On graviton production in braneworld cosmology

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We study braneworlds in a five dimensional bulk, where cosmological expansion is mimicked by motion through AdS_5 . We show that the five dimensional graviton reduces to the four dimensional one in the late time approximation of such braneworlds. Inserting a fixed regulator brane far from the physical brane, we investigate quantum graviton production due to the motion of the brane. We show that the massive Kaluza-Klein modes decouple completely from the massless mode and they are not generated at all in the limit where the regulator brane position goes to infinity. In the low energy limit, the massless four dimensional graviton obeys the usual 4d equation and is therefore also not generated in a radiation-dominated universe.

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I. INTRODUCTION

In recent years, cosmological models where our Universe is represented as a hypersurface moving through a higher dimensional spacetime have received a lot of attention. The reasons for that are twofold. First, string theory, the presently most promising candidate of a theory for quantum gravity is consistent only in more than $3+1$, namely in $9+1$ or (for M-theory) $10+1$ dimensions [1, 2]. Second, the hierarchy problem *i.e.* the unnatural disparity of the electroweak scale ~ 1TeV and the Planck scale $\sim 10^{16}$TeV can be addressed in the context of extra-dimensions [3]. For this, it is necessary that one or several of the extra-dimensions be much larger than the Planck scale. The 4-dimensional effective Planck mass then becomes $M_4 \simeq \sqrt{M_{4+n}^{2+n} L^n}$, where n denotes the number of extra dimensions and L is their size. These extra dimensions can be large and nevertheless undetected if standard model particles are confined to a lower dimensional hypersurface and cannot probe them. String theory predicts the existence of such Dp–branes onto which standard model particles are confined [4]. The bulk spacetime around the Dp-brane can then only be probed by gravity. Gravity has been tested only down to scales of about 0.1mm and therefore allows for $L \lesssim 0.1$mm. It has been shown that confinement of gravity to a region of size $L \lesssim 0.1$mm around a brane with one extra-dimension can also be achieved by a non-compact Anti-de Sitter spacetime [5]. The cosmological situation of an expanding universe is then obtained by a brane moving through a 5-dimensional Anti-de Sitter spacetime (AdS_5). At low energy, this setup leads to the usual Friedmann equations for the expansion of the Universe [6, 7]. This is the case which we consider in this paper.

We want to address the following issue: a brane moving through AdS_5 spacetime leads to time-varying boundary conditions for quantum fields living in the bulk. In quantum physics it is well known that moving boundaries yield particle creation from vacuum, so-called motion induced radiation [8]. For instance, photons are produced from vacuum in dynamical cavities (dynamical Casimir effect) [9] as well as by a single moving mirror [10]. Here we study the same mechanism in brane world cosmology, namely graviton generation provoked by the motion of the brane. Apart from the massless graviton, braneworlds allow for a tower of massive Kaluza-Klein gravitons which might also be produced by the moving brane. Such massive modes have potentially devastating effects as they would eventually dominate the energy density of the universe and spoil the phenomenology if their production is sufficiently copious. Here we shall show explicitly that the dangerous massive modes are not produced in the single brane case. Also the production of massless 4-dimensional gravitons is strongly suppressed at low energy.

In the next section we present our setup and derive the equations of motion for the graviton modes. We especially discuss the nature of the coupling matrix which describes the coupling of the different modes due to the time dependence of the boundary (the brane). The formalism laid out is very general and we plan to apply it to other setups (high energy, 2-brane models) in the future. We also show that at low energy and for a regulator brane placed sufficiently far, the massless graviton mode obeys precisely the 4d equation of motion for gravitons in a Friedmann

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universe and decouples from the massive modes. In Section III we brielfly estimate the produced gravitons and their spectrum analytically, focusing on the dominant 0-mode. We summarize our conclusions in Section IV.

II. THE MODE EQUATIONS FOR GRAVITONS IN MOVING BRANEWORLDS

A. The background

We consider the cosmological Randall-Sundrum-II model where we have AdS_5 geometry in the bulk with one physical brane at a time-dependent position $y_\mathrm{b}(t)$. In Poincaré coordinates, the bulk metric is given by

$$\mathrm{d}s^2 = g_{AB}\mathrm{d}x^A\mathrm{d}x^B = \frac{L^2}{y^2}\left[-\mathrm{d}t^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j + \mathrm{d}y^2\right] \ . \tag{2.1}$$

Capital Latin indices A, B run from 0 to 4 and lower case Latin indices i, j from 1 to 3. Four-dimensional indices running from 0 to 3 will be denoted by lower case Greek letters. L is the AdS_5 curvature radius and is related to the negative cosmological constant Λ by $\Lambda = -6/L^2$ to solve the bulk Einstein equations

$$G_{AB} + \Lambda g_{AB} = 0 \ . \tag{2.2}$$

We now introduce a brane at $y = y_\mathrm{b}(t)$ and replace the "left hand side", $0 < y < y_\mathrm{b}(t)$, of AdS_5 by a second copy of the "right hand side". We use the superscripts "$>$" and "$<$" for the bulk sides with $y > y_\mathrm{b}$ and $y < -y_\mathrm{b}$, respectively. In terms of the coordinate y, the value of y decreases continuously from ∞ to y_b and then jumps to $-y_\mathrm{b}$ over the brane whereafter it continues to decrease. At the brane position, $y_\mathrm{b}^> = y_\mathrm{b}(t)$, $y_\mathrm{b}^< = -y_\mathrm{b}(t)$, the metric function $(L/y)^2$ has a kink. The Einstein equations at the brane position are singular, they contain a Dirac–delta function,

$$G_{AB} + \Lambda g_{AB} = \kappa_5 T_{AB}^{\mathrm{brane}}\delta(y - y_\mathrm{b}) \ . \tag{2.3}$$

The delta function confines the energy momentum tensor from Standard Model fields to the brane. To avoid the delta function, one can integrate Eq. (2.3) over the extra dimension which leads to the so-called Israel-Darmois junction conditions [11, 12, 13, 14] at the brane position. These read [15]

$$g_{\mu\nu}^> - g_{\mu\nu}^< = 0 \ , \tag{2.4}$$

$$K_{\mu\nu}^> - K_{\mu\nu}^< = \kappa_5\left(S_{\mu\nu} - \frac{1}{3}Sq_{\mu\nu}\right) \equiv \kappa_5\widehat{S}_{\mu\nu} \ , \tag{2.5}$$

where $S_{\mu\nu}$ is the energy-momentum tensor on the brane with trace S, and

$$\kappa_5 \equiv 6\pi^2 G_5 = \frac{1}{M_5^3} \ . \tag{2.6}$$

M_5 and G_5 are the five-dimensional (fundamental) reduced Planck mass and Newton constant, respectively. $K_{\mu\nu}$ is the extrinsic curvature of the brane and $q_{\mu\nu}$ is the induced metric on the brane. The first junction condition (2.4) simply states that the induced metric, the first fundamental form,

$$q_{\mu\nu} = e_\mu^A e_\nu^B g_{AB} \ , \tag{2.7}$$

be continuous across the brane. Here the vectors e_μ are tangent to the brane. If we parametrize the brane by coordinates (z^μ) and its position in the bulk is given by functions $X_\mathrm{b}^A(z^\mu)$, the vectors e_μ can be defined by

$$e_\mu^A = \partial_\mu X_\mathrm{b}^A(z) \ . \tag{2.8}$$

Denoting the brane normal by n, we have $g_{AB}e_\mu^A n^B = 0$. The extrinsic curvature can be expressed purely in terms of the internal brane coordinates [16, 17], $K = K_{\mu\nu}dz^\nu dz^\mu$, with

$$K_{\mu\nu} = -\frac{1}{2}\left[g_{AB}\left(e_\mu^A\partial_\nu n^B + e_\nu^A\partial_\mu n^B\right) + e_\mu^A e_\nu^B n^C g_{AB,C}\right] \ . \tag{2.9}$$

The link between the extrinsic curvature of the hypersurface and the brane energy-momentum tensor is established by the second junction condition (2.5), which replaces the 4d Einstein equation.

A homogeneous and isotropic brane moving through AdS_5 with brane position $y_\mathrm{b}(\eta)$ and bulk time given by $t_\mathrm{b}(\eta)$ has the metric

$$\mathrm{d}s^2 = \frac{L^2}{y_\mathrm{b}^2(\eta)}\left[-\left(1-\left(\frac{\mathrm{d}y_\mathrm{b}}{\mathrm{d}t}\right)^2\right)\mathrm{d}t^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right] = a^2(\eta)\left[-\mathrm{d}\eta^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right] , \quad (2.10)$$

where $a = L/y_\mathrm{b}$ is the scale factor and η denotes the "conformal time" of an observer on the brane,

$$\mathrm{d}\eta = \sqrt{1-\left(\frac{\mathrm{d}y_\mathrm{b}}{\mathrm{d}t}\right)^2}\,dt \equiv \gamma^{-1}\mathrm{d}t\ . \quad (2.11)$$

The brane motion induces the γ-factor which relates the (conformal) eigentime η of the brane to the (conformal) coordinate time t of the bulk. From now on an overdot indicates a derivative w.r.t. conformal time η on the brane. The brane normal is then given by

$$n_0 = -\frac{\dot{y}_\mathrm{b}}{\dot{t}_\mathrm{b}}n_4\ , \qquad n_4^2 = \left(\frac{L}{y_\mathrm{b}}\right)^2 \dot{t}_\mathrm{b}^2\left(\dot{t}_\mathrm{b}^2 - \dot{y}_\mathrm{b}^2\right)^{-1}\ . \quad (2.12)$$

We consider a homogeneous and isotropic total energy momentum tensor on the brane, $S^\mu_\nu = T^\mu_\nu - \mathcal{T}\delta^\mu_\nu$. Here $\mathcal{T}$ is the brane tension and T^μ_ν is the energy momentum tensor of particles and fields confined on the brane given by $T^0_0 = -\rho$, $T^i_j = P\delta^i_j$. The second junction conditions now become

$$\kappa_5(\rho + \mathcal{T}) = 6\frac{\sqrt{1+L^2H^2}}{L}\ , \quad (2.13)$$

$$\kappa_5(\rho + P) = -\frac{2L\dot{H}}{a\sqrt{1+L^2H^2}}\ , \quad (2.14)$$

$$\dot{\rho} = -3Ha(\rho + P)\ , \quad (2.15)$$

$$H^2 = \frac{\kappa_5^2}{18}\mathcal{T}\rho\left(1+\frac{\rho}{2\mathcal{T}}\right) + \frac{\kappa_5^2\mathcal{T}^2}{36} - \frac{1}{L^2}\ , \quad (2.16)$$

where $H \equiv \dot{a}/a^2$. Equations (2.13) to (2.16) form the basis of brane cosmology and have been discussed at length in the literature (for a review, see [18] or [19]). The last equation is called the "modified Friedmann equation" for brane cosmology [7]. For usual matter with $\rho + P > 0$, ρ decreases during expansion and at sufficiently late time $\rho \ll \mathcal{T}$. The ordinary 4-dimensional Friedmann equation is then recovered if we set

$$\frac{\kappa_5^2\mathcal{T}^2}{12} - \frac{3}{L^2} = \Lambda_4 \quad \text{and} \quad \kappa_4 = 8\pi G_4 = \frac{\kappa_5^2\mathcal{T}}{6}\ . \quad (2.17)$$

Neglecting the 4-dimensional cosmological constant, $\Lambda_4 \simeq 0$, we obtain in addition

$$L = \frac{6}{\kappa_5\mathcal{T}} \quad \text{and} \quad \kappa_4 = \frac{\kappa_5}{L}\ . \quad (2.18)$$

Note that, although for a de Sitter brane the density is simply constant, $\rho = -P = \mathrm{cst}$ and there is no late time approximation, Eq. (2.14) implies that the Hubble rate remains constant and we reproduce the usual exponential expansion. Only the relation between the expansion rate H and the brane density ρ is modified.

B. Tensor perturbations

A quantum field in the bulk is generically expected to be modified by the moving brane which acts as a moving boundary of the 5-dimensional bulk spacetime. We want to study this effect for bulk gravitons. With this in mind, we now linearly perturb the bulk metric allowing for tensor modes[1],

$$\mathrm{d}s^2 = \frac{L^2}{y^2}\left[-\mathrm{d}t^2 + (\delta_{ij} + 2h_{ij})\mathrm{d}x^i\mathrm{d}x^j + \mathrm{d}y^2\right]\ . \quad (2.19)$$

[1] The word "tensor" here has to be understood as spin 2 degree of freedom w.r.t. rotations of the 3-dimensional homogeneous and isotropic slices normal to the extra-dimension y.

Tensor modes satisfy the traceless and transverse conditions, $h^i_i = \partial_i h^i_j = 0$. We then decompose h_{ij} into spatial Fourier modes,

$$h_{ij}(t, \boldsymbol{x}, y) = \int \frac{d^3k}{(2\pi)^{3/2}} \sum_{\bullet=+,\times} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} e^{\bullet}_{ij} h_\bullet(t, y; k) \ , \tag{2.20}$$

where $e^{\bullet}_{ij}$ are unitary constant transverse-traceless polarization tensors which form a base of the two polarization states $\bullet = +$ and $\bullet = \times$. The perturbed Einstein equations now yield the equation of motion for the mode function $h_\bullet$, which obeys the Klein-Gordon equation for minimally coupled massless scalar fields in AdS_5 [20, 21, 22]

$$\left[\partial_t^2 + k^2 - \partial_y^2 + \frac{3}{y}\partial_y\right] h_\bullet(t, y; k) = 0 \ . \tag{2.21}$$

In addition to the bulk equation of motion the modes also satisfy a boundary condition at the brane coming from the second junction conditions,

$$\left[LH\partial_t h_\bullet - \sqrt{1 + L^2H^2}\partial_y h_\bullet\right]\Big|_{y_\mathrm{b}} = \gamma^{-1}\left(\mathrm{v}\partial_t - \partial_y\right) h_\bullet\big|_{y_\mathrm{b}} = \frac{\kappa_5}{2} a P \Pi^{(T)}_\bullet \ . \tag{2.22}$$

Here $\Pi^{(T)}_\bullet$ denotes possible anisotropic stress perturbations in the brane energy-momentum tensor and we have introduced the brane velocity $\mathrm{v} \equiv \frac{LH}{\sqrt{1+L^2H^2}}$. For simplicity we set $\Pi^{(T)}_\bullet = 0$ in this work. (Some of the difficulties which appear when $\Pi^{(T)}_\bullet \neq 0$ are discussed in [23]. The wave equation (2.21) with boundary condition (2.22) cannot be solved analytically except if the background metric functions are separable, and this only happens for maximally symmetric branes, i.e., branes with constant Hubble rate. This includes the Randall-Sundrum case $H = 0$. A cosmologically relevant case is the de-Sitter brane, $0 < H = \mathrm{cst}$. The spectrum of gravitational waves generated during de Sitter brane inflation can be calculated [22, 24, 25, 26].

C. Late time approximation

We want to investigate late time and see whether we have graviton production also at late time, after inflation. We therefore consider the limit

$$\mathrm{v} \equiv \frac{LH}{\sqrt{1 + L^2H^2}} \ll 1 \ . \tag{2.23}$$

In this limit the boundary condition (2.22) reduces to

$$\partial_y h_\bullet\big|_{y_\mathrm{b}} = 0 \ . \tag{2.24}$$

If the position of the brane is fixed, the solutions of the system formed by Eq. (2.21) and Eq. (2.24) are well known. These are the Bessel functions,

$$h_\bullet = A \exp\left(\pm i\omega t\right)(\mathrm{m}y)^2 \left[J_2(\mathrm{m}y) + BY_2(\mathrm{m}y)\right] \ , \tag{2.25}$$

where $\omega = \sqrt{\mathrm{m}^2 + k^2}$. The junction condition (2.24) requires

$$B = -\frac{J_1(\mathrm{m}y_\mathrm{b})}{Y_1(\mathrm{m}y_\mathrm{b})} \simeq \frac{\pi}{4}(\mathrm{m}y_\mathrm{b})^2 \ , \tag{2.26}$$

where the last expression is a good approximation for $\mathrm{m}y_\mathrm{b} \ll 1$. This is precisely the result of Randall and Sundrum [5] for a static AdS_5 brane. Since the extra dimension y is not compact, the mass-spectrum is continuous, m^2 can take any non-negative value.

Allowing for a single moving brane is not well suited for a numerical treatment. We therefore introduce a second, so-called "regulator brane" far away from the physical brane, at the position $y = y_\mathrm{r}$ which we let tend to infinity at the end of our calculation. We assume the regular brane to be empty and fixed. The boundary condition at the regulator brane is thus

$$\partial_y h_\bullet\big|_{y_\mathrm{r}} = 0 \ . \tag{2.27}$$

For a solution of the form (2.25) this implies the additional constraint

$$J_1(\mathrm{m}y_\mathrm{r}) + BY_1(\mathrm{m}y_\mathrm{r}) = 0 \ . \tag{2.28}$$

This condition is satisfied only for a discrete series of mass eigenvalues m_α. With B also m_α depends on the position of the physical brane and therefore on time.

The evolution equation (2.21) together with the boundary conditions form a Sturm-Liouville problem, with eigenvalue equation

$$\left[-\partial_y^2 + \frac{3}{y}\partial_y\right]\phi_\alpha = -y^3\partial_y\left[y^{-3}\partial_y\phi_\alpha\right] = \mathrm{m}_\alpha^2\phi_\alpha \ . \tag{2.29}$$

At any given time, Eq. (2.29) yields an orthonormal system of "instantaneous solutions"

$$\phi_0 = A_0 + B_0 y^4 \ , \tag{2.30}$$

$$\phi_i = A_i(\mathrm{m}_i y)^2[J_2(\mathrm{m}_i y) + B_i Y_2(\mathrm{m}_i y)] \equiv A_i(\mathrm{m}_i y)^2 C_2(\mathrm{m}_i y) \ , \tag{2.31}$$

where we have introduced $C_\nu(\mathrm{m}_i y) \equiv J_\nu(\mathrm{m}_i y) + B_i Y_\nu(\mathrm{m}_i y)$. Here and below we use indices i and j to denote all the massive modes and indices α and β to denote all the modes, including the massless mode. The boundary conditions (2.24) and (2.27) then require

$$B_0 = 0 \quad \text{and} \quad C_1(\mathrm{m}_i y_\mathrm{b}) = C_1(\mathrm{m}_i y_\mathrm{r}) = 0 \ . \tag{2.32}$$

Furthermore, the solutions form an orthonormal system of functions on the Hilbert space $\mathscr{H} = \mathscr{L}^2\left([y_\mathrm{b}, y_\mathrm{r}], y^{-3}\mathrm{d}y\right)$

$$(\phi_\alpha, \phi_\beta) \equiv \int_{y_\mathrm{b}}^{y_\mathrm{r}} y^{-3}\phi_\alpha(t,y)\phi_\beta(t,y)\mathrm{d}y = \delta_{\alpha\beta} \ . \tag{2.33}$$

or, explicitly,

$$(\phi_i, \phi_j) = \frac{A_i A_j \mathrm{m}_i^2 \mathrm{m}_j^2}{\mathrm{m}_i^2 - \mathrm{m}_j^2}\left[\mathrm{m}_j y C_2(\mathrm{m}_i y)C_1(\mathrm{m}_j y) - \mathrm{m}_i y C_1(\mathrm{m}_i y)C_2(\mathrm{m}_j y)\right]\Big|_{y_\mathrm{b}}^{y_\mathrm{r}} = 0 \ , \tag{2.34}$$

$$(\phi_i, \phi_0) = -A_i\mathrm{m}_i\left[y^{-1}\left\{A_0 C_1(\mathrm{m}_i y) - B_0 y^4 C_3(\mathrm{m}_i y)\right\}\right]\Big|_{y_\mathrm{b}}^{y_\mathrm{r}} = 0 \ , \tag{2.35}$$

$$(\phi_i, \phi_i) = \frac{1}{2}\left(A_i\mathrm{m}_i\right)^2\left[(\mathrm{m}_i y)^2\left\{C_2^2(\mathrm{m}_i y) - C_1(\mathrm{m}_i y)C_3(\mathrm{m}_i y)\right\}\right]\Big|_{y_\mathrm{b}}^{y_\mathrm{r}} = 1 \ , \tag{2.36}$$

$$(\phi_0, \phi_0) = A_0^2\frac{y_\mathrm{r}^2 - y_\mathrm{b}^2}{2y_\mathrm{r}^2 y_\mathrm{b}^2} + A_0 B_0(y_\mathrm{r}^2 - y_\mathrm{b}^2) + \frac{1}{6}B_0^2(y_\mathrm{r}^6 - y_\mathrm{b}^6) = 1 \ . \tag{2.37}$$

The orthogonality relations are trivially satisfied with the boundary conditions (2.32), whereas the normalization conditions fix the constants A_α,

$$\frac{1}{2}\left(A_i\mathrm{m}_i\right)^2(\mathrm{m}_i y)^2 C_2^2(\mathrm{m}_i y)\Big|_{y_\mathrm{b}}^{y_\mathrm{r}} = 1 \ , \tag{2.38}$$

$$A_0^2\frac{y_\mathrm{r}^2 - y_\mathrm{b}^2}{2y_\mathrm{r}^2 y_\mathrm{b}^2} = 1 \ . \tag{2.39}$$

All the constants depend over the brane position $y_\mathrm{b}(t)$ on time. Since the orthonormal set $\{\phi_\alpha(t,y)\}_{\alpha=0}^\infty$ is complete in $\mathscr{H}$, we can expand a generic solution in the form

$$h_\bullet(t,y;k) = \sum_{\alpha=0}^\infty \tilde{\mathrm{q}}_\alpha(t;k)\phi_\alpha(t,y) \ , \qquad \tilde{\mathrm{q}}_\alpha(t;k) = (\phi_\alpha, h_\bullet) \ . \tag{2.40}$$

This enables us to write the second order action in $h_\bullet$ leading to the equation of motion for $h_\bullet$ as an action for the canonically normalized coefficients q_α. Denoting the second order perturbation of the gravitational Lagrangian by

$\delta(\sqrt{g}R)$ we obtain for the action

$$\begin{aligned}
\mathcal{S}(k) &= \frac{1}{2\kappa_5}\int \mathrm{d}t\mathrm{d}y\ \delta(\sqrt{g}R) \\
&= \frac{L^3}{2\kappa_5}\int \mathrm{d}t\mathrm{d}y\, y^{-3}\left[(\partial_t h_\bullet)^2 - (\partial_y h_\bullet)^2 - k^2 h_\bullet^2\right] \\
&= \frac{1}{2}\int \mathrm{d}t \sum_{\alpha,\beta}\left[(\partial_t \mathrm{q}_\alpha)(\partial_t \mathrm{q}_\beta)\delta_{\alpha\beta} + 2M_{\alpha\beta}\mathrm{q}_\alpha(\partial_t \mathrm{q}_\beta) + \left\{N_{\alpha\beta} - (\partial_y\phi_\alpha, \partial_y\phi_\beta) - k^2\delta_{\alpha\beta}\right\}\mathrm{q}_\alpha \mathrm{q}_\beta\right] \\
&= \frac{1}{2}\int \mathrm{d}t \sum_{\alpha}\Big[(\partial_t \mathrm{q}_\alpha)^2 - \omega_\alpha^2 \mathrm{q}_\alpha^2 + \sum_\beta \left\{2M_{\alpha\beta}\mathrm{q}_\alpha \partial_t \mathrm{q}_\beta + N_{\alpha\beta}\mathrm{q}_\alpha \mathrm{q}_\beta\right\}\Big]\ ,
\end{aligned} \tag{2.41}$$

where we have introduced coupling matrices

$$M_{\alpha\beta}(t) \equiv (\partial_t\phi_\alpha, \phi_\beta)\ , \qquad N_{\alpha\beta}(t) \equiv (\partial_t\phi_\alpha, \partial_t\phi_\beta)\ , \tag{2.42}$$

and the canonically normalized coefficients

$$\mathrm{q}_\alpha = \sqrt{\frac{L^3}{\kappa_5}}\,\tilde{\mathrm{q}}_\alpha\ . \tag{2.43}$$

As before, the frequency is given by

$$\omega_\alpha^2(t;k) \equiv k^2 + \mathrm{m}_\alpha^2\ . \tag{2.44}$$

The equations of motion for the Fourier coefficients $\mathrm{q}_\alpha(t;k)$ then simply follow from the Euler-Lagrange equations for the action (2.41),

$$\partial_t^2 \mathrm{q}_\alpha + \omega_\alpha^2 \mathrm{q}_\alpha + \sum_\beta \left[M_{\beta\alpha} - M_{\alpha\beta}\right]\partial_t \mathrm{q}_\beta + \sum_\beta \left[\partial_t M_{\alpha\beta} - N_{\alpha\beta}\right]\mathrm{q}_\beta = 0\ . \tag{2.45}$$

Since $(\phi_\alpha, \phi_\beta)$ = constant, one might naively expect that $M_{\alpha\beta}$ be antisymmetric. But this is not the case, since $M_{\alpha\beta}$ depends on time not only via ϕ_α and ϕ_β, but also via the integration boundary which enters the definition of the inner product (2.33). On the other hand, the completeness of the eigenfunctions ϕ_α implies

$$\sum_\gamma \phi_\gamma(y)\phi_\gamma(\tilde{y}) = \delta(y-\tilde{y})y^3\ , \tag{2.46}$$

so that

$$\sum_\gamma M_{\alpha\gamma}M_{\beta\gamma} = \sum_\gamma \left[(\partial_t\phi_\alpha, \phi_\gamma)(\partial_t\phi_\beta, \phi_\gamma)\right] = (\partial_t\phi_\alpha, \partial_t\phi_\beta) = N_{\alpha\beta}\ . \tag{2.47}$$

The system of coupled second-order differential equations (2.45) thus solely depends on the time-dependent frequency ω_α^2 and the coupling matrix $M_{\alpha\beta}$. Numerical solutions of this system in other situations (oscillating cavities) have been obtained in Refs. [27, 28]. Also the results obtained in this work are tested with the code described in [28]. We plan to apply this numerical technique in future work.

The matrix elements are given in terms of the instantaneous solutions ϕ_α. The use of several identities of Bessel functions finally leads to

$$M_{00} = \widehat{y}_\mathrm{b}\frac{y_\mathrm{r}^2}{y_\mathrm{r}^2 - y_\mathrm{b}^2}\ , \tag{2.48}$$

$$M_{0j} = 0\ , \tag{2.49}$$

$$M_{i0} = 2\sqrt{\widehat{\mathrm{m}}_i M_{00}}\ , \tag{2.50}$$

$$M_{ii} = \widehat{\mathrm{m}}_i\ , \tag{2.51}$$

$$\begin{aligned}
M_{ij} = &- A_i A_j \widehat{\mathrm{m}}_i \mathrm{m}_i^3 \mathrm{m}_j^2 \int_{y_\mathrm{b}}^{y_\mathrm{r}} y^2 C_1(\mathrm{m}_i y) C_0(\mathrm{m}_j y)\mathrm{d}y \\
&- \frac{2\mathrm{m}_i^2}{\mathrm{m}_i^2 - \mathrm{m}_j^2}\sqrt{\widehat{\mathrm{m}}_i \widehat{\mathrm{m}}_j}\left[1 + \frac{\widehat{\mathrm{m}}_i}{\widehat{y}_\mathrm{b}}\right]\left[\sqrt{\frac{\widehat{\mathrm{m}}_i(\widehat{y}_\mathrm{b} + \widehat{\mathrm{m}}_j)}{\widehat{\mathrm{m}}_j(\widehat{y}_\mathrm{b} + \widehat{\mathrm{m}}_i)}} - 1\right]\ ,
\end{aligned} \tag{2.52}$$

where we have set

$$\widehat{\mathrm{m}}_i \equiv \partial_t \ln(\mathrm{m}_i) = \widehat{y}_{\mathrm{b}} \left[\frac{Y_1^2(\mathrm{m}_i y_{\mathrm{b}})}{Y_1^2(\mathrm{m}_i y_{\mathrm{r}})} - 1\right]^{-1} \qquad \text{and} \tag{2.53}$$

$$\widehat{y}_{\mathrm{b}}(t) \equiv \partial_t \ln(y_{\mathrm{b}}) = -\mathrm{v} y_{\mathrm{b}}^{-1} \simeq -Ha = -\frac{\dot{a}}{a} \equiv -\mathcal{H} \ . \tag{2.54}$$

Numerics indicate that both terms appearing in M_{ij} have similar amplitudes, thus the second term can give us an order of magnitude estimate of Eq. (2.52). We are mainly interested in the limit $y_{\mathrm{r}} \to \infty$. Setting $\epsilon = y_{\mathrm{b}}/y_{\mathrm{r}}$ we find $\widehat{\mathrm{m}}_i \simeq \widehat{y}_{\mathrm{b}}\epsilon^2$ for $\epsilon \to 0$. To lowest order in ϵ, the limit $\mathrm{v} \ll 1$ the matrix elements are given by

$$M_{00} = -\mathcal{H}[1 + \mathcal{O}(\epsilon)] \ , \tag{2.55}$$

$$M_{0j} = 0 \ , \tag{2.56}$$

$$M_{i0} = \mathcal{H}\mathcal{O}(\epsilon) \ , \tag{2.57}$$

$$M_{ii} = \mathcal{H}\mathcal{O}(\epsilon^2) \ , \tag{2.58}$$

$$M_{ij} = \mathcal{H}\mathcal{O}(\epsilon^2) \ . \tag{2.59}$$

This leads to the following interesting result. At late times $t \simeq \eta$ and in the 1-brane limit $\epsilon \to 0$, the Kaluza-Klein modes with non-vanishing mass evolve trivially and only the massless mode is coupled to the brane motion[2]

$$\ddot{\mathrm{q}}_0 + \left[k^2 - \dot{\mathcal{H}} - \mathcal{H}^2\right] \mathrm{q}_0 = 0 \ , \tag{2.60}$$

$$\ddot{\mathrm{q}}_i + \left[k^2 + \mathrm{m}_i^2\right] \mathrm{q}_i = 0 \ , \quad \text{for} \quad i \neq 0 \ . \tag{2.61}$$

Therefore bulk tensor perturbations $h_\bullet \simeq h_{\bullet 0} = \sqrt{2}(L/a)\tilde{\mathrm{q}}_0$ satisfy the equation

$$\ddot{h}_\bullet + 2\mathcal{H}\dot{h}_\bullet + k^2 h_\bullet = 0 \ . \tag{2.62}$$

This equation is valid everywhere in the bulk, and in particular on the physical brane where it reproduces exactly the equation for a 4-dimensional gravity wave in a Friedmann universe. This explicitly proofs that at low energy (late times) the homogeneous tensor perturbation equation in brane cosmology reduces to the 4-dimensional tensor perturbation equation. The massive Kaluza-Klein modes decouple completely from the massless mode in the 1-brane limit. The friction term arises from the time-dependent boundary condition, hence from the motion of the brane, and is not due to the excitation of the massive modes as claimed in [29].

This is our first main result. In the cosmological 1-brane model (RSII with a moving brane), the 4-dimensional graviton obeys the usual equation of motion and the (canonical) massive modes behave like massive particles in Minkowski space. Contrary to the massless mode, they are not affected by the motion of the brane. However note that k and m are the co-moving momentum and mass.

III. QUANTUM GRAVITON PRODUCTION

So far, we have considered the classical problem of solving a partial differential equation with time-dependent boundary conditions. Now we go on to consider the quantum production of gravitons. Let us assume (for the sake of simplicity) that the physical brane y_{b} has been at rest up to some initial time $t_{\mathrm{in}} \simeq \eta_{\mathrm{in}} \geqslant L/a_{\mathrm{in}}$, and that at this initial time there have been no gravitons. The graviton field has been in the vacuum state. Now the brane starts moving and (as seen from an observer on the brane) the universe expands. Most probably, at some later time t_{fin}, the graviton field will no longer be in the vacuum state, gravitons have been generated.

Of course since the expansion of a radiation-dominated universe is decelerating already prior to t_{in} the universe is more likely to expand faster than to be at rest, hence our initial conditions are not realistic. To study the quantum generation of gravitons we now consider the field $h_\bullet(k)$. In addition to Eq. (2.40), we have a similar mode decomposition for its canonical momentum,

$$\pi_\bullet(t, y; k) = \sqrt{\frac{L^3}{\kappa_5}} \frac{\partial \mathcal{L}}{\partial(\partial_t h_\bullet)} = \sqrt{\frac{L^3}{\kappa_5}} \sum_{\alpha=0}^{\infty} \tilde{\mathrm{p}}_\alpha(t; k)\phi_\alpha(t, y) = \sum_{\alpha=0}^{\infty} \mathrm{p}_\alpha(t; k)\phi_\alpha(t, y) \ . \tag{3.1}$$

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[2] It is easy to see that $\epsilon^{-1}\sum_\alpha M_{\alpha j}$ is bounded for all values of ϵ and therefore we do not have to fear that the infinite sum might contribute in the limit $\epsilon \to 0$.

The last equality defines the expansion coefficients p_α of the momentum $\pi_\bullet$. In quantizing the field, we promote both the field $h_\bullet$ and its conjugate momentum $\pi_\bullet$ to operators. We then impose the equal time canonical commutation relations for bosonic fields,

$$\left[\hat{h}_\bullet(t,y;k),\hat{h}_\bullet(t,y';k)\right] = 0 \ , \tag{3.2}$$

$$[\hat{\pi}_\bullet(t,y;k),\hat{\pi}_\bullet(t,y';k)] = 0 \ , \tag{3.3}$$

$$\left[\hat{h}_\bullet(t,y;k),\hat{\pi}_\bullet(t,y';k)\right] = i\delta(y-y') \ . \tag{3.4}$$

The operator valued coefficients $\hat{\mathrm{q}}_\alpha, \ \hat{\mathrm{p}}_\alpha$ thus satisfy

$$[\hat{\mathrm{q}}_\alpha(t;k),\hat{\mathrm{q}}_\beta(t;k)] = 0 \ , \tag{3.5}$$

$$[\hat{\mathrm{p}}_\alpha(t;k),\hat{\mathrm{p}}_\beta(t;k)] = 0 \ , \tag{3.6}$$

$$[\hat{\mathrm{q}}_\alpha(t;k),\hat{\mathrm{p}}_\beta(t;k)] = i\delta_{\alpha\beta} \ . \tag{3.7}$$

As usual in quantum field theory, we work in the Heisenberg picture. The operator valued coefficients $\hat{\mathrm{q}}_\alpha(t;k)$ still satisfy the classical equation of motion (2.45). But now we also know the initial state. Before $t_{\rm in}$ the coefficient operators are given by

$$\hat{\mathrm{q}}_\alpha(t;k) = \frac{1}{\sqrt{2\omega_\alpha}}\left[\hat{a}_\alpha e^{-it\omega_\alpha} + \hat{a}_\alpha^\dagger e^{it\omega_\alpha}\right] \quad \text{for} \quad t < t_{\rm in} \ . \tag{3.8}$$

Here $\hat{a}_\alpha(k)$ and $\hat{a}_\alpha^\dagger(k)$ are the annihilation and creation operators associated with the initial vacuum state. At time t, we define the vacuum state $|0,t\rangle$ as the state annihilated by all annihilation operators, $\hat{a}_\alpha(k,t)\,|0,t\rangle = 0$, $\forall\, k$ and α.

We now consider a fixed wavenumber k, as before we work in the late time limit, hence $\eta \simeq t$ and we assume a barotropic fluid $P = w\rho$ with $w = \text{cst}$ on the brane, so that the scale factor is $a \simeq (\eta/L)^{2/(1+3w)}$. We are mainly interested in a radiation-dominated universe, $w = 1/3$. In the limit where $\epsilon \to 0$, we may neglect the intermode coupling. The massive Kaluza-Klein modes then obey the massive wave equation (2.61) with constant mass, hence they do not experience any particle generation. The massless mode evolves according to equation

$$\ddot{\mathrm{q}}_0 + \left[k^2 - \left(\nu^2 - \tfrac{1}{4}\right)\eta^{-2}\right]\mathrm{q}_0 = 0 \ , \qquad \nu = \frac{3(1-w)}{2(1+3w)} \ . \tag{3.9}$$

This is a standard harmonic oscillator equation with a time-dependent mass term $m^2 = -\left(\nu^2 - \frac{1}{4}\right)\eta^{-2} = \dot{\mathcal{H}} + \mathcal{H}^2 = \ddot{a}/a$. In a radiation-dominated universe $\nu = 1/2$, the mass vanishes identically (the expansion velocity $\dot{a}$ is constant) and there is no particle creation. During a matter dominated universe $m^2(\eta)$ is negative and time-dependent, so there is particle creation, but one easily estimates that the produced energy density is of the order $\rho_h(\eta)/\rho_c(\eta) \simeq \ell_4^2/(a\eta_{\rm eq})^2 \simeq (\ell_4/\tau_{\rm eq})^2 \sim 10^{-112}$, which is vanishingly small. Here $\ell_4 = \sqrt{\kappa_4}$ denotes the 4-dimensional Planck scale and τ_{eq} is the age of the universe at equality.

IV. CONCLUSION

We have shown that in the single brane cosmological Randall-Sundrum model at late time, no massive gravitons are produced due to the motion of the brane through AdS_5 . Even though we have only shown this for the 4-dimensional spin-2 mode, the same result is expected for the spin-1 "gravi-photon" and the spin-0 "gravi-scalar" modes which obey the same equations (they are the three additional helicity states of a massive spin-2 particle). Also the massless gravi-photon and gravi-scalar are not produced in the single brane limit since they become non-normalizable. The only particle which can be produced in this model is the massless spin-2 graviton.

At late time, the massless spin-2 graviton obeys exactly the 4-dimensional equation of motion. Therefore, as in 4-dimensional cosmology, there is no graviton production during the radiation-dominated phase of the universe. This is a very general result in 4-dimensional cosmology: during the radiation era, minimally coupled massless particles are not generated, however, if the universe obeys a different expansion law they are.

Models with two branes which approach each other and later move apart again (e.g., the ekpyrotic model), might be more interesting for the study of graviton production. Due to mode coupling, we expect particle production also for a radiation-dominated expansion law in this case. Furthermore, in two brane models also massive modes are produced, which can lead to stringent constraints, as they soon come to dominate the universe. Unfortunately, the high energy regime, where most of the particle production takes place, depends on the details of the model under consideration. We will discuss the two brane case in a forthcoming paper [30].

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