A no-go theorem for k-essence dark energy

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We demonstrate that if k-essence can solve the coincidence problem and play the role of dark energy in the universe, the fluctuations of the field have to propagate superluminally at some stage. We argue that this implies that successful k-essence models violate causality. It is not possible to define a time ordered succession of events in a Lorentz invariant way. Therefore, k-essence cannot arise as low energy effective field theory of a causal, consistent high energy theory.

Cosmological observations indicate that the expansion of the Universe is presently in an accelerating phase [1]. In a homogeneous and isotropic universe, this can be obtained, if the energy density is dominated by a component x with $w_x = P_x/\rho_x < -1/3$; here ρ_x is the energy density of the component x and P_x is its pressure. The simplest example of such a component which is compatible with observations is a cosmological constant of the order of $\Lambda \simeq 2H_0^2$, where H_0 is the present value of the Hubble parameter. Apart from the smallness of this value which cannot be explained by any sensible theory of fundamental interactions, it is perturbing that the value of the cosmological constant should just be such that it comes to dominate today.

In order to alleviate this coincidence problem quintessence [2] and k-essence [3] have been proposed. In these models a scalar field has the property that at early times, in the radiation dominated universe, its energy density 'tracks' the one of the cosmic fluid and therefore naturally provides a sizeable fraction of the energy density of the universe. Quintessence also tracks the matter energy density during matter domination, but the mechanism which leads to the domination of quintessence today is not clearly identified.

In the case of k-essence the situation is different. Within a certain range of initial conditions, the energy density sharply drops after the beginning of the matter dominated era and assumes an equation of state $P_k \simeq -\rho_k$ (de Sitter phase). Afterwards its contribution can either rise to dominate the energy density with an equation of state of $w_k = \text{constant} < 0$, or become comparable to that of matter and start to 'track' the matter. The radiation tracker, de Sitter phase and k-essence domination (or matter tracker) are all attractor solutions of the k-essence evolution equation. The k-essence field is driven from one to the other by the evolution of the universe [3–5]. This looks promising as a solution to the coincidence problem [17].

However, in this letter we shall show that a k-essence field which behaves in the way described above cannot emerge as the low energy limit of a consistent, causal high energy theory (be this a quantum field theory or string theory, see Ref. [6]). The problem of acausalities in scalar field theories with non-quadratic Lagrangian has also been addressed in Ref. [7].

The k-essence model is characterized by non-standard kinetic energy terms [3, 4]. The action of k-essence is given by

$$S = \int d^4x \sqrt{-g}\left[-\frac{R}{6} + P(\phi, X)\right] \;, \qquad (1)$$

were ϕ is the k-essence field and $X = \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi$. We use units with $\frac{8\pi G}{3} = 1$ and the metric signature is $(+,-,-,-)$. Furthermore, one assumes that the Lagrangian can be factorized $P(\phi, X) = K(\phi)p(X)$ with $K(\phi) > 0$. A standard scalar field with $P(\phi, X) = X$ does of course not have the behaviour we are looking for: one therefore allows p to be an arbitrary, monotonically growing function of X [4].

Varying the above action with respect to the metric one obtains the energy momentum tensor,

$$T_{\mu\nu} = \frac{\partial P(\phi, X)}{\partial X}\nabla_\mu\phi\nabla_\nu\phi - P(\phi, X)g_{\mu\nu} \qquad (2)$$

$$= (\rho_k + P_k)u_\mu u_\nu - P_k g_{\mu\nu} \qquad (3)$$

with $u_\mu = (2X)^{-1/2}\nabla_\mu\phi$, $\rho_k = 2X\frac{\partial P}{\partial X} - P$ and $P_k = P$ Note that in a homogeneous and isotropic universe $\nabla_\mu\phi$ is time-like, $X > 0$.

The idea is of course, that ϕ is a low energy effective degree of freedom of some fundamental high energy theory [3, 4, 8] which should satisfy basic criteria: among them, most importantly, Lorentz invariance and causality. No information should propagate faster than the speed of light $c = 1$. Let us translate this basic requirement to the low energy effective degree of freedom ϕ. We consider the cosmic background solution with small fluctuations, $\phi = \phi_0(t) + \delta\phi(t, \mathbf{x})$. It is then easy to derive an equation of motion for the fluctuations $\delta\phi$ which is of the generic form [18]

$$\ddot{\delta\phi} + \alpha\dot{\delta\phi} + \beta\delta\phi + c_s^2\triangle\delta\phi = 0 \quad \text{where} \qquad (4)$$

$$\triangle \delta\phi = g^{ij}\partial_i\partial_j(\delta\phi) \ .$$

Here an over-dot is a derivative w.r.t. physical time t and α, β and c_s^2 are functions of t. In the case of the k-essence field, from the action (1) one finds [8]

$$c_s^2 = \frac{p'}{2Xp''+p'} \qquad ' = \frac{d}{dX} \ . \tag{5}$$

If for some time $c_s^2 > 1$, the fluctuations $\delta\phi$ (or equivalently the Bardeen potential) propagate faster than the speed of light and are therefore acausal. Indeed, equation (4) can be rewritten as

$$(G^{-1})^{\mu\nu}\partial_\mu(\delta\phi)\partial_\nu(\delta\phi) + \alpha\dot{\delta\phi} + \beta\delta\phi = 0 \tag{6}$$

where we have defined

$$(G^{-1})^{\mu\nu} = g^{\mu\nu} - (1-c_s^2)g^{ij}\delta_i^\mu\delta_j^\nu \tag{7}$$

as the inverse of the metric which governs the propagation of the k-essence field. The characteristic cones of (6) are given by $(G^{-1})^{\mu\nu}$ [9], and the rays by the metric

$$G_{\mu\nu} = g_{\mu\nu} + \frac{1-c_s^2}{c_s^2}g_{ij}\delta_\mu^i\delta_\nu^j \ . \tag{8}$$

If $c_s^2 > 0$, $G_{\mu\nu}$ is Lorentzian [10]. However, if we consider a vector n^μ lying on the light cone defined by the Einstein metric $g_{\mu\nu}$, with respect to $G_{\mu\nu}$ we have

$$G_{\mu\nu}n^\mu n^\nu = \frac{1-c_s^2}{c_s^2}g_{ij}n^i n^j \ . \tag{9}$$

Since $g_{ij}n^i n^j < 0$, $c_s^2 > 1$ implies $G_{\mu\nu}n^\mu n^\nu > 0$, i.e. n^μ is time-like with respect to $G_{\mu\nu}$. Therefore, the characteristic cone given by $G_{\mu\nu}$ is wider than the light cone of causality defined by $g_{\mu\nu}$.

But a k-essence field value at some event $q_0 = (\eta_0, \mathbf{x}_0)$ can be affected by the values of all points inside the past characteristic cone defined by $G_{\mu\nu}$ [11]. Let us now consider a point $q_1 = (\eta_1, \mathbf{x}_1)$ which is inside the past characteristic cone, but outside the past light cone of q_0. Since $c_s^2 > 1$ such points exist and, in general, the field value at q_1 influences the value at q_0. However, since q_1 is outside the past light cone of q_0, the distance $q_0 - q_1$ is space-like and there exists a boost such that the boosted event q_1' is in the future of q_0'. In other words, the value of the field at q_0' can be affected by its values in the future; an evident a-causality. This is the well known consequence of relativity. Whenever an event in q is affected by something outside its past light cone (defined by the metric which determines causality, i.e. the propagation of light and other standard model particles), the present is affected by the future in some boosted reference frame. In order for causality to be respected, it is therefore not enough that the k-essence field propagates inside the light cone of $G_{\mu\nu}$ (as suggested in [12]). If this cone is wider than the one of the Einstein metric, this

leads to superluminal propagation of the k-essence field perturbations which are not acceptable.

Similar arguments in the more complicated case of multi-component fields have led Velo and Zwanziger to the exclusion of generic higher spin theories [13]. On the other hand, Gibbons [14] has analyzed the tachyon in the effective field model proposed by Sen [15], and by the same argument has concluded that "the tachyon is not a tachyon", because the characteristic cone of the tachyon lies inside or on the light cone of the Einstein metric. The tachyon is unstable but it does not violate causality. The causality argument is also at the basis of Ref. [6] where it is used to exclude certain Lagrangians as possible low energy approximations of a sound high energy field theory. There it is also shown that this argument is not alleviated if we allow the high energy theory to be a string theory. Therefore, the fluctuation equation (4) leads to a-causalities if c_s^2 becomes larger than one.

In the rest of the letter, we show that this is exactly what happens in the case of successful k-essence models. We first present two examples from the literature and then formulate a general proof showing that for a successful k-essence model c_s^2 must become larger than one for some time.

The equation of motion of the k-essence field is derived from the action (1). In order to have tracking solutions one must require $K(\phi) = 1/\phi^2$ [4]. Moreover, to describe the dynamics of the k-essence field it is useful to consider the new variable $y \equiv 1/\sqrt{X}$ and introduce the function $g(y) = p(X)/\sqrt{X}$. In the new variables the energy density becomes $\rho_k = 2X\frac{\partial P}{\partial X} - P = -g'/\phi^2$. Here a prime denotes differentiation with respect to y. Since ρ_k has to be positive, g is monotonically decreasing, $dg/dy = g' < 0$. Other useful relations are [3]

$$w_k = \frac{P_k}{\rho_k} = \frac{p}{2Xp'-p} = \frac{-g}{yg'} \quad \text{and} \tag{10}$$

$$c_s^2 = \frac{p'}{2Xp''+p'} = \frac{g-g'y}{g''y^2} \tag{11}$$

(note that a prime on p stands for derivatives w.r.t. X while a prime on g indicates derivatives w.r.t. y). The stability condition $c_s^2 > 0$ requires $g'' > 0$ so that g is convex [4].

In a Friedman universe with $H^2 = \rho_{\text{tot}} = \rho_r + \rho_d + \rho_k$ and $\Omega_k = \rho_k/\rho_{\text{tot}}$ (ρ_r being the energy density of radiation and ρ_d that of pressure-less matter, dust), the k-essence equation of motion can be written in the form [3]

$$\dot{y} = \frac{3(w_k(y)-1)}{2r'(y)}\left[r(y) - \sqrt{\Omega_k}\right] \tag{12}$$

$$\dot{\Omega}_k = 3\Omega_k(1-\Omega_k)(w_m - w_k(y)) \ . \tag{13}$$

Here a dot indicates the derivative w.r.t $N = \ln(a)$, where a is the scale factor, w_m is the ratio

$$w_m = \frac{1}{3}\frac{\rho_r}{\rho_r+\rho_d} \ , \quad \text{and} \tag{14}$$

$$r(y) \equiv \frac{3}{2\sqrt{2}}\sqrt{-g'}y(1+w_k) = \frac{3}{2\sqrt{2}}\frac{(g-g'y)}{\sqrt{-g'}} \ . \ (15)$$

Fixed points, y_f of the above system of equation are given by $r^2(y_f) = \Omega_k = $ constant, and either $w_k(y_f) = w_m$, or $\Omega_k = 0$ or $\Omega_k = 1$. These either are stable or can be made stable with small changes in the function g [4].

The evolution of the universe drives the k-essence field from one fixed point to another. At early times, within a suitable range of initial conditions, k-essence quickly approaches the radiation fixed point y_r. In order not to violate the nucleosynthesis bound one requires $r(y_r) = \sqrt{\Omega_k} \lesssim 0.1$. When the universe becomes matter dominated, the radiation fixed point is lost and the k-essence energy density decreases rapidly until the de Sitter attractor, y_s with $0 \simeq \Omega_k = r(y_s) \ll r(y_r)$ is reached. From there, the field evolves to the k-essence attractor y_k with $r(y_k) = \sqrt{\Omega_k} \simeq 1$ and $-1 < w_k(y_k) < 0$ or, if this attractor does not exist, it evolves on to the dust attractor y_d with $w_k(y_d) = w_m = 0$. At present, the field is on its way from the de Sitter fixed point up to either the k-essence or dust attractor.

Examples of Lagrangians $P(\phi, X)$ that can be found in the literature are [3, 4]

$$P(\phi, X) = \frac{1}{\phi^2}\left(-2.01+2\sqrt{1+X}+3\cdot10^{-17}X^3-10^{-24}X^4\right)$$
$$(16)$$

and

$$P(\phi, X) = \frac{1}{\phi'^2}\left(-2.05+2\sqrt{1+f(X)}\right) \quad \text{where}$$
$$f(X) = X - 10^{-8}X^2 + 10^{-12}X^3 - 10^{-16}X^4$$
$$+10^{-20}X^5 - 10^{-24}X^6/2^6 \ . \quad (17)$$

The evolution of interesting physical quantities for the Lagrangian (16) are shown in Figs. 1 and 2. Example (17) behaves similarly. In these examples, k-essence evolves to the final stage of k-essence domination.
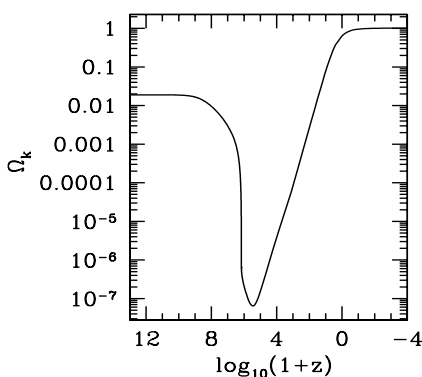

FIG. 1: The ratio of k-essence to the total energy density Ω_k as function of $1 + z$ for the example (16).

Fig. 2 shows that c_s^2 becomes larger than 1 during the evolution from the radiation fixed point to the de Sit-
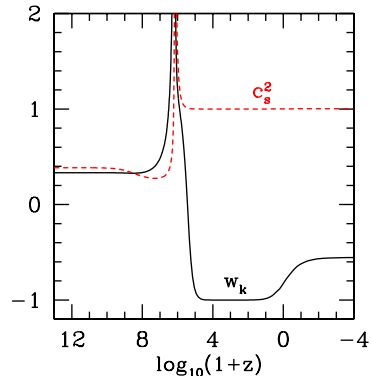

FIG. 2: The equation of state parameter w_k and sound velocity c_s^2 as functions of $1 + z$ for the example (16).

ter fixed point and remains slightly larger than 1 in the future, when k-essence reaches the k-attractor.

In Ref. [6], it is shown that for small values of X superluminal propagation of perturbations around a nontrivial background is related to the sign of the coefficient in front of the leading higher-dimension operator in the Lagrangian: the absence of superluminal propagation requires a positive coefficient. In both examples, expansion for small X leads to $p(X) = a + bX + dX^2 + \mathcal{O}(X^3)$, with $d < 0$. For small X, $p''(X) = 2d$ is negative, and Eq. (5) hence gives $c_s^2 > 1$. In both examples, the theory is acausal. We note however that no problem arises in k-inflation, where the coefficient in front of the term X^2 in the Lagrangian is positive [8].

We now demonstrate that $c_s^2 > 1$ is mandatory in any k-essence model that aims to solve the coincidence problem and leads to accelerated expansion of the universe today. From Eq. (11), $c_s^2 > 1$ is equivalent to $g''y^2 < g - g'y$. Using

$$w_k = \frac{-g}{yg'} \ , \quad w_k' = \frac{gg' + gg''y - g'^2y}{(g'y)^2} \ , \quad (18)$$

and remembering that $g' < 0$, we conclude that $w_k > 1$ is equivalent to $g + yg' > 0$ and $w_k' < 0$ is equivalent to $gg' + gg''y - g'^2y < 0$. If both these conditions are fulfilled, we necessarily have $0 > gg' + gg''y - g'^2y > -g'(-g + g''y^2 + g'y)$. In the last unequal sign we have used $g > -yg'$ and $g'' > 0$. Since $g' < 0$ this implies $g''y^2 < g - g'y$ and therefore $c_s^2 > 1$.

We now show that such a situation always arises in k-essence models which solve the coincidence problem. We first consider the evolution of k-essence from the radiation fixed point y_r to the k-essence fixed point y_k. We then show that the same arguments hold if the k-attractor is replaced by a late dust-attractor.

We remind that, at a fixed point, $r(y_f) = \sqrt{\Omega_k}$. Moreover one always has $y_k > y_r$ since g is monotonically decreasing, and $w_k(y_k) < 0$, hence $g(y_k) < 0$ while

$g(y_r) > 0$ (see Eq. (18)). Eq. (15) gives

$$\frac{dr}{dy} = \frac{3}{2\sqrt{8}} \frac{g''y}{\sqrt{-g'}} (w_k(y) - 1) , \qquad (19)$$

and since $g'' > 0$, $r(y)$ can increase only if $w_k > 1$. Since $r(y_k) > r(y_r)$, r has to increase from y_r to y_k. This implies that there exists an interval $y_0 < y < y_2$, with $y_r < y_0 < y_2 < y_k$, in which $w_k(y) > 1$. Since $w_k(y_k) < 0$ we can choose without loss of generality $w_k(y_2) = 1$. For some part of the interval, say in $]y_1, y_2[$, w_k has to decay: $w_k' < 0$. Therefore, both conditions necessary for $c_s^2 > 1$ are satisfied in $]y_1, y_2[$. In other words, between two points $y_a < y_b$ with $r(y_a) < r(y_b)$ and $w(y_a) < 1$, $w(y_b) < 1$ there exists necessarily an interval with $c_s^2(y) > 1$.

If the k-attractor is replaced by a late dust-attractor, the situation is alike. Indeed, since $w_k(y_d) = 0$ hence $g(y_d) = 0 < g(y_r)$ we must have $y_r < y_d$. Furthermore, in order to have a period of accelerated expansion with $w_k < -1/3$, we need $r(y_d) > r(y_r)$. If $r(y_d) < r(y_r)$, the accelerating phase is avoided because the k-essence fluid is attracted immediately to the dust-attractor after matter-radiation equality [4]. Therefore $r'(y)$ must be positive in an interval between y_r and y_d and the demonstration above holds also in this case.

The only behavior relevant for this result is the existence of a radiation tracker which goes over into an accelerating phase with $w_k < -1/3$ and a relatively large value of Ω_k, as we observe it today. During such a phase Ω_k must increase according to (13) and, if it is to reach a fixed point y_f with $\Omega_k = r^2(y_f)$ and $y_f > y_r$, the function $r(y)$ is bound to increase somewhere, which is sufficient for $c_s^2 > 1$ as we have shown above. On its way from the radiation fixed point, $w_k(y_r) = 1/3$, $\Omega_k = r^2(y_r) \ll 1$ to the k-essence fixed point with $w_k(y_k) < 0$, $\Omega_k = r^2(y_k) \simeq 1$ (or to the dust fixed point with $w_k(y_d) = 0$, $\Omega_k = r^2(y_d) > r^2(y_r)$) the k-essence fluid has to pass through an interval where $c_s^2 > 1$.

The fact that w_k has to be larger than 1 in some interval for a successful k-essence model was already pointed out in Ref. [4]. This means that there exists observers which see an energy flow which is faster than the speed of light. But this does not necessarily pose a problem for causality since the energy flow does not carry information. However, c_s^2 represents the propagation velocity of the perturbations, at least in the WKB limit which is always justified for large enough wave numbers, and therefore it really means that information can travel faster than light.

We have shown that k-essence which has the capacity to play the role of dark energy and, especially, to address the coincidence problem cannot result as a low energy effective theory from some meaningful, causal high energy field theory, because it necessarily undergoes phases where $c_s^2 > 1$. In the examples presented here, $c_s^2 > 1$ also at late times. This means that k-essence models which solve the coincidence problem are ruled out as serious candidates for dark energy. However, the k-essence model proposed in Ref. [16], which does not solve the coincidence problem, does not have $c_s^2 > 1$. Also the form of the Lagrangian needed for successful k-inflation usually does not suffer from this problem and has causally propagating fluctuations.

[1] A. Riess et al., *Astronomical J.* **116**, 1009 (1998); S. Perlmutter et al. *Astrophys. J.* **517**, 565 (1999); N. Bahcall, J.P. Ostriker, S. Perlmutter and P.J. Steinhardt, *Science* **284**, 1481 (1999); A. Conley et al., *Astrophys. J.* **644**, 1 (2006); R.A. Knop et al., *Astrophys. J.* **598**, 102 (2003).

[2] P.J.E. Peebles and B. Ratra, *Astrophys. J. Lett.* **325**, L17 (1988); B. Ratra and P.J.E. Peebles, *Phys. Rev.* **D37**, 3406 (1988); C. Wetterich, *Nucl. Phys.* **B302**, 668 (1988); R.R. Caldwell, R. Dave and P.J. Steinhardt, *Phys. Rev. Lett.* **80**, 1582 (1998).

[3] C. Armendariz-Picon, V. Mukhanov and P.J. Steinhardt, *Phys. Rev. Lett.* **85** 4438 (2000).

[4] C. Armendariz-Picon, V. Mukhanov and P.J. Steinhardt, *Phys. Rev.* **D63** 103510 (2001).

[5] M. Malquarti, E. Copeland and A. Liddle, *Phys. Rev.* **D68**, 023512 (2003).

[6] A. Adams et al. hep-th/0602178.

[7] J. Bekenstein, *Phys. Rev.* **D25**, 1527 (1982).

[8] C. Armendariz-Picon, T. Damour and V. Mukhanov, *Phys. Lett.* **B458**, 209 (1999).

[9] R. Courant and D. Hilbert, *Methods of Mathematical Physics, Vol. II*, Wiley Classics Edition (1989).

[10] A.D. Rendall, *Class. Quant. Grav.* **23**, 1557 (2006).

[11] Y. Choquet-Bruhat, C. De Witt-Morette and M. Dillard-Bleick, *Analysis, Manifods and Physics*, North-Holland (1977).

[12] C. Armendariz-Picon and E.A. Lim, *JCAP* **0508**, 007 (2005)

[13] G. Velo and D. Zwanzinger, *Phys. Rev.* **188**, 5 (1969).

[14] G. W. Gibbons, *Class. Quant. Grav.* **20**, S321-S346 (2003).

[15] A. Sen, *JHEP* 0207, 065 (2002).

[16] R. Scherrer, *Phys. Rev. Lett.* **93**, 011301 (2004).

[17] In Ref. [5] it is, however, argued that the bassin of attraction of the examples of Refs. [3, 4] is disturbingly small

[18] In principle we would have to write the evolution equation for a gauge invariant quantity, like the Bardeen potential, but it has the same form and, more importantly, the same velocity of propagation, c_s^2.