$

On large scales T grows with scale at most as Ψ or Ψ'.

The reason for this is that we want that the $\tilde\eta = \eta_1$ hypersurface does not arbitrarily diverge from the $\eta = \eta_1$ hypersurface on large scales. In other words, we require the time at which the bounce happens to be stable under large scale perturbations. It is clear that this assumption is not entirely trivial. It limits somewhat the large scale power of the 'new physics' which is needed to convert contraction into expansion. This new physics may not induce very strong infrared perturbations, which is very reasonable and confirmed by numerical examples on pre-big bang models [44].

Under this assumption the anisotropic term on the right hand side of Eq. (59), $\partial^i\partial_j T$, is negligible on large scales and we shall not discuss the possible, but subdominant, anisotropic surface stresses in what follows. On super horizon scales the perturbation of the extrinsic curvature is dominated by the trace part, $\tilde{\delta}K^i_j = (\tilde{\delta}K)\delta^i_j$ with

$$\tilde{\delta}K = \frac{1}{a}\left\{\Psi' + \mathcal{H}\Psi + (\mathcal{H}' - \mathcal{H}^2)T\right\}. \tag{64}$$

The matching conditions for the perturbations become Eqs. (61), and

$$\left[\Psi' + \mathcal{H}\Psi + (\mathcal{H}' - \mathcal{H}^2)T\right]_\pm = \kappa^2 a_\pm \tilde{\delta}P_s, \tag{65}$$

where $\tilde{\delta}P_s$ is the perturbation of the surface tension.

The condition posed in Eq. (63) has the following important consequences: from Eq. (64) we see that with T not being 'redder' than Ψ and Ψ', also $\tilde{\delta}K$ has typically the same k-dependence as Ψ or Ψ'. Therefore it remains small (of the same order as Ψ or Ψ' in k) when $k\eta_1$ tends to 0,

$$\tilde{\delta}K/(\mathcal{H}\Psi),\ \tilde{\delta}K/\Psi' \stackrel{k\to 0}{\longrightarrow} \text{finite}. \tag{66}$$

From Eq. (65) we then infer that $\tilde{\delta}P_s$ may as well have a non-trivial k-behavior but it remains small on large scales,

$$\tilde{\delta}P_s/(\mathcal{H}\Psi),\ \tilde{\delta}P_s/\Psi' \stackrel{k\to 0}{\longrightarrow} \text{finite}. \tag{67}$$

The k-dependence of $\tilde{\delta}P_s$ may become important when matching the perturbations but it cannot dominate on large scales.

The assumptions (63) and its consequences (66) and (67) become important in Sec. IV D where we try to derive a general result from these matching conditions. First, let us discuss some examples.

C. Two examples

The matching conditions (61) and (65), which the unknown details of the transition have to determine, fix the coefficients $A_+(k)$ and $B_+(k)$. So far, in the literature, for inflation [43] as well as for the ekpyrotic scenario [27–30,45], the hypersurface on which the matching has been performed was always chosen to be the constant energy hypersurface, $\rho + \delta\rho =$ constant. In this case, $T = \delta\rho/\rho'$.

The perturbed Einstein equations give (see *e.g.* [39], Eqs. (2.45) and (2.46), and use $\delta\rho = \rho D_s$ in longitudinal gauge),

$$\begin{aligned}\frac{\delta\rho}{\rho} &= -\frac{2}{\mathcal{H}^2}\left\{(3k^2 + \mathcal{H}^2)\Psi + \mathcal{H}\Psi'\right\} \\ &\simeq -2\left(\Psi + \mathcal{H}^{-1}\Psi'\right),\end{aligned} \tag{68}$$

on super horizon scales. With $\rho' = 2\rho(\mathcal{H}' - \mathcal{H}^2)/\mathcal{H}$ we have

$$T = \delta\rho/\rho' \simeq \frac{-1}{\mathcal{H}' - \mathcal{H}^2}(\mathcal{H}\Psi + \Psi'). \tag{69}$$

Eq. (61) then leads to

$$\left[\Psi - \frac{\mathcal{H}}{\mathcal{H}' - \mathcal{H}^2}(\mathcal{H}\Psi + \Psi')\right]_\pm \equiv [\zeta]_\pm = 0, \tag{70}$$

where ζ is the curvature perturbation introduced by Bardeen [40]. Furthermore, using Eq. (69), one finds that $\tilde{\delta}K_{ij} = 0$ on large scales and we obtain $[\tilde{\delta}K]_\pm \equiv 0$. Hence, this matching condition can be satisfied only if the surface tension P_s is unperturbed, $\tilde{\delta}P_s \equiv 0$.

These matching conditions are often used in inflationary models to go from the inflationary phase to the Friedmann radiation dominated phase. The difference with inflationary models is that here $\mathcal{H}$ jumps. Furthermore, Ψ in general will not be continuous at the transition, since even if T is continuous, $\mathcal{H}T$ is not. Notice that, even though $\mathcal{H}$ jumps at the transition from contraction to expansion, and hence $\mathcal{H}'$ contains a Dirac delta-function, $[T]_\pm$ is well defined as it is a continuous, bounded, monotonic function in some open intervals $(-\eta_2, -\eta_1)$ and (η_1, η_2).

Inserting ansatz (43) and (46) in the continuity condition for the metric, Eq. (70), yields

$$B_+\left(\frac{\mathcal{H}'_+ - 2\mathcal{H}^2_+}{\mathcal{H}'_+ - \mathcal{H}^2_+}\right) = B_-\left(\frac{\mathcal{H}'_+ - 2\mathcal{H}^2_-}{\mathcal{H}'_- - \mathcal{H}^2_-}\right). \tag{71}$$

Clearly, since B_+ couples only to B_- it inherits the blue spectrum of B_-. This is the main argument of Refs. [27–32] against the ekpyrotic model. As we shall see below, this is also the matching condition which leads to the $n = 4$ spectrum in the pre-big bang model given in Ref. [21].

There are two subtleties which have been left out in this argument. The first one is obvious: the surface tension P_s, the only ingredient of the high energy theory in this approach, may well also have a perturbation $\tilde{\delta}P_s$, requiring $[\tilde{\delta}K]_\pm = \kappa^2\tilde{\delta}P_s \neq 0$. If this is the case, the matching cannot be defined on the constant energy hypersurfaces, $T = \delta\rho/\rho'$. Secondly, and more importantly, in this model where contraction goes over to expansion, a transition surface with a physical surface tension is required and this surface does need not to agree with the $\rho + \delta\rho =$ constant!

As a concrete example, let us simply assume that this matching surface is given by the condition that its shear vanishes. This is actually just the $\eta =$ constant surface in longitudinal gauge, hence we have $T = 0$ in Eqs. (61) and (65). The junction conditions on super horizon scales then become

$$[\Psi]_\pm = 0, \tag{72}$$

$$[\mathcal{H}\Psi + \Psi']_\pm = a_\pm\kappa^2\delta P_s. \tag{73}$$

For our general solutions (43) and (46) this gives

$$A_+ = \frac{\mathcal{H}_-}{\mathcal{H}_+}A_- + \frac{a_\pm^2}{\mathcal{H}_+}(B_- - B_+) \tag{74}$$

$$\begin{aligned} B_+ = &\left(\frac{\mathcal{H}_+(\mathcal{H}'_-/\mathcal{H}_- - \mathcal{H}_-) - \mathcal{H}'_+ + \mathcal{H}_+^2}{2\mathcal{H}_+^2 - \mathcal{H}'_+}\right)\frac{\mathcal{H}_-}{a_\pm^2}A_- \\ &+ \left(1 + \frac{\mathcal{H}_-\mathcal{H}_+ - \mathcal{H}_+^2}{2\mathcal{H}_+^2 - \mathcal{H}'_+}\right)B_- \\ &+ \frac{\mathcal{H}_+}{2\mathcal{H}_+^2 - \mathcal{H}'_+}\kappa^2 a_\pm \delta P_s. \end{aligned} \tag{75}$$

Alternatively, we can express the matching conditions in terms of ζ given in Eq. (70) and its canonically conjugate variable Π defined in Ref. [46], by

$$\Pi = 2M_P^2 k^2 \frac{a^2}{\mathcal{H}}\Psi. \tag{76}$$

On super horizons scales we have

$$\zeta = \left(1 - \frac{\mathcal{H}^2}{\mathcal{H}' - \mathcal{H}^2}\right)B(k), \tag{77}$$

$$\Pi = 2M_P^2 k^2 \left(A(k) + \frac{a^2}{\mathcal{H}}B(k)\right). \tag{78}$$

The perturbation variable ζ is constant and proportional to the constant $B(k)$ while its conjugate momentum Π is proportional to $A(k)k^2$ and constant up to a decaying part proportional to $B(k)$ which will be negligible at the time $-\eta_1$, when we impose the matching conditions.

On the zero shear hypersurface we can write the matching conditions of the perturbations in terms of ζ and Π as

$$[\mathcal{H}\Pi]_\pm = 0, \tag{79}$$

$$\left[(\mathcal{H}' - \mathcal{H}^2)\left(\frac{\kappa^2}{2k^2a^2}\Pi - \frac{\zeta}{\mathcal{H}}\right)\right]_\pm = a\kappa^2\delta P_s. \tag{80}$$

Therefore we have

$$\Pi_+ = \frac{\mathcal{H}_+}{\mathcal{H}_-}\Pi_-, \tag{81}$$

$$\begin{aligned} \zeta_+ = &\frac{\mathcal{H}_+}{\mathcal{H}_-}\left(\frac{\mathcal{H}'_- - \mathcal{H}_-^2}{\mathcal{H}'_+ - \mathcal{H}_+^2}\right)\zeta_- \\ &+ \frac{\kappa^2}{2k^2a_\pm^2}\left(\mathcal{H}_- - \frac{\mathcal{H}'_- - \mathcal{H}_-^2}{\mathcal{H}'_+ - \mathcal{H}_+^2}\mathcal{H}_+\right)\Pi_- \\ &- \frac{\mathcal{H}_+}{\mathcal{H}'_+ - \mathcal{H}_+^2}a_\pm\kappa^2\delta P_s. \end{aligned} \tag{82}$$

Hence, using matching conditions on the zero shear hypersurface, ζ acquires, in the radiation dominated era, a mode $\propto \Pi_- k^{-2} \propto A_-$ which has a spectral index $n = 1 - 2q$ of A_-. In terms of A_+ and B_+ this leads again to Eqs. (74) and (75).

As A_- represents the growing mode during the contracting phase, $|A_-\mathcal{H}/a^2|$ is much larger than $|B_-|$, and the spectrum of B_+ inherits the scale invariant spectrum of A_-. It is easy to see from pure sign considerations that the pre-factor of A_- in Eq. (75) does not vanish.

D. A more general treatment

As we have seen, the important question is to determine the correct matching hypersurface and the perturbation of its tension. This can only be done by studying the high energy corrections of a specific model. Nevertheless, we now want to provide an argument why we think that a scale invariant spectrum is obtained in models where the collapsing phase is characterized by $a \propto (-\eta)^q$ with $q \ll 1$.

As we have seen in the above examples, the matching conditions are fixed by T, given as some combination of Ψ and Ψ', and determine Ψ_+ in terms of Ψ_-, Ψ'_-, and of the surface stress perturbation δP_s. The general result we are about to derive is based on one important assumption, the smallness of T, as given in Eq. (63). As explained there, this assumption precisely limits the 'infrared power' of the 'new physics' needed to convert contraction into expansion. As we have seen [Eqs. (66) and (67)], as a consequence the extrinsic curvature and tension perturbations, $\tilde{\delta}K$ and $\tilde{\delta}P_s$, have the same k-dependence as Ψ and Ψ'.

This assumption fixes completely the final spectrum, avoiding any arbitrariness such as the one found in [31] for the ekpyrotic scenario. Then, in Eqs. (61) and (65) the k-dependence is given entirely in terms of the coefficients A and B. As a result, the k-dependence of the coefficients A_+ and B_+ is a mixture of the k-dependence of A_- and B_- given by Eqs. (44) and (45),

$$A_+(k) = \alpha_A k^{-(1+\mu)} + \beta_A k^{-1+\mu}, \tag{83}$$

$$B_+(k) = \alpha_B k^{-(1+\mu)} + \beta_B k^{-1+\mu}, \tag{84}$$

where the α-terms come from the A_--mode and the β-terms come from the B_--mode. According to our assumption, the coefficients $\alpha_\bullet$ and $\beta_\bullet$ generically contain a constant and positive powers of $k\eta_1$. The A_+-mode is decaying and we may neglect it soon after the matching. Generically we expect, according to the amplitudes of the A_- and B_--modes, that α_A and α_B are much larger than β_A and β_B. Comparing the A_- and B_--modes we expect

$$\mathcal{O}\left(\alpha k^{-1-\mu}\right) \sim \mathcal{O}\left((k\eta_1)^{-2\mu}\,\beta k^{-1+\mu}\right), \tag{85}$$

hence, for super horizon modes, $k\eta_1 \ll 1$, we expect $\alpha k^{-1-\mu} \gg \beta k^{-1+\mu}$, as long as $\mu = q + 1/2$ is positive. Therefore, one typically inherits the spectrum of the α-terms in the radiation era, leading to

$$\mathcal{P}_\Psi = |\Psi|^2 k^3 = |\alpha_B|^2 k^{1-2\mu} \quad \left(\propto k^{n-1}\right). \tag{86}$$

In this generic situation, we obtain a scale invariant spectrum $1 \simeq n = 2 - 2\mu = 1 - 2q$ if q is close to zero, as in the ekpyrotic and modified pre-big bang case.

Only if the matching conditions are such that the α_B-term is suppressed by a factor smaller than $(k\eta_1)^{2\mu}$, the β_B-term comes to dominate and the spectrum becomes

$$\mathcal{P}_\Psi = |\Psi|^2 k^3 = |\beta_B|^2 k^{1+2\mu} \quad \left(\propto k^{n-1}\right). \tag{87}$$

Then, the spectral index $n = 2 + 2\mu = 3 + 2q$ results.

As an estimate, for scales of order the present Hubble parameter, relevant for the perturbations in the cosmic microwave background, $k = k/a_\pm \sim H_0$, and for $1/\eta_1 = 1/(a_\pm \eta_1) \sim 10^{17}$ GeV, we have $k\eta_1 \sim 10^{-59}$! Hence we typically expect the β-terms to be about 10^{59} times smaller than the α-terms on cosmologically relevant scales, $\alpha_\bullet k^{-1-\mu} \sim 10^{59} \beta_\bullet k^{-1+\mu}$.

For the constant energy hypersurface we have obtained $\alpha_B \equiv 0$ and hence the generic inequality $\alpha k^{-1-\mu} \gg \beta k^{-1+\mu}$ is violated. But if the matching hypersurface deviates by more than about $\sim 10^{-59}$ from the $\rho =$ constant hypersurface, we expect the A_--term, $\alpha k^{-1-\mu}$, to dominate in the Bardeen potential and to determine the final spectrum.

For a scalar field without potential, as in the original pre-big bang model, we have $q = 1/2$ which in the 'generic case' leads to a spectral index $n = 1 - 2q = 0$ and only under very special matching conditions, like matching on the constant energy hypersurface with $\delta P_s \equiv 0$, the spectral index $n = 4$ is obtained.

In the case of ordinary inflation, $q \sim -1$, where $\mu = 1/2 + q$ is negative, the situation is quite different. There, the A_--mode is decaying and the Bardeen potential at the end of inflation is dominated by the constant B_--mode. Hence, we generically expect to inherit in the radiation phase the spectral index from the B_--mode with $n = 3 + 2q$, leading to a scale invariant spectrum for ordinary inflation, $q \sim -1$. This is also the spectrum obtained when matching on the constant energy hypersurface.

In Ref. [47], a radiation dominated contracting phase is connected smoothly to a radiation dominated expanding phase, via a scalar field with negative energy density which comes to dominate in the high curvature regime. Here a $n = -1$ spectrum of perturbation is found with analytical arguments and via numerical simulation. This agrees with our result. In this case, in fact, $q = 1$ and according to our argument we would generically expect $n = 1 - 2q = -1$, as obtained in Ref. [47]. It is interesting to note that the matching conditions of Ref. [47] corresponds to the matching on the hypersurfaces determined by $T = -\mathcal{H}^{-1}\Psi$ from longitudinal gauge. According to Eq. (57), this corresponds to the gauge with $\tilde{\delta} g_{ij} = 0$, $i \neq j$, the 'off-diagonal gauge', which has also been considered in Ref. [21] as the gauge in which perturbations remain small during the pre-big bang phase.

This is our main result: When matching a collapsing universe to an expanding one, we expect the Bardeen potential in the expanding phase to inherit the spectrum of the mode which grows during the collapse phase, leading to

$$\mathcal{P}_\Psi \propto k^{-2q}, \quad n = 1 - 2q, \tag{88}$$

where q is the exponent with which the scale factor contracts in conformal time, $a \propto |\eta|^q$. Remind that this result holds only if we assume, as explained in Sec. IV B, that T is small on large scales [see Eq. (63)].

V. USING Ψ OR ζ ?

In the above discussion we have used mainly the Bardeen potential Ψ. Several authors [28–30,45] use the curvature perturbation ζ given in Eq. (70). In particular, Ref. [45] has found

$$\zeta \propto \frac{|\eta|^{1/2}}{a} H_\nu^{(2)}(k\eta), \qquad \nu = |q - 1/2|. \tag{89}$$

This also follows from the definition of ζ [see Eq. (70)], together with the solution (41) for Ψ. During the pre-big bang phase, $\eta < -\eta_1$, this leads to the following spectrum for ζ on super horizon scales, modulo logarithmic corrections,

$$\mathcal{P}_\zeta = |\zeta|^2 k^3 \propto \begin{cases} k^{4-2q} |\eta|^{2-4q} & \text{for } q > 1/2, \\ k^{2+2q} & \text{for } q < 1/2, \end{cases} \tag{90}$$

giving a spectral index for the variable ζ,

$$n_\zeta = \begin{cases} 5 - 2q & \text{for } q > 1/2, \\ 3 + 2q & \text{for } q < 1/2. \end{cases} \tag{91}$$

Since in Ref. [45] the matching condition $[\zeta]_\pm = 0$ is used, the spectral index of ζ translates directly into the spectral index of scalar perturbations in the radiation era, where ζ and Ψ essentially agree on super horizon scales. This is the reason why these authors obtain a scale invariant spectrum also for $q = 2$ (while they obtain $n = 3$ for the ekpyrotic model).

We have found the following behavior of the Ψ spectrum on super horizon scales during the pre-big bang phase (see Eq. (41) in the limit $k|\eta| \ll 1$),

$$\mathcal{P}_\Psi \propto \begin{cases} k^{-2q} |\eta|^{-(2+4q)} & \text{for } q > -1/2, \\ k^{2+2q} & \text{for } q < -1/2. \end{cases} \tag{92}$$

This leads to the spectral index of Ψ,

$$n_\Psi = \begin{cases} 1 - 2q & \text{for } q > -1/2, \\ 3 + 2q & \text{for } q < -1/2. \end{cases} \tag{93}$$

Comparing Eq. (90) and Eq. (92) we see that

$$\mathcal{P}_\zeta \simeq |k\eta|^{2\gamma} \mathcal{P}_\Psi \leq \mathcal{P}_\Psi, \tag{94}$$

with

$$\gamma = \begin{cases} 0 & \text{for } q < -1/2, \\ 1 + 2q & \text{for } -1/2 < q < 1/2, \\ 2 & \text{for } q > 1/2. \end{cases} \tag{95}$$

As we have mentioned above, for cosmologically relevant scales, the factor $|k\eta|$ becomes of the order of 10^{-59} at the matching surface. We have argued in the previous

subsection that the larger variable Ψ should be relevant at the matching surface, and only under very special special matching conditions the spectral index of ζ is inherited after the big bang. Generically we therefore expect $n = n_\Psi$ to be the spectral index in the radiation era. If $q < -1/2$, Ψ and ζ agree up to a constant pre-factor, and this distinction becomes irrelevant for the spectral index. This is exactly what happens in 'ordinary inflation' where $q \sim -1$. The functions n_Ψ and n_ζ are shown in Fig. 1.

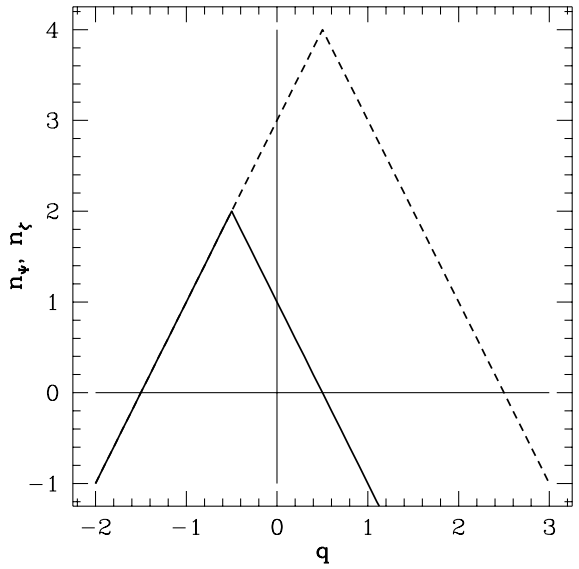

FIG. 1. The spectral indices n_Ψ (solid) and n_ζ (dashed) are shown as a function of $q = \mathcal{H}\eta$. As argued in the text, we expect a resulting spectral index $n = n_\Psi$ in the radiation era.

Finally, for completeness, we want to emphasize that the Bardeen potential in a radiation dominated universe really determines the spectral index n via $\mathcal{P}_\Psi = |\Psi|^2 k^3 \propto k^{n-1}$. A scale invariant spectrum is defined as one for which $\left\langle (\delta M/M)^2 \right\rangle_{\rm h.c.}$ is scale independent, where the brackets denote spatial average and the subscript h.c. indicates the scale of horizon crossing. Therefore, the spectral index is defined by $\left\langle (\delta M/M)^2 \right\rangle_{\rm h.c.} \propto k^{n-1}$, so that $n = 1$ represents a scale-invariant spectrum.

On the other hand

$$\left\langle \left(\frac{\delta M}{M} \right)^2 \right\rangle = k^3 \left| \frac{\delta\rho}{\rho} \right|^2 . \tag{96}$$

On sub horizon scales and also at horizon crossing, $\delta\rho/\rho$ is not strongly gauge dependent, so we may choose whatever gauge we please. We use comoving gauge (it is a simple estimate to verify the same behavior, *e.g.* for longitudinal gauge). In comoving gauge we have the constraint equation [39],

$$k^2\Psi = \frac{3\mathcal{H}^2}{2} \left(\frac{\delta\rho}{\rho} \right)_{\rm com.} . \tag{97}$$

Using that $\mathcal{H} \simeq k$ at horizon crossing and that Ψ is time independent on super horizon scales, we get

$$\left| \left(\frac{\delta\rho}{\rho} \right)_{\rm com.} \right|^2_{\rm h.c.} \simeq |\Psi|^2, \tag{98}$$

hence

$$\left\langle |\delta M/M|^2 \right\rangle_{\rm h.c.} \simeq k^3 |\Psi|^2 = \mathcal{P}_\Psi \quad \propto k^{n-1}.$$

In the radiation dominated era ζ is roughly equal to Ψ and the above equation therefore holds also for $\mathcal{P}_\zeta$.

VI. CONCLUSIONS

We have discussed the matching from a collapsing to an expanding Friedmann universe. We have noted that a non-vanishing surface tension at the matching surface is needed to turn the pre-big bang collapse into expansion. This surface tension and its perturbation have to be specified by the high energy corrections of the theory. It is this surface tension which determines the correct matching surface and it will generically not be parallel to the $\rho + \delta\rho =$ constant surfaces.

We have found that, if the matching is performed at the $\rho + \delta\rho =$ constant hypersurface, the growing mode from the pre-big bang phase is converted entirely into the decaying mode in the radiation phase. In this case the spectral index $n = 3 + 2q$ is obtained, leading to $n = 3$ for the ekpyrotic and modified pre-big bang model, and $n = 4$ for the original pre-big bang model. However, if the matching hypersurface is chosen to be somewhat different from $\rho + \delta\rho =$ constant, one obtains $n = 1 - 2q$. Hence, the ekpyrotic and the modified pre-big bang model can lead to a scale invariant spectrum of scalar perturbations.

Our result is based on the assumption that perturbing our background bouncing universe does not change completely the time and duration of the bounce on large scales. We have formulated this requirement precisely by restricting the allowed 'infrared power' of T.

Notice that the spectral index resulting from our matching conditions of a pre-big bang transition, is never blue, $n \leq 1$. This is not so surprising: On sub-horizon scales, the perturbations are in their vacuum state. They start growing as soon as they exit the horizon until the end of the pre-big bang phase. Hence large scales, which exit earlier, have more time to grow.

Often, as a heuristic approach to obtain the spectrum of fluctuations, one considered $|\Psi|^2 k^3$ at horizon crossing requiring that this behaves like k^{n-1}. Applying this procedure during the pre-big bang at the first horizon crossing (exit), one obtains the blue spectra $n = 3$ for the ekpyrotic or the modified pre-big bang model and $n = 4$ for the original pre-big bang respectively. However, if one determines the same quantity at the *second* horizon crossing (re-entry), during the radiation dominated

phase, one obtains the correct spectral indices $n = 1$ and $n = 0$ respectively. Since in an expanding universe the Bardeen potential does not grow on super horizon scales, it does not matter at which horizon crossing, exit or reentry, the spectrum is determined in the case of ordinary inflation. In a pre-big bang model however, this difference is crucial as we have seen.

The discussion presented in this paper does not affect the gravity wave spectrum [48] which still leads to the spectral index $n_T = 3$ for both models and is a potentially important observable to discriminate them from ordinary inflation.

The main open problem when studying this bouncing models remains the high energy transition from the pre- to the post-big bang. There, corrections should become important, and we have assumed here that for super horizon scales they can be summarized into a tension on the matching surface. Furthermore, it has not yet been shown from string theory that the dilaton can obtain an exponential potential (in the modified pre-big bang model) or that the brane distance simply obeys the equation of motion of a minimally coupled scalar field with exponential potential from the brane point of view for the ekpyrotic model.

Also the quantum production of other modes possible in these models, *e.g.* the axions and moduli in the modified pre-big bang, or the 'graviphoton' and 'graviscalar' coming from the extra-dimension in the ekpyrotic model, have to be investigated.

Nevertheless, we conclude that models where high energy corrections lead a slowly collapsing universe over into an expanding radiation dominated phase may represent viable alternatives to usual 'potential inflation', in generating a scale invariant spectrum of perturbations. However, many open questions, especially concerning the high energy corrections, and flatness, still have to be properly addressed.

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