ZU–TH14/92

GAUGE INVARIANT COSMOLOGICAL PERTURBATION THEORY

A General Study and its application to the Texture Scenario of Structure Formation

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Abstract

After an introduction to the problem of cosmological structure formation, we develop gauge invariant cosmological perturbation theory. We derive the first order perturbation equations of Einstein's equations and energy momentum "conservation". Furthermore, the perturbations of Liouville's equation for collisionless particles and Boltzmann's equation for Compton scattering are worked out. We fully discuss the propagation of photons in a perturbed Friedmann universe, calculating the Sachs–Wolfe effect and light deflection. The perturbation equations are extended to accommodate also perturbations induced by seeds.

With these general results we discuss some of the main aspects of the texture model for the formation of large scale structure in the Universe (galaxies, clusters, sheets, voids). In this model, perturbations in the dark matter are induced by texture seeds. The gravitational effects of a spherically symmetric collapsing texture on dark matter, baryonic matter and photons are calculated in first order perturbation theory. We study the characteristic signature of the microwave background fluctuations induced in this scenario and compare it with the COBE observations.

Contents

Introduction

Within standard cosmology, the formation of large scale structure remains one of the biggest unsolved problems, despite of great efforts. The most natural scenario, where structure forms by growth of adiabatic perturbations in a baryon dominated universe is clearly ruled out. Other extensively worked out scenarios like isocurvature baryons or baryons and collisionless matter (cold dark matter or hot dark matter) face severe difficulties [Geller and Huchra, 1989, Maddox et al., 1990, Saunders et al., 1991, Gouda et al., 1991] (for a short review see [Peebles and Silk, 1990]).

On the other hand, there is the interesting possibility to induce perturbations in the baryonic and dark matter by seeds. Seeds are an inhomogeneously distributed form of energy which contributes only a small fraction of the total energy density of the universe. Thus, linear perturbation theory can be used to calculate the induced fluctuations and their time evolution. Gauge–invariant linear perturbation theory [Bardeen, 1980] is superior to gauge dependent methods, since it is not plagued by gauge modes, and it leads in all known cases to the simplest systems of equations.

Examples of seeds are primordial black holes, boson stars, a first generation of stars, cosmic strings, global monopoles, or global texture. These last three are especially appealing since they can originate in a natural way from phase transitions in the early universe [Kibble, 1980]. If they indeed play an important role in structure formation they would link the smallest microscopical scales never probed by particle accelerators (down to $10^{-16}GeV^{-1}$) and the largest structures (up to 100Mpc and maybe more)! Another attractive feature of topological defects is that the gravitational effects of each class of them (cosmic strings are π_1 defects, monopoles are π_2 defects and texture are π_3 defects) are quite insensitive to the detailed symmetry breaking mechanism and only depend on the symmetry breaking scale η, which we cast in the dimensionless quantity $\epsilon = 16\pi G\eta^2$. This quantity determines the amplitude but not the shape or the time evolution of the perturbations.

Here, we mainly consider the π_3 defect global texture which was first proposed by Turok [1989] as a seed for large scale structure. Several subsequent investigations of this scenario gave promising results: The spatial and angular correlation functions, the large scale velocity fields and other statistical quantities obtained by numerical simulations agree roughly with observations (see Pen et al. [1993], Gooding et al. [1992], Spergel et al. [1991] and references therein).

My objective in this text is to fully develop gauge invariant linear perturbation theory to treat models with seeds. I then want to show in some detail, how all the linear perturbation theory aspects of a scenario of large scale structure can be investigated with these tools. As an example, we discuss the texture scenario. I choose this scenario not because I think it is the solution of the problem of cosmological structure formation. But it is the simplest worked out scenario, where initial fluctuations are induced by topological defects of a symmetry breaking phase transition, and I believe that this class of models deserves thorough investigation as an alternative to models with initial perturbations from inflation.

In the first chapter we set a frame for this review with a non–technical overview of the problem of cosmological structure formation. We also very briefly discuss some of the presently considered scenarios. A reader already familiar with the problem of structure formation may skip it. In the second chapter gauge–invariant cosmological perturbation theory [Hawking, 1966, Bardeen, 1980, Kodama and Sasaki, 1984, Durrer and Straumann, 1988, Mukhanov et al., 1992] is presented in a (hopefully) pedagogical fashion. Although, some familiarity with general relativity and basic concepts of differential geometry are needed to follow the derivations in this chapter. We extensively discuss scalar, vector and tensor perturbations for fluids and for collisionless matter. Furthermore, general formulae for the deflection of light in a perturbed Friedmann universe are derived. This part is new.

Finally, seeds as initial perturbations are introduced [Durrer, 1990]. Here it is assumed that non–gravitational interactions of the seeds with the surrounding matter are negligible. For topological defects, this is certainly a good approximation soon after the phase transition.

In Chapter 3, some simple but important applications of cosmological perturbation theory are described. We first discuss the ideal fluid [Bardeen, 1980, Kodama and Sasaki, 1984]. Then, a gauge invariant form of Boltzmann's equation for Compton scattering is derived. We study it in the limit of many collisions to obtain an approximation for the damping of cosmic microwave background (CMB) fluctuations by photon diffusion in a reionized universe. We then list all mechanisms proposed to induce anisotropies in the cosmic microwave background. Finally, two applications of light deflection are discussed: Lensing by global monopoles [Barriola and Vilenkin, 1989] and light deflection due to a passing gravitational wave. This effect is new and might be an alternative way to detect gravitational waves of very far away sources.

In Chapter 4 we review the concept of texture defects and present the exact flat space solution found by Turok and Spergel [1990]. We then calculate the gravitational potentials induced by this solution and apply the formalism developed in Chapter 2 to derive photon redshift, light deflection and perturbations of baryons and dark matter. All results can be obtained analytically and provide a nice application of gauge invariant perturbation theory. They are, however, cosmologically relevant only for scales substantially beyond horizon scale. Most of the results of this chapter were originally derived by Durrer [1990] and Durrer et al. [1992a], but the calculation of light deflection is new and the work on collisionless particles is presented in a much simpler way and a physically more sensible limit is performed.

We conclude with a chapter on textures in expanding space. Here we present a method to calculate the induced cosmic microwave background fluctuations by statistically distributing individual textures which are modeled spherically symmetric. The detailed numerical results of these investigations will be published elsewhere [Durrer et al., 1993].

In Appendix A the basics of 3+1 formalism of general relativity are outlined. In Appendix B, we provide a glossary of the variables used in this review.

Chapter 1

The Problem of Large Scale Structure Formation

1.1 The Standard Cosmological Model

We begin with some facts on the standard model of cosmology. Extensive treatments of this subject can be found in the books by Tolman [1934], Bondi [1960], Sciama [1971], Peebles [1971], Weinberg [1971], Zel'dovich and Novikov [1983], Börner [1988], Kolb and Turner [1990], Peebles [1993].

The assumption that the universe is homogeneous and isotropic on very large scales, leads to a very special class of solutions to the gravitational field equations, the Friedmann–Lemaître universes. Accepting furthermore the cosmological origin of quasars and/or the cosmic background radiation, one can reject the possibility of a so called bouncing solution and show that our universe necessarily started in a big bang [Ehlers and Rindler, 1989]. Due to homogeneity and isotropy, the energy momentum tensor is described by the total energy density ρ and pressure p which are functions of time only. The metric is of the form

$$ds^2 = a^2(-dt^2 + \gamma_{ij}dx^i dx^j) \ ,$$

where γ is a 3-d metric of constant curvature k. The simply connected 3–spaces corresponding to γ depend on the sign of the curvature: For $k > 0$, the three–space is a sphere, for $k < 0$ a pseudo sphere and for $k = 0$ flat, Euclidean space. (Note, however, that the metric γ cannot decide on the topological structure of 3–space. For $k = 0$, e.g., it may well be, that three space is topologically equivalent to $\mathbf{R}^3/\mathbf{Z}^3$, i.e., a torus with finite volume. The often stated phrase that for $k \leq 0$ 3–space is infinite is thus wrong.)

The time dependent function a is the scale factor and the physical time (proper time) τ of an observer at rest is given by $d\tau = adt$. The time coordinate t is called conformal time.

Einstein's equations imply the Friedmann equation which determines the scale factor as a function of the density:

$$\left(\frac{\dot{a}}{a}\right)^2 + k = \frac{8\pi}{3}G\rho a^2 + \Lambda a^2/3 \ . \tag{1.1}$$

Here G is Newton's gravitational constant, and Λ is the famous cosmological constant which has been resurrected several times in the past [Weinberg, 1989].

Important quantities in Friedmann cosmology are the Hubble parameter, H and the density parameter Ω, which are defined by

$$H = \dot{a}/a^2 \quad \text{and} \quad \Omega = \rho/\rho_c \ , \quad \text{with} \quad \rho_c = 3H^2/(8\pi G)$$

. Here ρ_c is the critical density, the density of a universe with $k = \Lambda = 0$. The present value of the Hubble parameter, H_0, is usually parameterized in the form $H_0 = h \times 100 km/s/Mpc$. Observations limit the value of H_0 in the range

$$0.4 \leq h \leq 1 \quad \text{and}$$

$$0.05 \leq \Omega_0 \leq 2 \ .$$

From estimates of the mass to light ratio one obtains a value for the amount of luminous matter, $\Omega_{lum} \approx 0.007$ and from velocity measurements one estimates the amount of 'clustered matter' to be $\Omega_{dyn} \approx 0.1$. These values result from many difficult observations and are correspondingly uncertain [Kolb and Turner, 1990]. Unfortunately, the measurements which lead to them can not provide information about a dark, poorly clustered matter contribution and therefore do not yield an upper limit for the density parameter. The upper bound comes from a comparison of lower limits to the age of the universe with the Hubble parameter.

Using the energy "conservation" equation

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p) \ , \tag{1.2}$$

one finds that for $p \geq 0$, ρ grows at least like a^{-3} for $a \to 0$ (i.e. approaching the big bang). Therefore, at an early enough epoch the density term in (1.1) always dominates over the curvature term and the cosmological constant, and Ω becomes arbitrarily close to 1. In other words, the evolution of a Friedmann universe always "starts" very close to the unstable fixpoint $\Omega = 1$.

On the other hand, (1.1) shows, if the cosmological constant Λ once dominates the expansion of the universe, the curvature term k and the energy density ρ become less and less important. The universe then expands approximately exponentially in terms of physical time τ

$$a(\tau) = a(\tau_i) \exp(\sqrt{\Lambda/3} \int_{\tau_i}^{\tau} d\tau') \quad .$$

This rapid expansion can only be stopped if some or all of the vacuum energy inherent in the cosmological constant is radiated into particles. Such an intermediate phase of Λ dominated expansion is the basic idea of most scenarios of inflation.

Today, the cosmological constant is severely limited from observations:

$$\Omega_\Lambda \equiv |\frac{\Lambda}{8\pi G H_0^2}| \leq \mathcal{O}(1) \ .$$

This is very much smaller than the values expected from particle physics, but might still be enough to influence the expansion of the universe and structure formation substantially [Holtzman, 1989]. It can, e.g., lead to a 'loitering period' during which the universe is nearly non–expanding [Durrer and Kovner, 1990], and thus fluctuations grow exponentially. Since such a small value for Λ requires an enormous fine tuning, we set $\Lambda = 0$ in the following.

For $\Lambda = 0$, one has $\Omega > 1$ in a closed universe (3–sphere) and $\Omega < 1$ in a negatively curved (open) universe.

Later, we also use the following consequences of (1.1) and (1.2) (setting $w = p/\rho$ and $c_s^2 = \dot{p}/\dot{\rho}$):

$$\ddot{a}/a = -\frac{1}{2}[(3w-1)(\frac{\dot{a}}{a})^2 + (3w+1)k] \quad (1.3)$$

$$\dot{w} = 3(\dot{a}/a)(w - c_s^2)(1+w) \ . \quad (1.4)$$

Like the standard model of particle physics, the standard model of cosmology has many impressive successes. The most important ones are:

- Uniform Hubble expansion (Detected by Wirtz [1918], Slipher [1920] and Hubble [1927]).
- The prediction of Gamov [1946] and detection by Penzias and Wilson [1965] of the cosmic microwave background radiation, its 'perfect' black body spectrum [Mather et al., 1990] and its extraordinary uniformity (see Figs. 1 and 2).
- The abundance of the light elements (H, ^2H, ^3H, ^3He, ^4He, ^7Li) can be calculated, and the comparison with observational estimates predicts $0.02 \le \Omega_B \le 0.1$ which is consistent with direct determinations of Ω_0. This calculations were originally performed by Alpher et al. [1948], Wagoner et al. [1967] and by many others later. A comprehensive review is Boesgaard and Steigman [1985]. For more recent developments, see e.g. Kurki–Suonio et al. [1988] and Walker et al. [1991].

However, many questions are left open. Most of them can be cast in terms of very improbable initial conditions:

- The question of the cosmological constant, Λ: Why is it so small? i.e., so much smaller than typical vacuum energies arising from particle physics [Weinberg, 1989].
- The flatness / oldness / entropy problem: why is the universe so old, $t \gg t_{pl}$, and still $\mathcal{O}(\Omega) = 1$? Here, t_{pl} is the Planck time, $t_{pl} = (\hbar G/c^5)^{1/2} = 5.39 \times 10^{-44}$sec .
- The horizon problem: how can different patches of the universe have been at the same temperature long before they ever where in causal contact?

So far, the most successful approach to answer the second and third questions is based on the hypothesis of an inflationary phase during which expansion is dominated by a 'fluid' with negative effective pressure, $p \le -\rho/3$ (e.g. a cosmological term, for which $p = -\rho$) and physical distances thus grow faster than the size of the effective particle horizon, $(\dot{a}/a^2)^{-1}$. I do not explain how inflation answers these questions, but refer the reader to some important publications on the subject: [Guth, 1981, Albrecht and Steinhardt, 1982, Linde, 1983, Linde, 1984, Linde, 1990, La and Steinhardt, 1989, Olive, 1990].

These open problems provide some of the very few and important hints to new fundamental physics beyond the standard models of cosmology and of particle physics.

1.2 Structure Formation

Besides the fundamental issues, related to the very early universe and unknown basic physics, there is the important problem of structure formation which, to some extent, should be solvable within the standard model. This is the problem I want to address here:

How did cosmological structures like galaxies, quasars, clusters, voids, sheets... form?

From difficult and expensive determinations of the three–dimensional distributions of optical and infrared galaxies on scales up to $150h^{-1}$Mpc [De Lapparent et al., 1986, De Lapparent et al., 1988, Geller and Huchra, 1989, Broadhurst et al., 1990, Saunders et al., 1991, Loveday et al., 1992a], [Fisher et al., 1993], we know that galaxies are arranged in sheets which surround seemingly empty voids of sizes up to $50h^{-1}Mpc$. Observations also show that the distribution of clusters of galaxies [Bahcall and Soneira, 1983, Bahcall, 1989, West and van den Bergh, 1991] is inhomogeneous with

$$\frac{\delta n_c}{n_c} \geq 1 \qquad \text{on scales up to 25 Mpc.}$$

In addition, it is important to observe that not all galaxies are young. There are quasars with redshift up to $z \approx 5$ and galaxies with $z \approx 3$. On the other hand, there are indications that the galaxy luminosity function is still evolving considerably in the recent past, i.e., that galaxy formation still continues [Loveday et al., 1992b].

From a naive Newtonian point of view one might say: *"Gravity is an unstable interaction. Once the slightest perturbations (e.g. thermal) are present they start growing and eventually form all these observed structures. The details of this process might be complicated but there seems to be no basic difficulty."*

This is true in a static space where small density fluctuations on large enough scales grow exponentially. But, as we shall see in Chapter 2, in an expanding space, this growth is reduced to a power law:

$$\frac{\delta\rho}{\rho} \propto \begin{cases} a & \text{for pressureless matter, dust} \\ \text{const.} & \text{for radiation, relativistic particles.} \end{cases}$$

In a radiation dominated universe, pressure prohibits any substantial growth of density perturbations. Once the universe is matter dominated $\delta\rho/\rho$ grows proportionally to the scale factor, but the gravitational potential

$$\Psi \propto \int (\delta\rho/r) r^2 dr = \int \delta\rho r dr$$

remains constant. This led Lifshitz [1946], who first investigated the general relativistic theory of cosmological density fluctuations, to the statement: *"We can apparently conclude that gravitational instability is not the source of condensation of matter into separate nebulae"*. Only 20 years later Novikov [1965] pointed out that Lifshitz was not quite right. But to some extent Lifshitz's point remains valid: Gravity cannot be the whole story. Imagine an $\Omega = 1$ universe which was matter dominated after $z \approx 2 \times 10^4$. Since density perturbations in cold matter evolve like $1/(z+1)$, they have grown by less than 10^5. But today there are galaxies with masses of $M \approx 10^{12} M_\odot \approx 10^{69} m_N$. Statistical inhomogeneities of these N nucleons have an amplitude of $1/\sqrt{N} \approx 10^{-34}$, which would have grown to less than $10^{-29} \ll 1$ today. Therefore, there must have been some non–thermal *initial fluctuations* present with an amplitude on the order of $10^{-5} - 10^{-4}$. We also know that the amplitude of these initial perturbations did not exceed this amount because of the high degree of isotropy of the cosmic microwave background radiation (CMB). In fact, only recently CMB anisotropies have been found in the COBE experiment [Wright et al., 1992, Smoot et al., 1992]. They are on the level

$$\frac{\Delta T}{T} \approx 10^{-5} \quad \text{or} \quad \delta\rho_{(rad)}/\rho_{(rad)} \approx 4 \times 10^{-5}$$

on all angular scales larger than 10^o and compatible with a scale invariant Harrison–Zel'dovich spectrum. On smaller angular scales other experiments led to limits, $\Delta T/T \leq$ a few$\times 10^{-5}$ (see Fig. 2). We disregard the dipole anisotropy which is due to our motion with about 600km/s with respect to the microwave background radiation.

There are scenarios of cosmological structure formation which are not based on the gravitational instability picture. The most recent of them is based on explosions (of supernovae or super conducting cosmic strings) [Ostriker and Cowie, 1981] which are supposed to sweep away baryons, producing shock fronts in the form of sheets surrounding roughly spherical voids. Usually such scenarios face enormous difficulties in producing the energy required to account for the large scale observations and to satisfy at the same time the limits for perturbations of the CMB on small angular scales. None of them has thus been worked out in detail. In fact, the explosion scenario also needs gravitational instability to amplify perturbations by a significant factor [McKee and Ostriker, 1988]. In this paper, we concentrate on scenarios which rely on gravitational instability. The fact, that the anisotropies measured by COBE just coincide with the amount of growth necessary to form structures today is taken as a hint that the gravitational instability picture may be correct.

1.3 The General Strategy

We now want to outline in some generality the ingredients that go into a model of structure formation which is based on the gravitational instability picture.

1.3.1 Initial fluctuations

We saw in the last section that small density fluctuations in a Friedmann universe may have grown (by gravitational instability) by about a factor of 2×10^4 during the era of matter domination. Therefore, a complete scenario of structure formation must lead to initial matter density fluctuations with amplitudes on the order of 10^{-4}. Two possibilities to obtain these initial fluctuations are primarily investigated.

A) Initial perturbations produced during inflation: Here it is assumed that density fluctuations are generated during an inflationary phase [Guth, 1981, Albrecht and Steinhardt, 1982, Linde, 1982, Linde, 1983, Linde, 1984, La and Steinhardt, 1989, Linde, 1990] from initial quantum fluctuations of scalar fields [Bunch and Davies, 1978, Hawking, 1982, Starobinsky, 1982], [Guth and Pi, 1982, Fischler et al., 1985]. Due to the nature of quantum fluctuations, the distribution of the amplitudes of these initial perturbations is usually Gaussian.

There are several different more or less convincing models of inflation. One divides them into: Standard (or old) inflation [Guth, 1981], new inflation [Albrecht and Steinhardt, 1982, Linde, 1982], chaotic inflation [Linde, 1983] and the most recently proposed possibilities of extended and hyper–extended inflation [La and Steinhardt, 1989]. Reviews on the subject are found in Linde [1984], Linde [1990], Olive [1990] and Steinhardt [1993]. All these models of inflation differ substantially, but the mechanism to produce initial fluctuations is basically the same:
The scales l of interest ($l \leq l_H(t_o) \approx 3000h^{-1}Mpc$) are smaller than the effective particle horizon at the beginning of the inflationary phase. The scalar field that drives inflation therefore experiences quantum fluctuations on all these scales. Their amplitude for a minimally coupled scalar field ϕ in

de Sitter universe[1] within a volume V can be calculated [Bunch and Davies, 1978]:

$$(\Delta\phi)_k^2 = \frac{Vk^3}{2\pi^2}|\delta\phi_k|^2 = (H/2\pi)^2 ,$$

with

$$\delta\phi_k = V^{-1}\int \phi(\boldsymbol{x})\exp(i\boldsymbol{k}\cdot\boldsymbol{x})d^3x .$$

In the course of inflation, the interesting scales inflate outside the horizon and quantum fluctuations 'freeze in' as classical fluctuations of the scalar field. This leads to energy density perturbations according to

$$\delta\rho_\phi = \delta\phi\frac{d\mathcal{V}}{d\phi} ,$$

where $\mathcal{V}$ denotes the potential of the scalar field. By causality, on scales larger than the effective particle horizon, these perturbations cannot grow or decay by any physical mechanism. The spurious growth of super–horizon size density perturbations in certain gauges like, e.g., synchronous gauge, is a pure coordinate effect!

This mechanism yields density perturbations D_λ of a given size λ which have constant amplitude at the time they re–enter the horizon after inflation is completed:

$$D_\lambda(t=\lambda) = A .$$

A natural consequence is thus the well known scale invariant Harrison–Zel'dovich spectrum [Harrison, 1970, Zel'dovich, 1972]. Under certain conditions on, e.g., the potential of the scalar field (which may require fine tuning of coupling constants), one can achieve that these fluctuations have the required amplitude of $A \approx 10^{-4}$.

Recently, the failure to account for the largest scale structure of the otherwise so successful cold dark matter (CDM) model has caused some effort to find inflationary models with spectra that differ substantially from Harrison–Zel'dovich on large scales [Polarski and Starobinsky, 1992]. Although of principle interest, this work may turn out to be unimportant after the new COBE results have shown such a striking consistency with a scale invariant spectrum.

B) Seeds as initial perturbations, topological defects: Initial fluctuations might have been triggered by seeds, i.e., by an inhomogeneously distributed matter component which contributes only a small fraction to the total energy density of the universe. Examples of seeds are a first generation of stars, primordial black holes, bosonic stars, cosmic strings, global monopoles and textures. We restrict our discussion to the latter three, the so called topological defects, which can arise naturally during phase transitions in the early universe [Kibble, 1980].

To understand how they form, consider a symmetry group G which is broken by a scalar field ϕ to a subgroup H at a temperature T_c. The vacuum manifold of the cooler phase is then generally given by $\mathcal{M}_0 = G/H$. Since the order parameter field ϕ (Higgs field) has a finite correlation length $\xi \leq l_H$ ($l_H \equiv$ horizon size) which is limited from above by the size of the horizon, the field varies in $\mathcal{M}_0$ if compared over distances larger than ξ. If the topology of $\mathcal{M}_0$ is non–trivial, the scalar field can vary in such a way that there are points in spacetime where, by continuity reasons, ϕ has to leave $\mathcal{M}_0$ and assume values of higher energy. This is the Kibble mechanism [Kibble, 1978].

[1] Except for extended and hyper–extended inflation, the universe during the inflationary phase is a de Sitter universe, expansion is driven by a cosmological constant.

The set of points with higher energy forms a connected sub–manifold without boundary in four dimensional spacetime. The dimensionality, d, of this sub–manifold is determined by the order, r, of the corresponding homotopy group $\pi_r(\mathcal{M}_0)$: $\quad d = 3 - r$, for $r \leq 3$. The points of higher energy of ϕ are often just called 'singularities' or 'defects'.

For illustration, let us look at the simplest example, where $\mathcal{M}_0$ is not connected, $\pi_0(\mathcal{M}_0) \neq 0$: At different positions in space with distances larger than the correlation length ξ, the field can then assume values which belong to disconnected parts of $\mathcal{M}_0$ and thus, by continuity, ϕ has to leave $\mathcal{M}_0$ somewhere in between. The sub–manifold of points of higher energy is three dimensional (in spacetime) and called a **domain wall**. Domain walls are disastrous for cosmology except if they originate from late time phase transitions.

A non–simply connected vacuum manifold, $\pi_1(\mathcal{M}_0) \neq 0$, leads to the formation of two dimensional defects, **cosmic strings**. Domain walls and cosmic strings are either infinite or closed.

If $\mathcal{M}_0$ admits topologically non–trivial (i.e. not shrinkable to a point) continuous maps from the two sphere, $\phi : \mathbf{S}^2 \to \mathcal{M}_0$, then $\pi_2(\mathcal{M}_0) \neq 0$, one dimensional defects, **monopoles**, form.

Finally, continuous mappings from the three sphere determine π_3. If $\pi_3(\mathcal{M}_0) \neq 0$, zero dimensional **textures** appear which are events of non–zero potential energy. Using Derricks theorem [Derrick, 1964], one can argue that a scalar field configuration with non–trivial π_3 winding number (i.e. a texture knot) contracts and eventually unwinds, producing a point of higher energy for one instant of time. This type of defect is discussed in more detail in Chapters 4 and 5.

Topological defects are very well known in solid state physics. Important examples are vortices in a super conductor or the vorticity lines in a super fluid. All four types of defects discussed above can also be found in liquid crystals see e.g. Chuang et al. [1991] and references therein.

Depending on the nature of the broken symmetry, defects can either be **local**, if the symmetry is gauged, or **global**, from a global symmetry like, e.g., in the Peccei–Quinn mechanism. In the case of local defects, gradients in the scalar field can be compensated by the gauge field and the energy density of the defect is confined to the defect manifold which has a thickness given roughly by the inverse symmetry breaking scale. On the other hand, the energy density of global defects is dominated by gradient energy, with a typical scale given by the horizon size at defect formation. The extension of the induced energy density perturbation is thus about the horizon size at its formation. This leads to a Harrison–Zel'dovich initial spectrum. The difference to initial perturbations arising from inflation is that density fluctuations induced by topological defects are not Gaussian distributed.

Local monopoles would dominate the energy density of the universe by far ($\Omega_M \approx 10^{11}$!!) and must be excluded. Local textures are not energetic enough to seed large scale structure. But global monopoles and global textures are quite promising candidates. Cosmic strings, both global or local, are interesting. One of the models which we introduce in the next section is based on local cosmic strings.

We will find in Chapter 4, that the properties of the scalar field ϕ other than the induced homotopy groups, e.g., the specific form of the Higgs potential, are of no importance for the defect dynamics. But the probability of defect formation via the Kibble mechanism might well depend on $\mathcal{M}_0$. The symmetry breaking scale just determines the energy of the defects, i.e., the amplitude of the fluctuations. From simulations of large scale structure formation one finds that successful models requires $T_c^2 G \approx m_c^2/m_{pl}^2 \approx 10^{-6}$, which corresponds to a typical GUT scale of $\sim 10^{16}$GeV. This can also be understood analytically: From the perturbation equations derived in Chapter 2, we shall see that defect induced structure formation leads to initial perturbations with amplitudes $A \sim 16\pi G T_c^2$.

1.3.2 Linear perturbation theory

Since the amplitudes of the initial fluctuations are tiny, at early times their evolution can be calculated within linear cosmological perturbation theory.

Chapter 2 is devoted to a thorough discussion of this subject, we thus skip it here. The reader just interested in the results is referred to Chapter 3 where some important consequences from cosmological perturbation theory are discussed.

1.3.3 N–body simulations

On relatively small scales (up to about 20Mpc or so) perturbations become non–linear at late times. Structure formation must then be followed by non–linear numerical simulations, whose input is the perturbation spectrum resulting from linear perturbation theory. Since at the time when the non–linear regime takes over, the corresponding scales are much smaller than the horizon size, and since the gravitational fields and velocities are small, Newtonian gravity is sufficient for these simulations. For a realistic calculation of the process of galaxy formation, the hydrodynamical evolution of baryons has to be included. The different heating and cooling processes and the production of radiation which might partially provide the X–ray background and might reionize the intergalactic medium have to be accounted for. Furthermore, the onset of nuclear burning which produces the light in galaxies has to be modeled, in order to obtain a mass to light ratio or the bias parameter (which may well be scale dependent).

The inclusion of the heating and cooling processes of baryons into N–body simulations is a very complex computational task and the results published so far are still preliminary [Cen et al., 1990, Cen, 1992, Cen and Ostriker, 1992, Cen et al., 1991]. Only very recently, the first calculations taking into account nuclear burning have been carried out [Ostriker, 1992]. High quality simulations which only contain collisionless particles have been performed to very good accuracy [Centrella and Melott, 1982, White, 1986, Davis et al., 1985/87], but unfortunately they leave open the question how light traces mass and therefore, how these results are to be compared with observations of galaxy clustering.

I shall not discuss this important and difficult part of structure formation in this review and refer the reader to the references given above.

1.3.4 Comparison with observations

Of course the final goal of all the effort is to confront the results of a given model with observations. The easiest and least uncertain part is to analyse the induced microwave background fluctuations. They can be calculated reliably within linear perturbation theory. Observations of the microwave background are also a measurement of the initial spectrum and it is very remarkable that the new COBE results are compatible with a Harrison–Zel'dovich spectrum [Wright et al., 1992, Smoot et al., 1992].

On smaller scales ($\theta \le 6^o$), the CMB anisotropies depend crucially on the question whether the universe has been reionized at early times ($z \ge 100$) and therefore indirectly on the formation of non–linearities in the matter density perturbation (which would provide the ionizing radiation). The different possibilities how the process of structure formation can induce anisotropies in the cosmic microwave background are discussed in detail in Chapter 3.

On scales up to about 20Mpc, the up today most extensively used tool to compare models with observations is the galaxy galaxy correlation function, $\xi_{GG}(r)$,

$$\xi_{GG}(r) = \frac{1}{\langle n\rangle^2}\int n(\vec{x})n(\vec{x}+r\vec{e})d^3x d\Omega_{\vec{e}} \ ,$$

where n is the number density of galaxies. Usually, the amplitudes of initial fluctuations in a given scenario are normalized by $J_3(R = 10Mpc)$:

$$J_3(R) = \int \xi_{GG}(r)W_R(r)r^2dr \ ,$$

with an observational value of $J_3(10Mpc)/(10Mpc)^3 \approx 0.27$. Here $W_R(r)$ is a (Gaussian or top hat) window function windowing scales smaller than R. Since the new COBE results now provide the amplitude of fluctuations on very large scales which are not influenced by non–linearities, future calculations will clearly be normalized on these scales. Observations fix the amplitude and slope of the galaxy galaxy and cluster cluster correlation functions to be

$$\xi_{GG}(r) \approx (\frac{r}{r_0})^{-1.8} \ , \tag{1.5}$$

$$\xi_{CC}(r) \approx (\frac{r}{r_0})^{-1.8} \ . \tag{1.6}$$

For galaxies one finds $r_0 \approx 5.4h^{-1}Mpc$ [Peebles, 1988], whereas for clusters r_0 depends on the "richness class" of the clusters considered. For rich clusters $r_0 \approx (20-25)h^{-1}Mpc$ [Bahcall and Soneira, 1983]. Recently, a smaller amplitude for the cluster cluster correlation function has been found from the APM survey, $\xi_{CC} = 4\xi_{GG}$ and $\xi_{CC} = 2\xi_{CG}$ [Dalten et al., 1991].

A disadvantage of the correlation function is its insensitivity to lower dimensional structures like sheets and filaments[2].

There are various other statistical tests which one can perform and compare with the sparse observations. I just mention a few:
The Mach number, which gives the ratio between the average velocity and the velocity dispersion on a given length scale, $M =< v >^2 /\sigma^2$ [Ostriker and Suto, 1990].
The genus test, where one calculates the number of holes minus the number of islands in an iso–density contour [Gott et al., 1986].
The 3–point or 4–point correlation functions, which determine the deviation from Gaussian statistics of the distribution of perturbations on a given length scale [Peebles, 1980]. They can be cast in the skewness, $< \delta\rho^3 > / < \delta\rho^2 >^{3/2}$ and the kurtosis $< \delta\rho^4 > / < \delta\rho^2 >^2$.

Additional more qualitative results are: - The earliest galaxies must have formed at $z \geq 5$ but that there still must be substantial galaxy formation going on at $z \approx 2-1$.

- From observations [De Lapparent et al., 1986, De Lapparent et al., 1988], [Geller and Huchra, 1989] we can conclude that galaxies are arranged in sheetlike structures around seemingly empty voids.

- Velocity observations have found large ($\approx 100h^{-1}Mpc$) coherent velocity fields with $\langle v\rangle \approx 500km/s$.

- A successful model should of course also obtain flat rotation curves of galaxies which have been observed with increasing accuracy since the seventies [Rubin, 1983].

[2]Recently the amplitudes of the correlation function have been criticized to depend crucially on the sample size and thus to be physically meaningless. New analyses [Einasto et al., 1986, Pietronero, 1987, Davis et al., 1988, Coleman and Pietronero, 1992] have shown that r_0 depends on the sample size, hinting that the distribution of galaxies may be fractal up to the largest scales presently accessible in volume limited samples, $R_{\max} \approx 30h^{-1}$Mpc. If this objection is justified, the normalization procedure with the help of the correlation function is useless!

1.4 Models

The most simple picture, a universe with $\Omega h^2 = \Omega_B h^2 \approx 0.1$ and adiabatic initial fluctuations is definitely ruled out by the limits of the microwave background fluctuations [Gouda et al., 1989, Gouda et al., 1991]. This is a first, very important result in the discussion of different models which might account for the formation of large scale structure.

To give the reader a taste of the presently favored scenarios, we present here five cases. A sixth possibility, the "texture scenario", will be discussed in Chapters 4 and 5. It is clear that a mixture of the hot dark matter (HDM) and cold dark matter (CDM) models presented here, as well as defects with HDM or any of the proposed scenarios with addition of a cosmological constant might lead to models that fit the observations better. The ones presented here are partly chosen by reasons of simplicity. The first attempt to treat a class of models systematically is given by Holtzman [1989], where 94 different combinations of $(\Omega_\Lambda, \Omega_{CDM}, \Omega_{HDM}, \Omega_B)$ are investigated.

In this section we give a short description of the models chosen, and compare them with some observational phenomena. Our aim is only to give a sketchy overview of these models; readers interested in more details are referred to the literature given in the text.

1.4.1 The isocurvature baryon model

As the most conservative alternative to the adiabatic baryon model, Peebles [1987] pursued the question, whether it is possible to construct a viable scenario of structure formation without the assumption of any exotic, i.e. up to now unobserved, form of energy, a universe with $\Omega_0 = \Omega_B \approx 0.1$, which is still marginally consistent with nucleosynthesis limits on Ω_B.

In order not to overproduce CMB fluctuations, one has to assume isocurvature initial perturbations, i.e. no perturbations in the geometry on cosmologically relevant scales (see Section 3.1). From this one can conclude $\delta\rho/\rho \approx 0$ on scales which are larger than the size of the horizon. Since the universe is radiation dominated initially, $\rho_r \gg \rho_B$ this yields $|\delta\rho_r/\rho_r| \ll |\delta\rho_B/\rho_B|$, i.e. isothermal fluctuations.

Isocurvature perturbations allow for relatively high initial values of $\delta\rho_B$ and therefore lead to early structure formation. Galaxies form at $z_g \geq 10$. The even earlier formation of small objects reionizes the universe. Small scale anisotropies in the CMB can then be damped by photon diffusion. For photon diffusion to be effective, reionization must take place before the universe becomes optically thin. This provides a lower limit to a 'useful' reionization redshift (see Chapter 3):

$$z_i \geq z_{dec} \approx 100(\frac{\Omega_B h}{.03})^{2/3} .$$

Photon diffusion then damps fluctuations on all scales smaller than the horizon scale at decoupling, $l_H(z_{dec})$ and correspondingly all angular scales with

$$\theta \leq \theta_{dec} \approx 6^o\sqrt{\Omega(100/z_{dec})} .$$

The problems of this scenario are twofold: First, the quadrupole anisotropy of the CMB turns out to be unacceptably large [Gouda et al., 1991]. A way out of this problem is a steep initial perturbation spectrum, but then it is difficult to reproduce the large amplitudes of the galaxy correlations on large scales.(Although, for steep a enough spectrum, even quadrupole fluctuations may be damped by photon diffusion, see Section 3.2.3!)

Secondly, observations hint that galaxy formation might peak around $z \approx 2$ or, at least, is still going on around $z \approx 1$, whereas in the isocurvature baryon model the process of galaxy formation is most probably over before $z = 3$.

1.4.2 Hot dark matter

Massive neutrinos with $\sum m_i \approx 200eVh^2$ they can provide the dark matter of the universe and dominate the total energy density with $1 = \Omega_{tot} \approx \Omega_\nu$ [Doroshkevich et al., 1980, Bond et al., 1980]. Large scale structure then develops as follows:

Initial fluctuations from inflation give rise to a scale invariant spectrum of Gaussian fluctuations. These initial fluctuations are constant until they 'reenter the horizon' (i.e. their scale becomes smaller than the size of the horizon, $l < l_H$). Thereafter they decay by free streaming if the universe is still radiation dominated; if the universe is matter dominated ($z \leq 2 \times 10^4$) they grow in proportion to the scale factor [Bond and Szalay, 1983, Durrer, 1989a]. This leads to a short wavelength cutoff of the linear perturbation spectrum at $l_{FS} \approx 40(m_\nu/30eV)^{-1}h^{-1}Mpc$. The corresponding cutoff mass is $M_{FS} \approx 10^{15}(30eV/m_\nu)^2 M_\odot$. The linear spectrum for HDM is given in Fig. 3.

In this model, large objects with mass $\approx M_{FS}$ (large clusters) form first . They then fragment into galaxies. Gravitational interaction of collisionless particles generates sheets, pancakes [Zel'dovich, 1970], and galaxies are thought to lie on the intersections of these sheets. This leads to a filamentary structure. Simulations of HDM show [Centrella and Melott, 1982, White et al., 1983] that in order to obtain the correct galaxy correlation function today, the large scale structure becomes heavily overdeveloped (see Fig. 4). The other main problem is that galaxy formation starts only very recently ($z \approx 1$). The model has serious difficulties to account for quasars with redshifts $z \geq 3 - 4$. In addition to these grave objections, CMB fluctuations turn out to be too large in this model. Since galaxy formation is only a secondary process, initial fluctuations which determine the amplitudes of CMB anisotropies must be rather large.

Because their thermal velocities are relatively high, massive neutrinos can only marginally provide the dark matter of galaxies but they cannot be bound to dwarf galaxies. If neutrinos are to constitute the dark matter of a virialized object with velocity dispersion σ and size r, their mass is limited from below by the requirement [Tremaine and Gunn, 1979]

$$m_\nu \geq 30eV(\frac{200km/s}{\sigma})^{1/4}(\frac{r}{10kpc})^{1/2} .$$

Since also dwarf galaxies do contain substantial amounts of dark matter [Carignan and Freeman, 1988], this is another serious constraint for the HDM model.

1.4.3 Cold dark matter

In this scenario one assumes the existence of a cold dark matter particle which at present dominates the universe with $1 = \Omega_{tot} \approx \Omega_{CDM}$. Particle physics candidates for such a matter component are the axion or the lightest super–symmetric particle. A more extended list can be found in Kolb and Turner [1990].

Again it is assumed that an inflationary phase leads to a scale invariant spectrum of Gaussian initial fluctuations. After the universe becomes matter dominated at $z_{eq} \approx 2 \times 10^4 h^2$, perturbations smaller than the horizon, $l \leq l_{eq} \approx 10(\Omega h^2)^{-1}$. start growing. Damping due to free streaming is negligible. Because of the logarithmic growth of matter fluctuations in the radiation dominated era (see Section 3.1), the spectrum is slightly enhanced on smaller scales $l < l_{eq}$.

According to this linear spectrum, sub–galactic objects form first. Once the perturbations become non–linear, these objects virialize and develop flat rotation curves. Only very recently, big structures begin to form through tidal interactions and mergers [Davis et al., 1988].

To obtain a mass distribution with $\Omega_{dyn} \approx 0.1 - 0.2 < \Omega = 1$, it is necessary that most of the mass is in the form of a dark background which is substantially less clustered than the luminous matter. This can be achieved with the idea of biasing: Luminous galaxies only form at high peaks of the density distribution. Since for a Gaussian distribution high peaks are more strongly clustered than average, this simple prescription has the desired effect [Kaiser, 1985, Bardeen et al., 1986]. Usually one introduces a bias parameter b and requires that galaxies form only in peaks of height $b\sigma$, where σ is the variance of the Gaussian distribution of density peaks. The best results are obtained for a bias parameter $b \approx 1.5 - 2$. Clearly, once the correct hydrodynamical treatment of baryons is included in the numerical simulations, such a bias parameter (which otherwise is just an assumption of how light traces mass) could be calculated. First preliminary results of such calculations [Cen et al., 1990, Ostriker, 1992] indicate that the above assumption is probably quite reasonable. A problem of the biasing hypothesis is the prediction of many dwarf galaxies in the voids which have not been found despite extensive searches.

The large scale structure obtained in this scenario looks at first sight rather realistic (see Fig. 5). It leads to the right galaxy galaxy correlation function up to 10Mpc. But the recently detected huge structures like the great wall (see Geller and Huchra [1989]) are very unlikely in this model. This situation has been quantified by comparing the angular correlation function from the deep IRAS survey with the one predicted by CDM [Maddox et al., 1990]. There, a substantial excess of power (as compared to the CDM model) on large angular scales, $\theta > 2^o$ is found (see Fig. 6).

Since galaxies form relatively late (at $z \approx 2$), it might also be difficult to produce the very high redshift quasars ($z \approx 5$). But since their statistics are still so low, and since very little is known on the ratio of normal galaxies to quasars at high redshift, this may not be a real problem.

1.4.4 Cosmic strings

Here, initial fluctuations are seeded by cosmic strings which form via the Kibble mechanism (see Section 3) after a phase transition at $T_c \approx 10^{15} GeV$ [Vilenkin, 1980].

Inter–commutation and gravitational radiation of cosmic string loops (see Vachaspati and Vilenkin [1984], Durrer [1989b]) determine the evolution of a network of (non–superconducting) cosmic strings. Numerical simulations support analytical arguments for a scaling law, $\rho_{string} \propto 1/(at)^2 \propto \rho$, for the energy density of a cosmic string network. In contrast to gauge monopoles or domain walls, strings do not dominate the energy density of the universe and are cosmologically allowed [Albrecht and Turok, 1989].

It is well known that a static straight cosmic string does not accrete matter, whereas a cosmic string loop from far away acts like a point mass [Vilenkin, 1980]. High resolution simulations of cosmic string networks [Bennett and Bouchet, 1990] have shown that the loops that chop off the network are too small ($l_{loop} \leq 0.01 l_H$) for efficient accretion. But moving long strings produce large accretion wakes behind them which might provide the sheets and walls observed in the universe [Bertschinger, 1987, Perivolaropoulos et al., 1990]. In this scenario galaxies form via fragmentation and/or accretion onto loops.

A relatively new idea is that chopping off small loops could lead to wiggles on the long strings. Cosmic strings with such small scale wiggles give rise to strings with an effective tension which is smaller than its effective energy density. In contrast to the original cosmic strings, wiggly strings can accrete matter even if they are static [Carter, 1990, Vilenkin, 1990, Vachaspati and Vilenkin, 1991]. In order for the large scale structure not to be 'drowned', in these small scale structures, the scenario works best if the dark matter is hot, massive neutrinos (HDM), so that the small scale fluctuations are damped by free streaming.

The requirement for successful structure formation on one hand and the difficulty not to overproduce microwave background anisotropies [Stebbins, 1988] on the other hand tightly constrain the possible value of the symmetry breaking scale

$$\mu = G\eta^2 \approx (1 \ - \ 2) \times 10^{-6} \ .$$

Recent work using new cosmic string simulations in which these estimates were redone to compare the CMB anisotropies with the COBE measurements led to a similar value [Bennett and Bouchet, 1992, Perivolaropoulos, 1993].

1.4.5 Global monopoles

Like for cosmic strings also the energy density of global monopoles produced by the Kibble mechanism obeys a scaling law. Therefore, they are candidates for a model of structure formation. In contrast to local monopoles, the gradient energy of global monopoles introduces a long range interaction, so that monopole anti–monopole pairs annihilate, leaving always only a few per horizon [Barriola and Vilenkin, 1989, Bennett and Rhie, 1990].

One assumes, like in the CDM model, that the matter content of the universe is dominated by cold dark matter with $1 = \Omega \approx \Omega_{CDM}$. The large scale structure induced by global monopoles seems to look quite similar to the texture scenario. Galaxy formation starts relatively early, the galaxy correlation function and large scale velocity field are in agreement with observations. The CMB fluctuations are similar to those obtained for the texture scenario (see Chapter 5) [Bennett and Rhie, 1992].

Recent investigations [Pen et al., 1993] claim that all models with global defects and CDM miss some large scale power on scales $l \geq 20$Mpc.

Chapter 2

Gauge Invariant Perturbation Theory

For linear cosmological perturbation theory to apply, we must assume that the spacetime manifold $\mathcal{M}$ with metric g and energy momentum tensor T of "the real universe" is somehow close to a Friedmann universe, i.e., the manifold $\mathcal{M}$ with a Robertson–Walker metric $\bar{g}$ and a homogeneous and isotropic energy momentum tensor $\overline{T}$. It is an interesting, non–trivial unsolved problem how to construct $\bar{g}$ and $\overline{T}$ from the physical fields g and T in practice. There are two main difficulties: Spatial averaging procedures depend on the choice of a hyper–surface and do not commute with derivatives, so that the averaged fields $\bar{g}$ and $\overline{T}$ will in general not satisfy Einstein's equations. Furthermore, averaging is in practice impossible over super–horizon scales.

We now assume that there exists an averaging procedure which leads to a Friedmann universe with spatially averaged tensor fields $\overline{Q}$, such that the deviations $(T_{\mu\nu} - \overline{T}_{\mu\nu})/\max_{\{\alpha\beta\}}\{|\overline{T}_{\alpha\beta}|\}$ and $(g_{\mu\nu} - \overline{g}_{\mu\nu})/\max_{\{\alpha\beta\}}\{\overline{g}_{\alpha\beta}\}$ are small, and $\bar{g}$ and $\overline{T}$ satisfy Friedmann's equations. Let us call such an averaging procedure 'admissible'. There might also be another admissible averaging procedure (e.g. over a different hyper–surface) leading to a slightly different Friedmann background $(\bar{g}_2, \overline{T}_2)$. In this case, the averaging procedures are isomorphic via an isomorphism ϕ on $\mathcal{M}$ which is close to unity:

$$
\begin{aligned}
\bar{g}_2 &= \phi_* \bar{g} \\
\overline{T}_2 &= \phi_* \overline{T} \ ,
\end{aligned}
$$

where $\phi_*(Q)$ denotes the pushforward of the tensor field Q under ϕ. The isomorphism ϕ can be represented as the infinitesimal flow of a vector field X, $\phi = \phi^X_\epsilon$. Remember the definition of the flow: For the integral curve $\gamma_x(s)$ of X with starting point x, i.e., $\gamma_x(s = 0) = x$ we have $\phi^X_s(x) = \gamma_x(s)$. In terms of the vector field X, the relation of the two averaging procedures is given by

$$
\bar{g}_2 = \phi_* \bar{g} = \bar{g} - \epsilon L_X \bar{g} + \mathcal{O}(\epsilon^2) \tag{2.1}
$$

$$
\overline{T}_2 = \phi_* \overline{T} = \overline{T} - \epsilon L_X \overline{T} + \mathcal{O}(\epsilon^2) \ . \tag{2.2}
$$

In the context of cosmological perturbation theory, the isomorphism ϕ is called a gauge transformation. And the choice of a background $(\bar{g}, \overline{T})$ corresponds to a choice of gauge. The above relation is of course true for all averaged tensor fields $\overline{Q}$ and $\overline{Q}_2$. Separating Q into a background component and a small perturbation, $Q = \overline{Q} + \epsilon Q^{(1)} = \overline{Q}_2 + \epsilon Q^{(2)}$, we obtain the following relation:

$$
Q^{(2)} = Q^{(1)} + L_X \overline{Q} \ . \tag{2.3}
$$

Since each vector field X generates a gauge transformation $\phi = \phi_\epsilon^X$, we can conclude that only perturbations of tensor fields with $L_X\overline{Q} = 0$ for all vector fields X, i.e., with vanishing (or constant) 'background contribution' are gauge invariant. This simple result is sometimes referred to as the 'Stewart Lemma' [Stewart and Walker, 1977].

The gauge dependence of perturbations has caused many controversies in the literature, since it is often difficult to extract the physical meaning of gauge dependent perturbations, especially on super–horizon scales. This has led to the development of gauge invariant perturbation theory which we are going to present in this chapter. The advantage of the gauge–invariant fromalism is that the variables used have simple geometric and physical meanings and are not plagued by gauge modes. Although the derivation requires somewhat more work, the final system of perturbation variables is usually simple and well suited for numerical treatment. We shall also see, that on subhorizon scales, the gauge invariant matter perturbations variables approach the usual, gauge dependent ones, and one of the geometrical variables approaches the Newtonian potential, so that the Newtonian limit can very easily be performed.

There are two review articles on the subject [Kodama and Sasaki, 1984, Mukhanov et al., 1992]. Our treatment will be in some way complementary. Collisionless particles, photon propagation and seeds (Sections 2.3, 2.4, 2.5) are not discussed in these reviews. On the other hand, we do not investigate perturbation theory of scalar fields which is presented extensively in the other publications, since it is needed to treat fluctuations induced by inflation. We want to discuss perturbations induced by topological defects. Here, the scalar field itself is a small perturbation on a matter or radiation dominated background. This issue will become more clear later.

First we note that since all relativistic equations are covariant (i.e. can be written in the form $Q = 0$ for some tensor field Q), it is always possible to express the corresponding perturbation equations in terms of gauge invariant variables.

2.1 Gauge Invariant Perturbation Variables

In this section we introduce gauge invariant variables which describe the perturbations of the metric and the energy momentum tensor in a Friedmann background.

There are two main approaches to find gauge invariant quantities: One possibility is to make full use of the above statement that tensor fields with vanishing background contribution are gauge invariant and use them to define gauge invariant perturbation variables. Examples are the Weyl tensor, the acceleration of the energy velocity field, anisotropic stresses, shear and vorticity of the energy velocity field. This covariant approach was originally proposed by Hawking [1966] and later extended by Ellis and Bruni [1989], Ellis et al. [1989], Hwang and Vishniac [1989], Bruni et al. [1992b], Dunsby [1992] and others.

The other possibility is to arbitrarily parametrize the perturbations of the metric, the energy momentum tensor, the distribution function, a scalar field... and discuss the transformation properties of these gauge dependent variables under gauge transformations. One can then combine them into gauge invariant quantities. This way was initiated by Gerlach and Sengupta [1978] and Bardeen [1980] and later continued by Kodama and Sasaki [1984],Kasai and Tomita [1986], Durrer [1988], Durrer and Straumann [1988], Durrer [1989a], Durrer [1990], Mukhanov et al. [1992] and others. In this approach one usually performs a harmonic analysis and the gauge invariant perturbation variables found in this way may be acausal (e.g., they may require inverse Laplacians over spatial hyper–surfaces).

As in the second approach we divide the perturbations into scalar, vector and tensor contributions, but we do not perform the harmonic analysis. Our perturbation variables are thus space (and time) dependent functions and not just amplitudes of harmonics. In order to obtain unique solutions, we require (for non–positive spatial curvature) all the perturbation variables to vanish at infinity. I partly relate the two approaches by originally performing the second, but in many cases identify gauge invariant variables with tensor fields which vanish in an unperturbed Friedmann universe. It could not be avoided to use a somewhat extensive vocubulary for all the variables used in this text. To help the reader not to get lost, I have included a glossary in Appendix B.

Like in Chapter 1, the unperturbed line element is given by

$$ds^2 = a^2(-dt^2 + \gamma_{ij}dx^i dx^j) \ ,$$

where $\boldsymbol{\gamma}$ is the metric of a three space with constant curvature $k = \pm 1$ or 0 and overdot denotes derivatives w.r.t conformal time t. We first define scalar perturbations of the lapse function α, the shift vector $\boldsymbol{\beta}$ and the 3-metric $\boldsymbol{g}$ of the slices of constant time[1] by

$$\alpha = a(1+A) \tag{2.4}$$

$$\boldsymbol{\beta} = \alpha B^{|i}\partial_i \tag{2.5}$$

$$\boldsymbol{g} = a^2[(1+2H_L - (2/3)l^2\triangle H_T)\gamma_{ij} + 2l^2 H_{T|ij}]dx^i dx^j \ . \tag{2.6}$$

We denote 3–dimensional vector and tensor fields by bold face letters; $|$ and $\triangle$ denote the covariant derivative and Laplacian with respect to the metric $\boldsymbol{\gamma}$. The variables $A,\ B,\ H_L$, and H_T are arbitrary functions of space and time. To keep them dimensionless, we have introduced a length l which, in applications, will be chosen to be the typical scale of the perturbations so that, e.g., $\mathcal{O}(B) \approx \mathcal{O}(lB_{|i})$.

By choosing the metric perturbations in the form (2.4–2.6), we restrict ourselves to scalar type perturbations, but we do not perform a harmonic analysis. Vector perturbations of the geometry are of the form

$$\beta = aB^i\partial_i \tag{2.7}$$

$$\boldsymbol{g} = a^2[\boldsymbol{\gamma}_{ij} + lH_{i|j} + lH_{j|i}]dx^i dx^j \ , \tag{2.8}$$

where $B^i\partial_i$ and $H^i\partial_i$ are divergencefree vector fields which vanish at infinity.

Tensor perturbations are given by

$$\boldsymbol{g} = a^2[\boldsymbol{\gamma}_{ij} + 2H_{ij}]dx^i dx^j \ . \tag{2.9}$$

Here H_{ij} is a traceless, divergencefree, symmetric tensor field on the slices of constant time.

Writing the 4–dimensional metric in the form

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + a^2 h_{\mu\nu} \ , \tag{2.10}$$

the above definitions of the perturbation variables for scalar perturbations yield

$$h^{(S)} = -2A(dt)^2 + 2lB_{,i}\,dtdx^i + 2(H_L - \frac{l^2}{3}\triangle H_T)\gamma_{ij}dx^i dx^j + 2l^2 H_{T|ij}dx^i dx^j \ . \tag{2.11}$$

For vector and tensor perturbations one obtains correspondingly

$$h^{(V)} = 2B_i dx^i dt + l(H_{i|j} + H_{j|i})dx^i dx^j \quad \text{and} \tag{2.12}$$

$$h^{(T)} = 2H_{ij}dx^i dx^j \ . \tag{2.13}$$

[1] A short description of the 3+1 formalism of general relativity is given in Appendix A

From (2.4), (2.5), (2.6), (2.7) and (2.8), one can calculate the 3–dimensional Riemann scalar and the extrinsic curvature given by $K_{ij} = -n_{i;j}$. Palatini's identity (see Straumann [1985]) yields the general formula

$$\delta R(g + \delta g) = (\delta g_i^j)|_{|j}^{|i} - \triangle \delta g_i^i \ .$$

This leads to the following result for scalar perturbations

$$\delta \boldsymbol{R} = -4a^{-2}(\triangle + 3k)\mathcal{R} \ , \qquad \mathcal{R} = H_L - \frac{l^2}{3}\triangle H_T \ , \tag{2.14}$$

$$K^{(aniso)} = -[2\dot{a}l^2(H_{T|ij} - \frac{1}{3}\gamma_{ij}\triangle H_T) + a(l\sigma_{|ij} - \frac{l}{3}\gamma_{ij}\triangle\sigma)]dx^i dx^j \ , \quad l\sigma = l^2\dot{H}_T - lB \ . \tag{2.15}$$

For vector perturbations one obtains

$$\delta \mathbf{R}^V = 0 \tag{2.16}$$

$$K^{V(aniso)} = -[\dot{a}(H_{i|j} + H_{j|i}) + \frac{a}{2}(\sigma_{i|j} + \sigma_{j|i})]dx^i dx^j \quad \text{with} \ \ \sigma_i = l\dot{H}_i - B_i \ . \tag{2.17}$$

$K^{(aniso)}$ is the traceless contribution to the extrinsic curvature of the slices of constant time or, what amounts to the same thing, the shear of the normal to the slices.

We now investigate the gauge transformation properties of the variables defined above. We introduce the vector field describing the gauge transformation by

$$X = T\partial_t + L^i\partial_i \ .$$

Using simple identities like $L_X(df) = d(L_X f)$ and $(\boldsymbol{L_X\gamma})_{ij} = \boldsymbol{X}_{i|j} + \boldsymbol{X}_{j|i}$, we obtain the Lie derivative of the unperturbed metric

$$L_X\bar{g} = a^2[-2(\frac{\dot{a}}{a}T + \dot{T})dt^2 + 2(\dot{L}_i - T_{|i})dx^i dt + (2\frac{\dot{a}}{a}T\gamma_{ij} + L_{i|j} + L_{j|i})dx^i dx^j] \ .$$

For scalar type gauge transformations, L^i is of the form $L^i = lL^{|i}$ for some scalar function L. Inserting this above and comparing with the parametrization of the perturbed metric $h^{(S)}$, eq. (2.11), we find the following transformation laws:

$$A \rightarrow A + \frac{\dot{a}}{a}T + \dot{T} \tag{2.18}$$

$$lB \rightarrow lB - T + l\dot{L} \tag{2.19}$$

$$H_L \rightarrow H_L + \frac{\dot{a}}{a}T + (l/3)\triangle L \tag{2.20}$$

$$l^2 H_T \rightarrow l^2 H_T + lL \ . \tag{2.21}$$

(2.22)

This yields the transformation properties for $\mathcal{R}$ and σ

$$\mathcal{R} \rightarrow \mathcal{R} + \frac{\dot{a}}{a}T \ ; \qquad \sigma \rightarrow \sigma + T \ . \tag{2.23}$$

The following scalar perturbation variables, the so called Bardeen potentials, are thus gauge invariant (see [Bardeen, 1980, Kodama and Sasaki, 1984] and [Durrer and Straumann, 1988]):

$$\Phi = \mathcal{R} - (\dot{a}/a)l\sigma \tag{2.24}$$

$$\Psi = A - (\dot{a}/a)l\sigma - l\dot{\sigma} \ . \tag{2.25}$$

We shall later see that Ψ is a relativistic analog of the Newtonian potential.

For vector type gauge transformations, where $T = 0$ and L^i is a divergence free vector field on the hyper–surfaces of constant time, one immediately sees that σ_i is gauge-invariant.

Tensor perturbations are of course always gauge invariant (there are no tensor type gauge transformations).

Gauge transformations remove two scalar and one vector degrees of freedom, so that geometrical perturbations are fully characterized by six degrees of freedom which we parametrize in the gauge invariant variables Ψ, Φ, σ_i and H_{ij}.

Some combinations of these quantities have simple geometric interpretations. From the Stewart Lemma, we know that the Weyl tensor, which vanishes in an unperturbed Friedmann universe, is gauge invariant. A somewhat lengthy calculation shows that the electric and magnetic components of the Weyl tensor, are given by

$$\begin{aligned} E_{ij} &\equiv \frac{1}{a^2}C_{i0j0} \\ &= \frac{1}{2}[(\Phi - \Psi)_{|ij} - \frac{1}{3}\triangle(\Phi - \Psi)\gamma_{ij} + \dot{\sigma}_{ij}] \end{aligned} \tag{2.26}$$

$$\begin{aligned} B_{ij} &\equiv \frac{-1}{2a^2}\epsilon_{ilm}C_{lmj0} \\ &= \epsilon_{(ilm}[\sigma_{j)m|l} - \frac{1}{3}\gamma_{j)l}\sigma^{s}_{m|s}] \end{aligned} \tag{2.27}$$

where $\quad \sigma_{lm} = \sigma^{(V)}_{lm} + \dot{H}_{lm}$ and (i..j) denotes symmetrization in i and j

is the sum of vector and tensor contributions to the extrinsic curvature (this result is obtained in Bruni et al. [1992b]).

We now discuss perturbations of the energy momentum tensor. We define the perturbed energy density ρ and energy velocity field u as the timelike eigenvalue and eigenvector of the energy momentum tensor (note that, apart from symmetry, we do not make any assumptions on the nature of $T_\mu{}^\nu$):

$$T_\mu{}^\nu u^\mu = -\rho u^\nu \ , \ \ u^2 = -1 \ .$$

We then define the perturbations in the energy density and energy velocity field by

$$\rho = \overline{\rho}(1 + \delta) \ , \tag{2.28}$$

$$u = u^0\partial_t + u^i\partial_i \ ; \tag{2.29}$$

u^0 is fixed by the normalization condition, $u^0 = a^{-1}(1 - A)$. In the 3–space orthogonal to u we define the stress tensor by

$$\tau^{\mu\nu} \equiv P^\mu{}_\alpha P^\nu{}_\beta T^{\alpha\beta} \ , \tag{2.30}$$

where $P = u \otimes u + g$ is the projection onto the sub–space of $T_q\mathcal{M}$ normal to $u(q)$. One obtains

$$\tau_0^0 = \tau_i^0 = \tau_0^i = 0 \ .$$

The perturbations of pressure and anisotropic stresses can be parametrized by

$$\tau_i{}^j = \bar{p}[(1+\pi_L)\delta_i^j + \pi_i{}^j] \quad , \text{with} \ \ \pi_i^i = 0 \ . \tag{2.31}$$

Again, we decompose these perturbations into scalar, vector and tensor contributions. For scalar perturbations one can set

$$u^0 = (1-A) \ , \ \frac{u^i}{u^0} = -lv^{,i} \quad \text{and } \pi^i{}_j = l^2(\pi^{|i}{}_{|j} - \frac{1}{3}\triangle\pi\delta^i{}_j) \ .$$

As before, the Lie derivatives of the unperturbed quantities $\bar{\rho}$, $\bar{u} = a^{-1}\partial_t$, $\bar{\tau} = \bar{p}\partial_i \otimes dx^i$ determine the transformation laws of the perturbation variables. One finds

$$L_X\bar{\rho} \ = \ T\dot{\bar{\rho}} = -3(1+w)\frac{\dot{a}}{a}T\bar{\rho} \tag{2.32}$$

$$L_X\bar{u} \ = \ [X,\bar{u}] = -a^{-1}[(\frac{\dot{a}}{a}T + \dot{T})\partial_t - L^i\partial_i] \tag{2.33}$$

$$\mathrm{L}_X\bar{\tau} \ = \ L_X\bar{p}\partial_i \otimes dx^i = T\dot{\bar{p}}\partial_i \otimes dx^i - 3\frac{c_s^2}{w}(1+w)\frac{\dot{a}}{a}T\bar{p}\partial_i \otimes dx^i \ , \tag{2.34}$$

where we have used the background energy equation,

$$\dot{\bar{\rho}} = -3(1+w)\frac{\dot{a}}{a}\bar{\rho} \ ,$$

and the definitions

$$w = \bar{p}/\bar{\rho} \ , \quad c_s^2 = \dot{\bar{p}}/\dot{\bar{\rho}} \ .$$

From eqs. (2.32) – (2.34) we obtain the transformation laws

$$\begin{aligned} \delta \ &\rightarrow \ \delta - 3(1+w)\frac{\dot{a}}{a}T \\ lv \ &\rightarrow \ lv + l\dot{L} \\ \pi_L \ &\rightarrow \ \pi_L - 3\frac{c_s^2}{w}(1+w)\frac{\dot{a}}{a}T \\ \pi \ &\rightarrow \ \pi \ . \end{aligned}$$

A first gauge invariant variable is therefore

$$\Pi = \pi \ ,$$

the scalar potential for anisotropic stresses. The other gauge invariant combination which can be constructed from matter variables alone is

$$? \ = \pi_L - (c_s^2/w)\delta \ .$$

Defining an entropy flux S^μ of the perturbations in the sense of small deviations from thermal equilibrium [Straumann, 1985, Appendix B], one finds for the entropy production rate of the perturbation [Durrer and Straumann, 1988]

$$S^\mu{}_{;\mu} = 3(\dot{a}/a^2T)? \ ,$$

where T denotes the temperature of the system. The variable ? thus measures the entropy production rate.

We now split the covariant derivative of the velocity field in the usual way into acceleration, expansion, shear and vorticity:

$$u_{\mu;\nu} = a_\mu \otimes u_\nu + P_{\mu\nu}\theta + \sigma_{\mu\nu} + \omega_{\mu\nu} \ .$$

Here

$$a = \nabla_u u \quad \text{is the acceleration,}$$

$$P_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$$

denotes the projection onto the sub–space of tangent space normal to u, $\theta = u^\mu_{;\mu}$ is the expansion,

$$\sigma_{\mu\nu} = \frac{1}{2}P^\lambda_\mu P^\rho_\nu(u_{\lambda;\rho} + u_{\rho;\lambda}) - P_{\mu\nu}\theta$$

is the shear of the vector field u and

$$\omega_{\mu\nu} = \frac{1}{2}P^\lambda_\mu P^\rho_\nu(u_{\lambda;\rho} - u_{\rho;\lambda})$$

is the vorticity. A short calculation shows $\omega_{\mu\nu} = 0$, $\sigma_{0\mu} = 0$ and

$$\sigma_{ij} = \frac{a}{l}V_{|ij} \quad \text{with} \quad lV = lv - l^2\dot{H}_T \ . \tag{2.35}$$

We choose V as gauge invariant scalar velocity variable. For the acceleration one obtains $a^0 = 0$ and

$$a_i \equiv \mathcal{A}_{,i} = \Psi_{,i} + \dot{V}_i + \frac{\dot{a}}{a}V_i \ ,$$

which shows for the first time the analogy of Ψ with the Newtonian potential.

There are several different useful choices of gauge invariant density perturbation variables:

$$D_s = \delta + 3(1+w)(\dot{a}/a)l\sigma \tag{2.36}$$

$$D_g = \delta + 3(1+w)\mathcal{R} = D_s + 3(1+w)\Phi \tag{2.37}$$

$$D = D_s + 3(1+w)(\dot{a}/a)lV \ . \tag{2.38}$$

For a physical interpretation of these variables note that

$$D_{,i} = P^\mu_i \rho_{,\mu} \tag{2.39}$$

$$(D_s)_{(ij)} + 3(1+w)\Psi_{(ij)} = P^\mu_i P^\nu_j \rho_{;\mu\nu} \tag{2.40}$$

$$(D_g)_{(ij)} + 3(1+w)(\Psi - \Phi)_{(ij)} = P^\mu_i P^\nu_j \rho_{;\mu\nu} \ . \tag{2.41}$$

Here we have set $S_{(ij)} \equiv S_{|ij} - (1/3)\triangle S\gamma_{ij}$ for an arbitrary scalar quantity S.
Therefore, D and $D_s + 3(1+w)\Psi$ are potentials for the first and second "spatial derivatives" of the energy density.

For vector perturbations only

$$u^i = \frac{1}{a}v^i \quad \text{and } \pi^i{}_j = \frac{l}{2}(\pi^i_{|j} + \pi_j{}^{|i})$$

survive. Vector type gauge transformations yield the transformation laws

$$\begin{aligned} v^i &\rightarrow v^i + \dot{L}^i \\ \pi^i &\rightarrow \pi^i \ . \end{aligned}$$

In addition to the anisotropic stress potential $\pi^i \equiv \Pi^i$, two interesting gauge invariant quantities are the shear and vorticity of the vector field u:

$$u_{i;j} + u_{j;i} = a(V_{i|j} + V_{j|i}) \ , \qquad \text{with} \quad V^i = v^i - l\dot{H}^i \ , \tag{2.42}$$

$$u_{i;j} - u_{j;i} = a(\Omega_{i|j} - \Omega_{j|i}) \ , \qquad \text{with} \quad \Omega^i = v^i - B^i \ . \tag{2.43}$$

For tensor perturbations the only variable $\pi^i{}_j \equiv \Pi^i{}_j$ is of course gauge invariant.

We now show that for perturbations which are small compared to the horizon distance, l_H, in a generic gauge the gauge invariant combinations V and $D_{(.)}$ approach the original v and δ. Let us choose our free length scale l to be the typical size of a given perturbation. From the above equation it is then clear that for $l \ll l_H = t$, $D \approx D_s$.

Noting that perturbations of the Einstein tensor are given by second derivatives of the metric perturbations (Palatini's identity, see e.g. Straumann [1985]), we obtain the following order of magnitude equation:

$$\mathcal{O}(\frac{\delta T}{T})\mathcal{O}(8\pi G T_{\mu\nu}) = \mathcal{O}(t^{-2}\frac{\delta g}{g} + (lt)^{-1}\frac{\delta g}{g} + l^{-2}\frac{\delta g}{g}) \ . \tag{2.44}$$

Using Friedmann's equation

$$\mathcal{O}(8\pi G T_{\mu\nu}) = \mathcal{O}(\dot{a}/a)^2 = \mathcal{O}(1/t^2) \equiv \mathcal{O}(1/l_H^2)$$

this yields

$$\mathcal{O}(\frac{\delta T}{T}) = \mathcal{O}(\frac{\delta g}{g} + (l_H/l)\frac{\delta g}{g} + (l_H/l)^2\frac{\delta g}{g}) \ . \tag{2.45}$$

On sub–horizon scales, $l_H \gg l$ the metric perturbations are thus generically much smaller than the matter perturbations and the difference between the gauge invariant quantities V, $D_{(.)}$, V^i, $|Om^i$ and v, δ, v^i becomes negligible.

2.2 The Basic Equations

In this section we write down the perturbation equations resulting from Einstein's equation, and energy momentum "conservation" in a form which will be convenient later. All these equations are most easily derived using the 3+1 formalism of gravity (see Appendix A) as we shall demonstrate for a few examples.

The perturbations of Einstein's equations and energy momentum conservation can be expressed in terms of the gauge invariant variables defined above. (A simple derivation of the equations for scalar perturbations is given in Durrer and Straumann [1988].) To simplify the notation we now drop the bar over background density and pressure.

A) Constraint equations

$$4\pi G a^2 \rho D = -(\triangle + 3k)\Phi \tag{2.46}$$

$$4\pi G a^2 (\rho + p) l V = (\dot{a}/a)\Psi - \dot{\Phi} \ . \tag{2.47}$$

B) Dynamical equations

$$-8\pi G a^2 p l^2 \triangle \Pi = \triangle(\Phi + \Psi) \tag{2.48}$$

$$\begin{aligned} 8\pi G a^2 p(? + (c_s^2/w)D_g - (2/3)l^2\triangle\Pi) &= (\dot{a}/a)\{\dot{\Psi} - [(1/a)(\frac{a^2\Phi}{\dot{a}})^{\cdot}]^{\cdot}\} + \\ \{2a(\dot{a}/a^2)^{\cdot} + 3(\dot{a}/a^2)^2\}[\Psi - 1/a(\frac{a^2\Phi}{\dot{a}})^{\cdot}] \ . \end{aligned} \tag{2.49}$$

Since vector and tensor type perturbations are not treated in Durrer and Straumann [1988] and Durrer [1990], we present an explicit derivation of the vector perturbation equations, making use of the 3+1 formalism of general relativity (see Appendix A and Durrer and Straumann [1988]). For vector perturbations, the unit normal to the equal time slices is given by

$$n = \alpha^{-1}(\partial_t - \boldsymbol{\beta}) = a^{-1}(\partial_t - B^i\partial_i) \ .$$

We now decompose the energy momentum tensor in the form

$$T = \epsilon n \otimes n + n \otimes \boldsymbol{S} + \boldsymbol{S} \otimes n + \boldsymbol{T}$$

where, as before, bold type vector and tensor fields, $\boldsymbol{S}$ and $\mathbf{T}$, are tangent to the equal time hyper–surfaces. Using the Gauss–Codazzi–Mainardi formulas to express the four dimensional curvature in terms of the three dimensional and the second fundamental form, one can derive the following 3+1 split of Einsteins equations (Appendix A4):

$$\boldsymbol{R} + (tr\boldsymbol{K})^2 - tr(\boldsymbol{K}^2) = 16\pi G\epsilon \tag{2.50}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{K} - \boldsymbol{\nabla} tr(\boldsymbol{K}) = 8\pi G\boldsymbol{S} \tag{2.51}$$

$$\begin{aligned} &\partial_t\boldsymbol{K} - \boldsymbol{L}_{\boldsymbol{\beta}}\boldsymbol{K} + \mathbf{Hess}(\alpha) - \\ &-\alpha[\mathbf{Ricci} - 2\boldsymbol{K}\cdot\boldsymbol{K} + (tr\boldsymbol{K})\boldsymbol{K}] = -4\pi G\alpha[2\boldsymbol{T} + \boldsymbol{g}(\epsilon - tr\boldsymbol{T})] \ , \end{aligned} \tag{2.52}$$

where $\boldsymbol{K} = \frac{1}{2\alpha}(\boldsymbol{L}_{\boldsymbol{\beta}}\boldsymbol{g} - \partial_t\boldsymbol{g})$ is the second fundamental form.

For vector perturbations only the second constraint equation and the traceless part of the dynamical equation contribute. From our definiton of vector perturbations of the energy momentum tensor, one finds

$$\boldsymbol{S} = a(\rho + p)(v^i - B^i)\partial_i \ .$$

In what follows we use the notation $X_{ij}^{(V)}$ for the symmetric, traceless tensor constructed from the vector field X, i.e., $X_{ij}^{(V)} = \frac{1}{2}(X_{i|j} + X_{j|i})$. We then have

$$\boldsymbol{g} = a^2(\gamma_{ij} + 2lH_{ij}^{(V)})dx^i dx^j$$

and with (2.17) the second fundamental form is given by

$$\boldsymbol{K} = -a[\frac{\dot{a}}{a}(\gamma_{ij} + 2lH_{ij}^{(V)}) + \sigma_{ij}^{(V)}]dx^i dx^j \ .$$

This leads to

$$\boldsymbol{\nabla} \cdot \boldsymbol{K} = -\frac{1}{2a}(\triangle\sigma^i + \sigma_j{}^{|ij})\partial_i = -\frac{1}{2a}(\triangle + 2k)\sigma^i\partial_i$$

The constraint equation (2.51) thus results in

$$-\frac{1}{2}(\triangle + 2k)\sigma^i = 8\pi G(\rho + p)a^2\Omega^i \ . \tag{2.53}$$

Let us now proceed to the dynamical equation. Up to first order we obtain the following expressions for the terms in (2.52):

$$\begin{aligned}
\boldsymbol{L_\beta K} &= -2\dot{a}B_{ij}^{(V)}dx^i dx^j \\
\boldsymbol{K}^2 &= [(\frac{\dot{a}}{a})^2(\gamma_{ij} + 2lH_{ij}^{(V)}) + 2\frac{\dot{a}}{a}\sigma_{ij}^{(V)}]dx^i dx^j \\
\boldsymbol{Ricci(g)} &= 2k(\gamma_{ij} + 2lH_{ij}^{(V)})dx^i dx^j \\
8\pi G\boldsymbol{T} &= 8\pi G a^2 p(\gamma_{ij} + 2lH_{ij}^{(V)} + l\pi_{ij}^{(V)})dx^i dx^j \\
4\pi G\boldsymbol{g}(\epsilon - tr\boldsymbol{T}) &= 4\pi G(\rho - 3p)a^2(\gamma_{ij} + 2lH_{ij}^{(V)})dx^i dx^j \\
\partial_t\boldsymbol{K} &= -[\ddot{a}(\gamma_{ij} + 2lH_{ij}^{(V)}) + \dot{a}(\sigma_{ij}^{(V)} + 2l\dot{H}_{ij}^{(V)}) + a\dot{\sigma}_{ij}^{(V)}]dx^i dx^j
\end{aligned}$$

The result for $\boldsymbol{Ricci(g)}$ is again easily derived using Palatini's identity [Straumann, 1985].

With the help of the background relation

$$4\pi G a^2(\rho - p) = \frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2 + 2k \ ,$$

equation (2.52) then yields

$$\dot{\sigma}_{ij}^{(V)} + 2(\frac{\dot{a}}{a})\sigma_{ij}^{(V)} = 8\pi G a^2 p l \Pi_{ij}^{(V)} \ .$$

Since we require $\lim_{r\to\infty}\sigma_i = \lim_{r\to\infty}\Pi_i = 0$, the tensor fields, $\sigma_{ij}^{(V)}$ and $\pi_{ij}^{(V)}$ uniquely determine the corresponding vector fields:

$$\dot{\sigma}_i + 2(\frac{\dot{a}}{a})\sigma_i = 8\pi G a^2 p l \Pi_i \ , \tag{2.54}$$

$$\text{or} \quad (a^2\sigma_i)^{\cdot} = 8\pi G a^4 p l \Pi_i \ . \tag{2.55}$$

In the absence of anisotropic stresses, vector anisotropies in the extrinsic curvature thus decay like $1/a^2$.

In a similar way one finds the tensor perturbation equation

$$\ddot{H}_{ij} + 2\frac{\dot{a}}{a}\dot{H}_{ij} + (2k - \triangle)H_{ij} = 8\pi G a^2 p\Pi_{ij} \ . \tag{2.56}$$

This is a wave equation with source term. It describes the creation, propagation and damping of gravitational waves in a Friedmann background .

C) Conservation equations

The energy and the momentum conservation equations of each independent type of matter (i.e. each matter component which does not interact other than gravitationally with the rest) yields the following equations of motion for the scalar perturbation variables D_α and V_α, where the index $_\alpha$ denotes the different matter components (e.g. radiation, dark matter, baryons ...):

$$\begin{aligned} \dot{D}_\alpha - 3w_\alpha(\dot{a}/a)D_\alpha &= (\triangle + 3k)[(1+w_\alpha)lV_\alpha + 2(\dot{a}/a)w_\alpha l^2\Pi_\alpha] \\ &\quad +3(1+w_\alpha)4\pi G a^2(\rho+p)(lV - lV_\alpha) \ , \end{aligned} \tag{2.57}$$

$$l\dot{V}_\alpha + (\dot{a}/a)lV_\alpha = \frac{c_\alpha^2}{1+w_\alpha}D_\alpha + \frac{w_\alpha}{1+w_\alpha}?_\alpha + \Psi + 2/3(\triangle + 3k)\frac{w_\alpha}{1+w_\alpha}l^2\Pi_\alpha \ . \tag{2.58}$$

The total perturbations are defined as the sums:

$$\rho D = \sum_\alpha \rho_\alpha D_\alpha \ , \quad (\rho+p)V = \sum_\alpha (\rho_\alpha + p_\alpha)V_\alpha \ . \tag{2.59}$$

The adiabatic sound speed, c_α and enthalpy, w_α are

$$c_\alpha^2 = \dot{p_\alpha}/\dot{\rho_\alpha} \quad , \qquad w_\alpha = p_\alpha/\rho_\alpha \ \text{ and } \ w = \frac{\sum_\alpha p_\alpha}{\sum_\alpha \rho_\alpha} \equiv p/\rho \ , \quad c_s^2 = \dot{p}/\dot{\rho}.$$

The corresponding equations for interacting matter components are derived in [Kodama and Sasaki, 1984].

We shall later also use equations (2.57) and (2.58) for a one component fluid in terms of the density perturbation variable D_g and V. One easily finds

$$\dot{D}_g + 3(c_s^2 - w)(\dot{a}/a)D_g - (1+w)l\triangle V + 3w(\dot{a}/a)? = 0 \tag{2.60}$$

$$l\dot{V} + (\dot{a}/a)(1-3c_s^2)lV = (\Psi - 3c_s^2\Phi) + \frac{c_s^2}{1+w}D_g + \frac{w}{1+w}[? + \frac{2}{3}(\triangle + 3k)l^2\Pi] \tag{2.61}$$

Again we derive the conservation equation for for vector perturbations in some detail. We start with the spatial part of the 3+1 split of $T^{\mu\nu}{}_{;\nu} = 0$ (see Appendix A.3):

$$\frac{1}{\alpha}(\partial_t - \boldsymbol{L_\beta})\boldsymbol{S} = -\boldsymbol{\nabla}(\ln\alpha)\epsilon + 2\boldsymbol{KS} + (tr\boldsymbol{K})\boldsymbol{S} - \frac{1}{\alpha}\boldsymbol{\nabla}\cdot(\alpha\boldsymbol{T}) \ . \tag{2.62}$$

All the terms in this equation are readily calculated. One obtains for each matter component (suppressing the index α)

$$\begin{aligned} \boldsymbol{S} &= (\rho+p)a^{-1}(v^i - B^i)\partial_i \\ \frac{1}{\alpha}\boldsymbol{\nabla}(\alpha\boldsymbol{T}) &= \boldsymbol{\nabla}\cdot\boldsymbol{T} = \frac{1}{2a^2}p(\triangle l\Pi^i + 2kl\Pi^i)\partial_i \end{aligned}$$

$$
\begin{aligned}
2\boldsymbol{K}\boldsymbol{S} &= -2\frac{\dot a}{a^3}(\rho+p)(v^i-B^i)\partial_i \\
(\mathrm{tr}\boldsymbol{K})\boldsymbol{S} &= -3\frac{\dot a}{a^3}(\rho+p)(v^i-B^i)\partial_i \\
\boldsymbol{L}_\beta\boldsymbol{S} &= 0 \quad \text{(in first order)} \\
\partial_t\boldsymbol{S} &= [-\dot a a^{-2}(\rho+p)(v^i-B^i)+a^{-1}(\dot\rho+\dot p)(v^i-B^i)+a^{-1}(\rho+p)(\dot v^i-\dot B^i)]\partial^i \\
&= a^{-1}(\rho+p)[\dot v_i-\dot B_i-(4+3c_s^2)\frac{\dot a}{a}(v_i-B_i)]\partial_i \ .
\end{aligned}
$$

For the last equality sign, we made use of the background equation $\dot\rho+\dot p=-3(\dot a/a)(1+c_s^2)(\rho+p)$. Inserting all these results, eq. (2.62) becomes

$$\dot\Omega^i+(1-3c_s^2)\frac{\dot a}{a}\Omega^i=-\frac{p}{2(\rho+p)}(\triangle+2k)l\Pi^i \ . \tag{2.63}$$

If there are no sources present, $\Pi^i=0$, and if $c_s^2=w=$ constant, the amplitude of the vorticity is proportional to $a^{3c_s^2-1}$. In comparison to the expansion velocity, $\dot a/a$, the vorticity behaves like

$$|\Omega|/\frac{\dot a}{a}\propto a^{0.5(9w-1)} \tag{2.64}$$

(as long as curvature is negligible, i.e. for $k\ll 1/t^2$). Especially, an initial vorticity in a radiation dominated universe ($w=1/3$) grows relative to the expansion velocity in the course of expansion.

In addition to Einstein's field equation and the conservation equations which are of course a consequence of them, we have to add matter equations to fully describe the system. If the fluid description is justified, these can be given in the form

$$? = ?(D,V) \ , \qquad \Pi=\Pi(D,V) \ .$$

In Section 3.1 we discuss, for illustration, the simplest possibility, adiabatic perturbations of an ideal fluid, where we just set $\Pi = ? = 0$

There are however situations where the description of matter as a fluid is not sufficient. One then has to resort to the matter equations of more fundamental quantities, e.g. scalar fields and/or gauge fields.

2.3 Collisonless Matter

In this section we discuss another approximate description of matter which can be used for weakly interacting particles like photons in a recombined universe or massive neutrinos which might constitute the dark matter. Here, the basic quantity is the one–particle distribution function f which lives on the mass bundle,

$$P_m=\{(p,x)\in T\mathcal{M}|\ g(x)(p,p)=-m^2\} \ .$$

When collisions can be neglected, the matter equation is the one particle Liouville equation (for a thorough treatment of the kinetic approach in general relativity see Stewart [1971] and references therein). Choosing coordinates (x^μ, p^i) on P_m (where $p = p^i\partial_i + p^0(x^\mu, p^i)\partial_0$), Liouville's equation reads

$$(p^\mu \partial_\mu - ?^i_{\alpha\beta} p^\alpha p^\beta \frac{\partial}{\partial p^i}) f = 0 \ .$$

In an unperturbed Friedmann universe, this equation is equivalent to the requirement that the distribution function, $\bar f$ is a function of the redshift corrected momentum, $v := a\sqrt{\bar g_{ij} p^i p^j}$ alone.

2.3.1 A gauge invariant variable for perturbations of the distribution function

We want to split the distribution function f in a perturbed Friedmann universe into a background component and a perturbation. We cannot do this directly since the background distribution function, $\bar f$ is not defined on P_m but on the background mass bundle $\bar P_m = \{(p,x)| \ \bar g(x)(p,p) = -m^2\}$. Therefore, to split f, we first have to choose an isomorphism $\iota : P_m \to \bar P_m : (x,p) \mapsto (x, \bar p)$. Then we can define the perturbation according to

$$f = (\bar f + F) \circ \iota \ .$$

It is clear that the perturbation F in general depends on the isomorphism ι which deviates in first order from identity. Choosing two basis (e_μ) and $(\bar e_\mu)$ which are tetrads with respect to g and $\bar g$, we may set for $p = p^\mu\partial_\mu = \pi^\mu e_\mu$

$$\iota(x, \pi^\mu e_\mu) = (x, \pi^\mu \bar e_\mu) \ .$$

On the other hand, every isomorphism ι is of this form (i.e. $\bar p(\pi^\mu e_\mu) = \pi^\mu \bar e_\mu$) for suitably chosen tetrads (e_μ) and $(\bar e_\mu)$. The tetrad e_μ can be defined by

$$e_\mu = \bar e_\mu + R_\mu^{\ \nu} \bar e_\nu \ , \tag{2.65}$$

where the symmetrical part of R is determined by the orthogonality condition:

$$R_{\mu\nu} + R_{\nu\mu} = -a^2 h(\bar e_\mu, \bar e_\nu) \quad (g = \bar g + a^2 h) \ .$$

To determine how F transforms under gauge transformations, we consider a vector field X which defines a gauge transformation. The flow of X on $\mathcal{M}$, ϕ^X_s, induces the flow $T\phi^X_s$ on $T\mathcal{M}$. The natural lift, TX, of X to $T\mathcal{M}$ is defined by $T\phi^X_s = \phi^{TX}_s$. A short calculation shows that for $X = X^\mu\partial_\mu$

$$TX = (X^\mu\partial_\mu, X^\mu{}_{,\nu} p^\nu \frac{\partial}{\partial p^\mu})$$

for a coordinate basis $(x^\mu, p = p^\mu\partial_\mu)$. For the full distribution function this leads to the transformation law

$$[f : P_m \to \mathbf{R}] \ \longrightarrow [(T\phi)_* f : T\phi(P_m) \to \mathbf{R}] \ . \tag{2.66}$$

In linearized form this yields

$$f \to f + L_{TX} f \ .$$

We now want to use our split of f to obtain a transformation law for F. The first problem to note here is that $\bar f$ is not defined on all of $T\mathcal{M}$ but only on $\bar P_m$, and since TX is in general not tangent to $\bar P_m$, $L_{TX}\bar f$ is not well defined. But of course it is possible to extend the definition of $\bar f$ to an open sub–set of $T\mathcal{M}$ containing $\bar{P}_m$. We thus do not have to bother about this technical point. We just keep in mind that the gauge transformation properties of F should not depend on such an extension.

More important is that F also depends on the choice of the isomorphism ι, the transformation properties of which have also to be taken into account. We now choose two admissible splittings of f given by

$$f = \bar{f}_1 \circ \iota_1 + F_1 = \bar{f}_2 \circ \iota_2 + F_2 \ .$$

Then, there exists a gauge transformation given by a vector field X, such that $\bar{f}_2 = \bar{f}_1 - L_{TX}\bar{f}_1$, and therefore

$$\bar{f}_1 \circ \iota_1 + F_1 = \bar{f}_1 \circ \iota_2 + F_2 - L_{TX}\bar{f}_1 \ .$$

The change of F under a gauge transformation is thus given by

$$F_2 - F_1 = \bar{f} \circ \iota_1 - \bar{f} \circ \iota_2 + L_{TX}\bar{f} \equiv L_{(TX)_{||}} F_1 \ ,$$

where we have dropped the index 1 on $\bar{f}$.

If $\iota_{1,2}$ are specified by the tetrads $e_{(1,2)\mu}$ which are related to the background tetrad $\bar{e}_\mu$ according to eq. (2.65) with matrices $R_\mu^{(1,2)\nu}$, we obtain

$$\bar{f} \circ \iota_1 - \bar{f} \circ \iota_2 = \frac{\partial \bar{f}}{\partial \pi^\nu} \pi^\mu (R_\mu^{(2)\nu} - R_\mu^{(1)\nu}) \equiv -(TX)_- \bar{f} \ , \tag{2.67}$$

where we have introduced $\quad (TX)_- = (R_\mu^{(2)\nu} - R_\mu^{(1)\nu})\pi^\mu \dfrac{\partial}{\partial \pi^\nu}$.

For $p = p^\mu \partial_\mu = \pi^\mu e_\mu$ we find

$$(TX)_{||} = TX - (TX)_- = X^\mu \partial_\mu + X^\mu{}_{,\nu} p^\nu \frac{\partial}{\partial p^\mu} - (R_\mu^{(2)\nu} - R_\mu^{(1)\nu})\pi^\mu \frac{\partial}{\partial \pi^\nu} \ .$$

It is easy to see that $(TX)_{||}(\bar{g}(p,p)) = 0$ which shows that $(TX)_{||}$ is parallel to $\bar{P}_m$, i.e., $(TX)_{||}\bar{f}$ does not depend on the extension of $\bar{f}$ which is necessary to make $(TX)\bar{f}$ and $(TX)_-\bar{f}$ well defined.

We now explicitly calculate $(TX)_{||}$ for $X = T\partial_t + L^i\partial_i$ and a tetrad (e_μ) which is adapted to the splitting of spacetime into $\{t = \text{const.}\}$ hyper–surfaces , Σ_t:

$$e_0 = n \ , \ \ \bar{e}_0 = \bar{n} = a^{-1}\partial_t \ , \ \ g(\boldsymbol{e}_i, n) = 0 \ .$$

The triad $(\boldsymbol{e}_i)$ is an orthogonal basis on the slices Σ_t. We furthermore use the fact that there exist coordinates such that the metric of a space of constant curvature k is given by

$$\gamma_{ij} = (1 + \frac{k}{4}r^2)^{-2}\delta_{ij} \equiv \lambda^2 \delta_{ij} \ .$$

In this coordinates we can choose $\bar{e}_i = (a\lambda)^{-1}\partial_i$. According to our definition of the perturbed lapse function, shift vector and three metric, we then have

$$\begin{aligned} e_0 &= \alpha^{-1}(\partial_t - \boldsymbol{\beta}) = (1 - A)\bar{e}_0 - B^i \lambda \bar{e}_i \\ \boldsymbol{e}_i &= (1 - H_L)\bar{e}_i - H_i{}^j \bar{e}_j \ , \end{aligned}$$

where the indices of the perturbation variables are, as usual raised and lowered with the metric $\boldsymbol{\gamma}$ and H_i^j is traceless, but may contain scalar, vector and tensor contributions. Using the gauge transformation properties of these variables and the definition of $(TX)_-$ we obtain

$$\begin{aligned}(TX)_- &= [(\dot{a}/a)T + \dot{T}]\pi^0\frac{\partial}{\partial\pi^0} - \lambda(T^{|i} - \dot{L}^i)\pi^0\frac{\partial}{\partial\pi^i} \\ &\quad +[(\dot{a}/a) - T\delta_i^{\ j} + 1/2(L_i^{\ |j} + L^j_{\ |i})]\pi^i\frac{\partial}{\partial\pi^j} \ .\end{aligned}$$

Inserting the expression for TX we find

$$\begin{aligned}(TX)_{||} &= T\partial_t + L^i\partial_i + T_{,i}\,p^i\frac{\partial}{\partial p^0} + L^i{}_{,j}\,p^j\frac{\partial}{\partial p^i} + \dot{L}^i p^0\frac{\partial}{\partial p^i} - (\dot{a}/a)T\pi^0\frac{\partial}{\partial\pi^0} \\ &\quad +\lambda T'^i\pi^0\frac{\partial}{\partial\pi^i} - \lambda\dot{L}^i\pi^o\frac{\partial}{\partial\pi^i} + 1/2(L_i^{\ |j} + L^j_{\ |i})]\pi^i\frac{\partial}{\partial\pi^j} + (\frac{\dot{a}}{a}T\pi^i\frac{\partial}{\partial\pi^i} \ .\end{aligned}$$

We now use that $\bar{f}$ is only a function of

$$v = a\sqrt{\boldsymbol{g}(\boldsymbol{p},\boldsymbol{p})} = a\sqrt{\sum_i \pi_i\pi_i} = a^2\lambda\sqrt{\sum_i p_ip_i} \ .$$

Denoting the direction cosines of the momentum by ϵ^i we obtain ($\epsilon^i = p^i/\sqrt{\sum_i p_ip_i} = \pi^i/\sqrt{\sum_i \pi_i\pi_i}$),

$$\begin{aligned}\frac{\partial\bar{f}}{\partial p^i} &= a^2\lambda\epsilon^i\frac{d\bar{f}}{dv} \\ \frac{\partial\bar{f}}{\partial\pi^i} &= a\epsilon^i\frac{d\bar{f}}{dv} \\ \left(\frac{\partial\bar{f}}{\partial t}\right)_{p^i} &= 2(\dot{a}/a)v\frac{d\bar{f}}{dv} \\ \left(\frac{\partial\bar{f}}{\partial x^j}\right)_{p^i} &= (\lambda_{,j}/\lambda)v\frac{d\bar{f}}{dv} \\ p^j\frac{\partial\bar{f}}{\partial p^i} &= \pi^j\frac{\partial\bar{f}}{\partial\pi^i} = \epsilon^j\epsilon^i v\frac{d\bar{f}}{dv} \\ \frac{\partial\bar{f}}{\partial p^0} &= \frac{\partial\bar{f}}{\partial\pi^0} = 0 \ .\end{aligned}$$

Furthermore,

$$1/2((L_i^{\ |j} + L^j_{\ |i})\epsilon^i\epsilon^j v\frac{d\bar{f}}{dv} = (L^j{}_{,i}\,\epsilon^i\epsilon^j + (\lambda_{,j}/\lambda)L^j)v\frac{d\bar{f}}{dv} \ .$$

Thus, the terms containing L^i in $(TX)_{||}\bar{f}$ cancel. Introducing $q = \lambda\sqrt{v^2 + m^2a^2}$ we finally obtain

$$(TX)_{||}\bar{f} = [v(\dot{a}/a)T + q\epsilon^iT_{,i}]\frac{d\bar{f}}{dv} \ .$$

This leads to the transformation law

$$F \to F\frac{d\bar{f}}{dv}[v(\dot{a}/a) + q\epsilon^i\partial_i]T \ . \tag{2.68}$$

We first note that F is invariant under vector type gauge transformations. Comparing (2.68) with the transformation properties of $\mathcal{R}$ and σ given in (2.23) we find the following gauge invariant combination:

$$\mathcal{F} = F - [v\mathcal{R} + ql\epsilon^i\partial_i\sigma]\frac{d\bar{f}}{dv} \ . \tag{2.69}$$

2.3.2 The perturbation of Liouville's equation

Choosing an arbitrary basis of vector fields (e_μ) and corresponding momentum coordinates, $p = \pi^\mu e_\mu$ on $T\mathcal{M}$, Liouville's equation is given by

$$L_{X_g} f = \pi^\mu e_\mu(f) - \omega^i{}_\mu(p)\pi^\mu \frac{\partial f}{\partial \pi^i} = 0 \ . \tag{2.70}$$

If we select, as above, a tetrad adapted to the slicing of $\mathcal{M}$ into slices Σ_t of constant time,

$$e_0 = n = \alpha^{-1}(\partial_t - \boldsymbol{\beta}) \ \ , \qquad \boldsymbol{e}_i \in T\Sigma_t \ ,$$

we find (see Appendix A5)

$$\begin{aligned} X_g &= \frac{\pi^0}{\alpha}(\partial_t - \boldsymbol{\beta}) + \boldsymbol{p} \\ &\quad -[\boldsymbol{\omega}^i{}_j(\boldsymbol{p} - \frac{\pi^0}{\alpha}\boldsymbol{\beta})\pi^j + (\pi^0)^2\frac{\alpha^{|i}}{\alpha} - \alpha^{-1}(\beta_j^{|i} - c^i{}_j)\pi^0\pi^j]\frac{\partial}{\partial \pi^i} \ , \end{aligned}$$

where $\boldsymbol{p} = \pi^i\boldsymbol{e}_i$ is the component of p tangent to the slices and $c^i{}_j$ is defined by

$$\partial_t\boldsymbol{\vartheta}^i = c^i{}_j\boldsymbol{\vartheta}^j \ .$$

The $(\boldsymbol{\vartheta}^i)$ are the basis of one forms dual to the vector fields $(\boldsymbol{e}^i)$ on Σ_t. More details are found in Appendix A. We now rewrite $X_g f$ in terms of the variables
t, $\boldsymbol{x}$, $v = a\sqrt{\pi_i\pi^i}$, $\epsilon^i = a\pi^i/v = \epsilon_i$ and $q = a\lambda\pi^0$. For this we use the following easily established identities:

$$\begin{aligned} \boldsymbol{\beta} &= B^i\partial_i = a\lambda B^i\boldsymbol{e}_i \\ \alpha &= a(1+A) \ , \quad \alpha^{-1} = a^{-1}(1-A) \\ c^i{}_j &= (\dot{a}/a)\delta^i_j + \dot{H}_L\delta^i_j + \dot{H}^i_{Tj} \\ \beta^j_{|i} &= \beta_j^{|i} = \boldsymbol{e}_i(\beta^j) + \bar{\boldsymbol{\omega}}^i{}_l(\boldsymbol{e}_j)\beta^l = B^j{}_{,i} + \frac{k\lambda}{2}(B^ix^j - B^jx^i - B^lx^l\delta^j_i) \ . \end{aligned}$$

Since the background distribution function only depends on v, i.e., $\frac{\partial}{\partial\pi^i}\bar{f} = a\epsilon^i\frac{d\bar{f}}{dv}$, only the symmetrical part of $\beta_j{}^{|j}$ contributes to the Liouville equation in first order and the term $\boldsymbol{\omega}^i{}_j(\boldsymbol{\beta})$ gives no contribution in this approximation. With the help of the splitting $f = (\bar{f}+F)\circ\iota$ and the background Liouville equation for $\bar{f}$ we finally obtain

$$q\partial_t F + v\epsilon^i\partial_i F - vk\lambda/2(x^i - x^j\epsilon^j\epsilon^i)\frac{\partial F}{\partial\epsilon^i} = [q^2\epsilon^i A_{,i} - qv\epsilon^i\epsilon_j(B^j_{|i} - \dot{H}^j_i) + v^2 H_L]\frac{df}{dv} \ .$$

For the gauge invariant tensor and vector contributions to F this yields, setting $F^{(T)} \equiv \mathcal{F}^{(T)}$ and $F^{(V)} \equiv \mathcal{F}^{(V)}$

$$q\partial_t \mathcal{F}^{(T)} + v\epsilon^i \partial_i \mathcal{F}^{(T)} - v?^{\,i}_{\,jk}\epsilon^j\epsilon^k \frac{\partial \mathcal{F}^{(T)}}{\partial \epsilon^i} = qv\epsilon^i\epsilon_j \dot{H}^i_j \frac{d\bar{f}}{dv} \tag{2.71}$$

$$q\partial_t \mathcal{F}^{(V)} + v\epsilon^i \partial_i \mathcal{F}^{(V)} - v?^{\,i}_{\,jk}\epsilon^j\epsilon^k \frac{\partial \mathcal{F}^{(V)}}{\partial \epsilon^i} = qv\epsilon^i\epsilon_j \sigma^{(V)i}_{|j} \frac{d\bar{f}}{dv} \ . \tag{2.72}$$

For scalar perturbations the situation is somewhat more complicated. Since the scalar contribution $F^{(S)}$ to F is not gauge invariant we want to express the Liouville equation in terms of the gauge invariant combination

$$\mathcal{F}^{(S)} = F^{(S)} - [v\mathcal{R} + ql\epsilon^i\sigma_{,i}\,]\frac{d\bar{f}}{dv} \ .$$

After carefully calculating $\partial_t[v\mathcal{R} + ql\epsilon^i\sigma_{,i}\,]$, $\partial_i[v\mathcal{R} + ql\epsilon^i\sigma_{,i}\,]$ and $\frac{\partial}{\partial\epsilon^i}[v\mathcal{R} + ql\epsilon^i\sigma_{,i}\,]$, we finally obtain the Liouville equation for scalar perturbations in a Friedmann universe [Durrer, 1990]:

$$q\partial_t \mathcal{F}^{(S)} + v\epsilon^i \partial_i \mathcal{F}^{(S)} - v\mathbf{\Gamma}^k_{ij}\epsilon^i\epsilon^j \frac{\partial \mathcal{F}^{(S)}}{\partial \epsilon^k} = (q^2\partial_i\Psi - v^2\partial_i\Phi)\epsilon^i \frac{d\bar{f}}{dv} \ . \tag{2.73}$$

2.3.3 Momentum Integrals

To connect this equation of motion to Einstein's field equations, we calculate the energy momentum tensor from f, which is given by

$$T^{\mu\nu} = \int_{P_m(x)} p^\mu p^\nu f \mu(p) \ , \tag{2.74}$$

where μ is an invariant measure on $P_m(x)$ (for a general definition see Stewart [1971]). With respect to a tetrad $p = \pi^\nu e_\nu$, μ looks like in special relativity

$$\mu(p) = \frac{d\pi^1 \wedge d\pi^2 \wedge d\pi^3}{|\pi^0|} = \frac{\lambda v^2}{a^2 q} dv d\Omega \ , \tag{2.75}$$

where we have used the definition of v and q, and $d\Omega$ is the usual surface element on the 2-sphere integrating over the momentum directions $\boldsymbol{\epsilon}$. Let us, as an example, consider T^0_0:

$$T^0_0 = \bar{\rho}(1+\delta) = \frac{\lambda}{a^2}\int \frac{p^0 p_0 v^2}{q}(\bar{f}(v) + F) dv d\Omega \ .$$

Expressing p^0 and p_0 in terms of v and q and separating into a background and first order contribution yields

$$\bar{\rho} = \frac{4\pi}{\lambda a^4}\int v^2 q\bar{f} dv \quad , \quad \rho\delta = \frac{1}{\lambda a^4}\int v^2 qF dv d\Omega \ .$$

Using this we obtain

$$\frac{1}{\rho\lambda a^4}\int v^2 q\mathcal{F}^{(S)} dv d\Omega = \delta - \frac{\mathcal{R}}{\rho\lambda a^4}\int v^3 q\frac{d\bar{f}}{dv} dv d\Omega \ .$$

After a partial integration, inserting $\frac{dq}{dv} = \lambda^2 v/q$ and

$$\bar{p} = \frac{4\pi\lambda}{3a^4}\int (v^4/q)\bar{f} dv$$

we end up with

$$\frac{1}{\rho\lambda a^4}\int v^2 q\mathcal{F}^{(S)}dvd\Omega = \delta + 3(1+w)\mathcal{R} = D_g \ . \qquad (2.76)$$

For the last equality sign, the definition of the gauge invariant density perturbation variable D_g is inserted.

In a similar way all the other momentum integrals are obtained:

$$\begin{aligned} l\triangle V &= \frac{-1}{a^4(\rho+p)}\int v^3\epsilon^i\partial_i\mathcal{F}^{(S)}dvd\Omega & (2.77)\\ l^2(\Pi_{|ij} - 1/3\gamma_{ij}\triangle\Pi) &= \frac{\lambda}{a^4 p}\int\frac{v^4}{q}(\epsilon^i\epsilon^j - (1/3)\delta_{ij})\mathcal{F}^{(S)}dvd\Omega & (2.78)\\ ? &= \frac{1}{a^4 p}\int(\frac{v^4\lambda}{3q} - \frac{c_s^2 v^2 q}{\lambda})\mathcal{F}^{(S)}dvd\Omega \ . & (2.79)\end{aligned}$$

For the vector and tensor perturbations one finds

$$\begin{aligned} V^{(V)i} &= \frac{-1}{a^4(\rho+p)}\int v^3\epsilon^i\mathcal{F}^{(V)}dvd\Omega & (2.80)\\ (l/2)(\Pi^{(V)}_{i|j} - \Pi^{(V)}_{j|i}) &= \frac{\lambda}{a^4 p}\int\frac{v^4}{q}(\epsilon^i\epsilon^j - (1/3)\delta_{ij})\mathcal{F}^{(V)}dvd\Omega & (2.81)\\ \Pi^{(T)}_{ij} &= \frac{\lambda}{a^4 p}\int\frac{v^4}{q}(\epsilon^i\epsilon^j - (1/3)\delta_{ij})\mathcal{F}^{(T)}dvd\Omega \ . & (2.82)\end{aligned}$$

These matter variables inserted in Einstein's equations (2.46) (2.48), (2.53) and (2.56) yield the geometrical perturbations Ψ, Φ, σ^i and $H^{(T)}_{ij}$ which enter in (2.73,2.72,2.71). In Section 5, we discuss how this closed system is altered in the presence of seeds.

2.3.4 The ultrarelativistic limit

Here we briefly investigate the special case of extremely relativistic particles for which we can set $m = 0$. Since curvature only may play a role in the late, matter dominated stages of the universe, we neglect it here, $k = 0, \lambda = 1$, so that $q = v$. (The generalization to $k \neq 0$ is straight forward.) In the extremely relativistic case all the integrals above contain the energy integral $\cdot \int v^3\mathcal{F}dvd\Omega$. Therefore, it makes sense to introduce the perturbation of the energy integrated distribution function, the brightness:

$$\mathcal{M} \equiv \frac{4\pi}{\bar\rho a^4}\int_0^\infty \mathcal{F}v^3 dv \ , \qquad (2.83)$$

which is a function of the momentum directions ϵ^i only. Defining

$$\iota = \frac{4\pi}{\bar\rho a^4}\int_0^\infty F v^3 dv \ ,$$

one finds, using (2.69) and the gauge invariance of $F^{(V)}$ and $F^{(T)}$

$$\begin{aligned} \mathcal{M}^{(S)} &= \iota^{(S)} + 4\mathcal{R} + 4l\epsilon^i\partial_i\sigma & (2.84)\\ \mathcal{M}^{(V)} &= \iota^{(V)} & (2.85)\\ \mathcal{M}^{(T)} &= \iota^{(T)} \ . & (2.86)\end{aligned}$$

In the case where $\mathcal{M}$ describes thermal radiation, we may interpret the perturbation in the distribution function as a perturbation of the temperature:

$$f = \bar{f}\left(\frac{\boldsymbol{g}(\boldsymbol{p},\boldsymbol{p})^{1/2}}{T(x^\mu,\boldsymbol{\epsilon})}\right) = \bar{f}\left(\frac{v}{aT(x^\mu,\boldsymbol{\epsilon})}\right) \quad \text{with} \quad T(x^\mu,\boldsymbol{\epsilon}) = \bar{T}(t) + \delta T(x^\mu,\boldsymbol{\epsilon}) \ . \tag{2.87}$$

Inserting this form of f one obtains $F = -v\frac{d\bar{f}}{dv}\cdot\frac{\delta T}{T}$. The integral (2.83) then yields

$$\begin{aligned}
\frac{1}{4}\mathcal{M}^{(S)} &= \frac{\delta T^{(S)}}{\overline{T}} + \mathcal{R} + l\epsilon^i\sigma_i \\
\frac{1}{4}\mathcal{M}^{(V)} &= \frac{\delta T^{(V)}}{\overline{T}} \\
\frac{1}{4}\mathcal{M}^{(T)} &= \frac{\delta T^{(T)}}{\overline{T}} \ .
\end{aligned}$$

Therefore, $(1/4)\mathcal{M}$ can be interpreted as a gauge invariant variable for the temperature perturbation.[2]

In terms of $\mathcal{M}$ the perturbation equations (2.73,2.72, 2.71) become (for $k=0$)

$$\dot{\mathcal{M}}^{(S)} + \epsilon^i\partial_i\mathcal{M}^{(S)} = 4\epsilon^i\partial_i(\Phi-\Psi) \tag{2.88}$$

$$\dot{\mathcal{M}}^{(V)} + \epsilon^i\partial_i\mathcal{M}^{(V)} = -4\epsilon^i\epsilon_j\sigma^{(V)i}_{|j} \tag{2.89}$$

$$\dot{\mathcal{M}}^{(T)} + \epsilon^i\partial_i\mathcal{M}^{(T)} = -4\epsilon^i\epsilon^j\dot{H}_{ij} \ . \tag{2.90}$$

The evolution of the distribution of massless particles only depends on the Weyl part of the curvature. This is geometrically very reasonable since null geodesics are conformally invariant.

By similar calculations like in the preceding paragraph, one finds the perturbations of the energy momentum tensor for extremely relativistic particles

$$D_g = \frac{1}{4\pi}\int \mathcal{M}d\Omega \tag{2.91}$$

$$l\triangle V = \frac{-3}{16\pi}\int \epsilon^i\partial_i\mathcal{M}d\Omega \tag{2.92}$$

$$l^2(\Pi_{|ij} - 1/3\epsilon_{ij}\triangle\Pi) = \frac{3}{4\pi}\int(\epsilon^i\epsilon^j - \frac{1}{3}\delta_{ij})\mathcal{M}d\Omega \tag{2.93}$$

$$? = 0 \tag{2.94}$$

$$V^{(V)i} = \frac{1}{4\pi}\int \epsilon^i\mathcal{M}^{(V)}d\Omega \tag{2.95}$$

$$(l/2)(\Pi^{(V)}_{i|j} - \Pi^{(V)}_{j|i}) = \frac{3}{4\pi}\int(\epsilon^i\epsilon^j - (1/3)\delta_{ij})\mathcal{M}^{(V)}d\Omega \tag{2.96}$$

$$\Pi^{(T)}_{ij} = \frac{3}{4\pi}\int(\epsilon^i\epsilon^j - (1/3)\delta_{ij})\mathcal{M}^{(T)}dvd\Omega \ . \tag{2.97}$$

[2]Note that even though $\mathcal{F}$, $\mathcal{M}$ and δT are scalar functions they do in general contain vector and tensor perturbations, since they depend not only on position but also on momentum or momentum direction. They may contain terms of the form $\alpha^i\epsilon_i$ or $\tau^{ij}\epsilon_i\epsilon_j$ where α is a divergence free vector field and τ is a traceless, divergencefree tensor field. These are the type of contributions which we indicate with $^{(V)}$ and $^{(T)}$.

We use Liouville's equation for massless particles for the numerical calculation of perturbations of the cosmic microwave background in Chapter 5. In Section 3.2 we use (2.88) to derive the Boltzmann equation for photons in an electron proton plasma.

2.4 The Propagation of Photons in a Perturbed Friedmann Universe

On their way from the last scattering surface into our antennas, the microwave photons travel through a perturbed Friedmann geometry. Thus, even if the photon temperature was completely uniform at the last scattering surface, we would receive it slightly perturbed [Sachs and Wolfe, 1967]. In addition, a photon traveling through a perturbed universe is in general deflected. In this section we calculate both these effects in first order perturbation theory. I present the calculation rather explicitly, since I haven't found a complete gauge invariant treatment of this problem anywhere in the literature. For sake of simplicity we restrict ourselves to $k = 0$.

As already mentioned, two metrics which are conformally equivalent,

$$d\tilde{s}^2 = a^2 ds^2 \ ,$$

have the same lightlike geodesics, only the corresponding affine parameters are different. We may thus discuss the propagation of light in a perturbed Minkowski geometry. This simplifies things greatly. We denote the affine parameters by $\tilde{\lambda}$ and λ respectively and the tangent vectors to the geodesic by

$$n = \frac{dx}{d\lambda} \ \text{and} \ \tilde{n} = \frac{dx}{d\tilde{\lambda}} \ , \quad n^2 = \tilde{n}^2 = 0 \ ,$$

with unperturbed values $n^0 = 1$ and $\boldsymbol{n}^2 = 1$. If the tangent vector of the perturbed geodesic is $(1, \boldsymbol{n}) + \delta n$, the geodesic equation for the metric

$$ds^2 = (\eta_{\alpha\nu} + h_{\alpha\nu}) dx^\alpha dx^\nu$$

yields to first order

$$\delta n^\mu|_i^f = -\eta^{\mu\nu}[h_{\nu 0} + h_{\nu i} n^i]_i^f + \frac{\eta^{\mu\sigma}}{2} \int_i^f h_{\rho\nu,\sigma} n^\rho n^\nu d\lambda \ , \tag{2.98}$$

where the integral is along the unperturbed photon trajectory and the unperturbed values for n^μ can be inserted. Starting from this general relation, let us first discuss the photon redshift. The ratio of the energy of a photon measured by some observer at t_f to the energy emitted at t_i is given by

$$E_f/E_i = \frac{(\tilde{n} \cdot u)_f}{(\tilde{n} \cdot u)_i} = (T_f/T_i) \frac{(n \cdot u)_f}{(n \cdot u)_i} \ , \tag{2.99}$$

where u_f and u_i are the four velocities of the observer and the emitter respectively and the factor T_f/T_i is the usual redshift which relates n and $\tilde{n}$. We write T_f/T_i and not a_f/a_i here, since also this redshift is slightly perturbed in general, and we want a to denote the unperturbed background expansion factor.

Since this is a physical, intrinsically defined quantity it is independent of coordinates. It must thus be possible to write it in terms of gauge invariant variables. We now calculate the gauge invariant expression for E_f/E_i. The observer and emitter are comoving with the cosmic fluid. We have

$$u = (1 - A)\partial_t - l v^{,i} \partial_i \ .$$

Furthermore, since the photon density $\rho^{(r)} \propto T^4$ may itself be perturbed

$$T_f/T_i = (a_i/a_f)(1 + \frac{\delta T_f}{T_f} - \frac{\delta T_i}{T_i}) = a_i/a_f(1 + (1/4)\delta^{(r)}|_i^f) \ ,$$

where $\delta^{(r)}$ is the intrinsic density perturbation in the radiation. This term was neglected in the original analysis of Sachs and Wolfe, but since it is gauge dependent, doing so violates gauge invariance. We therefore have to include $\delta^{(r)}$ to obtain a gauge invariant expression. Inserting all this and (2.98) into (2.99) yields

$$E_f/E_i = (a_i/a_f)[1 + n^j v,_j|_i^f + A|_i^f + (1/4)\delta^{(r)}|_i^f - 1/2\int_i^f \dot{h}_{\mu\nu}n^\mu n^\mu d\lambda] \ . \tag{2.100}$$

With the help of equation (2.11) for the definition of $h_{\mu\nu}$ one finds for **scalar** perturbations after several integrations by part

$$\begin{aligned} (E_f/E_i)^{(S)} &= (a_i/a_f)\{1 + [(1/4)D_s^{(r)} + lV_{|j}^{(m)}n^j + \Psi]|_i^f - \int_i^f (\dot{\Phi} - \dot{\Psi})d\lambda\} & (2.101) \\ &= (a_i/a_f)\{1 + [(1/4)D_g^{(r)} + lV_{|j}^{(m)}n^j + \Psi - \Phi]_i^f - \int_i^f (\dot{\Phi} - \dot{\Psi})d\lambda\} \ . & (2.102) \end{aligned}$$

Here $D_s^{(r)}$, $D_g^{(r)}$ denote the gauge invariant density perturbation in the radiation field and $V^{(m)}$ is the peculiar velocity of the matter component (the emitter and observer of radiation). From the second of these equations one sees explicitly that the geometrical part of the perturbation of the photon redshift depends on the Weyl curvature only (specialize eq. (2.26) to purely scalar perturbations), i.e., is conformally invariant.

For a discussion of the Sachs–Wolfe effect alone we neglect the intrinsic density perturbation of the radiation, i.e., we set $D_g^{(r)} = 0$, which now is a gauge invariant statement (but a bad approximation in many circumstances like, e.g. for adiabatic CDM perturbations). $V^{(m)}$ is a Doppler term due to the relative motion of emitter and receiver. The $\Psi - \Phi$ – term accounts for the redshift due to the difference of the gravitational field at the place of the emitter and receiver and the integral is a path dependent contribution to the redshift.

For **vector** perturbations $\delta^{(r)}$ and A vanish and eq. (2.100) leads to

$$(E_f/E_i)^{(V)} = (a_i/a_f)[1 - V_j^{(m)}n^j|_i^f + \int_i^f \dot{\sigma}_j n^j d\lambda] \ . \tag{2.103}$$

Again we obtain a Doppler term and a gravitational contribution. For **tensor** perturbations, i.e. gravitational waves, only the gravitational part remains:

$$(E_f/E_i)^{(T)} = (a_i/a_f)[1 - \int_i^f \dot{H}_{lj}n^l n^j d\lambda] \ . \tag{2.104}$$

Equations (2.101), (2.103) and (2.104) are the manifestly gauge invariant results for the Sachs–Wolfe effect for scalar vector and tensor perturbations.

In addition to redshift, photons in a perturbed Friedmann universe also experiences deflection. We now calculate this effect.

The direction of the light ray with respect to a comoving observer is given by the direction of the spacelike vector

$$n_{(3)} = n + (un)u$$

which lives on the sub–space of tangent space normal to u. Let us also define the vector field $n^{||}_{(3)}$, which coincides with $n_{(3)}$ initially (i.e. at t_i) and is Fermi transported along u, i.e.

$$\nabla_u n^{||}_{(3)} = (n^{||}_{(3)} \nabla_u u) u \ . \qquad (2.105)$$

Note, that we have to require Fermi transport and not parallel transport since u is in general not a geodesic and therefore $(u n^{||}_{(3)}) = 0$ is not conserved under parallel transport! (For an explanation of Fermi transport, see e.g. Straumann [1985]). $n^{||}_{(3)} = (0, \boldsymbol{n}) + \delta n^{||}_{(3)}$, where $\delta n^{||}_{(3)}$ is determined by the Fermi transport equation (2.105). Since the observer Fermi transports her frame of reference with respect to which angles are measured, she would consider the light ray as not being deflected if $n^{||}_{(3)}(t_f)$ is parallel to $n_{(3)}(t_f)$. The difference between the direction of these two vectors is thus the light deflection:

$$\varphi e = \left[n_{(3)} - \frac{(n^{||}_{(3)} \cdot n_{(3)})}{(n^{||}_{(3)} \cdot n^{||}_{(3)})} n^{||}_{(3)} \right] (t_f) \ . \qquad (2.106)$$

Here e is a spacelike unit vector normal to u and normal to $n^{||}_{(3)}$ which determines the direction of the deflection and φ is the deflection angle. (Note that (2.106) is the general formula for light deflection in an arbitrary gravitational field. Up to this point we did not make any assumptions about the strength of the field.) For a spherically symmetric problem, as we shall encounter when discussing the collapsing texture, e is uniquely determined by the above conditions since the path of a light ray is confined to the plane normal to the angular momentum. In the general case, when angular momentum is not conserved e still has one degree of freedom. We now calculate φe perturbatively. Let us recall and define the perturbed quantities:

$$\begin{aligned} n &= (1, \boldsymbol{n}) + \delta n \quad \text{with} \quad \boldsymbol{n}^2 = 1 \\ u &= (1, 0) + \delta u \ = \ (1 + \frac{1}{2} h_{00}, \boldsymbol{v}) \\ n_{(3)} &= (0, \boldsymbol{n}) + \delta n_{(3)} \quad \text{and} \\ n^{||}_{(3)} &= (0, \boldsymbol{n}) + \delta n^{||}_{(3)} \ . \end{aligned}$$

The perturbation δn is given in (2.98). Furthermore, we obtain

$$\delta n_{(3)} = \epsilon u + \delta n - \delta u \ , \qquad (2.107)$$

with

$$\epsilon = [n^i v^i - \delta n^0 + \frac{1}{2} h_{00} + n^i h_{i0}]$$

The Fermi transport equation leads to

$$\delta (n^{||}_{(3)})^0 \ = \ n^i (h_{i0} + v_i) \qquad (2.108)$$

$$\delta (n^{||}_{(3)})^j \ = \ -\frac{1}{2} [h_{lj} n^l)|_i^f + \int_i^f dt (h_{j0,l} - h_{l0,j}) n^l \ . \qquad (2.109)$$

So that $(n^{||}_{(3)})^2 = 1$ and

$$\varphi e = \delta n_{(3)} - (n^{||}_{(3)} \cdot n_{(3)}) n^{||}_{(3)} = \epsilon u + \delta n - \delta u - \delta n^{||}_{(3)} - [h_{ji} n^i n^j + n^i (\delta (n^{||}_{(3)})^i + \delta n^i - v^i)] \boldsymbol{n} \ .$$

Inserting (2.107–2.109) into (2.106) we find

$$\varphi e^0 = 0 \qquad (2.110)$$

$$\varphi e^i = \delta^i - (\boldsymbol{\delta} \cdot \boldsymbol{n}) n^i \ , \quad \text{with} \qquad (2.111)$$

$$\delta_j = [\delta n_j - v_j + \frac{1}{2} h_{jk} n^k]|_i^f + \frac{1}{2} \int_i^f (h_{0j,k} - h_{0k,j}) n^k dt \ . \qquad (2.112)$$

This quantitiy is observable and thus gauge invariant. For scalar perturbations one finds (after integrations by parts)

$$(\delta_j)^{(S)} = V_{,j}|_i^f + \int_i^f (\Phi - \Psi)_{,j} d\lambda \ . \qquad (2.113)$$

For spherically symmetric perturbations, where e is uniquely defined, we can write this result in the form

$$\varphi = V_{,i} e^i|_i^f + \int_i^f (\Phi - \Psi)_{,i} e^i d\lambda \ . \qquad (2.114)$$

The first term here denotes the special relativistic spherical aberration. The second term represents the gravitational light deflection. Here again one sees that gravitational light deflection, which of course is conformally invariant, only depends on the Weyl part of the curvature. As an easy test we insert the Schwarzschild weak field approximation: $\Psi = -\Phi = -\frac{GM}{r}$. The unperturbed geodesic is given by $x = (\lambda, \boldsymbol{n}\lambda + \boldsymbol{e}d)$, where d denotes the impact parameter of the photon. Inserting this into (2.114) yields Einstein's well known result

$$\varphi = \frac{4GM}{d} \ .$$

For vector perturbations we obtain from (2.112)

$$(\delta_j)^{(V)} = \Omega_j|_i^f - \frac{1}{2}[\int_i^f (\sigma_{j,k} - \sigma_{k,j}) n^k dt + \int_i^f \sigma_{k,j} n^k d\lambda] \ . \qquad (2.115)$$

This result can be expressed in three dimensional notation as follows:

$$\varphi \boldsymbol{e} = \boldsymbol{\delta} - (\boldsymbol{\delta} \cdot \boldsymbol{n}) \boldsymbol{n} =$$
$$-(\boldsymbol{\Omega} \wedge \boldsymbol{n}) \wedge \boldsymbol{n}|_i^f + \frac{1}{2} \int_i^f (\boldsymbol{\nabla} \wedge \boldsymbol{\sigma}) \wedge \boldsymbol{n} dt - \int_i^f (\boldsymbol{\nabla}(\boldsymbol{\sigma} \cdot \boldsymbol{n}) \wedge \boldsymbol{n}) \wedge \boldsymbol{n} d\lambda \ .$$

The first term is again a special relativistic "frame dragging" effect. The second term is the change of frame due to the gravitational field along the path of the observer and the third term gives the gravitational light deflection. This formula could be used to obtain in first order the light deflection in the vicinity of a rotating neutron star or a Kerr black hole. The special relativistic Thomas precession is not recovered with this formula since it is of order v^2.

For tensor perturbations we find

$$(\delta_j)^{(T)} = -H_{jk} n^k|_i^f + \int_i^f H_{lk,j} n^l n^k d\lambda \ ,$$

or, after an integration by parts,

$$(\varphi e_j)^{(T)} = -H_{jk}n^k|_i^f + \int_i^f (H_{lk,j} + \dot{H}_{kl}n_j)n^l n^k d\lambda \ . \tag{2.116}$$

Only the gravitational effects remain. The first contribution comes from the difference of the metric before and after the passage of the gravitational wave. Usually this term is negligible. The second term accumulates along the path of the photon.

2.5 Gauge Invariant Perturbation Theory in the Presence of Seeds

In this section we add an inhomogeneous term to the perturbation equations. Perturbations can then be generated even starting from an initially unperturbed spacetime. Seeds produce this inhomogeneous term in a natural way. By seeds we mean density perturbations originating from an inhomogeneously distributed form of energy whose mean density is much smaller than the density of the Friedmann background. We assume that, once they are produced, these seeds do not interact with the rest of the matter other than gravitationally.

2.5.1 The energy momentum tensor

Since the energy momentum tensor of the seeds, $T^{\mu\nu}_{(s)}$, has no homogeneous background contribution, it is gauge invariant by itself according to (2.3).

$T^{\mu\nu}_{(s)}$ can be calculated by solving the matter equations for the seeds in the Friedmann *background* geometry. (Since $T^{\mu\nu}_{(s)}$ has no background component it satisfies the unperturbed matter and "conservation" equations.) We again decompose $T^{\mu\nu}_{(s)}$ into scalar, vector and tensor contributions. They decouple within linear perturbation theory and it is thus possible to write the equations for each of these contributions separately. However, this decomposition is acausal. It requires $T_{\mu\nu}(t, \boldsymbol{x})$ at a given time t to be known for all positions $\boldsymbol{x}$ and not only within a causally connected region. We just ignore this problem now and, nevertheless, work with this decomposition. We parametrize the scalar (S) vector (V) and tensor (T) contributions to $T^{\mu\nu}_{(s)}$ in the form

$$\begin{aligned}
T^{(sS)}_{00} &= a^2\rho^{(s)} = (M^2/l^2)f_\rho && (2.117)\\
T^{(sS)}_{i0} &= a^2 l v^{(s)}_{|i} = (M^2/l)f_{v\ |i} && (2.118)\\
T^{(sS)}_{ij} &= a^2[(p^{(s)} - (l^2/3)\triangle\Pi^{(s)})\gamma_{ij} + l^2\Pi^{(s)}_{|ij}] \\
&= M^2[(f_p/l^2 - (1/3)\triangle f_\pi)\gamma_{ij} + f_{\pi\ |ij}] && (2.119)\\
T^{(sV)}_{i0} &= a^2 v^{(s)}_i = (M^2/l^2)w^{(v)}_i && (2.120)\\
T^{(sV)}_{ij} &= \frac{a^2 l}{2}[\Pi^{(s)}_{i|j} + \Pi^{(s)}_{j|i}] = (M^2/2l)(w^{(\pi)}_{i|j} + w^{(\pi)}_{j|i}) && (2.121)\\
T^{(sT)}_{ij} &= a^2\Pi^{(s)}_{ij} = (M^2/l^2)\tau^{(\pi)}_{ij} \ . && (2.122)
\end{aligned}$$

As before, l is introduced merely to keep the functions $f_\cdot$, the vector fields $w^\cdot$ and the tensor field τ^π dimensionless. It may be chosen as a typical size of the seeds. M denotes a typical mass of the seeds. (It is of course possible to choose $l = M^{-1}$.)

If we are given the full energy momentum tensor $T^{(s)}_{\mu\nu}$ which may contain scalar, vector and tensor contributions, the scalar parts f_v and f_π are in general determined by the non–local identities

$$T^{(s)\,|j}_{0j} = (M^2/l)\triangle f_v \ ,$$

$$(T^{(s)}_{ij} - 1/3\gamma_{ij}\gamma^{kl}T_{kl})^{|ij} = \frac{2}{3}M^2(\triangle + 3k)\triangle f_\pi \ .$$

On the other hand $\triangle f_v$ and $\triangle(\triangle + 3k)f_\pi$ are also determined in terms of f_ρ and f_p by the "conservation" equations:

$$\dot{f}_\rho - l\triangle f_v + (\dot{a}/a)(f_\rho + 3f_p) = 0 \ , \qquad (2.123)$$

$$-l\dot{f}_v - 2(\dot{a}/a)lf_v + f_p + (2/3)l^2(\triangle + 3k)f_\pi = 0 \ . \qquad (2.124)$$

Once f_v is known it is easy to get $w^v_i = l^2/M^2(T_{0i}) - lf_{v,i}$. To obtain w^π_i we use

$$T^{(s)|j}_{ij} - T^{(sS)|j}_{ij} = \frac{M^2}{l}(\triangle + 2k)w^{(\pi)}_i \ .$$

Again $w^{(\pi)}_i$ can also be obtained in terms of $w^{(v)}_i$ with the help of the "conservation" euation:

$$\dot{w}^{(v)}_i + 2(\frac{\dot{a}}{a})w^{(v)}_i - \frac{l}{2}(\triangle + 2k)w^{(\pi)}_i = 0 \ . \qquad (2.125)$$

2.5.2 Perturbation equations

The energy momentum tensor of the seeds is determined by the unperturbed equations of motion. The gravitational interaction with the perturbations of other components does not contribute to first order. We assume non–gravitational interactions with other components can be neglected. This is certainly a good approximation soon after the phase transition and thus can only affect the initial conditions. (Situations where non–gravitational interactions must not be neglected are discussed by Magueijo [1992].) The geometrical perturbations can then be separated into a part induced by the seeds and a part caused by the perturbations in the remaining matter components:

$$\begin{aligned}\Psi &= \Psi_s + \Psi_m \quad , \quad \Phi = \Phi_s + \Phi_m \\ \sigma_i &= \sigma^{(s)}_i + \sigma^{(m)}_i \\ H_{ij} &= H^{(s)}_{ij} + H^{(m)}_{ij} \ .\end{aligned}$$

Using Einstein's equations, we can calculate the geometry perturbations induced by the seeds:

$$-(\triangle + 3k)\Phi_s = \epsilon(f_\rho/l^2 + 3(\dot{a}/a)f_v/l) \qquad (2.126)$$

$$(\dot{a}/a)\Psi_s - \dot{\Phi}_s = \epsilon f_v/l \qquad (2.127)$$

$$\triangle(\Phi_s + \Psi_s) = -2\epsilon\triangle f_\pi \qquad (2.128)$$

$$
\begin{aligned}
(\dot a/a)\{\dot\Psi_s - [(1/a)(\frac{a^2\Phi_s}{\dot a})^{\cdot}]^{\cdot}\}+ & \\
\{2a(\dot a/a^2)^{\cdot} + 3(\dot a/a^2)^2\}[\Psi_s - 1/a(\frac{a^2\Phi_s}{\dot a})^{\cdot}] &= 2\epsilon(f_p/l^2 - (2/3)\triangle f_\pi) \qquad (2.129)\\
(\triangle + 2k)\sigma_i^{(s)} &= 2\epsilon w_i^{(v)}/l^2 \qquad (2.130)\\
2\frac{\dot a}{a}\sigma_i^{(s)} + \dot\sigma_i^{(s)} &= -2\epsilon w_i^{(\pi)}/l \qquad (2.131)\\
\ddot H_{ij}^{(s)} + \frac{\dot a}{a}\dot H_{ij}^{(s)} + (2k-\triangle)H_{ij}^{(s)} &= 2\epsilon\tau_{ij}^{(\pi)}/l^2\ . \qquad (2.132)
\end{aligned}
$$

We assume that $\epsilon \equiv 4\pi GM^2$ is much smaller than 1, so that linear perturbation analysis is justified. As before, the part of the geometrical perturbations induced by the matter are determined by equations (2.46) to (2.48). But in the conservation equations and in any matter equations the full perturbations, Ψ, Φ, σ_i and H_{ij} have to be inserted.

We now discuss the example of a single fluid with only scalar perturbations where Π and ? are given in terms of D and V. We assume that in addition to the seeds we have one perturbed matter component which we indicate by a subscript m. Other components which contribute to the background, but whose perturbations can be neglected, may also be present. The conservation equation (2.57) then reads

$$
\dot D_m - 3w_m(\dot a/a)D_m = (\triangle + 3k)[(1+w_m)lV_m + 2(\dot a/a)w_m l^2\Pi_m] - 3(1+w_m)\epsilon f_v/l\ . \qquad (2.133)
$$

The last term describes the influence of the seeds.

Solving this equation for $(\triangle + 3k)lV_m$ and inserting the result and its time derivative into (2.58) yields a second order equation for D_m. Using

$$
(\triangle + 3k)\Psi = 4\pi G\rho_m a^2(D_m - 2w_m l^2(\triangle + 3k)\Pi_m) + \epsilon(f_\rho/l^2 + 3\frac{\dot a}{a}f_v/l - 2(\triangle + 3k)f_\pi)
$$

and the conservation equation (2.124) we find

$$
\begin{aligned}
&\ddot D - (\triangle + 3k)c_s^2 D + (1 + 3c_s^2 - 6w)(\dot a/a)\dot D - \{3w(\ddot a/a) - 9(\dot a/a)^2(c_s^2 - w) + \\
&+(1+w)4\pi G\rho a^2\}D \ = \\
&(\triangle + 3k)w? + 2(\dot a/a)wl^2(\triangle + 3k)\dot\Pi \\
&+\{2(\ddot a/a)w - 6(\dot a/a)^2(c_s^2 - w) + (1+w)8\pi Ga^2p + 2/3(\triangle + 3k)w\}l^2(\triangle + 3k)\Pi \\
&+(1+w)\epsilon(f_\rho + 3f_p)/l^2\ , \qquad (2.134)
\end{aligned}
$$

where we have dropped the subscript m.
This equation describes the behavior of density perturbations in the presence of seeds in an arbitrary Friedmann background. We have not used Friedmann's equations to express $\ddot a/a$ in terms of w and $\dot a/a$, or ρ in terms of $(\dot a/a)^2$ so that (2.134) is valid also if there are unperturbed components which contribute to the expansion but not to the perturbation. Note that within this gauge invariant treatment the source term is, up to a factor $(1+w)$, just the naively expected term $4\pi Ga^2(\rho^{(s)} + 3p^{(s)})$ for all types of fluids.

We now simplify equation (2.134) in the case where $\Pi = ? = 0$ (adiabatic perturbations and no anisotropic stresses) and $k = 0$: If one chooses a realistic density parameter $0.2 \leq \Omega_0 \leq 2$, the curvature term can always be neglected at early times, e.g., for redshifts $z \geq 5$. It is of the order $(\max(l, l_H)/l_k)^2$ as compared to the other contributions. (Here l and $l_k = k^{-1/2}$ denote the typical size of the perturbation and the radius of curvature, respectively.) Under these assumptions, eq. (2.134) becomes

$$\begin{aligned} \ddot{D} - c_s^2 \triangle D + (1 + 3c_s^2 - 6w)(\dot{a}/a)\dot{D} - & \\ 3[w(\ddot{a}/a) - 3(\dot{a}/a)^2(c_s^2 - w) + (1+w)(4\pi/3)G\rho a^2]D &= S \,, \end{aligned} \tag{2.135}$$

where $S = (1+w)\epsilon(f_\rho + 3f_p)/l^2$.

We Fourier transform (2.135) (and denote the Fourier transform of D with the same letter):

$$\begin{aligned} \ddot{D} + k^2 c_s^2 D + (1 + 3c_s^2 - 6w)(\dot{a}/a)\dot{D} & \\ -3[w(\ddot{a}/a) - 3(\dot{a}/a)^2(c_s^2 - w) + (1+w)(4\pi/3)G\rho a^2]D &= \tilde{S} \,. \end{aligned} \tag{2.136}$$

$\tilde{S} = (1+w)\epsilon(\tilde{f}_\rho + 3\tilde{f}_p)/l^2$ is the Fourier transform of S.

From the homogeneous solutions D_1 and D_2 of (2.136), we can find the perturbation induced by S with the Wronskian method:

$$D = c_1 D_1 + c_2 D_2 \;; \tag{2.137}$$

$$c_1 = -\int (\tilde{S} D_2/W)dt \quad , \quad c_2 = \int (\tilde{S} D_1/W)dt \quad , \tag{2.138}$$

where $W = D_1\dot{D}_2 - \dot{D}_1 D_2$ is the Wronskian determinant of the homogeneous solution.
This leads to the following general behavior: If the time dependence of D_1, D_2 and $\tilde{S}$ can be approximated by power laws, D behaves like $D \propto \tilde{S}$ as long as $\tilde{S} \neq 0$. If D_1 and D_2 are waves with approximately constant amplitude and frequency ω, D can be approximated by a wave with amplitude proportional to $\omega^{-1}\int e^{i\omega t}\tilde{S}dt$. Thus, only typical frequencies of the source finally survive.

As a second example, we consider collisionless particles (again only scalar perturbations and $k = 0$ are considered). The source term on the r.h.s. of Liouville's equation (2.73) can be separated as above into a part due to the collisionless component and a part induced by the seeds. Equation (2.73) then becomes

$$(q\partial_t + v^k\partial_k)\mathcal{F} = \frac{d\bar{f}}{dv}[(q/v)v^k\partial_k\Psi_m - (v/q)v^k\partial_k\Phi_m] + \mathcal{S} \,, \tag{2.139}$$

with

$$\mathcal{S} = \frac{d\bar{f}}{dv}[(q/v)v^k\partial_k\Psi_s - (v/q)v^k\partial_k\Phi_s] \,. \tag{2.140}$$

In the same way, one obtains for massless particles

$$\partial_t\mathcal{M} + \gamma^i\partial_i\mathcal{M} = 4\gamma^i\partial_i(\Phi_m - \Psi_m) + \mathcal{S} \,, \tag{2.141}$$

with corresponding source term

$$\mathcal{S} = 4\gamma^i\partial_i(\Phi_s - \Psi_s) \,.$$

With the integrals for the fluid variables D_g, V, ? and Π given in Section 3 and Einstein's equations (2.46) to (2.48) for the geometrical perturbations Ψ_m and Φ_m induced by the collisionless component, this forms a closed system.

Chapter 3

Some Applications of Cosmological Perturbation Theory

3.1 Fluctuations of a Perfect Fluid

A perfect fluid is free of anisotropic stresses, i.e. $\Pi = 0$, $\Pi_i = 0$, $\Pi_{ij} = 0$ for scalar, vector and tensor perturbations, respectively. For scalar perturbations the dynamical equation (2.48) then relates the Bardeen potentials:

$$\Psi = -\Phi \ .$$

For vector perturbations we obtain from (2.54)

$$a^2\sigma_i = \text{const.} \ , \text{ i.e. } \ \sigma_i = (a_*/a)^2\sigma_i^* \ .$$

The vector contribution to the shear of the equal time hyper–sufaces of perfect fluid perturbations thus decays like $1/a^2$. For tensor perturbations eq. (2.56) yields

$$\ddot{H}_{ij} + 2(\dot{a}/a)\dot{H}_{ij} + (2k - \triangle)H_{ij} = 0 \ ,$$

which describes a damped gravitational wave with damping scale $\dot{a}/a = t^{-1}$.

The often used notion of *isocurvature fluctuations* is defined by $\Psi = \Phi = 0$ (for scalar perturbations), $\sigma_i = 0$ (for vector perturbations) and $H_{ij} = 0$ (for tensor perturbations) on super–horizon scales. *Adiabatic fluctuations* require ? = 0. In a coupled baryon/photon universe this reduces to $D_g^{(B)} = (4/3)D_g^{(r)}$. Note that fluctuations from topological defects are always isocurvature, since they emerge in a causal way from an initially homogeneous and isotropic Friedmann universe.

Let us now solve the perturbation equations for adiabatic perturbations of a one component perfect fluid with $w = c_s^2$ =const. and negligible spatial curvature. This simplification is a good approximation during some periods of time (e.g. in the radiation dominated epoch). The very simple behavior of vector perturbations in this case is given by equation (2.64). For scalar perturbations equations (2.57–2.58) reduce to

$$\dot{D} - 3w(\dot{a}/a)D = (1+w)l\triangle V \tag{3.1}$$

$$l\dot{V} + (\dot{a}/a)lV = \frac{c_s^2}{w+1}D + \Psi \ . \tag{3.2}$$

Taking the Laplacian of (3.2) (using (2.46) and $\Phi = -\Psi$) and inserting eq. (3.1) we obtain a second order equation for D

$$\ddot{D} + (\dot{a}/a)(1-3w)\dot{D} - [3w(\ddot{a}/a) + (1+w)4\pi G\rho a^2 - c_s^2\triangle]D = 0\ . \tag{3.3}$$

For a spatially flat universe with $-1/3 < w = c_s^2$ =const., the scale factor obeys a power law

$$a = (\nu\sqrt{C}t)^\nu\ , \quad \text{with} \quad \nu = \frac{2}{3w+1} \quad \text{and} \quad C = (8\pi G/3)\rho a^{2(\nu+1)/\nu} = \text{ const.}$$

We now Fourier transform D, so that the Laplacian is replaced by a factor $-k^2$. Equation (3.3) can then be expressed as ordinary differential equation in the dimensionless variable $\eta = kt$. The regime $\eta \ll 1$ describes perturbations with wavelength substantially larger than the size of the horizon and the regime $\eta \gg 1$ describes perturbations with wavelength much smaller than the size of the horizon. Denoting the Fourier transform of D again with D we find

$$D'' + \frac{2(\nu-1)}{\eta}D' - \frac{(2-\nu)(\nu-1) - \nu(\nu+1)}{\eta^2}D + wD = 0\ .$$

In terms of $f = D\eta^{\nu-2} \propto \rho a^3 D$, this equation becomes

$$f'' + \frac{2}{\eta}f' + [w + \frac{\nu(\nu+1)}{\eta^2}]f = 0\ . \tag{3.4}$$

For $w = c_s^2 \neq 0$ this is the well known Bessel differential equation whose general solution is

$$f = Aj_\nu(c_s\eta) + Bn_\nu(c_s\eta) \equiv Z_\nu(c_s\eta) \tag{3.5}$$

(see Abramowitz and Stegun [1970]). For our perturbation variables D, Ψ and V this yields

$$D = \eta^{2-\nu}Z_\nu(c_s\eta) \tag{3.6}$$

$$\Psi = -\frac{2}{3}\nu^2\eta^{-\nu}Z_\nu(c_s\eta) \tag{3.7}$$

$$V = \frac{2}{3}\nu[\eta^{1-\nu}Z_\nu(c_s\eta) - \frac{c_s}{1-\nu}\eta^{2-\nu}Z_{\nu-1}(c_s\eta)]\ , \tag{3.8}$$

where we have used $Z'_\nu = c_s Z_{\nu-1} - (\nu + 1/\eta)Z_\nu$ [Abramowitz and Stegun, 1970]. This solution was originally obtained by Bardeen [1980]. The asymptotic behavior of Bessel functions yields

$$Z_\nu = C\eta^\nu + E\eta^{-(\nu+1)}\ , \qquad \text{for } c_s\eta \ll 1$$

and

$$Z_\nu = \tfrac{A}{\eta}\cos(c_s\eta - \alpha_\nu) + \tfrac{B}{\eta}\sin(c_s\eta - \alpha_\nu)\ , \quad \text{for } c_s\eta \gg 1$$

with $\alpha_\nu = \pi(\nu+1)/2$. Therefore, the "growing" mode behaves like

$$\left.\begin{array}{rcl} \Psi &=& \Psi_0 \\ D &=& D_0\eta^2 \\ V &=& V_0\eta \end{array}\right\} \text{ for } c_s\eta \ll 1 \tag{3.9}$$

and

$$\left.\begin{array}{rcl} \Psi &=& \Psi_0\eta^{-(\nu+1)}\cos(c_s\eta - \alpha_\nu) \\ D &=& D_0\eta^{1-\nu}\cos(c_s\eta - \alpha_\nu) \\ V &=& V_0\eta^{1-\nu}\cos(c_s\eta - \alpha_{\nu-1}) \end{array}\right\} \text{ for } c_s\eta \gg 1\ . \tag{3.10}$$

On scales below the sound horizon, $c_s\eta \gg 1$, density and velocity perturbations grow only if $\nu < 1$, i.e. for $w > 1/3$ (or $w < -1/2$) and they decay for $-1/2 < w < 1/3$. Radiation ($w = 1/3$) represents the limiting case where the amplitude of density perturbations remains constant. Perturbations in the gravitational field always decay for $\nu \geq -1$, i.e. for $w \geq -2/3$. On scales substantially larger than the sound horizon ($c_s\eta \ll 1$) it might seem at first sight that density and velocity perturbations are growing. But one easily establishes that, e.g., the "growing" mode of the alternative gauge invariant density variable $D_g = D - 3(1+w)\Psi - 3(1+w)(\nu/\eta)V$ is constant. Therefore, in a coordinate system where D_g represents the density fluctuation (a slicing with $\mathcal{R} = 0$), density perturbations do not grow. This shows that the behavior of density perturbations on these scales crucially depends on the coordinates and can not be inferred from the growth of the particular gauge invariant variable D. This leads us to define the perturbation amplitude $\mathcal{A}$ on super–horizon scales as the amplitude of the largest perturbation variable in a gauge where this quantity is a minimum. For scalar perturbations thus

$$\mathcal{A} = \min_{\{\text{gauges}\}}(\max\{A, B, H_l, H_T, \delta, v, \pi_L, \pi_T\}) \ .$$

It is clear that this quantity is of the same order of magnitude as the largest gauge invariant variable. In our case therefore

$$\mathcal{A} \approx |\Psi| \ ,$$

i.e., *super–horizon perturbations do not grow in amplitude*, as one would also expect for causality reasons.

Fortunately, this analysis, which shows that no perturbations with $0 < w \leq 1/3$ grow, does not hold for dust ($w = c_s^2 = 0$). In this case, equation (3.3) reduces to

$$\ddot{D} + (2/\eta)\dot{D} - (6/\eta^2)D = 0 \ ,$$

with the general solution $D = At^2 + Bt^{-3}$, yielding the growing mode solution

$$\begin{aligned} \Psi &= \Psi_0 \\ D &= D_0\eta^2 \qquad\qquad \eta = kt \\ V &= V_0\eta \ . \end{aligned}$$

Again on super–horizon scales, $\eta < 1$, the perturbation amplitude is given by Ψ_0 and is thus constant, whereas on sub–horizon scales, $\eta \gg 1$, the largest gauge invariant variable is $D = D_0\eta^2$, i.e., density perturbations do grow on sub–horizon scales.

An additional important case is given by dust perturbations in a radiation dominated universe with $\rho_r \gg \rho_m$ at times $t \ll t_{eq}$, where t_{eq} denotes the time when $\rho_r = \rho_m$, which happens at some time, because ρ_r decays faster ($\propto a^{-4}$) than $\rho_m \propto a^{-3}$. At $t \ll t_{eq}$ the scale factor grows according to $a \propto t$ but $4\pi G\rho_m a^2 \approx 4\pi G\rho_r a^2(t/t_{eq}) = (3/2)\frac{1}{tt_{eq}}$. Equation (3.3) then yields

$$\ddot{D} + (1/t)\dot{D} - \frac{3}{2tt_{eq}}D = 0 \ ,$$

which is approximately solved by

$$D = D_0 \log(t/t_{eq}) \ , \qquad t \ll t_{eq} \ . \tag{3.11}$$

This fact, that even dust perturbations cannot grow substantially in a radiation dominated universe, is called the Mézáros effect [Mézáros, 1974].

We can now draw the following conclusions: For adiabatic perturbations of a perfect fluid, in a universe where spatial curvature is negligible, the time evolution of D, Φ and V is given by

$$D \propto \begin{cases} a\ , & \text{in a matter dominated background,} \\ \log(t/t_*)\ , \quad \text{for } kt \gg 1, & \text{in a radiation dominated background,} \end{cases} \tag{3.12}$$

$$\Phi = -\Psi = \text{const.}\ . \tag{3.13}$$

For pure radiation perturbations ($p = (1/3)\rho$, $? = 0$, $\Pi = 0$) one obtains

$$D \quad \propto \quad \begin{cases} \text{const.}\ , & \text{for } kt \ll 1 \\ \exp(ik(x - \sqrt{1/3}t))\ , & \text{for } kt \gg 1\ , \end{cases} \tag{3.14}$$

$$\Phi = -\Psi \quad \propto \quad \begin{cases} \text{const.}\ , & \text{for } kt \ll 1 \\ \frac{1}{a^2}\exp(ik(x - \sqrt{1/3}t))\ , & \text{for } kt \gg 1\ . \end{cases} \tag{3.15}$$

On super–horizon scales, $kt \ll 1$ the total perturbation amplitude is given by $|\Psi| = |\Phi|$. Thus, even if one special variable like, e.g., D is growing the perturbation amplitude $\mathcal{A}$ as defined above remains constant.

We note the important result: The only substantial growth of linear perturbation is that of sub–horizon sized, pressureless matter density perturbations in a matter dominated universe. Then $D \propto a$ and the gravitational potential is constant.

3.2 The Perturbation of Boltzmann's Equation for Compton Scattering

In this section, we restrict ourselves to scalar type perturbations, i.e., vector and tensor fields can be derived from scalar potentials. Furthermore, we set $k = 0$.

Following Section 2.3, the perturbed photon distribution function is denoted by $f = (\bar{f} + F) \circ \iota$ and lives on the one particle zero mass phase space, $\mathcal{P}_0 = \{(x, p) \in T\mathcal{M} : g(x)(p, p) = 0\}$. Choosing coordinates (x^μ, p^i) on $\mathcal{P}_0$, Boltzmann's equation for f reads

$$p^\mu \partial_\mu f - ?^{\,i}_{\mu\nu} p^\mu p^\nu \frac{\partial f}{\partial p^i} = C[f]\ ,$$

where $C[f]$ is the collision integral depending on the cross section and angular dependence of the interactions considered.

In terms of the gauge invariant perturbation variable $\mathcal{M}$ (see section 2.3.4) the collisionless Boltzmann equation is (2.88)

$$\dot{\mathcal{M}} + \epsilon^i \partial_i \mathcal{M} = 4\epsilon^i \partial_i (\Phi - \Psi)\ , \tag{3.16}$$

with (2.83)

$$\begin{aligned} \mathcal{M} \quad &\equiv \quad \frac{4\pi}{\bar{\rho} a^4} \int_0^\infty \mathcal{F} v^3 dv \\ &= \quad \iota + 4\mathcal{R} + 4l\epsilon^i \partial_i \sigma\ . \end{aligned}$$

To this we have to add the collision term which is given by

$$C[\mathcal{M}] = \frac{4\pi}{\rho a^4}\int v^3 dv C[f] = \frac{4\pi}{\rho a^4}\int v^3 dv \frac{df_+}{dt} - \frac{df_-}{dt} \equiv \frac{d\iota(\boldsymbol{\epsilon})_+}{dt} - \frac{d\iota(\boldsymbol{\epsilon})_-}{dt} ,$$

where f_+ and f_- denote the distribution of photons scattered into respectively out of the beam due to Compton scattering.

In the matter (baryon/electron) rest frame, which we indicate by a prime, we know

$$\frac{df'_+}{dt'}(p, \boldsymbol{\epsilon}) = \frac{\sigma_T n_e}{4\pi}\int f'(p', \boldsymbol{\epsilon}')\omega(\boldsymbol{\epsilon}, \boldsymbol{\epsilon}')d\Omega' ,$$

where n_e denotes the electron number density, σ_T is the Thomson cross section, and ω is the normalized angular dependence of the Thomson cross section:

$$\omega(\boldsymbol{\epsilon}, \boldsymbol{\epsilon}') = 3/4[1 + (\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2] = 1 + \frac{3}{4}\epsilon_{ij}\epsilon'_{ij} \quad \text{with} \quad \epsilon_{ij} = \epsilon_i\epsilon_j - \frac{1}{3}\delta_{ij} .$$

In the baryon rest frame which moves with four velocity u, the photon energy is given by

$$p' = p_\mu u^\mu .$$

We denote by p the photon energy with respect to a tetrad adapted to the slicing of spacetime into $\{t = \text{constant}\}$ hyper–surfaces:

$$p = p_\mu n^\mu , \quad \text{with} \quad n = a^{-1}[(1-A)\partial_t + \beta^i\partial_i] , \ \text{see Chap. 2} .$$

The lapse function and the shift vector of the slicing are given by $\alpha = a(1+A)$ and $\boldsymbol{\beta} = -B^{,i}\partial_i$. In first order,

$$p_0 = ap(1+A) - ap\epsilon_i\beta^i \quad ,$$

and in zeroth order, clearly,

$$p_i = ap\epsilon_i .$$

Furthermore, the baryon four velocity has the form $u^0 = a^{-1}(1-A)$, $u^i = u^0 v^i$ up to first order. This yields

$$p' = p_\mu u^\mu = p(1 + \epsilon_i(v^i - \beta^i)) .$$

Using this identity and performing the integration over photon energies, we obtain

$$\rho_r\frac{d\iota_+(\epsilon)}{dt'} = \rho_r\sigma_T n_e[1 + 4\epsilon_i(v^i - \beta^i) + \frac{1}{4\pi}\int \iota(\epsilon')\omega(\epsilon, \epsilon')d\Omega'] .$$

The distribution of photons scattered out of the beam, has the well known form
(see e.g. Lifshitz and Pitajewski [1983])

$$\frac{df_-}{dt'} = \sigma_T n_e f'(p', \boldsymbol{\epsilon}) ,$$

so that we finally obtain

$$C' = \frac{4\pi}{\rho_r a^4}\int dp(\frac{df_+}{dt'} - \frac{df_-}{dt'})p^3 = \sigma_T n_e[\delta_r - \iota + 4\epsilon_i(v^i - \beta^i) + \frac{3}{16\pi}\epsilon_{ij}\int \iota(\epsilon')\epsilon'_{ij}d\Omega'] ,$$

where $\delta_r = (1/4\pi)\int \iota(\epsilon)d\Omega$ is the photon energy density perturbation.
Using the definitions of the gauge–invariant variables $\mathcal{M}$ and V, we can write C' in gauge–invariant form.

$$C' = \sigma_T n_e[D_g^{(r)} - \mathcal{M} + 4\epsilon_i l\partial_i V + \frac{1}{2}\epsilon_{ij}M^{ij}] \ , \tag{3.17}$$

with $D_g^{(r)} = (1/4\pi)\int \mathcal{M}d\Omega = \delta_r + 4\mathcal{R}$ and

$$M^{ij} = \frac{3}{8\pi}\int \mathcal{M}(\epsilon')\epsilon'_{ij}d\Omega'] \ .$$

Since the term in square brackets of (3.17) is already first order we have $C = \frac{dt'}{dt}C' = aC'$. So that the Boltzmann equation becomes

$$\dot{\mathcal{M}} + \epsilon^i\partial_i\mathcal{M} = 4\epsilon^i\partial_i(\Phi - \Psi) + a\sigma_T n_e[D_g^{(r)} - \mathcal{M} - 4\epsilon^i l\partial_i V + \frac{1}{2}\epsilon_{ij}M^{ij}] \ . \tag{3.18}$$

In the next two subsections we rewrite this equation for two special cases.

Note that perturbations of the electron density, $n_e = \bar{n}_e + \delta n_e$ do not contribute in first order for a homogeneous and isotropic background distribution of photons. The only first order term which accounts for the perturbations of baryons and therefore acts as a source term for the photon perturbations is the Doppler term $4\epsilon^i\partial_i V$ which is due to the relative motion of baryons and photons. A comparison of the numerical integration of equation (3.18) with and without collision term (with a gravitational potential $\Psi - \Phi$ originating from a collapsing texture) is shown in Fig. 7. There one sees that the collision term has two effects:

1) Damping of the perturbations by several orders of magnitude.

2) Broadening of the signal to about 7^o (FWHM) which corresponds to the horizon scale at the time when Compton scattering 'freezes out', i.e. $t_T \approx t$ and the mean free path of photons becomes larger than the size of the horizon. In general, there will always be the question whether the damping term, $D_g^{(r)} - \mathcal{M}$, or the source term, $4\epsilon^i\partial_i V$, in equation (3.18) wins. It seems that for standard CDM the source term is so strong, that no damping due to photon diffusion occurs [Efstathiou, private communication]. It is an interesting and partially unsolved problem, in which scenarios of structure formation, photon perturbations on small scales ($\theta \leq 7^o$) are effectively damped by reionization. It seems plausible to me, that models with topological defects, where perturbations are highly correlated, are affected more strongly than models with Gaussian perturbations.

The collision term above also appears in the equation of motion of the baryons as a drag. The Thomson drag force is given by

$$F_j = \frac{a\sigma_T n_e\rho_r}{4\pi}\int C[\mathcal{M}]\epsilon_j d\Omega = \frac{a\sigma_T n_e\rho_r}{3}(M_j + 4l\partial_i V) \ , \tag{3.19}$$

$$\text{with} \quad M_j = \frac{3}{4\pi}\int \epsilon_j\mathcal{M}d\Omega \ .$$

This yields the following baryon equation of motion in an ionized plasma

$$l\partial_j\dot{V} + (\dot{a}/a)l\partial_i V = \partial_i\Psi - \frac{a\sigma_T n_e\rho_r}{3\rho_b}(M_j + 4l\partial_i V) \ , \tag{3.20}$$

where we have added the drag force to eq. (2.58) with $w = c_s^2 = 0$.

We now want to discuss equations (3.18,3.20) in the limit of very many collisions. Clearly the photon mean free path is given by $t_T = l_T = (a\sigma_T n_e)^{-1}$. In lowest order t_T/t and l_T/l these equations reduce to

$$D_g^{(r)} + \frac{1}{2}\epsilon_{ij}M^{ij} + 4\epsilon^i l\partial_i V = \mathcal{M} \tag{3.21}$$

and

$$4l\triangle V = \partial_i M_i = 3\dot{D}_g^{(r)} \ , \tag{3.22}$$

where we made use of (3.21) and (3.28) below, for the last equal sign. Eq. (3.22) is equivalent to (2.60) for radiation. Using also (2.60) for baryons, $w = 0$, we obtain

$$\dot{D}_g^{(r)} = \frac{4}{3}l\triangle V = \frac{4}{3}\dot{D}_g^{(m)}.$$

This shows that entropy per baryon is conserved, $? = 0$. Inserting (3.21) in (3.18) we find up to first order in t_T

$$\begin{aligned}\mathcal{M} &= D_g^{(r)} - 4l\epsilon^i\partial_i V + \frac{1}{2}\epsilon_{ij}M^{ij} - t_T[\dot{D}_g^{(r)} - 4l\epsilon^i\partial_i\dot{V} + \frac{1}{2}\epsilon ij\dot{M}^{ij} \\ &\quad +\epsilon^j\partial_j D_g^{(r)} - 4l\epsilon^i\epsilon^j\partial_i\partial_j V + \frac{1}{2}\epsilon^i\epsilon kj\partial_i M^{kj} - 4\epsilon^j\partial_j(\Phi - \Psi)] \ .\end{aligned} \tag{3.23}$$

Using (3.23) to calculate the drag force yields

$$F_i = (\rho_r/3)[4l\partial_i V - \partial_i D_g^{(r)} + 4\partial_i(\Phi - \Psi)] \ .$$

Inserting F_i in (3.20), we obtain

$$(\rho_b + (4/3)\rho_r)l\partial_i\dot{V} + \rho_b(\dot{a}/a)l\partial_i V = (\rho_r/3)\partial_i D_g^{(r)} + (\rho_b + (4/3)\rho_r)\partial_i\Psi - (4\rho_r/3)\partial_i\Phi \ .$$

This is equivalent to (2.61) for $\rho = \rho_b + \rho_r$, $p = \rho_r/3$ and $? = \Pi = 0$, if we use

$$D_g^{(r)} = (4/3)D_g^{(m)} \quad \text{and} \quad D_g = \frac{\rho_r D_g^{(r)} + \rho_m D_g^{(m)}}{\rho_m + \rho_r} \ .$$

In this limit therefore, baryons and photons behave like a single fluid with density $\rho = \rho_r + \rho_m$ and pressure $p = \rho_r/3$.

From (2.60) and (2.61) we can derive a second order equation for D_g. To discuss the coupled matter radiation fluid we regard a plane wave $D = D(t)\exp(i\boldsymbol{k}\cdot\boldsymbol{x})$. We then obtain

$$\ddot{D} + c_s^2k^2D + (1 + 3c_s^2 - 6w)(\dot{a}/a)\dot{D} - 3[w(\ddot{a}/a) - (\dot{a}/a)(3(c_s^2 - w) - (1/2)(1 + w))]D = 0 \ .$$

For small wavelengths (which are required for the fluid approximation to be valid), $1/t_T \gg c_sk \gg 1/t$, we may drop the term in square brackets. The ansatz $D(t) = A(t)\exp(-i\int kc_s dt)$ then eliminates the terms of order $c_s^2k^2$. For the terms of order c_sk/t we obtain the equation

$$2\dot{A}/A + (1 - 3c_s^2 - 6w)(\dot{a}/a) + \dot{c}_s/c_s = 0 \ . \tag{3.24}$$

For the case $c_s^2 = w$ =const. , this equation is solved by $A \propto (kt)^{1-\nu}$ with $\nu = 2/(3w+1)$, i.e., the short wave limit (3.10). In our situation we have

$$\begin{aligned} w &= \frac{\rho_r}{3(\rho_r + \rho_m)} \\ c_s^2 &= \frac{\dot{\rho_r}}{3(\dot{\rho_r} + \dot{\rho_m})} = \frac{(4/3)\rho_r}{4\rho_r + 3\rho_m} \quad \text{and} \\ \dot{c}_s/c_s &= -3/2(\dot{a}/a)\frac{\rho_m}{4\rho_r + 3\rho_m} \ . \end{aligned}$$

Using all this, one finds that

$$A = \left(\frac{\rho_m + (4/3)\rho_r}{c_s(\rho_r + \rho_m)^2 a^4}\right)^{1/2} = \left(\frac{\rho + p}{c_s \rho^2 a^4}\right)^{1/2}$$

solves (3.24) exactly, so that we finally obtain the approximate solution for the, tightly coupled matter radiation fluid

$$D(t) \propto \left(\frac{\rho + p}{c_s \rho^2 a^4}\right)^{1/2} \exp(-ik\int c_s dt) \ . \tag{3.25}$$

Note that this short wave approximation is only valid in the limit $t \gg 1/(c_s k)$, thus the limit to the matter dominated regime, $c_s \to 0$ cannot be performed. In the limit to the radiation dominated regime, $c_s^2 \to 1/3$ and $\rho \propto a^{-4}$ we recover the acoustic waves with constant amplitude which we have already found in the last subsection. But also in this limit, we still need matter to ensure $t_T = 1/(a\sigma_T n_e) \ll t$. In the oppotite case, $t_T \gg t$, radiation does not behave like an ideal fluid but it becomes collisionless and has to be treated with Liouville's equation (3.16).

In the paragraph 3.2.3 we evolve $\mathcal{M}$ up to order t_T^2 and obtain the damping of $\mathcal{M}$ by photon diffusion.

3.2.1 Spherically symmetric Boltzmann equation

In the spherically symmetric case $\mathcal{M}$ depends on the momentum direction $\boldsymbol{\epsilon}$ only via the variable $\mu = (\boldsymbol{r} \cdot \boldsymbol{\epsilon})/r$. Integrating out the independent angle φ and choosing the coordinates in photon momentum space such that the third axis is parallel to $\boldsymbol{r}$, one finds

$$\frac{1}{2\pi}\int_0^{2\pi} \epsilon_{ij} d\varphi = \left\{ \begin{array}{ll} 0 \ , & i \neq j \\ \mu^2 - 1/3 \ , & (i,j) = (3,3) \\ \frac{1}{2}(1/3 - \mu^2) \ , & (i,j) = (1,1) \text{ or } (2,2) \ . \end{array} \right.$$

We thus obtain

$$\int_0^{2\pi} d\varphi[\epsilon_{ij} M^{ij}] = \frac{3}{4}(\mu^2 - 1/3)\int_{-1}^{1}(\mu'^2 - 1/3)\mathcal{M} d\mu' \equiv \frac{3}{2}(\mu^2 - \frac{1}{3})M_2 \ .$$

For $\mathcal{M}(t, r, \mu)$ we further have

$$\epsilon^i \partial_i \mathcal{M} = \mu \partial_r \mathcal{M} + \frac{1 - \mu^2}{r}\partial_\mu \mathcal{M}$$

so that Boltzmann's equation becomes

$$\dot{\mathcal{M}} + \mu\partial_r\mathcal{M} + \frac{1-\mu^2}{r}\partial_\mu\mathcal{M} = 4\mu\partial_r(\Phi-\Psi) +$$
$$a\sigma_T n_e[M - \mathcal{M} - 4\mu\partial_r V + (\mu^2 - 1/3)\frac{3}{2}M_2] \ . \quad (3.26)$$

This is the equation which we have integrated numerically, (in somewhat different coordinates, see Chap. 4) for a $\Phi-\Psi$ from a spherically symmetric texture, to produce Fig. 7.

3.2.2 Moment expansion

As long as collisions are efficient, $t > 1/(a\sigma_T n_e) \equiv t_T$, it is reasonable to truncate $\mathcal{M}$ at second moments

$$\mathcal{M} = D_g^{(r)} + \epsilon_i M^i + 5\epsilon_{ij}M^{ij} \ . \quad (3.27)$$

With this ansatz, the integration over directions of the zeroth, first and second moments of (3.18) yields

$$\dot{D}_g^{(r)} + \frac{1}{3}\partial_i M^i = 0 \ , \quad (3.28)$$

$$\dot{M}^j + \partial_j D_g^{(r)} + 2\partial_i M^{ij} = 4\partial_j(\Phi-\Psi) - a\sigma_T n_e[M^j + 4\partial_j lV] \ , \quad (3.29)$$

$$\dot{M}^{ij} + \frac{1}{15}(\partial_i M^j + \partial_j M^i) = -\frac{9}{10}a\sigma_T n_e M^{ij} \ . \quad (3.30)$$

Taking the total divergence of (3.29) and (3.30), we obtain with the help of (3.28)

$$\ddot{D}_g^{(r)} - \frac{1}{3}\triangle D_g^{(r)} - \frac{2}{3}\alpha = -\frac{4}{3}\triangle(\Phi-\Psi) + a\sigma_T n_e[\dot{D}_g^{(r)} - \frac{4}{3}\triangle lV] \quad (3.31)$$

and

$$\dot{\alpha} - \frac{2}{5}\triangle\dot{D}_g^{(r)} = -\frac{9}{10}a\sigma_T n_e\alpha \ , \quad (3.32)$$

where we have set $\alpha = \partial_i\partial_j M^{ij}$.

3.2.3 Damping by photon diffusion

In this subsection we want to estimate the damping of CBR fluctuations in an ionized plasma using our gauge invariant approach, as it was done by Peebles [1980] within synchronous gauge. We again consider eqs. (3.18) and (3.20), but since we are mainly interested in collisions which take place on time scales $t_T \ll t$, we neglect gravitational effects and the time dependence of the coefficients. We can then look for solutions of the form

$$V \propto \mathcal{M} \propto \exp(i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)) \ .$$

In (3.18) and (3.20) this yields (neglecting also the angular dependence of Compton scattering, i.e., the term $\epsilon_{ij}M^{ij}$)

$$\mathcal{M} = \frac{D_g^{(r)} - 4i\boldsymbol{k}\boldsymbol{\epsilon}lV}{1 - it_T(\omega - \boldsymbol{k}\cdot\boldsymbol{\epsilon})} \tag{3.33}$$

and

$$t_T\boldsymbol{k}\omega lV = (\rho_r/3\rho_m)(4i\boldsymbol{k}lV + \boldsymbol{M}) \ , \tag{3.34}$$

with $\boldsymbol{M} = (3/4\pi)\int \boldsymbol{\epsilon}\mathcal{M}d\Omega$. Integrating (3.18) over angles, one obtains $\dot{D}_g^{(r)} + (1/3)\partial_i M^i = 0$. With our ansatz therefore $\boldsymbol{k}\cdot\boldsymbol{M} = 3\omega D_g^{(r)}$. Using this after scalar multiplication of (3.34) with $\boldsymbol{k}$, we find, setting $R = 3\rho_m/4\rho_r$,

$$lV = \frac{(3/4)\omega D_g^{(r)}}{t_T k^2 R\omega - ik^2} \ .$$

Inserting this result for V in (3.33) leads to

$$\mathcal{M} = D_g^{(r)}\frac{1 + \frac{3\mu\omega/k}{1-it_T\omega R}}{1 - it_T(\omega - k\mu)} \ ,$$

where we have set $\mu = \boldsymbol{k}\cdot\boldsymbol{\epsilon}/k$. This is exactly the same result as in Peebles [1980], where this calculation is performed in synchronous gauge. Like in there (§92), one obtains in lowest order ωt_T the dispersion relation

$$\omega = \omega_0 - i\gamma \ \text{ with } \ \omega_0 = k/[3(1+R)]^{1/2} \ \text{ and } \ \gamma = (k^2t_T/6)\frac{R^2 + \frac{4}{5}(R+1)}{(R+1)^2} \ . \tag{3.35}$$

In the matter dominated regime, $R \gg 1$, therefore

$$\gamma \approx k^2t_T/6 \ . \tag{3.36}$$

We now consider a texture which collapses in a matter dominated universe ($a \propto t^2$) at time $t_c = t_0/(1+z_c)^{1/2}$. The total damping, $\exp(-f)$, which this perturbation experiences is given by the integral

$$f \approx \int_{t_c}^{t_{end}} \gamma(t)dt \ . \tag{3.37}$$

The end time t_{end} is the time, when the mean free path, t_T equals t_c, the size of the perturbation. We define z_{dec} as the redshift when photons and baryons decouple due to free streaming. This is about the time when the mean free path has grown up to the size of the horizon: $t_T(z_{dec}) \approx t(z_{dec})$. To obtain exponential damping ($t_c < t_{end}$), we thus need $z_c > z_{dec}$. In this case damping is effective until

$$1 + z_{end} = (1+z_{dec})^{3/4}(1+z_c)^{1/4} \ ,$$

and we obtain

$$f \approx 2\left(\frac{1+z_{dec}}{1+z_c}\right)^{3/2}\left[\left(\frac{1+z_c}{1+z_{dec}}\right)^{15/8} - 1\right] \ , \tag{3.38}$$

where we have set $k^2 = (2\pi/t_c)^2$. In terms of angles this yields

$$f(\theta) \approx 2(\theta/\theta_d)^3[(\theta_d/\theta)^{15/4} - 1] \ , \quad \theta < \theta_d$$

where $\theta_d = 1/\sqrt{1+z_{dec}} \approx 6^0$.

If the plasma ionizes at a redshift z_i, $z_{dec} < z_i < z_c$, after the texture has already collapsed, damping is only effective after z_i, if $z_{end} < z_i$. Instead of formula (3.38), we then obtain

$$f \approx 2\left(\frac{1+z_{dec}}{1+z_c}\right)^{3/2}\left[\left(\frac{1+z_c}{1+z_{dec}}\right)^{15/8} - \left(\frac{1+z_c}{1+z_i}\right)^{15/8}\right] \ . \tag{3.39}$$

This damping factor can again be converted in an angular damping factor in the above, obvious way. The CBR signal of textures is thus exponentially damped only if

$$z_i > z_{dec} \approx 100(0.05/h_{50}\Omega_B)^{2/3} \ ,$$

and in this case only textures with $1+z_{dec} < 1+z_c < (1+z_i)^2/(1+z_{dec})$ are affected. So, if $z_i \leq 50$, there is little exponential damping, even if $h_{50} = 2$ and $\Omega_B = 0.1$, but if $z_i = 200$, all textures which collapse at redshifts $z_i \geq z_c$ are damped by a factor 10 or more.

In this approximation, we have neglected the the amount of damping which still may occur after z_{end} and the induced fluctuations due to the source term $\propto lV$ in (3.18). A more accurate numerical treatment where just equation (3.18) is solved is shown in Fig. 7.

Astonishingly, even perturbations on large scales up to the quadrupole may be damped by photon diffusion if the spectrum is steep enough (Peebles, private communication): We assume perturbations of a given size l are uncorrelated and have an average amplitude A. On a larger scale $L = Nl$, they statistically induce perturbations with typical amplitude $AN^{-3/2} = A(l/L)^{3/2}$. (A cube of size L^3 contains N^3 cubes of size l. The statistical residual of N^3 perturbations with amplitude A is thus $AN^{-3/2}$.)

This simple argument has two interesting conclusions:

i) The effective spectrum $|D(k)|^2$ of Gaussian distributed fluctuations cannot decrease faster than $|D(k)|^2 \propto k^3$ towards large scales.

ii) In the limiting case, $|D(k)|^2 \propto k^3$, all the power in large scales is induced due to the statistical residuals of small scale perturbations. Therefore if small scale perturbations are damped so are large scale perturbations up to the size of the horizon.

It is clear that this mechanism crucially depends on the assumption of uncorrelated perturbations. It is thus not effective for fluctuations induced by topological defects. It also does not work for a Harrison–Zel'dovich spectrum, $|D(k)|^2 \propto k$.

3.3 Perturbations of the Microwave Background

The redshift of photons propagating in a perturbed Friedmann geometry is given by (for scalar perturbations) (2.102):

$$\frac{\delta E}{E} = [\frac{1}{4}D_g^{(r)} - V_j n^j + \Psi - \Phi]|_i^f - \int_i^f(\dot\Psi - \dot\Phi)d\tau \ . \tag{3.40}$$

The first of these terms is due to intrinsic fluctuations on the surface of last scattering, the second term is the usual, special relativistic Doppler shift, the third and the last terms are gravitational redshift contributions, the Sachs—Wolfe effect [Sachs and Wolfe, 1967]: The third term is the difference of the potential at the emitter and receiver and the last term is due to the time dependence of the gravitational potential along the path of the photon.

Since perturbations of the cosmic microwave background (CMB) are probably the most reliable observational tool for investigating the initial perturbation spectrum, and since they are calculable within linear perturbation analysis, we present them here in some detail. There are seven different physical mechanisms which perturb the microwave background on different scales. The first four of them are given by equation (3.40):

- Intrinsic inhomogeneities on the last scattering surface,

$$\frac{\Delta T}{T} = \frac{1}{4} D_r \ ,$$

- relative motions of emitter and observer,

$$\frac{\Delta T}{T} = -\boldsymbol{V} \cdot \boldsymbol{n}|_i^f \ ,$$

- the difference of the gravitational potential at the position of emitter and observer (Sachs–Wolfe I),

$$\frac{\Delta T}{T} = (\Psi - \Phi)|_i^f \ ,$$

- and the time dependence of the gravitational field along the path of the photon (Sachs–Wolfe II),

$$\frac{\Delta T}{T} = \int_i^f (\dot{\Psi} - \dot{\Phi}) d\tau \ .$$

- In an intergalactic ionized plasma, fluctuations are damped by photon diffusion (Section 3.2.3). As long as the mean free path of photons is considerably smaller than the size of the horizon, $l_T = 1/(\sigma_T n_e) \ll l_H$, this damping increases exponentially with the damping rate $\gamma \propto l_T/l^2$ and is thus very effective for inhomogeneities with sizes smaller or of the order of the Thompson mean free path, $l \leq l_T$

$$(\frac{\delta T}{T})_f = (\frac{\delta T}{T})_i \exp(-\int_i^f \gamma dt) \ .$$

  Clearly, damping by photon diffusion can only be efficient as long as $l_T < l_H$. At $z \approx 100$, $l_T \approx l_H$ (for $\Omega = 1$ and $\Omega_B h = 0.05$) and angular scales which are larger than the horizon scale at $z \approx 100$, which corresponds to about 6^o, are usually not affected (for an exception of this rule, see the note at the end of the last section).

- On the other hand, the passage of photons through a cloud of ionized gas at a temperature T_e different from the photon temperature T induces deviations in the black body spectrum, which can be cast in the so called Compton y parameter. This is the Sunyaev–Zel'dovich effect [Sunyaev and Zel'dovich, 1973/80]. In the Rayleigh–Jeans regime the change of the spectrum corresponds to a temperature shift according to

$$\frac{\Delta T}{T} \approx -2y \text{ , with } y = \int_i^f (\frac{T_e - T}{m_e}) n_e \sigma_T d\tau \ .$$

- Since the Bardeen potentials of linear dust perturbations are constant, they do not give rise to a path dependent contribution to the photon redshift. But once a dust perturbation has become non–linear and is virialized so that its density $\rho D \approx$ const. , the corresponding gravitational potential Ψ grows like a^2. Usually, the gravitational potential of such a dust cloud is still very small, so that the redshift of photons passing through it can be calculated in linear perturbation theory. Equation (3.40) then yields

$$\frac{\Delta T}{T} = -2(\Phi_i/a_i^2)(a_f^2 - a_i^2) \approx 4\Psi \frac{\Delta z}{1+z} \ ,$$

  where Δz denotes the redshift difference of the two ends of the structure and z is the average redshift of the perturbation. Typically not only Ψ but also Δz is very small. This non–linear contribution to $\Delta T/T$ is called the Rees–Sciama effect [Rees and Sciama, 1968].

From these seven effects, for a long time only the Doppler term, point two, was detected. It leads to the famous dipole anisotropy with amplitude $(\Delta T/T)_{dipole} \approx 2 \times 10^{-3}$ which shows that we are moving with a relative velocity of about $600km/s$ with respect to the microwave background. Now, the COBE team confirmed that also the spectrum of the dipole anisotropy is the derivative of a blackbody spectrum with $T = 2.731K$ to an accuracy better than 1% [Mather et al., 1993].

Recently, also the Sunyaev–Zel‘dovich effect has been confirmed [McHardy et al., 1990, Birkinshaw et al., 1991], and with the COBE satellite [Wright et al., 1992, Smoot et al., 1992], the sum of intrinsic fluctuations and the Sachs–Wolfe effect (points one, three and four) have been measured. With these recent observations, the cosmic microwave background has started to become a very successful tool for investigating cosmological perturbations on the linear level.

3.4 Light Deflection

We now want to present some applications of the effect of light deflection in a perturbed Friedmann universe. The general formulae are derived in Section 2.4.

3.4.1 Monopoles

As a first example, we discuss light deflection and lensing in the field of a global monopole, see also Barriola and Vilenkin [1989]. We discuss the simple, static hedgehog solution of a three component scalar field with $\phi^2 = \eta^2$, i.e., a non–linear sigma model on $\mathbf{S}^2$:

$$\phi^i = \eta x^i/r \ .$$

This is an infinite action and infinite energy solution and should thus not be taken seriously at large distances. In a cosmological context, when monopoles form via the Kibble mechanism during a symmetry breaking phase transition, the hedgehog solution may be approximately valid on distances small compared to the distance to the next monopole or antimonopole, which is about horizon distance. This is also the scale where the approximation of a static, i.e., non–expanding background, which we adopt here, breaks down.

The energy momentum tensor of the hedgehog solution is readily calculated:

$$\begin{aligned} T_{00} &= \eta^2/r^2 \\ T_{ij} &= -\eta^2 x^i x^j/r^4 \end{aligned}$$

Setting $\eta^2 = M^2$ we obtain, using the definitions (2.117) to (2.119),

$$f_\rho = l^2/r^2 \tag{3.41}$$

$$f_v = 0 \tag{3.42}$$

$$f_p = -(1/3) f_\rho \tag{3.43}$$

$$f_\pi = -(1/2)\log(r/l) \ . \tag{3.44}$$

It is interesting to note that the quantity $f_\rho + 3f_p$ which enters as source term in the evolution equation for density perturbations (2.134) vanishes, which shows that static global monopoles do not produce an attractive gravitational force, much like cosmic strings.

Setting $\epsilon = 8\pi G\eta^2$, we find from (2.126) and (2.128)

$$\Phi = -\epsilon\log(r/l) \ , \qquad \Psi = 0 \ . \tag{3.45}$$

The second of these equations again shows that the analog to the Newtonian potential vanishes for global monopoles, like it does for cosmic strings (exercise for the reader). The divergence of Φ at large distances reflects the infinite energy of our solution which needs a physical cutoff (at most at the distance to the next monopole).

We consider a photon passing the monopole at $t = 0$, say with an impact parameter b. Its unperturbed trajectory is then given by $\boldsymbol{x} = \lambda\boldsymbol{n} + b\boldsymbol{e}$. According to (2.114), neglecting spherical aberration, we obtain

$$\varphi = +\int_i^f (\Phi - \Psi)_{,i} e^i d\lambda = -\epsilon\int_i^f \frac{b}{\lambda^2 + b^2} d\lambda = -\epsilon\arctan(\lambda/b) \approx -\epsilon\pi = -8\pi^2 G\eta^2 \ . \tag{3.46}$$

This result was originally obtained (by completely different means) by Barriola and Vilenkin [1989].

We want to investigate the situation of gravitational lensing. First, we treat the special case, where source (S), monopole (M) and observer (O) are perfectly aligned at distances $\overline{SM} = s$ and $\overline{MO} = d$ from each other. If emitted at a small angle $\alpha = b/s$, a photon will reach the observer at an angle $\beta = b/d$, if the deflection angle $|\varphi| = \alpha + \beta = b(s + d)/sd$. This leads to an Einstein ring with opening angle

$$\beta = b/d = |\varphi| s/(s + d) = 8\pi^2 G\eta^2 s/(s + d) \ . \tag{3.47}$$

If observer and source are slightly misaligned by an angle less than $|\varphi|$, the ring is reduced to two points with the same angular separation.

For monopoles produced at a typical GUT scale, $\eta \approx 10^{16} GeV$, the deflection angle is $|\varphi| \approx 10$ arcsec and thus observable. Since the density of global monopoles is about one per horizon citeBR, roughly 10 monopoles present at a redshift $z = 4$ would be visible for us today. The probability that one of them is within less than 10 arcsec of a quasar with redshift $z > 4$, so that the lensing event discussed above could occur, is very small indeed.

3.4.2 Light deflection due to gravitational waves

We discuss formula (2.116) for light deflection due to a passing gravitational wave pulse for which the difference of the gravitational field before and after the passage of the wave is negligible:

$$\varphi e_j = \int_i^f (H_{lk},_j + \dot{H}_{lk} n_j) n^k n^l d\lambda \ . \tag{3.48}$$

We consider a plane wave,

$$H_{kl} = \Re(\epsilon_{kl} \exp(i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t))) \quad \text{with} \quad \epsilon_{kl} k^l = 0 \ .$$

For a photon with unperturbed trajectory $\boldsymbol{x} = \boldsymbol{x}_o + \lambda \boldsymbol{n}$, $t = \lambda$, we obtain

$$\varphi e_j = \begin{cases} \Re[ie^{(i\boldsymbol{k} \cdot \boldsymbol{x_o})} \epsilon_{lm} n^l n^m \frac{k_j - \omega n_j}{\boldsymbol{\omega - k \cdot n}} [e^{(i(\boldsymbol{k} \cdot \boldsymbol{n} - \omega)\lambda_f)} - e^{(i(\boldsymbol{k} \cdot \boldsymbol{n} - \omega)\lambda_i)}]] & \text{for} \quad \boldsymbol{k} \neq \omega \boldsymbol{n} \\ 0 & \text{for} \quad \boldsymbol{k} = \omega \boldsymbol{n} \ . \end{cases}$$

Setting $\boldsymbol{n} = (p/\omega)\boldsymbol{k} + q\boldsymbol{n}_-$ with $\boldsymbol{n}_-^2 = 1$ and $p^2 + q^2 = 1$, we have $\epsilon_{lm} n^l n^m = q^2 \epsilon_-$, where $\epsilon_- = \epsilon_{ij} n_-^i n_-^j$. Inserting this above, we find for the deflection angle an equation of the form

$$\varphi = \epsilon_- \frac{\sqrt{2} q^2}{\sqrt{1-p}} \cos(\alpha + \omega(p-1)t) \ . \tag{3.49}$$

Here ϵ_- is determined by the amplitude of the gravitational wave and q , $p^2 = 1 - q^2$ is determined by the intersection angle of the photon with the gravitational wave as explained above.

This effect for a gravitational wave from two coalescing black holes would be quite remarkable: Since for this (most prominent) event ϵ_- can be as large as $\approx 0.1(R_s/r)$, the rays of sources behind the black hole with impact parameters up to $b < 10^4 R_S$ would be deflected by a measurable amount:

$$\varphi \approx 2''(10^4 R_S/b) \ .$$

Setting the the source at distance d_{LS} from the coalescing black holes and at distance d_S from us, we observe a deviation angle

$$\alpha = \varphi d_{LS}/d_S \ .$$

The best source candidates would thus be quasars for which d_{LS}/d_S is of order unity for all coalescing black holes with, say $z < 1$. In the vicinity of the black holes ($r \leq 10R_S$), linear perturbation theory is of course not applicable and also the 'monopole' component of the gravitational field is not negligible. But in the wide range $10^7 R_S > b > 10 R_S$ (for radio sources) and $10^4 R_S > b > 10 R_S$ (for optical sources) our calculation is valid and leads to an effect that is in principle detectable.

A thorough investigation of the possibility of detecting gravitational waves of coalescing black holes out to cosmological distances by this effect may be worth while.

Chapter 4

Textures in Flat Space

4.1 The σ–Model Approximation for Texture Dynamics

Global texture occurs in every symmetry breaking phase transition where a global symmetry group G is broken to a sub–group H, such that $\pi_3(G/H) \neq 1$.

In the cooler, broken symmetry phase the Higgs field remains most of the time in the vacuum manifold $M \equiv G/H$, where the effective potential assumes its minimum value. It only leaves M if the gradients, i.e., the kinetic energy of the massless Goldstone modes, become comparable to the symmetry breaking scale η. In order to discuss the dynamics of the massless modes at temperatures $T \ll T_c \approx \eta$, it is thus sufficient to neglect the potential and, instead, fix the Higgs field in M with a Lagrange multiplier.

For illustration, and since we believe that the results remains valid at least qualitatively also for other symmetry groups, we discuss the simplest version, an $SU(2)$ symmetry which is completely broken by a $\mathbf{C}^2$ valued scalar field ϕ. In this case, clearly $M \equiv SU(2) \equiv \mathbf{S}^3$. Here $\mathbf{S}^n$ denotes the n–sphere and $\equiv$ means topological equivalence.

The Higgs field ϕ is described by the zero temperature Lagrangian

$$L = \frac{1}{2}\phi_{,\mu}\phi^{,\mu} - \lambda(\phi \cdot \phi - \eta^2)^2 \ , \tag{4.1}$$

with corresponding field equation

$$\Box\phi + 4\lambda\phi(\phi \cdot \phi - \eta^2) = 0 \ . \tag{4.2}$$

As argued above, on energy scales well below the symmetry breaking scale η we may fix $\phi^2 = \eta^2$ by the Lagrangian multiplier $\lambda(\phi \cdot \phi - \eta^2)$. Instead of the usual equations of motion, we then obtain the scale invariant equation

$$\Box\phi - \eta^{-2}(\phi \cdot \Box\phi)\phi = 0 \ , \tag{4.3}$$

i.e., ϕ describes a harmonic map from spacetime to $\mathbf{S}^3$. If ϕ is asymptotically parallel, $\phi(r) \rightarrow_{r\rightarrow\infty} \phi_0$, we can regard it (at fixed time) as a map from compactified three dimensional space ($\equiv \mathbf{S}^3$) into $\mathbf{S}^3$. The degree (winding number) of this map is called the texture winding number. It is given by the integral of the closed three–form

$$\omega = \frac{1}{12\pi^2}\epsilon_{abcd}\phi^a d\phi^b \wedge d\phi^c \wedge d\phi^d \quad ,$$

(ω is nothing else than the pullback of the volume form on $\mathbf{S}^3$). This integral over some region of space (e.g. a horizon volume) is of course well defined also if ϕ is not parallel at infinity and is often referred to as fractional texture winding number. Numerical simulations show that a texture starts collapsing as soon as the fractional winding inside the horizon exceeds about 0.5 [Leese and Prokopec, 1991, Borill et al., 1991].

In our case even an exact spherically symmetric solution to equation (4.3) is known: A spherically symmetric ansatz for ϕ is

$$\phi(\mathbf{r},t) = \eta(\hat{\mathbf{r}}\sin\chi, \cos\chi) \ , \tag{4.4}$$

where $\hat{\mathbf{r}}$ denotes the unit vector in direction of $\mathbf{r}$ and χ is an angular variable depending on r and t only. In flat space the field equations for ϕ yield

$$-\partial_t^2\chi + \partial_r^2\chi + \frac{2}{r}\partial_r\chi = \frac{\sin 2\chi}{r^2} \ . \tag{4.5}$$

We now look for solutions which depend only on the self similarity variable $y = (t-t_c)/r$, where t_c is an arbitrary time constant. In terms of y, equation (4.5) becomes

$$(y^2-1)\chi'' = \sin 2\chi \ ,$$

with the exact solutions $\chi = 2\arctan(\pm y) \pm n\pi$ which were found by Turok and Spergel [1990]. To describe a collapsing texture which has winding number 1 for $t < t_c$ and winding number 0 for $t > t_c$, we patch together these solutions in the following way:

$$\chi(y) = \left\{ \begin{array}{ll} 2\arctan y + \pi \ , & -\infty \le y \le 1 \\ 2\arctan(1/y) + \pi \ , & 1 \le y \le \infty \ . \end{array} \right. \tag{4.6}$$

It is straight forward to calculate the integral of the density ω given above for this example and one of course obtains

$$\int_{\mathbf{R}^3} \omega(t) = \left\{ \begin{array}{ll} 1, & \text{if } t < t_c \\ 0, & \text{if } t > t_c \ . \end{array} \right.$$

The kink of χ at $y = 1$ reflects the singularity of the σ–model approach at the unwinding event $t - t_c = r = 0$. There the gradient energy of the solution (4.6) diverges, i.e., becomes bigger than the symmetry breaking scale η. Therefore, the Higgs field leaves the vacuum manifold, unwinds and the kinetic energy can be radiated away in massless Goldstone modes. To remove the singularity at $r = 0,\ t = t_c$, we would have to evolve the innermost region of the texture ($r \le \eta^{-1}$) with the true field equation (4.2) during the collapse ($t_c - \eta^{-1} \le t \le t_c + \eta^{-1}$). It is well known [Derrick, 1964, Derrick's theorem], that a σ–model like (4.3) cannot have static finite energy solutions, but it is not proven that there is no static infinite energy solution. It is easy to see that (4.6) represents an infinite energy solution. We shall see that the infinite energy of the solution will require some 'renormalization' or, equivalently, a physical cutoff.

Remark: It seems quite natural that this scale invariant problem admits scale invariant solutions. But this is by no means guarantied. It is well known, e.g., that Yang Mills theories in general do not admit non–trivial spherically symmetric solutions. But also other σ–type models do not admit them. As an example we have studied $SO(3)$ broken to $SO(2)$ by a vector field (the Heisenberg model). The vacuum manifold in this example is $\mathbf{S}^2$ with $\pi_3(\mathbf{S}^2) = \mathbf{Z}$ (the Hopf fibration). A spherically symmetric ansatz for a texture configuration in this model is the Hopf map:

$$\mathcal{H} : \overline{\mathbf{R^3}} \to \mathbf{S}^2 : \mathbf{r} \to \mathbf{n} = R(\theta(r), \hat{\mathbf{r}})\mathbf{n}_o \ , \tag{4.7}$$

where $R(\theta, \mathbf{e})$ denotes a rotation around $\mathbf{e}$ with angle θ, $\hat{\mathbf{r}}$ is the unit vector in direction of $\mathbf{r}$ and $\mathbf{n}_o$ is an arbitrary but fixed unit vector. For the mapping (4.7) to be well defined, we must require the boundary conditions

$$\theta(r=0)=0\ , \quad \theta(r=\infty)=N2\pi\ . \tag{4.8}$$

N is the Hopf invariant, or the π_3 winding number of $\mathcal{H}$. The σ–model equations for $\mathbf{n}$ are

$$\Box\mathbf{n}-(\mathbf{n}\cdot\Box\mathbf{n})\mathbf{n}=\mathbf{0}. \tag{4.9}$$

With the ansatz (4.7) above, this gives rise to the following two equations for θ:

$$\Box\theta-\frac{2\sin\theta}{r^2} = 0 \tag{4.10}$$

$$-(\partial_t\theta)^2+(\partial_r\theta)^2-\frac{2(1-\cos\theta)}{r^2} = 0\ , \tag{4.11}$$

which have no common non–trivial solutions.

4.2 Gravitational Effects of Textures in Flat Space

As long as $\epsilon=16\pi G\eta^2$ is much smaller than 1, the gravitational field of a texture is weak and we can calculate it in first order perturbation theory. We apply the gauge invariant formalism developed in Chapter 2.

The energy momentum tensor,

$$T_{\mu\nu}=\phi,_\mu\,\phi,_\nu-1/2g_{\mu\nu}\phi,_\lambda\,\phi^{,\lambda}\ , \tag{4.12}$$

of solution (4.6) is readily calculated with the result

$$T_{00} = \frac{2\eta^2}{r^2}\frac{1+3y^2}{(1+y^2)^2} \tag{4.13}$$

$$T_{0i} = -\frac{4\eta^2}{r^2}\frac{y}{(1+y^2)^2}\hat{\mathbf{r}}_i \tag{4.14}$$

$$T_{ij} = \frac{2\eta^2}{r^2}\frac{1-y^2}{(1+y^2)^2}\delta_{ij}. \tag{4.15}$$

Clearly (4.6) represents a solution with infinite action and infinite energy,

$$E(R)=4\pi\int_0^R T_{00}r^2dr\propto R\ , \quad \text{for } R>>t-t_c\ ,$$

diverges linearly.

Since this texture solution is spherically symmetric, it only gives rise to scalar perturbations. We now set $M^2=4\eta^2$ and $l=t_c$, the 'radius' of the texture (for an expanding universe, t_c is the horizon scale at texture collapse). We then find

$$f_\rho = \frac{l^2}{2r^2}\frac{1+3y^2}{(1+y^2)^2} \tag{4.16}$$

$$f_v = \frac{l}{2r}\frac{y}{1+y^2} \tag{4.17}$$

$$f_p = \frac{l^2}{2r^2}\frac{1-y^2}{(1^2+y^2)^2} \tag{4.18}$$

$$f_\pi = 0\ . \tag{4.19}$$

From (4.16) to (4.19) and the equations for the gauge invariant Bardeen potentials of scalar seed perturbations (2.126, 2.127, 2.128), we obtain in flat space ($\dot{a} = 0\ ,\quad a = 1$)

$$\begin{aligned}
-\triangle\Phi_s &= \frac{\epsilon}{2}\frac{1+3y^2}{r^2(1+y^2)^2} \\
\dot{\Phi}_s &= \frac{-\epsilon}{2}\frac{y}{r(1+y^2)} \\
\triangle(\Phi_s+\Psi_s) &= 0\ ,
\end{aligned}$$

with the solution

$$\Phi_s = -\frac{\epsilon}{4}\ln((1+y^2)r^2/t_c^2) \tag{4.20}$$

$$\Psi_s = \frac{\epsilon}{4}\ln(\frac{1+y^2}{y^2})\ . \tag{4.21}$$

Ψ_s is only determined up to a function of time, which we have chosen to ensure $\Psi_s \to 0$, for $t \to \pm\infty$.

$$\Psi_s = \frac{1}{4}\epsilon\ln(\frac{r^2+(t-t_c)^2}{(t-t_c)^2})\ . \tag{4.22}$$

Of course, physical observables do not depend on this choice.

It might seem unphysical that the potentials Φ_s and Ψ_s do not vanish at infinity $r \to \infty$, in contrary, they diverge. This reflects again the infinite energy of our solution. Noting this divergence, one might fear that linear perturbation theory breaks down at large distances from the texture, but we find that the relevant geometrical quantities, like e.g. the 3–dimensional Riemann scalar on the surfaces of constant time,

$$\delta^3R = 4a^{-2}\triangle\mathcal{R} \approx 4a^{-2}\triangle\Phi = -\frac{2\epsilon}{a^2}\frac{r^2+3(t-t_c)^2}{(r^2+(t-t_c)^2)^2}\ , \tag{4.23}$$

do vanish at infinity. So that, far away from the collapsing texture and at early and late times, the solution does approach flat space. The validity of linear perturbation theory for textures is also confirmed in Durrer et al. [1991], where we find an exact solution for a texture coupled to gravity and show that, for $\epsilon < 0.1$ say, it deviates only very little from the flat space solution used here.

We now calculate the behavior of baryons and collisionless particles (dark matter or photons) and photons in this geometry.

4.2.1 Baryons around a collapsing texture

Let us briefly discuss the behavior of cosmic dust (baryons) in the field of a texture. Equation (2.134) for a flat dust universe ($c_s^2 = w = 0$, $\dot{a} = 0$) yields

$$\frac{d^2D}{dy^2} + 4\pi G\bar{\rho}t_c^2 D = S \ , \tag{4.24}$$

with

$$S = 2\epsilon \frac{1}{(1+y^2)^2} \ .$$

The term $4\pi G\bar{\rho}t_c^2 D$ is the coupling of the perturbation to its own gravitational field. It leads to exponential growth of perturbations which is a feature of the non–expanding universe only. But our approximation, neglecting expansion, means that all times involved are much smaller than Hubble time. This coincides with $4\pi G\bar{\rho}t_c^2 \ll 1$. Within our approximation, it is thus consistent to neglect the self gravitating term in (4.24). Direct integration then yields the solution

$$\begin{aligned} D &= \epsilon[y\mathrm{arctg}(y) + c_1 y + c_2] \\ &= \epsilon(t/r)[\mathrm{arctg}(t/r) + \pi/2] + \epsilon \ , \end{aligned} \tag{4.25}$$

where we have chosen the integration constants c_1 and c_2 such that D converges to 0 for large negative times, $D(t = -\infty) = 0$, and D converges to a constant for large radii, $D(t, r = \infty) = \epsilon$. Since D is only a function of the self similarity variable t/r, we cannot consistently choose both boundary conditions to be 0. For late times, $t/r \gg 1$, D grows linearly with time:

$$D = \epsilon\pi(t/r) \ ,$$

Near the time of collapse, $|t/r| \ll 1$, D is of the order of ϵ, $D(t = 0, r) = \epsilon$. A given time t_* after texture collapse which is small compared to the Hubble time, D has the following profile: For large radii $D \approx \epsilon$ and roughly at $r = t_*$ bends into $D \approx \epsilon\pi t_*/r$ and diverges for $r \to 0$. This divergence leads to early formation of non–linear structure on small scales. At time t_* perturbations on scales of the order of $r \leq r_{nl} = \epsilon\pi t_*$ have become non–linear.

The total mass accumulated around a texture diverges like the mass of the texture itself (see [4]). But in the real, expanding universe one has to cut it of at roughly the Hubble radius at the time when the texture collapses, l_H.

In this simple approximation, we end up with the following picture: Due to textures forming at a time t in the universe, objects of mass $M \approx 2\epsilon M_H(t)$, form at separations on the order of $p^{-1/3} l_H(t)$. Where M_H denotes the horizon mass at the time when the texture collapses and p is the probability that a four component vector field which is distributed in a completely uncorrelated manner over a 2-sphere winds around a 3-sphere (i.e. the probability of texture formation at the horizon). In Gooding et al. [1991], Åminneborg [1992] this probability has been found numerically to be about 1/25.

From matter conservation, (2.133) for $w = 0$ we obtain

$$\triangle V = \epsilon/r[\mathrm{arctg}(t/r) + \pi/2] - (\epsilon/2)\frac{t}{r^2 + t^2}$$

and therefore

$$v_i = -\partial_i V = -\frac{\epsilon}{2}\frac{r_i}{r}[\mathrm{arctg}(t/r) + \pi/2]. \tag{4.26}$$

The total change in a particle's velocity as the texture collapses is thus independent of the particles distance from it and is given by

$$\Delta v_j = v_j(\infty) - v_j(-\infty) = -(\epsilon\pi/2)\frac{r_j}{r} \ . \tag{4.27}$$

A result which was found by Turok and Spergel [1990] and Durrer [1990].

A numerical calculation for the distribution of texture in an expanding Friedmann universe, where the growth of density perturbations is given according to (2.135) with $w = c_s^2 = 0$, is presented in Gooding et al. [1991].

4.2.2 Collisionless particles in the gravitational field of a texture

Let us now calculate the perturbations in the distribution function of collisionless particles induced by a collapsing texture. Like for dust, we neglect self gravity. We start with the gauge invariant perturbation equation for Liouville's equation in a Friedmann universe with $k = 0$ (2.139):

$$q\partial_t\mathcal{F} + v^k\partial_k\mathcal{F} = \frac{d\bar{f}}{dv}[(q^2/v)v^k\partial_k\Psi_s - vv^k\partial_k\Phi_s] \equiv \mathcal{S} \ , \tag{4.28}$$

where Ψ_s and Φ_s are the metric perturbations due to the texture.

Making use of spherical symmetry and inserting the results (4.22) and (4.20) for Ψ and Φ yields

$$q\partial_t\mathcal{F} + v\mu\partial_r\mathcal{F} + \frac{v(1-\mu^2)}{r}\partial_\mu\mathcal{F} = \mathcal{S} \tag{4.29}$$

with

$$\mathcal{S} = (\epsilon/2)\frac{d\bar{f}}{dv}\cdot(q^2+v^2)\frac{\mu}{r(1+y^2)} \ .$$

The solution of (4.29) is easily found with the help of the physical coordinates

$$\tau = t - (q/v)\mu r \ , \quad b = r\sqrt{1-\mu^2} \quad \text{and} \quad t \ . \tag{4.30}$$

It is straight forward to see that τ is just the time of closest encounter of the given particle with the texture, the impact time, and b is the impact parameter. In these coordinates eq. (4.29) reduces to

$$q\partial_t\mathcal{F}(t,\tau,b) = \mathcal{S}(t,\tau,b) \quad \text{and thus}$$

$$\mathcal{F}(t,\tau,b) = \frac{1}{q}\int_{t_o}^{t}\mathcal{S}(t,\tau,b)dt \tag{4.31}$$

A short calculation gives

$$\frac{1}{q}\int_{t_1}^{t^2}\mathcal{S}dt = \frac{\epsilon}{2}\frac{d\bar{f}}{dv}\left\{\frac{v}{2}\ln[q^2(b^2+t^2)+v^2(t-\tau)^2] - \frac{\tau q v}{\sqrt{v^2\tau^2+(q^2+v^2)b^2}}\arctan\left(\frac{q^2t+v^2(t-\tau)}{q\sqrt{v^2\tau^2+(q^2+v^2)b^2}}\right)\right\}\Bigg|_1^2 \ . \tag{4.32}$$

We are interested in the total change of the distribution function due to the collapsing texture, i.e. in the limit $t_1 \to -\infty$ and $t_2 \to \infty$. The logarithmic term in eq. (4.32), let us call it L, diverges in this limit and the difference,

$$\lim_{\begin{pmatrix} t_1 \to -\infty \\ t_2 \to \infty \end{pmatrix}} [L(t_2) - L(t_1)] \ ,$$

crucially depends on how we perform the limit. It can take any value from $-\infty$ to ∞. This is due to the fact that we are dealing with an infinite energy solution, and we certainly have to 'renormalize' our results. If we would change the energy momentum tensor in a way that the texture would be 'born' some time in the finite past, or if we would compensate it in a consistent way, as we do it in the expanding universe, this problem would disappear. A physically intuitive procedure is to introduce a cutoff at some time $t_2 = -t_1 \gg b, |\tau|$. With such a cutoff the logarithmic term cancels, and we obtain

$$\mathcal{F}(t,\tau,b,v) = -\frac{\epsilon}{2}\frac{d\bar{f}}{dv}\frac{\tau q v}{\sqrt{v^2\tau^2 + (q^2+v^2)b^2}} \arctan\left(\frac{q^2 t + v^2(t-\tau)}{q\sqrt{v^2\tau^2+(q^2+v^2)b^2}}\right)\Bigg|_1^2 \ .$$

We then can remove the cutoff and obtain the change of the distribution function long after the corresponding particles have passed the texture

$$\mathcal{F}(t,\tau,b,v) = -\frac{\epsilon\pi}{2}\left(\frac{d\bar{f}}{dv}\right)\frac{\tau q v}{\sqrt{v^2\tau^2+(q^2+v^2)b^2}} \ . \tag{4.33}$$

Introducing our old variables

$$\begin{aligned} r^2 &= (t-\tau)^2 v^2/q^2 + b^2 \\ \mu &= (t-\tau)\frac{v}{qr} \ , \end{aligned}$$

we find

$$\mathcal{F}(t,r,\mu,v) = -\frac{\epsilon\pi}{2}\left(\frac{d\bar{f}}{dv}\right)\frac{q(vt - q\mu r)}{\sqrt{(vt-q\mu r)^2 + r^2(1-\mu^2)(q^2+v^2)}} \ . \tag{4.34}$$

This result can be inserted in equations (2.76) to (2.78) to obtain the induced perturbations of the energy momentum tensor. Here we just perform the non–relativistic and extremely relativistic limits.

Non–relativistic limit:
In the non–relativistic case we have $v \ll q$. The gravitational field of the texture is strong only for $r \sim |t|$. We can therefore expand (4.34) in the small quantity $vt/(qr)$. The first order approximation yields

$$\mathcal{F}(t,r,\mu,v) = (\epsilon\pi/2)\frac{d\bar{f}}{dv}q^2[\mu + (vt/qr)(\mu^2-1) + \mathcal{O}(vt/qr)^2] \ , \tag{4.35}$$

This leads to

$$V = \frac{2\pi}{\rho}\int \mathcal{F}v^3\mu dv d\mu = -\frac{\epsilon\pi}{2} \tag{4.36}$$

$$D = \frac{2\pi}{\rho}\int \mathcal{F}v^2 q dv d\mu = \epsilon\pi t/r \ , \tag{4.37}$$

the well known late time ($t \gg r$) results of the preceding subsection.

Extremely relativistic limit:
In this case we have $v = q$ and therefore

$$\mathcal{F}(t,r,\mu,v) = -\frac{\epsilon\pi}{2}\frac{\bar{f}}{dv}\frac{v(t-\mu r)}{\sqrt{(t-\mu r)^2 + 2r^2(1-\mu^2)}} \ . \tag{4.38}$$

Integrating $\mathcal{F}v^3$ over v and dividing by $\rho/4\pi$ yields the fractional perturbation of the brightness introduced in Chapter 2,

$$\mathcal{M}(t,r,\mu) = 2\epsilon\pi\frac{t-\mu r}{\sqrt{(t-\mu r)^2 + 2r^2(1-\mu^2)}} \ . \tag{4.39}$$

Since $\mathcal{M}$ describes the fluctuations in the energy density radiated in a given direction μ, the temperature fluctuation is given by $\Delta T/T = \Delta\mathcal{M}/4$ (see also the more explicit discussion of this point below eqn (2.83)). This coincides exactly with eq. (4.43) for photon redshift in the next paragraph.

In a flat, eternal universe the signal from a texture collapsing at $t = 0$, as seen from an observer at time t_o and distance r_o would thus be

$$\frac{\Delta T}{T}(\theta) = \frac{\pi\epsilon}{2}\frac{t_o - r_o\cos\theta}{\sqrt{(t_o - r_o\cos\theta)^2 + 2r_o^2\sin^2\theta}} \ . \tag{4.40}$$

A similar calculation using a specific gauge is presented in Durrer et al. [1992a]. Unlike in the expanding universe (see next chapter), there is no horizon present in this calculations. Photons that pass the texture long before or after collapse, $|t_o| \gg r_o$ are still influenced by it and yield even a maximum temperature shift,

$$\frac{\Delta T}{T} = \pm\frac{\pi}{2}\epsilon \ .$$

In the expanding universe of finite age we expect, because of the finite size of the event horizon, $\Delta T/T$ to achieve a maximum for $r_o \approx t_o - t_c$ and to vanish for $t_o - t_c \ll r_o$. The comparison of the flat space result (4.40) and the effect of a compensated texture in the expanding universe is shown in Fig. 8.

4.2.3 Redshift of photons in the texture metric

Let us first determine the energy shift which a photon experiences by passing a texture. Without loss of generality, we set $t_c = 0$ in this paragraph. If we neglect the distinctive dipole term and intrinsic density perturbations, equation (2.101) leads to

$$\left.\frac{\delta E}{E}\right|_i^f = \int_i^f(\dot{\Phi} - \dot{\Psi})d\lambda + \Psi|_i^f \ . \tag{4.41}$$

Denoting the impact parameter of the photon trajectory by b and the time when the photon passes the texture (the impact time) by τ, we get $r^2 = b^2 + (t-\tau)^2$. Eq. (4.41) then yields after the same "renormalization procedure which led to (4.33)

$$\left.\frac{\delta E}{E}\right|_i^f = \frac{\epsilon\tau}{2(\tau^2 + 2b^2)^{1/2}}[\mathrm{arctg}(\frac{2t-\tau}{\sqrt{\tau^2 + 2b^2}})]_i^f \ . \tag{4.42}$$

For $t_f,\ -t_i \gg \tau,\ b$, we obtain

$$\left.\frac{\delta E}{E}\right|_i^f \approx (\epsilon\pi/2)\frac{\tau}{(\tau^2+2b^2)^{1/2}} \ . \tag{4.43}$$

This result was first found by different methods by Turok and Spergel [1990]. Photons which pass the texture before it collapses, $\tau < 0$, are redshifted, and photons passing it after collapse, $\tau > 0$, are blueshifted. This produces a very distinctive hot spot — cold spot signal in the microwave sky wherever a texture has collapsed.

Of course our result is not strictly correct in the expanding universe, since we have neglected expansion in the calculation of Ψ_s and Φ_s. But the main contribution to the energy shift comes from times $|t| \le |\tau| + b$. Therefore, our approximation is reasonable also for the expanding case, if $|\tau| \le l_H$ and $b \le l_H$, where l_H denotes the horizon distance at the time of collapse, $t = 0$. On the other hand, by causality the texture cannot have a big effect on photon trajectories with $|\tau| > l_H$ or $b > l_H$. A first approximation to the situation in the expanding universe is thus

$$\left.\frac{\delta E}{E}\right|_i^f = \begin{cases} \frac{\epsilon\pi}{2}\frac{\tau}{\sqrt{\tau^2+2b^2)}} & , \text{ for } |\tau| < l_H \text{ and } b < l_H \\ 0 & , \text{ for } |\tau| > l_H \text{ or } b > l_H \ . \end{cases} \tag{4.44}$$

4.2.4 Light Deflection

To obtain the light deflection in the gravitational field of a spherically symmetric texture, we consider (as above) a photon passing the texture at impact time τ at a distance b (impact parameter). The trajectory of this photon is then given by $x(\lambda) = (\tau + \lambda, \lambda\boldsymbol{n} + b\boldsymbol{e})$. Making use of eq. (2.114), neglecting spherical abberation due to the relative motion of emitter and observer, we find

$$\begin{aligned} \varphi &= \int_i^f (\Phi - \Psi)_{,i} e^i d\lambda \\ &\approx \epsilon \int_{-\infty}^{\infty} \frac{b}{b^2 + 2\lambda^2 + 2\lambda\tau + \tau^2} d\lambda \\ &= \epsilon\pi \frac{b}{\sqrt{2b^2+\tau^2}} \ . \end{aligned} \tag{4.45}$$

For $\tau = 0$, the deflection angle assumes the maximum value, $\varphi_{\max} = \epsilon\pi/\sqrt{2}$. For $b = 0$ or $\tau \to \infty$, light deflection vanishes. This result was first obtained by different methods in [Durrer et al., 1992a].

In contrast to global monopoles and strings, the deflection angle is not independent of the impact parameter b of the photon trajectory (except for photons which pass the texture exactly at collapse time, $\tau = 0$). This leads to a qualitative difference to the lensing caused by global monopoles and cosmic strings [Barriola and Vilenkin, 1989, Vilenkin, 1984]: If a light source and an observer at distances L, respectively D behind, respectively in front of the texture are perfectly aligned with the texture, the observer sees an Einstein ring with a time dependent opening angle

$$\beta(t) = \sqrt{2\left(\frac{\epsilon\pi L}{L+D}\right)^2 - 2\left(\frac{t}{D} - 1\right)^2} \ . \tag{4.46}$$

This ring opens up at time $t = D - \epsilon\pi LD/(L+D)$, reaches a maximum opening angle $\beta_{\rm max} = \sqrt{2}\pi\epsilon L/(L+D)$ at $t = D$ and shrinks back to a point at $t = D + \epsilon\pi LD/(L+D)$. It exists over a time span $\Delta t = 2\pi\epsilon\frac{DL}{D+L}$. For realistic values of $\epsilon \sim 10^{-4}$ the maximum opening angle can become about $1'$.

If source and observer are not perfectly aligned, but deviate by an angle $\gamma < \varphi_{\rm max}$ from alignment the ring reduces to two points which first move apart and then together again within the time Δt and with a maximum separation angle $\beta_{\rm max}$. Unfortunately, even if textures exist, since late time textures are so rare, the probability of observing this effect is rather small. There is typically one texture which has collapsed after $z = 4$ and for which the photons that have passed it close to the time of collapse are visible for us now (see Durrer and Spergel [1991]). The probability that there is a quasar behind it with angular separation less than $\beta_{\rm max} \sim 10^{-3}$ is even smaller than for global monopoles (see Section 3.4.1).

Chapter 5

Textures in Expanding Space

We now want to discuss the effect of collapsing textures in a spatially flat, expanding Friedmann Universe. To be specific, we assume a universe which is dominated by cold dark matter (CDM) which contributes about 95% of the total matter density of the universe. The remaining 5% are baryonic. In the next section we write down the system of equations which describes a spherically symmetric collapsing texture in this background and the perturbations it induces in the dark matter, baryons and photons. We have solved this equations numerically. We then indicate how the single texture events can be distributed in space and time to obtain a full sky map of the cosmic microwave background. We report briefly on the results of this numerically obtained microwave sky and compare it with the COBE data. (A more extensive presentation of this numerical work, which was done in collaboration with A. Howard and Z.–H. Zhou will be presented elsewhere [Durrer et al., 1993].)

Numerical simulations and analytical estimates have already shown that the texture scenario leads to early formation of small objects which are likely to reionize the universe as early as $z \approx 100$ (see Sect. 4 and Durrer [1990], Spergel et al. [1991] and Gooding et al. [1991]). Photons and baryons then are coupled again via Compton scattering of electrons (see Section 3.2.3). They remain so until the electrons are too diluted to scatter effectively. This decoupling time is determined by the distance into the past, at which the optical depth becomes unity:

$$\tau(t_{dec}) = \int_{t_0}^{t_{dec}} dt a \sigma_T n_e = 1 \ .$$

In a flat universe, $\Omega = 1$ one obtains $1 + z_{dec} \approx 100 \left(\frac{0.05}{\Omega_b h_{50}}\right)^{-2/3}$, where Ω_b denotes the density parameter of baryons, and h_{50} is Hubble's constant in units of $50km/s/Mpc$. This last scattering surface is of course not as instantaneous as the recombination shell. The decoupling due to dilution is a rather gradual process, and the thickness of the last scattering surface is approximately equal to the horizon scale at $z_{dec} \approx t_{dec}$.

Due to reionization, perturbations smaller than t_{dec} but larger than the mean free path of the photons are exponentially damped by photon diffusion. A rough estimate of this effect is given in Section 3.2.3. Results of a numerical calculation of this damping are shown in Fig. 7.

5.1 Formalism

In this section, we present the equations for calculating the response of matter and radiation to the collapse of a single spherically symmetric texture. The background metric is given by

$$ds^2 = a^2(-dt^2 + d\boldsymbol{x}^2) \ .$$

The spherically symmetric ansatz for an $\mathbf{S}^3$ texture unwinding at a given time $t = t_c$ is, like in the preceding chapter

$$\boldsymbol{\phi} = \eta^2(\sin\chi\sin\theta\cos\varphi, \sin\chi\sin\theta\sin\varphi, \sin\chi\cos\theta, \cos\chi) \ , \tag{5.1}$$

where θ , φ are the usual polar angles, and $\chi(r,t)$ has the properties

$$\begin{aligned} \chi(r=0, t<t_c) &= 0 \\ \chi(r=0, t>t_c) &= \pi \\ \chi(r=\infty, t) &= \pi \ . \end{aligned}$$

In expanding space, the evolution equation (4.5) is replaced by

$$\partial_t^2\chi + 2(\dot{a}/a)\partial_t\chi - \partial_r^2\chi - \frac{2}{r}\partial_r\chi = -\frac{\sin 2\chi}{r^2} \ . \tag{5.2}$$

We parametrize the energy momentum tensor of the spherically symmetric texture field,

$$T_{\mu\nu} = \partial_\mu\boldsymbol{\phi}\cdot\partial_\nu\boldsymbol{\phi} - \frac{1}{2}g_{\mu\nu}\partial_\lambda\boldsymbol{\phi}\cdot\partial^\lambda\boldsymbol{\phi} \ , \tag{5.3}$$

in terms of scalar seed perturbations

$$T_{00} = \frac{\epsilon}{4\pi}f_\rho/l^2 \ , \quad T_{0i} = -\frac{\epsilon}{4\pi}f_{v,i}/l \ ,$$

$$T_{ij} = \frac{\epsilon}{4\pi}[(f_p/l^2 - \frac{1}{3}\triangle f_\pi)\delta_{ij} + f_{\pi,ij}] \ ,$$

where $\epsilon = 16\pi G\eta^2$. Inserting ansatz (5.1) in (5.3), one finds

$$f_\rho/l^2 = 1/8[(\partial_t\chi)^2 + (\partial_r\chi)^2 + \frac{2\sin^2\chi}{r^2}] \ , \tag{5.4}$$

$$f_p/l^2 = 1/8[(\partial_t\chi)^2 - \frac{1}{3}(\partial_r\chi)^2 - \frac{2\sin^2\chi}{3r^2}] \ , \tag{5.5}$$

$$f_v/l = \frac{1}{4}\int_r^\infty(\partial_t\chi)(\partial_r\chi)dr \ , \tag{5.6}$$

$$\triangle f_\pi = \frac{1}{4}[(\partial_r\chi)^2 - \frac{\sin^2\chi}{r^2}] + \frac{3}{4}\int_\infty^r[(\partial_r\chi)^2 - \frac{\sin^2\chi}{r^2}]\frac{dr}{r} \ . \tag{5.7}$$

The variables f_ρ/l^2 and f_p/l^2 denote the energy density and isotropic pressure of the texture field, f_v is the potential of the velocity field and f_π is the potential for anisotropic stresses. Spherically symmetric perturbations are of course always of scalar type. Due to the adoption of spherical symmetry, we loose, e.g., all information about gravitational waves produced during the collapse.

In addition to the texture, we want to describe dark matter, baryons and radiation. Since dark matter has zero pressure, we just need a variable describing its density perturbation.

By ρ_d we denote the density of dark matter and D is its gauge invariant density perturbation, as defined in Chapter 2. The evolution of the dark matter fluctuations is governed by (2.135) for $c_s^2 = w = 0$:

$$\ddot{D} + (\dot{a}/a)\dot{D} - 4\pi G\rho_d a^2 D = \epsilon(f_\rho + 3f_p)/l^2 \;, \tag{5.8}$$

To describe the baryon– photon system we need two additional variables: The potential for the baryon velocity V and the perturbation of the energy integrated photon distribution function $\mathcal{M}$. At times $t \le t_{dec}$, the collisionless Boltzmann equation for the photons and the equation of motion for the baryons must be modified to take into account scattering. The dominant effect is non–relativistic Compton scattering by free electrons. Let us denote the collision integral which is calculated in Section 3.2.1 by $C(\mathcal{M}, V)$. As in Chapter 3, $\boldsymbol{\epsilon}$ is the direction of the photon momentum. One finds (3.17)

$$C = a\sigma_T n_e[D_g^{(r)} - \mathcal{M} + 4\epsilon_i\partial_i V + \frac{1}{2}\epsilon_{ij}M^{ij}] \;, \tag{5.9}$$

with

$$M^{ij} = \frac{3}{2\pi}\int \mathcal{M}(\epsilon)\epsilon_{ij}d\Omega] \;;\quad \epsilon_{ij} \equiv \epsilon_i\epsilon_j - (1/3)\delta_{ij}$$

(We do not have to worry about the position of spatial indices of perturbation variables, they are raised and lowered with the Euclidean metric δ_{ij} since $k = 0$ in this chapter.)

The drag force due to Thomson drag of photons on the matter is given by (3.19)

$$F_i = -\frac{\rho_r}{4\pi}\int C\epsilon_i d\Omega = \frac{a\sigma_T n_e\rho_r}{3}(M_i + 4l\partial_i V) \;,$$

with $M_i = (3/4\pi)\int \epsilon_i\mathcal{M}d\Omega$. Including this drag force into the equation of motion for the baryons, (2.58) for $w = c_s^2 = 0$, one obtains (3.20)

$$l\partial_i\dot{V} + (\dot{a}/a)l\partial_i V = \partial_i\Psi - \frac{a\sigma_T n_e\rho_r}{3\rho_b}(4l\partial_i V + M_i) \quad \text{or}$$

$$l\triangle\dot{V} + (\dot{a}/a)l\triangle V = \triangle\Psi + a\sigma_T n_e\frac{\rho_r}{\rho_b}(\dot{M} - \frac{4}{3}l\triangle V) \;. \tag{5.10}$$

For the last equation we have used the zeroth moment of Boltzmann's equation, the continuity equation,

$$\dot{M} + (1/3)\partial_i M^i = 0 \;.$$

In our numerical computations, we have made the simplifying assumption $n_e = 0$ for $z > z_i$ and $n_e = n_B$ for $z < z_i$ for some ionization redshift $z_i \approx 200$.

The evolution of the photons is given by the perturbation of Boltzmann's equation, (3.26).

$$\dot{\mathcal{M}} + \mu\partial_r\mathcal{M} + \frac{1-\mu^2}{r}\partial_\mu\mathcal{M} =$$

$$4\mu\partial_r(\Phi - \Psi) + a\sigma_T n_e[M - \mathcal{M} - 4\mu\partial_r V + 3(\mu^2 - 1/3)M_2] \tag{5.11}$$

$$\text{where}\quad M_2(r,t) = \frac{1}{2}\int_{-1}^{1}\mathcal{M}(\mu')(\mu'^2 - 1/3)d\mu' \;, \tag{5.12}$$

where μ is the direction cosine of the photon momentum in the direction of $\mathbf{r}$.

As in Chapter 4, the photon evolution equation is more transparent in characteristic coordinates, (t, τ, b), where $b = r\sqrt{1-\mu^2}$ is the impact parameter, and $\tau = t - r\mu$ is the impact time. In these variables (5.12) simplifies:

$$\partial_t \mathcal{M}(t, \tau, b) = 4\mu\partial_r(\Phi - \Psi) + C(t, \tau, b) \ , \tag{5.13}$$

where $C(t, \tau, b)$ is the collision integral above, expressed in terms of the new variables.

In order to write down the perturbed Einstein equations, we have in principle to calculate the energy momentum tensor of radiation from $\mathcal{M}$. But since we are only interested in late times where density perturbations can grow, $\rho_d > \rho_r$, we may neglect the contribution of radiation to the density perturbation. The potential Φ is then determined by the texture and dark matter perturbations alone:

$$\triangle\Phi = -\epsilon(f_\rho/l^2 + 3(\dot{a}/a)f_v/l) - 4\pi G a^2 \rho_d D \ . \tag{5.14}$$

On the other hand, since dark matter does not give rise to anisotropic stresses, we have to take into account the contribution of radiation to the latter. To calculate the anisotropic stresses of the photons we recall the definition of the amplitude of anisotropic stresses, Π:

$$\delta T_i^j - \frac{1}{3}\delta T_l^l \delta_i^j = p[\Pi_{,i}^{,j} - \frac{1}{3}\triangle\Pi\delta_i^j] \ .$$

The anisotropic contributions to the energy momentum perturbations of the photons are given by

$$\delta T_i^j - \frac{1}{3}\delta T_l^l \delta_i^j = \frac{\rho_r}{4\pi}\int(\epsilon_i\epsilon^j - \frac{1}{3}\delta_i^j)\mathcal{M} d\Omega \ .$$

Using these equations and spherical symmetry, one finds

$$\Pi'' - \Pi'/r = \frac{9}{4}\int_{-1}^{1}(\mu^2 - \frac{1}{3})\mathcal{M} d\mu = \frac{9}{2}M_2(r,t) \ . \tag{5.15}$$

This anisotropy and the anisotropy of the texture field contribute to the sum of the two Bardeen potentials

$$\triangle(\Psi + \Phi) = -2\triangle(\epsilon f_\pi + (4/3)\pi G a^2 \rho_r \Pi) \ . \tag{5.16}$$

Let us choose initial conditions that are physically plausible. If the phase transition that produced texture occurred in an initially uniform universe, causality requires that there are no geometry fluctuations well outside the horizon [Veeraraghavan and Stebbins, 1990]. This implies that initially, at $t_i \ll t_c$, we must require $\Psi = \Phi = 0$. We want to compensate as much as possible of the initial texture fluctuations with an initial dark matter perturbation. Hence, we use as initial conditions for the density field,

$$D(r, t = t_i) = -\frac{\epsilon}{4\pi G a^2 \rho_d}(f_\rho/l^2 + 3(\dot{a}/a)f_v/l) \ . \tag{5.17}$$

This initial condition implies that metric fluctuations are induced by the differences between the texture equation of state, and the equation of state of the background matter. This choice yields $\Phi = 0$ at $t = t_i$, but not $\Psi = 0$. Due to its equation of state the dark matter cannot compensate anisotropic stresses of the texture. We thus must compensate them by an initial photon perturbation. For Ψ to vanish, we have to require according to (5.16)

$$\Pi = -\frac{3\epsilon}{4\pi G a^2 \rho_r} f_\pi \ .$$

This does not lead to a unique initial condition for $\mathcal{M}$, but if we, in addition, require the zeroth and first moments of $\mathcal{M}$ to vanish (which otherwise would interfere with eq. (5.17)) it is reasonable to set

$$\mathcal{M}(r,\mu,t=t_i) = -\frac{15\epsilon}{8\pi G a^2 \rho_r}(\mu^2 - \frac{1}{3})(\triangle f_\pi - \frac{3}{r} f'_\pi) \ . \tag{5.18}$$

Together with initial conditions for the texture field $\chi(r, t = t_i)$, the requirements (5.17) and (5.18) and the evolution equations (5.2) to (5.16) determine the system which we have solved numerically.

Figure 9 shows the microwave background fluctuations induced by the collapse of the texture as a function of τ for small impact parameter in the expanding universe for different times. At $t > t_c$ it is interesting to see the blueshift at $\tau \sim t$ of the photons which have fallen into the dark matter potential but have not yet climbed out of it again. (During their way out of the dark matter potential this blueshift will of course be exactly compensated. This is also visible in the figure.) Figure 10 shows the microwave background fluctuations induced by texture collapse as a function of impact parameter. Note that the temperature fluctuations are induced only for photons that pass within the event horizon of the texture.

5.2 Textures and the Microwave Sky

In the previous section, we described how the collapse of a single texture produces fluctuations in the photon temperature. In this section, we sum the contributions of many textures and describe how to construct microwave maps of the night sky.

Since COBE observations cover the entire celestial sphere, we construct a numerical grid consisting of 1 square degree patches. These patches are arranged so that they cover equal areas and each patch has roughly the same shape.

Since the texture fluctuations are in the linear regime, we assume that the contributions of each texture to the fluctuations at each point in the sky can be added independently. We randomly throw down textures everywhere within the event horizon using the texture density distribution [Spergel et al., 1991],

$$\frac{dn}{dt} = \frac{1}{25}\frac{1}{t^4}. \tag{5.19}$$

We then follow the collapse of each texture and sum their contributions. We include only textures that collapse after recombination. Textures that collapse earlier do not contribute significantly to microwave fluctuations on scales accessible to COBE.

In order to simulate the COBE observations, the map of the night sky is smoothed with a Gaussian beam with a FWHM of 7°, the angular resolution of the DMR detector on COBE [Smoot et al., 1991]. After computing and removing the intrinsic dipole contribution ($10^{-5} - 10^{-4}$) and any monopole fluctuations, we then compute the microwave quadrupole, the r.m.s. pixel-pixel fluctuations and other statistics of the microwave sky.

5.3 Results

Here we present our results for the not reionized universe ($z_i < z_{dec}$). We just compare the numerical results with the COBE observations on $\theta \geq 10^o$. In the texture scenario, these large angular scales are not affected by reionization.

The amplitude of the microwave background fluctuations depends upon the scale of symmetry breaking associated with the texture. If the symmetry breaking scale is normalized so that the scenario can reproduce the amplitude of the galaxy-galaxy correlation function, we have to set the value of the dimensionless parameter $\epsilon = 16\pi G\eta^2$ to $\epsilon \approx 5.7 \times 10^{-4} b^{-1}$, where b, the bias factor, is the ratio of the mass-mass correlation function to the galaxy-galaxy correlation function [Gooding et al., 1991]. Note that the definition of ϵ in this work is $2/\pi$ times the definition used in Gooding et al. [1991]. Comparison of the predictions of the texture model with observations of clusters [Bartlett et al., 1993] suggest that $b \approx 2$, a value compatible with hydrodynamical simulations of texture-seeded galaxy formation [Cen et al., 1991]. We normalize the microwave background fluctuations to this value and present our results in units of ϵ_0, where $\epsilon_0 = \epsilon/2.8 \times 10^{-4}$.

We numerically performed 100 realizations of the model. For illustration, a 'COBE map' produced from a typical simulation is shown in Fig. 11. Averaging over all realizations, we find an r.m.s. value for the quadrupole moment of

$$Q = (1.4 \pm 1.2) \times 10^{-5} \epsilon_0 \ .$$

Since only a handful of textures are the source of most of the large scale fluctuations, the quadrupole varies significantly from realization to realization.

To compare with the COBE result, we have smoothed our calculations over an angular scale of 10 degrees. The average pixel-to-pixel fluctuations of the smoothed simulations are

$$(\Delta T/T)_{rms}(10^o) = (3.8 \pm 2.6) \times 10^{-5} \epsilon_0$$

The distribution of temperature fluctuations are only mildly non–Gaussian, the skewness of the distribution is -4 ± 2 and the kurtosis of the distribution is 32 ± 29. The errors quoted are statistical 1σ deviations of one hundred realizations. One example for the pixel distribution of the fluctuations is shown in Fig. 12.

The results from the COBE differential microwave background radiometers [DMR] place strong constraints on CMB fluctuations on scales larger than 10^o. In this experiment [Wright et al., 1992, Smoot et al., 1992] a value of $Q \approx (0.6 \pm 0.2) \times 10^{-5}$ has been found for the microwave quadrupole and

$$(\Delta T/T)_{rms}(\theta = 10^o) = (1.1 \pm 0.18) \times 10^{-5}$$

is the result for the fluctuations at a scale of 10 degrees. The overall spectrum is compatible with a Harrison–Zel'dovich spectrum:

$$(\Delta T/T)(\theta) \approx 10^{-5} (\theta/10^o)^{-(0.1 \pm 0.5)/2} \ .$$

Our simulations with bias factor $b = 2$ are compatible with these results within one sigma. The $(\Delta T/T)$ of the simulations is somewhat large but the scatter is considerable.

We feel that the adoption of spherical symmetry may lead to underestimates of $(\Delta T/T)$ since contributions due to gravitational waves and random fluctuations in the scalar field which do not give rise to texture (i.e., topological winding number) have been neglected in this approach. If, e.g.,

$(\Delta T/T)(\theta = 10^o)$ is enhanced by a factor of 2 by these contributions, the COBE measurement is about 2.5σ below the average texture result.

The conclusions from these simulations can thus be put as follows: The CMB anisotropies from spherically symmetric texture collapse are slightly high but agree within 1σ with the COBE measurements. This is encouraging. But in the results given above only statistical fluctuations, (i.e., cosmic variance) have been taken into account. Due to the uncertainties in modeling the typical texture and the approximations inherent in modeling the texture as spherically symmetric, these estimates of the microwave background fluctuations are uncertain by at least a factor of ~ 2 (systematic error), probably underestimating the true induced fluctuations. A full $3d$ simulation, which takes into account also gravitational waves and non–topological fluctuations of the scalar field, is necessary to finally decide on the scenario. Such simulations have now been performed [Bennett and Rhie, 1992, Pen et al., 1993]. They obtain results which are higher than those obtained in the spherically symmetric approach by a factor 1.5 to 2.

Smaller scales, $\theta \approx 1^o$ lead to somewhat larger fluctuations and much smaller standard deviations (since many textures contribute to them):

$$(\Delta T/T)_{rms}(2^o) = (4 \pm 0.8) \times 10^{-5}\epsilon_0$$

If the new measurements which require $(\Delta T/T)(1^o) < 1.4 \times 10^{-5}$ [Gaier et al., 1992] are confirmed, we need reionization to damp small scale fluctuations in the texture scenario.

A rough estimate of the effects of reionization can be obtained by just smoothing each texture with a smoothing scale of about the horizon size at z_{dec},

$$z_{dec} = 100\left(\frac{0.05}{h_{50}\Omega_B}\right)^{2/3} \; .$$

This corresponds to an angular scale of

$$\theta_{smooth} = t(z_{dec})/t_0 = (1+z)^{-1/2} \approx 5.7^o \; . \tag{5.20}$$

If the formation of objects leads to reionization prior to z_{dec}, this would suppress microwave background fluctuations on scales $\theta \leq \theta_{dec} \sim 6^o$, but would not effect the fluctuations on larger scales which are discussed above.

5.4 Conclusions

The texture scenario of large scale structure formation has many attractive features. Its galaxy galaxy correlation function and the large scale velocity fields agree better with observations than in the standard cold dark matter model [Gooding et al., 1992]. Also a couple of other statistical parameters (Mach number, skewness, kurtosis) are in satisfactory agreement with observations. New simulations which used the COBE quadrupole to normalize the fluctuations [Pen et al., 1993] hint that, in contrary to earlier results, the scenario may have severe difficulties to reproduce the very large scale galaxy clustering observed in the infrared [Fisher et al., 1993], just like the standard CDM model. Due to the steep dark matter potential produced with texture, small scale structure can form very early. This may lead to early reionization of the universe. Calculations of the microwave background anisotropies show that reionization is necessary to reconcile with CMB anisotropies on small scales (up to about 5^o). Due to the unique signature and relatively large amplitude, the CMB anisotropies produced by textures are one of the most hopeful criterion for confirming or ruling out this scenario.

Another interesting observational test, lensing of background quasars by a foreground texture, is very improbable.

This scenario certainly deserves further work. Especially the investigation of the question if reionization as early as $z \sim 200 - 100$ is possible and a careful analysis of CMB anisotropies for the reionized model on angular scales around 1^o are important tasks.

Acknowledgement

I thank Dave Spergel, Neil Turok, Norbert Straumann, Slava Mukhanov, Zhi–Hong Zhou and Markus Meuwly for many stimulating discussions. I'm also much indebted to Norbert Straumann, Daniel Wyler and Othmar Brodbeck for reading parts of the manuscript.
This research was supported in parts by the Swiss National Science Foundation.

Appendix A

The $3+1$ Formalism

Since we have made use of $3+1$ split of general relativity in deriving the linear cosmological perturbation equations in this review, we want to derive the necessary tools in this appendix. A mathematically rigorous overview is given in Choquet–Bruhat and York [1980] and Fischer and Marsden [1979]. But explicit calculations are missing there, we shall thus be rather detailed. In this appendix we mainly follow the exposition in Durrer and Straumann [1988].

A.1 Generalities

We assume that spacetime $(\mathcal{M}, g)$ admits a slicing by slices Σ_t, i.e., there is a diffeomorphism $\phi : \mathcal{M} \to \Sigma \times I$, $I \subset \mathbf{R}$, such that the manifolds $\Sigma_t = \phi^{-1}(\Sigma \times \{t\})$ are spacelike and the curves $\phi^{-1}(\{x\} \times I)$ are timelike. These curves are what we call *preferred* timelike curves. They define a vector field ∂_t, which can be decomposed into normal and parallel components relative to the slicing (Figure 13):

$$\partial_t = \alpha n + \boldsymbol{\beta} \ . \tag{A1}$$

Here n is a unit normal field and $\boldsymbol{\beta}$ is tangent to the slices Σ_t. The function α is the *lapse function* and $\boldsymbol{\beta}$ is the *shift vector* field.

A coordinate system $\{x^i\}$ on Σ induces natural coordinates on $\mathcal{M}$: $\phi^{-1}(m, t)$ has coordinates (t, x^i) if $m \in \Sigma$ has coordinates x^i. The preferred timelike curves have constant spatial coordinates. Let us set $\boldsymbol{\beta} = \beta^i \partial_i$ ($\partial_i = \frac{\partial}{\partial x^i}$). From $g(n, \partial_i) = 0$ and (A1) we find

$$g(\partial_t, \partial_t) = -\alpha^2 + \beta^i \beta_i \ , \ \ g(\partial_t, \partial_i) = \beta_i \ .$$

In "comoving coordinates" thus

$$g = -(\alpha^2 - \beta^i \beta_i) dt^2 + 2\beta_i dx^i dt + g_{ij} dx^i dx^j \tag{A2}$$

or

$$g = -\alpha^2 dt^2 + g_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \ . \tag{A3}$$

This shows that the forms dt and $dx^i + \beta^i dt$ are orthogonal.

The tangent and cotangent spaces of $\mathcal{M}$ have two natural decompositions. One is defined by the slicing

$$T_p(\mathcal{M}) = H_p \oplus V_p \ , \tag{A4}$$

where the "horizontal" space H_p consists of the vectors tangent to the slice through p and the "vertical" sub–space is the 1-dimensional space spanned by $(\partial_t)_p$ (preferred direction). The dual decomposition of (A4) is

$$T_p^*(\mathcal{M}) = H_p^* \oplus V_p^* \ , \tag{A5}$$

with $H_p^* = \{\omega \in T_p^*(\mathcal{M}) : \langle \omega, \partial_t \rangle = 0\}$ and $V_p^* = \{\omega \in T_p^*(\mathcal{M}) : \langle \omega, H_p \rangle = 0\}$, which is spanned by $(dt)_p$.

The metric defines - through the normal field n - yet another decomposition

$$T_p(\mathcal{M}) = H_p \oplus H_p^- \ , \tag{A6}$$

where H_p^- is spanned by n, and dually

$$T_p^*(\mathcal{M}) = (V_p^*)^- \oplus V_p^* \ , \tag{A7}$$

Equation (A1) reflects the fact that in general the two directions V_p and H_p^- do not agree. Dually this implies that H_p^* and $(V_p^*)^-$ do not coincide. We have for $\omega^- \in (V_p^*)^-$ the following decomposition relative to (A5)

$$\omega^- = \mathrm{hor}(\omega^-) + \langle \omega^-, \boldsymbol{\beta} \rangle dt \ . \tag{A8}$$

The decompositions (A4) to (A7) induce two types of decompositions of arbitrary tensor fields on $\mathcal{M}$. We call a tensor field *horizontal* if it vanishes, whenever at least one argument is ∂_t or dt. Relative to a comoving coordinate system such a tensor has the form

$$\boldsymbol{S} = S^{i_1 \cdots i_r}_{j_1 \cdots j_s} \partial_{i_1} \otimes \cdots \otimes \partial_{i_r} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s} \ .$$

This shows that horizontal tensor fields can naturally be identified with families of tensor fields on Σ_t, or with time-dependent tensor fields on Σ ("absolute" space). We denote them with boldface letters (except ∂_i and dx^i).

As an often occurring example of a decomposition, we consider a horizontal p-form $\boldsymbol{\omega}$ and its exterior derivative $d\boldsymbol{\omega}$. We have

$$d\boldsymbol{\omega} = \boldsymbol{d}\boldsymbol{\omega} + dt \wedge \partial_t \boldsymbol{\omega} \ ,$$

where $\boldsymbol{d\omega}$ is again horizontal. In comoving coordinates $\boldsymbol{d}$ involves only the dx^i ($\boldsymbol{d} = dx^i \wedge \partial_i$) and $\partial_t \boldsymbol{\omega}$ is the partial time derivative. $\boldsymbol{d\omega}$ and $\partial_t \boldsymbol{\omega}$ are horizontal and can be interpreted as t-dependent forms on Σ. In this interpretation $\boldsymbol{d\omega}$ is just the exterior derivative of $\boldsymbol{\omega}$. Similarly, other differential operators (covariant derivative, Lie derivative, etc) can be decomposed. We use two types of bases of vector fields and 1-forms which are adapted to (A4) and (A5), respectively (A6) and (A7). Obviously, the dual pair $\{\partial_\mu\}$ and $\{dx^\mu\}$ for comoving coordinates $\{x^\mu\}$ are adapted to (A4) and (A5). On the other hand, equations (A1) and (A3) show that the dual pair

$$\{\partial_i, n\} \quad \text{and} \quad \{dx^i + \beta^i dt, \alpha dt\} \tag{A9}$$

is adapted to (A6) and (A7).

Instead of $\{\partial_i\}$ we also use an orthonormal horizontal basis $\{\mathbf{e}_i\}$ ($g(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij}$), together with the dual basis $\{\boldsymbol{\vartheta}^i\}$ instead of $\{dx^i\}$. Then we have the following two dual pairs, which are constantly used:

$$\{\mathbf{e}_i, \partial_t\} \ , \quad \{\boldsymbol{\vartheta}^i, dt\} \qquad \text{(adapted to slicing)}, \tag{A10}$$

$$\{\mathbf{e}_i, e_0 = n\} \qquad \{\theta^\mu\} \qquad \text{(adapted to (A6) and (A7))} \ , \tag{A11}$$

where the orthonormal tetrad $\{\theta^\mu\}$ is given by

$$\theta^0 = \alpha dt \ , \ \theta^i = \boldsymbol{\vartheta}^i + \beta^i dt \ , \tag{A12}$$

with β^i here defined by $\boldsymbol{\beta} = \beta^i \mathbf{e}_i$. We note also the relation

$$e_0 = n = \frac{1}{\alpha}(\partial_t - \beta^i \mathbf{e}_i) \ .$$

A.2 The connection and curvature forms

We now calculate the connection and curvature forms in the orthonormal basis introduced above.

A.2.1 The connection forms

From the first structure equation ,

$$d\theta^\mu + \omega^\mu_\nu \wedge \theta^\nu = 0 \ ,$$

and the definition of the second fundamental form:

$$K_{ij} = -n_{i;j} \ , \tag{A13}$$

where n denotes the normal field of the slicing, one finds immediately the Gauss' formulas : (remember $n = e_0$)

$$\omega^i_{\ k}(\mathbf{e}_j) = \boldsymbol{\omega}^i_{\ k}(\mathbf{e}_j) \tag{A14}$$

$$\omega^0_i(\mathbf{e}_j) = -K_{ij} \ . \tag{A15}$$

We define

$$\partial_t \boldsymbol{\vartheta^i} = c^i_{\ j} \boldsymbol{\vartheta^j} \ . \tag{A16}$$

Then we can calculate the following quantities :

$$\omega^0_i(e_0) = \alpha^{-1}\alpha_{|i} \ , \tag{A17}$$

$$\omega^i_{\ j}(e_0) = -\alpha^{-1}\boldsymbol{\omega}^i_{\ j}(\boldsymbol{\beta}) + \frac{1}{2\alpha}(\beta^i_{|j} - \beta^j_{|i} - c^i_{\ j} + c^j_{\ i}) \ , \tag{A18}$$

$$K_{ij} = \frac{1}{2\alpha}(\beta_{i|j} + \beta_{j|i} - c^i_{\ j} - c^j_{\ i}) \ , \tag{A19}$$

and thus,

$$\boldsymbol{K} = \frac{1}{2\alpha}[L_\beta \boldsymbol{g} - \partial_t \boldsymbol{g}] \ , \tag{A20}$$

where the vertical bar $_|$ denotes covariant derivation with respect to $\boldsymbol{g}$. Using the general relation

$$\partial_t(\det \boldsymbol{g}) = \mathrm{tr}(\partial_t \boldsymbol{g}) \det \boldsymbol{g} \ ,$$

we find from (A20)

$$\partial_t \mathrm{vol}(\boldsymbol{g}) = (\mathrm{div}\boldsymbol{\beta} - \alpha \mathrm{tr}\boldsymbol{K})\mathrm{vol}(\boldsymbol{g}) \ . \tag{A21}$$

Let's derive (A17), (A18) and (A19) briefly .

$$d\theta^0 = d(\alpha dt) = \boldsymbol{d}\alpha \wedge dt = \alpha_{|i}\boldsymbol{\vartheta}^i \wedge dt = \alpha^{-1}\alpha_{|i}\theta^i \wedge \theta^0 \ .$$

This together with the first structure equation results in (A17). (A18) and (A19) are obtained as follows: From the first structure equation and (A15) we conclude

$$\begin{aligned} i_{\mathbf{e}_l} i_{e_0} d\theta^i &= -i_{\mathbf{e}_l} i_{e_0}(\omega^i_{\ 0} \wedge \theta^0 + \omega^i_{\ j} \wedge \theta^j) \\ &= -(K_{il} + \omega^i_{\ l}(e_0)) \ . \end{aligned}$$

We calculate the left hand side of this equation:

$$\begin{aligned} i_{\mathbf{e}_j} i_{e_0} d\theta^i &= i_{\mathbf{e}_j} i_{e_0} d(\boldsymbol{\vartheta}^i + \beta^i dt) \\ &= i_{\mathbf{e}_j} i_{e_0}(\boldsymbol{d}\boldsymbol{\vartheta}^i + dt \wedge \partial_t \boldsymbol{\theta}^i + \boldsymbol{d}\beta^i \wedge dt) \\ &= \alpha^{-1} i_{\mathbf{e}_j}(i_{\boldsymbol{\beta}}(\boldsymbol{\omega}^i_{\ l} \wedge \boldsymbol{\vartheta}^l) + \partial_t \boldsymbol{\vartheta}^i - \boldsymbol{d}\beta^i) \\ &= \alpha^{-1}[\omega^i_{\ j}(\boldsymbol{\beta}) - \omega^i_{\ k}(\mathbf{e}_j)\beta^k - \boldsymbol{d}\beta^i(\mathbf{e}_j) + \partial_t \boldsymbol{\vartheta}^i(\mathbf{e}_j)] \\ &= \alpha^{-1}[\omega^i_{\ j}(\boldsymbol{\beta}) - \beta^i_{\ |j} + c^i_{\ j}] \ . \end{aligned}$$

The symmetric and antisymmetric contribution of the last identity yield the formulas (A19) and (A18) for K_{ij} and $\omega^i_{\ j}(e_0)$, respectively.

A.2.2 The curvature forms

We now want to calculate the $3+1$ split of R_{ij}, R_{0j} and G_{00}. From the second structure equation,

$$\Omega^\mu_{\ \nu} = d\omega^\mu_{\ \nu} + \omega^\mu_{\ \lambda} \wedge \omega^\lambda_{\ \nu} \ ,$$

and equations (A14) to (A19) one finds immediately

$$\Omega^i_{\ j}(\mathbf{e}_k, \mathbf{e}_l) = \boldsymbol{\Omega}^i_{\ j}(\mathbf{e}_k, \mathbf{e}_l) + K^i_{\ k}K_{jl} - K^i_{\ l}K_{jk} \qquad \text{(Gauss)} \tag{A22}$$

$$\Omega^0_{\ j}(\mathbf{e}_k, \mathbf{e}_l) = K_{jk|l} - K_{jl|k} \qquad \text{(Mainardi)} \ . \tag{A23}$$

We need also the normal components of $\Omega^0_{\ j}$. By the second structure equation we know

$$\Omega^i_{\ 0} = -d(K_{ij}\theta^j) + d(\alpha^{-1}\alpha_{|i}\theta^0) + \omega^i_{\ l} \wedge (K_{lj}\theta^j + \alpha^{-1}\alpha_{|l}\theta^0) \ .$$

A straight forward calculation leads to

$$\Omega^i_{\ 0} = \alpha^{-1}\alpha_{|ij}(\theta^j \wedge \theta^0) - dK^i_{\ j} \wedge \theta^j + K^i_{\ j}(\omega^j_{\ l} \wedge \theta^l - K^j_{\ l}\theta^l \wedge \theta^0) + \omega^i_{\ j}K^j_{\ l} \wedge \theta^l \ ,$$

which yields (A23) and the normal components of $\Omega^i_{\ 0}$:

$$\Omega^i_{\ 0}(\mathbf{e}_j, e_0) = \alpha^{-1}\alpha^{|i}_{\ |j} + dK^i_{\ j}(e_0) - K^i_s\omega^s_{\ j}(e_0) + K^{2i}_{\ j} - \omega^i_{\ s}(e_0)K^s_{\ j} \ . \tag{A24}$$

From equations (A22) to (A24) we can calculate the Ricci tensor with the result:

$$R_{\beta\sigma} = \Omega^\alpha_{\ \beta}(e_\alpha, e_\sigma) \ ,$$

$$R_{00} = \frac{1}{\alpha}\triangle\alpha + \alpha^{-1}(\partial_t \mathrm{tr}(K) - L_{\boldsymbol{\beta}}\mathrm{tr}(K_{ij})) + \mathrm{tr}\boldsymbol{K}^2 \ . \tag{A25}$$

With help of (A23) one finds

$$R_{0i} = (trK)_{|i} - K_i{}^j{}_{|j} \ . \tag{A26}$$

For the spatial components we obtain

$$R_{ij} = \Omega^0_i(e_0, \mathbf{e}_j) + \Omega^k_i(\mathbf{e}_k, \mathbf{e}_j) \ .$$

Using (A24) and (A22) for the curvature forms leads to

$$R_{ij} = \boldsymbol{R}_{ij} + \mathrm{tr}(K)K_{ij} - 2K^2_{ij} - \alpha^{-1}\alpha_{|ij} - \alpha^{-1}(\partial_t K_{ij} - L_{\boldsymbol{\beta}}K_{ij}) + K_{is}\omega^s_j(e_0) + K_{js}\omega^s_i(e_0) \ . \tag{A27}$$

With help of (A18), (A19) and (A20) one can bring (A27) into the form

$$\mathrm{hor}(Ricci(g)) = \boldsymbol{Ricci(g)} + \mathrm{tr}(\boldsymbol{K})\boldsymbol{K} - 2\boldsymbol{K}^2 - \alpha^{-1}(\partial_t\boldsymbol{K} - L_{\boldsymbol{\beta}}\boldsymbol{K}) - \alpha^{-1}\boldsymbol{Hess}(\alpha) \ . \tag{A28}$$

Using (A25) and (A27) we find

$$\begin{aligned} G_{00} &= 1/2(R_{00} + \textstyle\sum_i R_{ii}) \\ &= 1/2[\boldsymbol{R} + (\mathrm{tr}(K))^2 - \mathrm{tr}(K^2)] \ . \end{aligned} \tag{A29}$$

A.3 The $3+1$ split of hydrodynamics

Calculations similar to those in the last section lead quite rapidly to a $3+1$ split of hydrodynamics.

Let us decompose the energy-momentum tensor into horizontal and vertical components:

$$T = \epsilon e_0 \otimes e_0 + e_0 \otimes \boldsymbol{S} + \boldsymbol{S} \otimes e_0 + \boldsymbol{T} \ . \tag{A30}$$

For an ideal fluid with

$$T = (\rho + p)u \otimes u + pg^{\#} \tag{A31}$$

we find, setting as in special relativity $u = \gamma(e_0 + \boldsymbol{v})$, $\gamma = (1 - \boldsymbol{v^2})^{-1/2}$,

$$\epsilon = \gamma^2(\rho + p\boldsymbol{v^2}) \ , \tag{A32}$$

$$\boldsymbol{S} = (\rho + p)\gamma^2\boldsymbol{v} \ , \tag{A33}$$

$$\boldsymbol{T} = (\rho + p)\gamma^2\boldsymbol{v} \otimes \boldsymbol{v} + p\boldsymbol{g^{\#}} \ . \tag{A34}$$

Now we compute $\nabla \cdot T$ for an arbitrary T. From

$$\nabla_{e_0}(\epsilon e_0 \otimes e_0) = L_{e_0}(\epsilon)e_0 \otimes e_0 + \epsilon\omega^i{}_0(e_0)\boldsymbol{e}_i \otimes e_0 + \epsilon e_0 \otimes \omega^i{}_0(e_0)\boldsymbol{e}_i$$

and

$$\nabla_{\boldsymbol{e}_k}(\epsilon e_0 \otimes e_0) = L_{\boldsymbol{e}_k}(\epsilon)e_0 \otimes e_0 + \epsilon\omega^i{}_0(\boldsymbol{e}_k)\boldsymbol{e}_i \otimes e_0 + \epsilon e_0 \otimes \omega^i{}_0(\boldsymbol{e}_k)\boldsymbol{e}_i$$

we obtain

$$\nabla \cdot (\epsilon e_0 \otimes e_0) = L_{e_0}(\epsilon)e_0 + \epsilon\omega^i{}_0(e_0)\boldsymbol{e}_i + \epsilon\omega^i{}_0(\boldsymbol{e}_i)e_0 \ .$$

In the same manner one finds the other contributions with the result:

$$(\nabla \cdot T)^0 = L_{e_0}(\epsilon) + \epsilon\omega^i{}_0(\boldsymbol{e}_i) + \omega^0{}_j(e_0)S^j + S^k{}_{|k} + \omega^0{}_j(e_0)S^j + \omega^0{}_j(\boldsymbol{e}_i)T^{ij}$$

Inserting the expressions for the connection forms given in Section A1, leads to the following form of the energy equation:

$$\frac{1}{\alpha}(\partial_t - \boldsymbol{L}_{\boldsymbol{\beta}})\epsilon = -\boldsymbol{\nabla}\cdot\boldsymbol{S} - 2\boldsymbol{\nabla}(\ln\alpha)\cdot\boldsymbol{S} + \epsilon\mathrm{tr}(\boldsymbol{K}) + \mathrm{tr}(\boldsymbol{K}\cdot\boldsymbol{T}) \ . \tag{A35}$$

Similarely one finds

$$(\nabla \cdot T)^i = \omega^i{}_0(e_0)\epsilon + L_{e_0}(S^i) + [\omega^i{}_0(\boldsymbol{e}_j) + \omega^i{}_j(e_0)]S^j + \omega^j{}_0(\boldsymbol{e}_j)S^i + \omega^0{}_j(e_0)T^{ji} + T^{ij}{}_{|j}$$

and from this we obtain the momentum conservation

$$\frac{1}{\alpha}(\partial_t - \boldsymbol{L}_{\boldsymbol{\beta}})\boldsymbol{S} = -\boldsymbol{\nabla}(\ln\alpha)\epsilon + 2\boldsymbol{K}\cdot\boldsymbol{S} + \mathrm{tr}(\boldsymbol{K})\boldsymbol{S} - \alpha^{-1}\boldsymbol{\nabla}\cdot(\alpha\boldsymbol{T}) \ . \tag{A36}$$

This equation is used to derive the vector perturbation equation (2.63) in Chapter 2.

A.4 The $3+1$ split of Einsteins field equations

Here we discuss the often used $3+1$ split of the gravitational field equations. The calculation of the curvature forms relative to the basis (A12) is presented in Section A2. The reader will note that Cartan's calculus leads rather quickly to the required results.

We use the notation introduced in the previous section (A30) for the various projections of the energy-momentum tensor T into normal and horizontal components. From equations (A26), (A28) and (A29) for the Einstein and Ricci tensors, Einsteins field equations can be written in the form (recall that boldface letters always refer to the slices Σ_t):

$$\boldsymbol{R} + (\mathrm{tr}\boldsymbol{K})^2 - \mathrm{tr}\boldsymbol{K}^2 = 16\pi G\epsilon \ , \tag{A37}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{K} - \boldsymbol{\nabla}\cdot\mathrm{tr}(\boldsymbol{K}) = 8\pi G\boldsymbol{S} \ , \tag{A38}$$

$$\partial_t\mathbf{K} = \boldsymbol{L}_{\boldsymbol{\beta}}\boldsymbol{K} - \mathbf{Hess}(\alpha) + \alpha[\mathbf{Ric}(\boldsymbol{g}) - 2\boldsymbol{K}\cdot\boldsymbol{K} + (\mathrm{tr}\boldsymbol{K})\boldsymbol{K} - 8\pi G(\boldsymbol{T} - \frac{1}{2}\boldsymbol{g}(\epsilon - \mathrm{tr}\boldsymbol{T}))] \ . \tag{A39}$$

In addition to (A37), (A38) and (A39) we have the following relation (Section A2, equation (A20)) between $\boldsymbol{g}$ and the second fundamental form $\mathbf{K}$:

$$\partial_t\boldsymbol{g} = -2\alpha\boldsymbol{K} + \boldsymbol{L}_{\boldsymbol{\beta}}\boldsymbol{g} \ . \tag{A40}$$

Note that this decomposition into constraint equations (A37), (A38) and dynamical equations (A39), (A40) involves only horizontal quantities and those provides the $3+1$ split of the gravitational field equations.

In Chapter 2 we also use the following consequence of (A39) and (A37)

$$\partial_t\mathrm{tr}(\boldsymbol{K}) = -\triangle\alpha + \boldsymbol{L}_{\boldsymbol{\beta}}\mathrm{tr}(\boldsymbol{K}) + \alpha[\mathrm{tr}(\boldsymbol{K}^2) + 1/2(\epsilon + \mathrm{tr}\boldsymbol{T})] \ . \tag{A41}$$

Note that ∂_t and tr do not commute. With (A40) one shows easily

$$\mathrm{tr}(\partial_t\boldsymbol{K} - \boldsymbol{L}_{\boldsymbol{\beta}}\boldsymbol{K}) = \partial_t\mathrm{tr}(\boldsymbol{K}) - \boldsymbol{L}_{\boldsymbol{\beta}}\mathrm{tr}(\boldsymbol{K}) + 2\alpha\mathrm{tr}(\boldsymbol{K}^2) \ . \tag{A42}$$

To derive the perturbation equations for vector perturbations we mainly use (A38) and (A39).

A.5 The $3+1$ split of the Liouville operator for a geodesic spray

In this section we derive a useful form of the Liouville operator for a geodesic spray for an arbitrary $3+1$ split.

We start with some generalities. The metric g of the spacetime manifold $\mathcal{M}$ defines a natural diffeomorphism between the tangent bundle $T\mathcal{M}$ and the cotangent bundle $T^*\mathcal{M}$, which can be used to pull back the natural symplectic form on $T^*\mathcal{M}$. In terms of natural bundle coordinates the diffeomorphism is given by $(x^\mu, p^\mu) \mapsto (x^\mu, p_\mu = g_{\mu\nu}p^\nu)$ and those the induced symplectic 2-form on $T\mathcal{M}$ is

$$\omega = dx^\mu \wedge d(g_{\mu\nu}p^\nu) \ . \tag{A43}$$

The Lagrangian $L = \frac{1}{2}g_{\mu\nu}p^\mu p^\nu$ on $T\mathcal{M}$ defines a Hamiltonian vector field X_g on $T\mathcal{M}$, determined by

$$i_{X_g}\omega = dL \ .$$

In terms of natural bundle coordinates the geodesic spray X_g is given by

$$X_g = p^\mu \partial_\mu - ?^\mu{}_{\alpha\beta} p^\alpha p^\beta \frac{\partial}{\partial p^\mu} \ , \tag{A44}$$

where $?^\mu{}_{\alpha\beta}$ are the Christoffel symbols for $(\mathcal{M}, g)$. (For further details see [Stewart, 1971].)

The one-particle phase space for particles of mass m, i.e., the sub–bundle $\{p \in T\mathcal{M} : g(p,p) = -m^2\}$, is invariant under the geodesic flow and we denote the restriction of X_g to the one-particle phase space also by X_g.

Let f be a distribution function on the one-particle phase space. The Vlasov and Boltzmann equations for f involve the Lie derivative $L_{X_g}f$. If we consider the spatial components p^i, relative to an orthonormal tetrad $\{e^\mu\}$ as independent variables of f, then the Liouville operator L_{X_g} can be written as

$$L_{X_g}f = p^\mu e_\mu(f) - \omega^i{}_\alpha(p)p^\alpha \frac{\partial f}{\partial p^i} \ , \tag{A45}$$

where ω^μ_ν are the connection forms relative to the dual basis $\{\theta^\mu\}$.

We derive now a more explicit expression of (A45) for an arbitrary $3+1$ slicing. In order to do this, we need the connection forms relative to the basis $\{\theta^\mu\}$ calculated in Section A2 . They can be expressed in terms of $\alpha, \boldsymbol{\beta}, \boldsymbol{\omega}^i{}_j, c^i{}_j$. Using equations (A17), (A15), (A18) and (A14) we find ($\boldsymbol{p} = p^i\boldsymbol{e}_i$, $p^0 = \sqrt{\boldsymbol{p}^2 + m^2}$):

$$\begin{aligned}
\omega^i{}_\alpha(p)p^\alpha \tfrac{\partial}{\partial p^i} &= \omega^i{}_0(p)p^0 \tfrac{\partial}{\partial p^i} + \omega^i{}_j(p)p^j \tfrac{\partial}{\partial p^i} \\
&= [\omega^i{}_0(e^0)p^0 + \omega^i{}_0(\boldsymbol{p})]p^0 \tfrac{\partial}{\partial p^i} + [\omega^i{}_j(e^0)p^0 + \omega^i{}_j(\boldsymbol{p})]p^j \tfrac{\partial}{\partial p^i} \\
&= (p^0)^2\alpha^{-1}\alpha^{|i} \tfrac{\partial}{\partial p^i} - K^i{}_j p^0 p^j \tfrac{\partial}{\partial p^i} + \boldsymbol{\omega}^i{}_j(\boldsymbol{p})p^j \tfrac{\partial}{\partial p^i} + \omega^i{}_j(e^0)p^0p^j \tfrac{\partial}{\partial p^i} \\
&= (p^0)^2\alpha^{-1}\alpha^{|i} \tfrac{\partial}{\partial p^i} + \boldsymbol{\omega}^i{}_j(\boldsymbol{p} - \alpha^{-1}\boldsymbol{\beta}p^0)p^j \tfrac{\partial}{\partial p^i} - \tfrac{p^0}{\alpha}(\beta_j{}^{|i} - c_j{}^i)p^j \tfrac{\partial}{\partial p^i} \ .
\end{aligned}$$

Here $K^i{}_j$ are the components of the second fundamental form of Σ_t, for which we also use equation (A19) of Section A2.

This leads to the following useful $3+1$ split of the Liouville operator:

$$L_{X_g}f = [\frac{p^0}{\alpha}\partial_t + \boldsymbol{L}_{\boldsymbol{p} - \frac{p^0}{\alpha}\boldsymbol{\beta}}]f - [\boldsymbol{\omega}^i{}_j(\boldsymbol{p} - \frac{p^0}{\alpha}\boldsymbol{\beta})p^j + (p^0)^2(\ln\alpha)^{|i} - p^0 H^i{}_j p^j]\frac{\partial f}{\partial p^i} \ , \tag{A46}$$

where we have introduced the horizontal tensor field

$$H^i{}_j = \alpha^{-1}(\beta^i{}_j - c^i{}_j) \ . \tag{A47}$$

Equation (A46) is used in Section 2.3.

A.6 Glossary

In this appendix we provide a glossary of the variables used in the text. For most terms we give a short explanation and refer to the equation or section where this variable is first used. Usually it is defined there. If not, this should be a very common variable found, e.g., in most basic text books on general relativity (like the Christoffel symbols, the Riemann tensor and so on).

A Perturbation of the 00 component of the metric, respectively the lapse function (2.4),(2.11), Appendix A.

B Scalar perturbation of the $0i$ component of the metric, respectively the shift vector (2.5),(2.11).

B_i Vector perturbation of the $0i$ component of the metric, respectively the shift vector (2.7),(2.12).

B_{ij} Magnetic part of the Weyl tensor (2.27).

$C_{\alpha\mu\beta\nu}$ $= R_{\alpha\mu\beta\nu} - (1/2)(g_{\alpha\beta}R_{\mu\nu} - +g_{\mu\nu}R_{\beta\alpha} - g_{\mu\beta}R_{\nu\alpha} - g_{\alpha\nu}R_{\mu\beta}) + \frac{R}{6}(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\mu\beta})$, Weyl tensor (2.26,27).

$D^{(\alpha)}$ Gauge invariant density perturbation variable for the matter component α (2.38).

$D_g^{(\alpha)}$ Gauge invariant density perturbation variable for the matter component α (2.37).

$D_s^{(\alpha)}$ Gauge invariant density perturbation variable for the matter component α (2.36).

E_{ij} Electrical part of the Weyl tensor (2.26).

F Gauge dependent perturbation variable for the distribution function, paragraph (2.3.1).

$\mathcal{F}^{(S)}$ Gauge invariant perturbation variable for scalar perturbations of the distribution function (2.69).

$\mathcal{F}^{(T)}$ Gauge invariant perturbation variable for tensor perturbations of the distribution function, paragraph (2.3.2).

$\mathcal{F}^{(V)}$ Gauge invariant perturbation variable for vector perturbations of the distribution function, paragraph (2.3.2).

G Newtons constant, $G = 6.6720 \times 10^{-8} cm^3 g^{-1} sec^{-2}$.

$G_{\mu\nu}$ $= R_{\mu\nu} - (1/2)g_{\mu\nu}R$, Einstein tensor.

H_L Trace perturbation of the spatial part of the metric.

H_T Anisotropic scalar perturbation of the spatial part of the metric (2.6,11).

H_i Anisotropic vector perturbation of the spatial part of the metric (2.8,12).

H_{ij} Anisotropic tensor perturbation of the spatial part of the metric (2.9,13).

K_{ij} Second fundamental form, Section 2.1, Appendix A.

L^i Spatial components of the vector field X parametrizing a gauge transformation, Section 2.1, 2.3.

L_X Lie derivative w.r.t the vector field X, Section 2.3.

M Mass used to parametrize the energy momentum tensor of seed perturbations, Section 2.5.

$\mathcal{M}$ Gauge invariant perturbation variable for the energy integrated photon distribution (2.83).

$\mathcal{M}$ Spacetime manifold, Appendix A, Section 2.3

M_i $= \frac{3}{4\pi}\int d\Omega \epsilon_i \mathcal{M}$ The first moment of $\mathcal{M}$ (3.19).

M_{ij} $= \frac{3}{8\pi}\int d\Omega \epsilon_{ij} \mathcal{M}$ The second moment of $\mathcal{M}$ (3.17).

$P_\mu{}^\nu = u_\mu u^\nu + \delta_\mu^\nu$ The projection operator onto the 3–space orthogonal to u (2.30).

P_m The mass bundle, Section 2.3

R The Ricci scalar.

$R = 3\rho_m/4\rho_r$ Parameter used in paragraph 3.2.3.

$\mathcal{R}$ Perturbation of the scalar curvature on the slices of constant time (2.14).

$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$ The Ricci tensor.

$R^\beta{}_{\mu\alpha\nu}$ The Riemann tensor

T Temperature of the cosmic background radiation.

T Temporal component of the vector field X parametrizing a gauge transformation, Section 2.1, 2.3.

$T\mathcal{M}$ Tangent space to spacetime, Section 2.3.

TX Tangent vector field associated to the vector field X, Section 2.3.1

$T_{\mu\nu}^{(sS)}$ Scalar contribution to the energy momentum tensor of the seeds (2.117,118,119).

$T_{\mu\nu}^{(sT)}$ Tensor contribution to the energy momentum tensor of the seeds (2.122).

$T_{\mu\nu}^{(sV)}$ Vector contribution to the energy momentum tensor of the seeds (2.120,121).

V Gauge invariant variable for scalar perturbations of the velocity field (2.35).

V_i Gauge invariant variable for vector perturbations of the velocity field (2.42).

X Vector field parametrizing a gauge transformation, Section 2.1, 2.3.

$\partial_\mu = \frac{\partial}{\partial x^\mu}$ Partial derivative (vector field)

a Cosmic scale factor, Section 1.1.

b Impact parameter, Section 3.4, 4.2.2.

c Speed of light, usually set equal to 1 in this text.

$c_s = \sqrt{\dot{p}/\dot{\rho}}$, ($c_\alpha = \sqrt{\dot{p}_\alpha/\dot{\rho}_\alpha}$) Adiabatic sound speed (of matter component α), Section 1.1.

e_μ Tetrad vector field, Section 2.3, Appendix A.

f Distribution function of phase space, Section 2.3.

f_π Gauge invariant scalar potential parametrizing anisotropic stresses of seeds (2.119).

f_ρ Gauge invariant perturbation variable parametrizing the energy density of seeds (2.117).

f_p Gauge invariant perturbation variable parametrizing the pressure of seeds (2.119).

f_v Gauge invariant perturbation variable parametrizing the scalar velocity potential of seeds (2.118).

$g_{\mu\nu}$ Metric of spacetime, Chapter 2.

h Used to parametrize Hubble's constant $H_0 = h \times 100 \frac{km}{secMpc}$, Section 1.1.

$h_{\mu\nu}$ Metric perturbation (2.10).

$\hbar$ Planck's constant, $\hbar = 1.0546 \times 10^{-27} cm^2 g sec^{-1}$, usually set equal to 1 in this text.

k Spatial curvature of a Friedmann universe (1.1).

k Comoving wave number, Section 3.2, paragraph 3.4.2.

k_B Boltzmann's constant, $k_B = 1.3807 \times 10^{-16} erg/K$, usually set equal to 1 in this text.

l Length introduced to keep perturbation variables dimensionless, in applications it may be set equal to a typical scale of perturbations, Section 2.1.

$l_H = t$ Comoving size of the horizon, Section 1.3.

q Redshift corrected energy, paragraph 2.3.1.

t Conformal time, Section 1.1.

t_T Conformal Thomson mean free path, Section 3.2.

u Energy velocity field, Section 2.1

v Scalar velocity potential Section 2.1.

v Redshift corrected momentum, paragraph 2.3.1.

v^i Vector peculiar velocity field, Section 2.1.

w Enthalpy, Section 1.1.

$w_i^{(\pi)}$ Gauge invariant vector potential parametrizing anisotropic stresses of seeds (2.121).

$w_i^{(v)}$ Gauge invariant vector contribution to the energy flow of seeds (2.120).

z Cosmological redshift.

? Gauge invariant entropy perturbation variable, Section 2.1.

$?^{\mu}_{\nu\lambda}$ Christoffel symbols, Section 2.3.

Δ Laplacian.

Λ Cosmological constant (1.1).

Π Gauge invariant scalar potential for anisotropic stresses, Section 2.1.

Π_i Gauge invariant vector potential for anisotropic stresses, Section 2.1.

Π_{ij} Gauge invariant tensor contribution to anisotropic stresses, Section 2.1.

Σ Three dimensional spatial hypersurface, Appendix A.

Φ Gauge invariant scalar potential for geometry perturbations (2.24).

Ψ Gauge invariant scalar potential for geometry perturbations (2.25).

Ω^i Gauge invariant perturbation variable for the fluid vorticity (2.43).

$\Omega^{\mu}{}_{\nu}$ Curvature 2–form, Appendix A.

α Lapse function (2.4), Appendix A.

β Shift vector (2.5), Appendix A.

γ_{ij} Metric of a three space of constant curvature, Section 1.1.

δ Gauge dependent density perturbation (2.28).

$\boldsymbol{\epsilon}^i$ Spatial unit vector (e.g. denoting photon directions), Section 2.3.

$\epsilon = 4\pi GM^2$ Smallness parameter for the amplitude of seed perturbations, paragraph 2.5.2.

ϵ_{ijk} Three dimensional totally antisymmetric tensor (2.27).

$\epsilon_{ij} = \epsilon_i\epsilon_j - \gamma_{ij}$, Section 3.2.

η Symmetry breaking scale (4.1).

θ^{μ} Orthonormal tetrad of 1–forms, Appendix A.

ϑ^i Orthonormal triad of 1–forms on the hypersurfaces of constant time, Appendix A.

ι Isomorphism between the perturbed and unperturbed mass bundles, Section 2.3.

ι Gauge dependent perturbation variable for the energy integrated photon distribution paragraph (2.3.4).

λ Parameter in the scalar field potential (4.1).

$\lambda = (1 + kr^2/4)^{-1}$ Conformal factor for the metric of a 3 space of constant curvature, Section 2.3.

μ Cosine between the photon direction and the radial direction, paragraph 3.2.1.

π^μ Orthonormal momentum components Section 2.3.

π^i_j Anisotropic stresses (2.31).

π_L Gauge dependent pressure perturbation variable (2.31).

$\rho_{(\alpha)}$, $\bar{\rho}_{(\alpha)}$ Background energy density of component α.

σ Scalar potential for the shear of the equal time hypersurfaces, extrinsic curvature (2.15).

σ^i Vector potential for the shear of the equal time hypersurfaces, extrinsic curvature (2.17).

σ_T Thomson cross section, $\sigma_T = 6.6524 \times 10^{-25} cm^2$.

τ Physical time, Section 1.1.

τ Impact time (4.30)

τ Optical depth, Section 5.1.

$\tau_{\mu\nu}$ Stress tensor (2.30).

ϕ Scalar field, Chapter 4.

χ Variable parametrizing spherically symmetric scalar field configurations (4.4).

ω Winding number density of the scalar field, Section 4.1.

$\omega_\nu{}^\mu$ Connection forms, Appendix 1.

ω_{ij} Vorticity of a velocity field Section 2.1.

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FIGURE CAPTIONS

Fig. 1: The spectrum of the cosmic microwave background radiation as measures by COBE (Figure from Mather et al. [1990]).

Fig. 2: Limits on the CMB anisotropy on different angular scales. The COBE result is the only positive detection. All other marks represent 95% confidence upper limits.

Fig. 3: The linear perturbation spectrum of hot dark matter, for one (2) or three (1) types of massive neutrinos. The fluctuations are heavily damped on scales smaller than $\lambda_{FS} \approx m_{Pl}/m_\nu^2$ (Figure from Durrer [1989]).

Fig. 4: Simulations of structure formation with HDM (top right and bottom pictures) compared with the corresponding picture from the CfA survey (top left). Triangles are high density regions identified as galaxies. One sees that the simulations lead to highly over developed large scale structure (Figure from White [1986]).

Fig. 5: CDM simulations (a) and (b) compared with the CfA survey (c). No striking inconsistencies are visible at first sight (Figure from Kolb and Turner [1990]).

Fig. 6: The angular galaxy galaxy correlation function as measured by the IRAS survey (black dots) compared with the predictions from CDM models with $h = 0.4$ (black line) and $h = 0.5$ (dotted line). The open circles and squares are results from an older analysis of the Lick catalogue (Figure from Maddox et al. [1990]).

Fig. 7 The CMB anisotropy (in units of 10^{-3}) from a spherically symmetric texture collapsing at $z = 30$ (left) and $z = 200$ (right) respectively as a function of angular separation from the center of the texture. This figure is calculated for a universe which reionizes at $z = 200$. It shows how signals from small scale textures are substantially damped and broadened by photon diffusion.

Fig. 8: The hot spot—cold spot signal of a spherically symmetric collapsing texture in units of

$\epsilon \sim 2.8 \times 10^{-4}$. The horizontal variable $\tau = t - r\cos\theta$ denotes the 'impact time' of a photon arriving at a distance r from the texture at time t traveling with an angle θ with respect to the radial direction. The hot spot–cold spot is shown for photons with fixed impact parameter $b = r\sin\theta \approx 0.1t_c$ (t_c is the time of texture collapse). The signal from the expanding universe at $t = t_c$, line (1), and $t = 1.5t_c$, line (2), is compared with the flat space result (dashed curve). The second peak appearing at $t = 1.5t_c$ is due to the dark matter potential.

Fig. 9 As Fig. 8 but for different times with time steps $\Delta t \approx 0.25t_c$. One sees an outgoing wake of blue shift at $\tau \approx t$. This is caused by photons which have fallen into the dark matter potential but have not yet climbed out of it again. This blueshift will of course be completely compensated by the redshift these photons will acquire during their way out of the dark matter potential.

Fig. 10 The CMB perturbation in units of $\epsilon \sim 2.8 \times 10^{-4}$ as a function of the impact parameter b for fixed $\tau \sim 0.5t_c = 10$. The signal disappears at an impact parameter $b \sim 1.5t_c$ ($t_c = 20$ in the units chosen).

Fig. 11 A simulated COBE map as it might look in a scenario with texture + CDM. The color scheme goes from -4×10^{-4} (dark blue) to 1×10^{-4} (deep red). Monopole and dipole contributions are subtracted in this map. A description of how the map is produced (in collaboration with D.N. Spergel and A. Howard) is given in the text.

Fig. 12 The statistical distribution of microwave anisotropies in the texture scenario. The number of pixels showing a given anisotropy are counted for one realization of the CMB sky. The distribution is slightly non–Gaussian with skewness ≈ -1 and curtosis ≈ 3.

Fig. 13 A $3+1$ slicing of spacetime $\mathcal{M}$. The family of immersions of Σ into $\mathcal{M}$ is denoted by $i_t;\;\; i_t(m) = \phi^{-1}(m,t)$.