Anisotropic 'hairs' in string cosmology

Kerstin E. Kunze and Ruth Durrer

Département de Physique Théorique, Université de Genève, 24 quai Ernest Ansermet, CH-1211 Genève 4, Switzerland

In this letter we investigate whether the isotropy problem is naturally solved in inflationary cosmologies inspired by string theory, so called pre-big-bang cosmologies. We find that, in contrast to what happens in the more common 'potential inflation' models, initial anisotropies do not decay during pre-big-bang inflation.

In most models, inflation is driven by the potential energy of a scalar field. This has similar effects as adding vacuum energy or, equivalently, a positive cosmological constant. For these models Wald [1] proved a cosmic no hair theorem, which implies that an initial shear is inflated away [2].

A couple of years ago, a new mechanism of inflation has been proposed, where inflation is due to the kinetic term of the dilaton which is always present in the low energy effective action from a string theory. These string cosmologies are symmetric under the duality transformation, $t \rightarrow -t$ and $a \rightarrow 1/a$. Here t is cosmic time and a is the scale factor. An expanding solution at negative times is inflating [3]. Here, we want to study whether a 'no hair' theorem is also valid for these pre-big-bang inflationary solutions.

We investigate especially spatially homogeneous, but anisotropic cosmologies in the pre-big-bang scenario. The aim is to determine the evolution of a possible primordial shear. The pre-big-bang universe starts off at $t \rightarrow -\infty$ in a nearly Minkowski spacetime. But the hypothesis of low curvature and low coupling in the pre-big-bang scenario does not say anything about the possibility of starting with an anisotropic spacetime.

Recently, the initial conditions of the pre-big-bang scenario have been criticized not to be natural [4]. Buonanno et al. [5] have addressed this issue and have shown that the pre-big-bang inflationary phase in the string frame is equivalent to gravitational collapse in the Einstein frame. They therefore have concluded that the initial conditions for pre-big-bang inflation are as natural as those for gravitational collapse. However, the pre-big-bang bubble picture suggested in this work [5] seems to favor anisotropic Kasner solutions. It is thus important to investigate whether such initial anisotropies are inflated away during the subsequent inflationary evolution.

Below we show that this is not the case. We find that the behaviour of shear in pre-big-bang cosmology is quite different from its behavior in ordinary potential inflation and primordial shear is not inflated away.

We first briefly recall the expressions for the Ricci tensor of spatially homogeneous models in the orthonormal frame. (Latin indices run from 0..3 and Greek indices from 1..3.) The metric (g_{ab}) of spatially homogeneous models can be written as

$$ds^2 = -dt^2 + h_{\alpha\beta}(t)\omega^\alpha\omega^\beta \tag{1}$$

where $\{\omega^\alpha\}$ is an invariant basis of one-forms satisfying the algebra

$$d\omega^\alpha = \frac{1}{2}C^\alpha_{\mu\nu}\omega^\mu \wedge \omega^\nu,$$

where $C^\alpha_{\mu\nu}$ are the structure constants of the symmetry group of the corresponding homogeneous model. Bianchi models are divided into class A and B depending on properties of the group structure constants. Here we restrict ourselves to Bianchi class A models which are characterized by

$$C^\alpha_{\mu\alpha} = 0.$$

For these models $h_{\alpha\beta}(t)$ is diagonal and thus of the form

$$h_{\alpha\beta}(t) = diag\left(a_1^2(t), a_2^2(t), a_3^2(t)\right). \tag{2}$$

We also choose an orthonormal frame $\sigma^{(a)}$, so that

$$ds^2 = \eta_{(a)(b)}\sigma^{(a)}\sigma^{(b)} , \tag{3}$$

where $\eta_{(a)(b)} = diag(-1,+1,+1,+1)$. Indices in parentheses refer to the orthonormal frame.

The relation between the basis one-forms and the orthonormal frame is given by, $\sigma^{(\alpha)} = a_\alpha(t)\omega^\alpha$ (no sum over α).

The parameter t is chosen such that $g^{ab}n_a n_b = -1$, where $n = -\frac{\partial}{\partial t}$ is the normal to the (space-like) homogeneous hyper-surfaces. The expansion and shear tensors, θ_{ab} and σ_{ab} respectively, of the hyper-surfaces $\{t = const.\}$ are defined by [6]

$$n_{a;b} = \theta_{ab} \tag{4}$$

$$\sigma_{ab} = \theta_{ab} - \frac{1}{3}\theta(g_{ab} + n_a n_b) , \tag{5}$$

where $\theta = \theta^a_a$ is the expansion. In terms of the scale-factors $\theta = \sum_{\alpha=1}^{3}\frac{\dot{a}_\alpha}{a_\alpha}$. A dot indicates derivative with respect to t. The shear σ_{ab} is trace-free. It is convenient to define

$$\sigma^2 \equiv \frac{1}{2}\sigma_{ab}\sigma^{ab}. \tag{6}$$

With this notation the non-vanishing components of the Ricci tensor for Bianchi class A models in the orthonormal frame are given by (for useful formulae see [7] [8])

$$R_{(0)(0)} = -\dot{\theta} - \frac{1}{3}\theta^2 - 2\sigma^2$$

$$R_{(\alpha)(\alpha)} = \dot{\sigma}_{(\alpha)(\alpha)} + \theta\sigma_{(\alpha)(\alpha)} + \frac{1}{3}\dot{\theta} + \frac{1}{3}\theta^2 + F_{(\alpha)(\alpha)} \quad (7)$$

where there is no sum over α and $F_{(\alpha)(\alpha)}$ is a function of the scale factors defined by

$$F_{(\alpha)(\alpha)} = \gamma_{(\alpha)}{}^{(\mu)(\kappa)}\gamma_{(\kappa)(\mu)(\alpha)} - \gamma_{(\alpha)}{}^{(\mu)(\kappa)}\gamma_{(\alpha)(\kappa)(\mu)}$$
$$-\gamma_{(\kappa)(\mu)(\alpha)}\gamma_{(\alpha)}{}^{(\kappa)(\mu)} .$$

The Ricci rotation coefficients, γ are given by

$$\gamma^{(\alpha)}{}_{(\beta)(\mu)} = \frac{1}{2}\left[\frac{a_\alpha}{a_\beta a_\mu}C^\alpha_{\beta\mu} - \frac{a_\mu}{a_\alpha a_\beta}C^\mu_{\alpha\beta} + \frac{a_\beta}{a_\mu a_\alpha}C^\beta_{\mu\alpha}\right]$$
$$= \gamma_{(\alpha)(\beta)(\mu)} = \gamma_{(\alpha)}{}^{(\beta)(\mu)} = \ldots .$$

The group structure constants $C^\alpha_{\beta\mu}$ for the different Bianchi models can be found, for example, in [8].

For Bianchi class B models the Ricci tensor has off-diagonal components and additional terms in the diagonal components (see for example [9]). However, all these scale with the scale factors in such a way that they become sub-dominant during inflationary expansion. Therefore, the discussion of Bianchi class A models presented here is sufficient.

Let us now derive the equations of motion for pre-big-bang inflation in this background. The low energy effective action of string theory is given by

$$S = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}e^{-\phi}(R + \partial_\mu\phi\partial^\mu\phi) + S_{\mathrm{matter}}. \quad (8)$$

As matter source we include a perfect fluid. The equations of motion derived from (8) are then given by [10]

$$R^\nu_\mu + \nabla_\mu\nabla^\nu\phi = 8\pi Ge^\phi T^\nu_\mu \quad (9)$$

$$R - (\nabla_\mu\phi)^2 + 2\nabla_\mu\nabla^\mu\phi = 0 \quad (10)$$

$$\dot{\rho} + \theta(\rho + p) = 0. \quad (11)$$

The last equation already uses the form of the energy momentum tensor $T_{ab} = \rho(t)n_an_b + p(t)(g_{ab} + n_an_b)$. We assume that the perfect fluid satisfies the equation of state $p = \gamma\rho$.

In order to discuss Bianchi class A space-times we use the Ricci tensor given in Eq. (7). In the orthonormal frame defined above, these equations then read

$$\dot{\sigma}_{(\mu)(\mu)} + (\theta - \dot{\phi})\sigma_{(\mu)(\mu)} = -F_{(\mu)(\mu)} + \frac{1}{3}\sum_\alpha F_{(\alpha)(\alpha)} \quad (12)$$

$$\dot{\theta} + (\theta - \dot{\phi})\theta = -\sum_\mu F_{(\mu)(\mu)} + 8\pi Ge^\phi 3\gamma\rho \quad (13)$$

$$\ddot{\phi} + (\theta - \dot{\phi})\dot{\phi} = 8\pi Ge^\phi(3\gamma - 1)\rho \quad (14)$$

$$\dot{\rho} + \theta(\gamma + 1)\rho = 0 \quad (15)$$

$$\frac{1}{3}\theta^2 - \sigma^2 - \frac{1}{2}(2\theta - \dot{\phi})\dot{\phi} = -\frac{1}{2}\sum_\mu F_{(\mu)(\mu)} + 8\pi Ge^\phi\rho. \quad (16)$$

It is easy to see that for $\rho = F_{(\mu)(\mu)} = 0$ this system is invariant under the transformation

$$a_\alpha \to 1/a_\alpha , \quad t \to -t \quad \text{and} \quad \dot{\phi} \to 2\theta - \dot{\phi} , \quad (17)$$

the so called scale factor duality. Under these changes, an expanding decelerating solution in the post-big-bang era ($t > 0$) transforms into an inflating expanding solution in the pre-big-bang era ($t < 0$) [10]. For the following discussion about the pre-big-bang inflationary era, we have to keep in mind that t goes from $-\infty$ to 0.

The evolution equation for the matter energy density yields

$$\rho = \frac{\rho_0}{(a_1a_2a_3)^{\gamma+1}}. \quad (18)$$

We assume that inflation has started and ρ describes an 'ordinary' fluid with $\gamma > -1$. As we shall check at the end, it is then justified to neglect terms involving ρ. Furthermore, the terms $F_{(\alpha)(\alpha)}$ can be neglected. Also this hypothesis will be checked later for consistency.

With these approximations, Eqs. (12) to (16) reduce to

$$\dot{\sigma}_{(\mu)(\mu)} + (\theta - \dot{\phi})\sigma_{(\mu)(\mu)} \simeq 0 \quad (19)$$

$$\dot{\theta} + (\theta - \dot{\phi})\theta \simeq 0 \quad (20)$$

$$\ddot{\phi} + (\theta - \dot{\phi})\dot{\phi} \simeq 0 \quad (21)$$

$$\frac{1}{3}\theta^2 - \sigma^2 \simeq \frac{1}{2}(2\theta - \dot{\phi})\dot{\phi}. \quad (22)$$

But this set of equations can be readily solved and with scale factors of the form $a_\alpha = (t/t_0)^{-\lambda_\alpha}$ which implies

$$\theta = -\frac{\sum_\alpha \lambda_\alpha}{t} , \quad \dot{\phi} = -\frac{\sum_\alpha \lambda_\alpha + 1}{t} . \quad (23)$$

Since t is negative in our domain of interest and we want positive scale factors, we choose also t_0 negative. Expansion in all directions is then garanteed if $\lambda_\alpha > 0$.

The evolution of $\sigma_{(\alpha)(\alpha)}$ is given by

$$\sigma_{(\alpha)(\alpha)} = \frac{\frac{1}{3}\sum_\mu \lambda_\mu - \lambda_\alpha}{t} \quad (24)$$

The constraint equation (22) yields the Kasner constraint

$$\sum_\alpha \lambda_\alpha^2 = 1 .$$

We have thus found that the quantities we are interested in, the relative amplitudes of shear, i.e. $\frac{\sigma_{(\alpha)(\alpha)}}{\theta}$ and $\frac{\sigma^2}{\theta^2}$ remain constant. A primordial shear is not inflated away during pre-big-bang inflation,

$$\frac{\sigma_{(\alpha)(\alpha)}}{\theta} = \text{const.} \qquad \frac{\sigma^2}{\theta^2} = \text{const.} \quad (25)$$

This is our main result.

2

It remains to check that it is justified to neglect terms involving ρe^ϕ and the $F_{(\alpha)(\alpha)}$. The first expression is given by

$$\rho e^\phi \sim \left(\frac{t}{t_0}\right)^{\gamma \sum_\alpha \lambda_\alpha - 1}. \tag{26}$$

Consistency requires (cf. equations (13), (14)) that, with increasing time, this term becomes less and less important if compared, for example, with $\dot{\theta}$. In other words, $\gamma \sum_\alpha \lambda_\alpha - 1 > -2$. For positive γ this is always satisfied since $\lambda_\alpha > 0$. For $\gamma < 0$ it leads to the constraint

$$\sum_\alpha \lambda_\alpha < \frac{1}{|\gamma|}. \tag{27}$$

But the Kasner constraint which holds for the unperturbed solution, $\sum_\alpha \lambda_\alpha^2 = 1$, implies $\sum_\alpha \lambda_\alpha > \sum_\alpha \lambda_\alpha^2 = 1$. Hence the inequality (27) is satisfied for $\gamma > -1$.

Let us now discuss the behaviour of the functions $F_{(\alpha)(\alpha)}$. In particular, we want to address the question how the shear can be effected by a contribution from the $F_{(\alpha)(\alpha)}$. For self consistency, we just have to check that for a solution close to Kasner, the $F_{(\alpha)(\alpha)}$'s may be neglected. The dominant contribution to the Ricci rotation coefficients comes from a factor $\left(\frac{a_\mu}{a_\alpha a_\beta}\right)^2$ where a_μ expands fastest and a_α and a_β expand slowest. For a Kasner solution the contribution to the Ricci rotation coefficients grows like $\left(\frac{a_\mu}{a_\alpha a_\beta}\right)^2 \propto t^{2(\lambda_\alpha + \lambda_\beta - \lambda_\mu)}$. The term with minimal $\lambda_\alpha + \lambda_\beta - \lambda_\mu = \lambda_{\min}$ grows fastest. But the Kasner condition readily implies that $0 < \lambda_\alpha < 1$ so that $\lambda_{\min} > -1$. Therefore, if the deviation from the 'Kasner solution' is small, it decreases with time and will eventually be negligible. If at some given time during pre-big-bang inflation, the universe is close to a a Kasner solution, it will approach the Kasner solution during subsequent evolution. In that sense the Kasner solutions are (local) attractors of the Bianchi type A models with ordinary matter content.

Note also that, if the solution is reasonably close to isotropic, $\lambda_\alpha \sim 1/\sqrt{3}$, $\lambda_{\min}$ is even positive and the $F_{(\alpha)(\alpha)}$ are very strongly suppressed. This means that our argument applies and subsequent pre-big-bang evolution does not 'isotropize' the solution.

As a simple example we consider a Bianchi II string cosmology. We neglect a possible additional contribution from matter, $\rho = 0$. The exact 7 parameter family of solutions can be found in [11]. The evolution of the ratios $\frac{\sigma_{(\alpha)(\alpha)}}{\theta}$ for a particular choice of parameters is shown in Fig. 1. We have chosen the parameters such that the solution converges to the Kasner solution with scale factors $a_1 \propto (-t)^{-\lambda_1}$, $a_2 \propto (-t)^{-\lambda_2}$, and $a_3 \propto (-t)^{-\lambda_3}$, where $\lambda_1 = 0.68$, $\lambda_2 = 0.61$, and $\lambda_3 = 0.42$. These satisfy the string Kasner condition $\sum_1^3 \lambda_i^2 = 1$. The evolution of the shear parameters $\sigma_{(\alpha)(\alpha)}$ for this particular pre-big-bang inflationary solution is shown in Fig. 1. Clearly, $\sigma_{(\alpha)(\alpha)}/\theta$ approaches the Kasner value,

$$\sigma_{(\alpha)(\alpha)}/\theta \xrightarrow{t \to 0^-} \frac{\lambda_\alpha - \frac{1}{3}\sum_\mu \lambda_\mu}{\sum_\mu \lambda_\mu}.$$
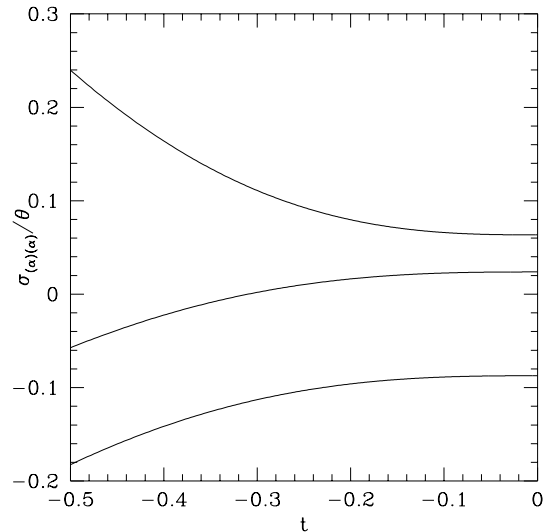

FIG. 1. Evolution of the ratios $\frac{\sigma_{(\alpha)(\alpha)}}{\theta}$ in a Bianchi II model. The parameters are chosen in order to admit an pre-big-bang inflationary solution with $\lambda_1 \sim 0.68$, $\lambda_2 \sim 0.61$ and $\lambda_3 \sim 0.42$.

The situation in pre-big-bang inflation can be contrasted with ordinary slow roll inflation. There the scale factor is expanding close to exponentially and $\theta \sim \text{const}$. The equation for $\sigma_{(\alpha)(\alpha)}$ in slow roll inflation is obtained from (12) by setting the dilation term, $\dot{\phi} = 0$. Neglecting again the right hand side of (12), we find in this case

$$\frac{\sigma_{(\alpha)(\alpha)}}{\theta} \sim e^{-\theta t}.$$

Primordial shear decays exponentially and very soon it becomes negligible.

In contrary, in pre-big-bang inflation the shear parameter is essentially determined by its initial value. This behaviour can also be understood by looking at the evolution in the Einstein frame in which usual general relativity is recovered. The dilaton provides a matter content which behaves as a stiff perfect fluid ($p = \rho$) whose energy density evolves like a^{-6}. The shear σ^2 also evolves as a^{-6}, and again their ratio is a constant.

It is known in vacuum general relativity that the singularity at $t = 0$ in spatially homogeneous models is either asymptotically velocity term dominated [12] (Kasnerlike) or Mixmaster like [13]. Adding a massless scalar field destroys the Mixmaster behaviour after a finite number of oscillations leaving just the Kasner behaviour [14]. Thus at sufficiently late times, $t \to -0$ terms due to spatial curvature like the $F_{(\alpha)(\alpha)}$ terms in the string frame become negligible.

We consider our result as quite important for the pre-big-bang model. It implies that the problem of

3

isotropization cannot be solved in the simplest version of pre-big-bang inflation. Especially, it cannot be solved in the early, classical, low coupling regime. We also doubt that this problem can be solved by quantum particle production back-reaction, a mechanism, which can be used to some extent to damp anisotropies in the very early post-big-bang universe [15]. In our case, the anisotropies are a very long range, low energy phenomena and it seems unlikely to us that they can be cured by particle creation at very negative times.

As $t \to -t_{string}$ copious particle production and also other mechanisms, like higher order corrections to the action as conjectured in [16] may damp anisotropies. But these corrections only become important close to the Planck time. During the entire pre-big-bang inflation the universe remains as anisotropic as in the initial conditions. This has significant implications on quantum particle creation during the pre-big-bang phase as has been investigated in [16].

Within string cosmology, cosmological fluctuations are due to coherent quantum particle production in the pre-big-bang phase [17]. It is still an open problem, to what extent anisotropic cosmological fluctuations would be visible in the perturbations of the post-big-bang universe, like *e.g.* as a preferred direction in the anisotropies of the cosmic microwave background. A study of this problem is in preparation.

[1] R.M. Wald, Phys. Rev. D **28**, 2118 (1983); J.D. Barrow, G. Götz, Phys. Lett. B **231**, 228 (1989).

[2] M.S. Turner, L.M. Widrow, Phys. Rev. Lett. **57**, 2237 (1986); L.G. Jensen, J.A. Stein-Schabes, Phys. Rev. D**34**, 931 (1986).

[3] G. Veneziano, Phys. Lett. B **265** (1991) 287; M. Gasperini and G. Veneziano, Astropart. Phys. **1**, 317 (1993); Mod. Phys. Lett. A **8**, 3701, (1993); Phys. Rev. D **50**, 2519 (1994). An updated collection of papers on the pre-big-bang scenario is available at http://www.to.infn.it/~gasperin.

[4] M.S. Turner, E.J. Weinberg, Phys. Rev. D**56**, 4604 (1997); N. Kaloper, A. Linde and R. Bousso, Phys. Rev. D**59**, 043508 (1999).

[5] A. Buonanno, T. Damour and G. Veneziano, Nucl. Phys. B **543**, 275 (1999).

[6] G.F.R. Ellis, M.A.H. MacCallum, Comm. math. Phys. **12**, 108 (1969).

[7] S. Chandrasekhar, "The Mathematical Theory of Black Holes", Oxford University Press, Oxford (1992).

[8] M.P. Ryan, L.C. Shepley, "Homogeneous Relativistic Cosmologies", Princeton University Press, Princeton (1975).

[9] H.H. v. Borzeszkowski, V. Müller, Ann. Phys. (Leipzig) **35**, 361 (1978); A. Harvey, D. Tsoubelis, Phys. Rev. D **15**, 2734 (1977).

[10] M. Gasperini, G. Veneziano, Astropart. Phys. **1**, 317 (1993).

[11] N.A. Batakis, A.A. Kehagias, Nucl. Phys. B**449**, 248 (1995).

[12] D. Eardley, E. Liang, R. Sachs, J. Math. Phys. **13**, 99 (1972); J.A. Isenberg, V. Moncrief, Ann. Phys. (N.Y.) **199**, 84 (1990).

[13] V.A. Belinskii, E.M. Lifshitz, I.M. Khalatnikov, Sov. Phys. Usp. **13**, 745 (1971); C.W. Misner, Phys. Rev. Let. **22**, 1071 (1969).

[14] V.A. Belinskii, L.M. Khalatnikov, Sov. Phys. JETP **36**, 591 (1973).

[15] B.L. Hu, L. Parker, Phys. Rev. D**17**, 933 (1978); Erratum ibid. **17**, 3292 (1978).

[16] M. Giovannini, Phys. Rev. D**59**, 123518 (1999).

[17] R. Durrer, M. Gasperini, M. Sakellariadou and G. Veneziano, Phys. Rev. D**59**, 043511 (1999).