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GENERAL RELATIVISTIC TEXTURES AND THEIR INTERACTIONS WITH MATTER AND RADIATION*

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Abstract

Time dependent solutions of the coupled Einstein σ-model equations are studied analytically and numerically. We analyse in detail a spherically symmetric self-similar solution which describes the general relativistic collapse of a global texture. In view of the cosmological implications of the texture scenario for large scale structure formation, we derive exact analytical formulae for the light deflection, photon red shift and density fluctuations in the gravitational field of this texture solution. We also discuss the behavior of collisionless particles. Numerical results of a parameter study of other texture solutions are presented and are compared with the special self-similar solution.

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1. INTRODUCTION

In a series of recent papers the viability of a new mechanism for producing large-scale inhomogeneities in an initially homogeneous Universe has been investigated [1, 2, 3, 4, 5]. The new proposal is based on the same general idea as the cosmic string scenario, namely that the inhomogeneities of the Universe might have been induced by topological defects which formed during a phase transition in the early Universe. It is, however, assumed that textures (π_3-defects) — instead of cosmic strings (π_1-defects) — were the original seeds for structure formation. This alternative was partly motivated by some serious difficulties encountered in the cosmic string picture. In particular, improved calculations have shown that the scale of the string network appears to be too small to lead to the formation of the observed structures.

At this early stage the texture scenario looks quite promising. One of the attractive features is the resulting power spectrum for the density perturbations [6, 7, 8], which has more power both on small and large scales than the standard CDM model with biasing, but leads to the same predictions on intermediate scales. This seems to be exactly what is needed on observational grounds [10]. (See also the recent review [11] of the observational contraints for the power spectrum of primordial density fluctuations.) Similar results might be obtained in a scenario with global monopoles (π_2–defects) [9].

As for any other model of galaxy-formation the upper limits on the microwave background anisotropies represent a crucial test for texture seeded structure formation. The predicted anisotropies are still under investigation. Preliminary results for CDM and dust show that small scale fluctuations are strongly suppressed, but on scales of 10 - 20 degrees peaks with amplitudes $\Delta T/T \approx (1-5)10^{-5}$ are predicted and about 5 to 10 such peaks are expected all over the sky [12]. In these calculations it is, however, assumed that recombination never took place, because it is expected that early formation of collapsed small objects would probably reionize matter at high red shifts (≥ 200). It may well be that the theoretical expectations will soon come into conflict with new observations. Without reionization the texture model would already be in serious difficulties.

The collapse of textures has first been studied in flat space and in an expanding Friedman Universe. In this context the metric perturbations were evaluated in linearized approximation for an exact spherically symmetric flat space texture solution [3]. As a first step this procedure is certainly justified, given the fact that the only parameter entering the coupled field equations is the small number $\kappa \equiv 8\pi(\eta/m_{Pl})^2$, where η is the vacuum expectation value of the Higgs field and m_{Pl} is the Planck mass. For grand-unified-theories typical values of η are of the order of 10^{16} GeV. Successful structure formation requires a value of $\kappa \approx 5 \cdot 10^{-4}$ [6] which yields η of this order.

The metric perturbations computed in Ref.[3], and also the gauge–invariant metric potentials given in Ref.[5] diverge logarithmically at large distances and\or large times, implying that the linearized approximation is no longer valid in this region. This fact lead us to look for solutions of the coupled Einstein-σ model equations and to calculate the interactions of the collapsing texture field with matter and radiation. We found two spherically symmetric collapse solutions [13], for which all dependences on the time t and the radial coordinate r are given by functions of r/t (for suitable gauge choices). The coupled field equations reduce to two, respectively, three ordinary differential equations, which can easily be solved numerically. Our suspicion, that the two solutions might be connected by a complicated coordinate transformation turned out to be true. Both forms are, however, useful for practical applications.
Independently Barriola an Vachaspati [14] derived the coupled field equations in another set of coordinates, which are also connected to our coordinate choice by a complicated transformation.

The present paper contains a more complete discussion of our self-similar solution and we study in detail the interaction of the gravitational texture field with matter and radiation. In addition, we have developed a code for numerical integration of the coupled partial differential equations that describe the spherically symmetric collapse of textures and some results of a parameter study will be presented. We hope that these are useful ingredients for more detailed investigations of the texture scenario.

The paper is organized as follows: In section 2 we give the basic equations and in section 3 we present the ordinary differential equations for self-similar configurations for two very different coordinate choices. Since both forms are useful we discuss in detail their equivalence. In addition, we give some results of a parameter study of numerical solutions of the partial differential equations describing spherically symmetric texture evolution. In section 4 the interaction of matter and radiation with the collapsing textures is studied. We calculate the deflection of light, the red shift for photons and derive an exact solution for the fluid equations of CDM and dust in the self-similar texture field. A similar discussion is presented for the Liouville equation describing collisionless matter. In all cases the validity of the linearized approximation can easily be controlled.

2. BASIC EQUATIONS

As in all papers on the texture scenario for large-scale structure formation [3], we consider here a non-linear O(4)-σ model in which the four component field $\vec{\phi} = (\phi^a)$ $(a = 1, .., 4)$ varies on the 3-sphere S^3. This should be considered as a limiting case of the four component Higgs field described by the action

$$S(\vec{\phi}) = \int (\frac{1}{2}\nabla_\mu\vec{\phi}\cdot\nabla^\mu\vec{\phi} \; - \frac{\lambda}{4}(\vec{\phi}^2 - \eta^2)^2)\sqrt{-g}\; d^4x. \tag{0.1}$$

In many circumstances it is a good approximation to replace the potential term in (0.1) by the constraint $\vec{\phi}\cdot\vec{\phi} = \eta^2$. The dynamics is then determined, subject to this constraint, by the first term of (0.1) and is highly non-trivial. Mathematically, $\vec{\phi}(x)$ describes harmonic maps from the spacetime manifold (with metric g and corresponding covariant derivative ∇_μ) into the vacuum manifold of (0.1), which is a 3-sphere. Such maps can be grouped into homotopy classes. In particular, the field configurations which have a well-defined limit at spacial infinity, fall into classes of the third homotopy group, $\pi_3(S^3) = \mathcal{Z}$, of the vacuum manifold. The integers of this group correspond to the winding number, defined by the induced map from the compactified three-dimensional space S^3 into the vacuum manifold.

In the σ–model approximation we can still use the action (0.1) but λ has now the meaning of a Lagrange multiplier. The field equations are then

$$\nabla^\mu\nabla_\mu\vec{\phi} + (\nabla_\mu\vec{\phi}\cdot\nabla^\mu\vec{\phi})\vec{\phi} = 0. \tag{0.2}$$

(We chose the units such that the vacuum expectation value η of the Higgs field is equal to one.) The energy-momentum tensor is

$$T_{\mu\nu} = \nabla_\mu\vec{\phi}\cdot\nabla_\nu\vec{\phi} - \frac{1}{2}g_{\mu\nu}\nabla_\lambda\vec{\phi}\cdot\nabla^\lambda\vec{\phi}. \tag{0.3}$$

Our basic dynamical equations are (0.2) and Einsteins field equations with expression (0.3) for the energy-momentum tensor. The only parameter, appearing in these coupled field equations, is the number κ defined earlier.

For later discussions, the following remark has to be kept in mind. The field equation (0.2) for the

matter field implies, of course, $\nabla_\nu T^{\mu\nu} = 0$, but the converse is *not* true, as can easily be seen.
For fixed time, any field configuration $\vec{\phi}(x)$ defines a closed three-form

$$\Omega = \frac{1}{12\pi^2}\epsilon_{abcd}\ \phi^a\ d\phi^b \wedge d\phi^c \wedge d\phi^d, \tag{0.4}$$

which does not involve the metric g and has a purely topological meaning. Its integral is a homotopy invariant. If the asymptotics is such that we can consider $\vec{\phi}$ as a map from the compactified three-dimensional space into S^3, then the integral of ω gives just the degree (winding number) of this map and thus has to be an integer.
We now look for spherically symmetric solutions of the coupled system. For $\vec{\phi}$ we make the hedgehog ansatz

$$\vec{\phi}(x) = \begin{pmatrix} \sin\chi(r,t)\cdot\hat{x} \\ \cos\chi(r,t) \end{pmatrix}, \tag{0.5}$$

where $\hat{x}$ denotes the unit vector in direction of $\mathbf{x}$. The metric is taken to be of the form

$$g = e^{2a(r,t)}dt^2 - [e^{2b(r,t)}dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)]. \tag{0.6}$$

For our choice of the radial coordinate, the most general spherically symmetric ansatz would also contain a term $g_{rt}drdt \neq 0$.
In what follows, spacetime indices always refer to the orthonormal frame

$$\Theta^0 = e^a dt, \quad \Theta^1 = e^b dr, \quad \Theta^2 = r d\vartheta, \quad \Theta^3 = r\sin\vartheta d\varphi. \tag{0.7}$$

$T^{\mu\nu}$ has the general form

$$(T^{\mu\nu}) = \begin{pmatrix} T^{00} & T^{01} & 0 & 0 \\ T^{10} & T^{11} & 0 & 0 \\ 0 & 0 & T^{22} & 0 \\ 0 & 0 & 0 & T^{33} \end{pmatrix}, \quad T^{22} = T^{33}, \tag{0.8}$$

and the equation $\nabla_\nu T^{\mu\nu} = 0$ imply the following two independent relations, which will turn out to be also very useful for the discussion of density fluctuations (see section 4.3)

$$(T^{00} + T^{11})D_r a + D_r T^{11} + D_t T^{01} + 2T^{01}D_t b - 2(T^{22} - T^{11})\frac{e^{-b}}{r} = 0, \tag{0.9}$$

$$D_t T^{00} + (T^{00} + T^{11})D_t b + D_r T^{01} + 2T^{01}D_r a + 2\frac{e^{-b}}{r}T^{01} = 0, \tag{0.10}$$

where $D_t = e^{-a}\partial_t$, $D_r = e^{-b}\partial_r$.

From these it is obvious that the matter equation (0.2) and the constraint equations $G_{00} = 8\pi G\, T_{00}$, $G_{01} = 8\pi G\, T_{01}$ imply all other components of the Einstein field equations.
Inserting (0.5) into the action (0.1) gives

$$S(\vec{\phi}) = const\int [e^{b-a}r^2\dot{\chi}^2 - e^{a-b}r^2\chi'^2 - 2e^{a+b}\sin^2\chi]drdt, \tag{0.11}$$

with $\dot{\chi} = \partial\chi/\partial t$, $\chi' = \partial\chi/\partial r$, and the corresponding Euler-Lagrange equation is

$$(e^{b-a}r^2\dot{\chi})^{\cdot} - (e^{a-b}r^2\chi')' + e^{a+b}\sin 2\chi = 0. \tag{0.12}$$

For flat space ($a = b = 0$) this reduces to

$$\ddot{\chi} - \chi'' - \frac{2}{r}\chi' + \frac{\sin 2\chi}{r^2} = 0. \tag{0.13}$$

This nonlinear partial differential equation has been studied extensively in Ref.[3], where an exact self-similar solution which describes the collapse of a texture is discussed. It was established that textures rapidly become quite spherical as they collapse. (A simple scaling argument, as in the standard proof of Derrick's theorem, shows that eq.(0.13) has no static solutions.)
The relevant components of the Einstein tensor and of the energy-momentum tensor are

$$G_{00} = \frac{1}{r^2} - e^{-2b}(\frac{1}{r^2} - \frac{2b'}{r}), \qquad G_{11} = -\frac{1}{r^2} + e^{-2b}(\frac{1}{r^2} + \frac{2a'}{r}), \tag{0.14}$$

$$G_{01} = \frac{2\dot{b}}{r}e^{-a-b}, \tag{0.15}$$

$$T_{00} = \frac{1}{2}[e^{-2a}\dot{\chi}^2 + e^{-2b}\chi'^2 + \frac{2}{r^2}\sin^2\chi], \qquad T_{11} = \frac{1}{2}[e^{-2a}\dot{\chi}^2 + e^{-2b}\chi'^2 - \frac{2}{r}\sin^2\chi], \tag{0.16}$$

$$T_{01} = e^{-a-b}\dot{\chi}\chi'. \tag{0.17}$$

3. SOLUTIONS OF THE EINSTEIN σ-MODEL EQUATIONS

In this section we investigate first in detail a self-similar solution of the coupled system and then present results of a numerical study of the partial differential equations for $\chi(r,t), a(r,t)$ and $b(r,t)$.

3.1. Self-similar solutions

Guided by the fact that a self-similar solution exists in flat spacetime, we tried the following ansatz , which remarkably turns out to be compatible with the coupled field equations:

$$\chi(r,t) = f(r/t),\ a(r,t) = A(r/t),\ b(r,t) = B(r/t). \tag{0.18}$$

Inserting this into (0.12) we find ($x \equiv r/t$) :

$$x^2[e^{B-A}x^2f']' - [e^{A-B}x^2f']' + e^{A+B}\sin 2f = 0, \tag{0.19}$$

(where now $' = d/dx$) and the Einstein field equations reduce to

$$\frac{1}{x^2} - e^{-2B}(\frac{1}{x^2} - \frac{2B'}{x}) = \frac{\kappa}{2}[(x^2e^{-2A} + e^{-2B})f'^2 + \frac{2}{x^2}\sin^2 f], \tag{0.20}$$

and

$$B' = \frac{\kappa}{2}xf'^2. \tag{0.21}$$

These equations imply, as noted earlier, in particular $G_{00} + G_{11} = 8\pi G(T_{00} + T_{11})$, which leads to the simple relation

$$(e^{2A})' = x^2 (e^{2B})'. \tag{0.22}$$

In constructing numerical solutions we used as independent equations (0.19), (0.21) and the following simple consequence of (0.21) and (0.22)

$$A' = \frac{\kappa}{2} x^3 f'^2 e^{2B-2A}. \tag{0.23}$$

For the numerical integration we need the behavior near $x = 0$. We note first that any $A(0) \neq 0$ can be absorbed into a redefinition of the time coordinate. Furthermore, $f(0) = 0$, otherwise $\vec{\phi}$ would be singular at the origin [see eq.(0.5)]. The differential equations then imply the following expansion

$$f(x) = f'(0)x + \alpha x^3 + \mathcal{O}(x^5),$$

$$A(x) = \beta x^4 + \mathcal{O}(x^5),$$

$$B(x) = \gamma x^2 + \delta x^4 + \mathcal{O}(x^5), \tag{0.24}$$

where

$$\alpha = \frac{f'(0)}{60}[12 + (9\kappa - 8)f'(0)^2],$$

$$\beta = \frac{\kappa}{8} f'(0)^2,$$

$$\gamma = \frac{\kappa}{4} f'(0)^2,$$

$$\delta = \kappa \frac{f'(0)^2}{80}[12 + (9\kappa - 8)f'(0)^2]. \tag{0.25}$$

Thus the expansion (0.24) contains only the parameter $f'(0)$. This slope at the origin is implicitely determined as follows. The differential equation (0.19) has a critical point at $x = x_c$, where

$$e^{-2B(x_c)} = x_c^2 e^{-2A(x_c)}. \tag{0.26}$$

The second derivative f'' remains only finite at this position if $f(x_c) = \pi/2$. From (0.22) and (0.26) we find $A'(x_c) = B'(x_c)$ and the constraint equation (0.20) then implies

$$B(x_c) = -\frac{1}{2}\ln(1 - \kappa), \tag{0.27}$$

which is positive for $\kappa < 1$. Our computer program searches for x_c and then determines the parameter $f'(0)$ such that $f(x_c) = \pi/2$. Afterwards, it controls eq.(0.27).
One of the interesting points will be a comparison of f with the flat space solution. For A=B=0, eq.(0.19) reduces to

$$(x^2 - 1)(x^2 f')' + \sin 2f = 0, \tag{0.28}$$

for which the critical point is at $x = 1$. A solution with $f(0) = 0$ anf $f(1) = \pi/2$ was recently discovered by Spergel and Turok [3]:

$$f = 2\arctan\ x \qquad (x > 0). \tag{0.29}$$

For $x \to \infty$ this solution approaches the value π, which means that $\vec{\phi}$ in (0.5) has the constant limit $\vec{\phi}(\underline{0}, -1)$ at spacial infinity. Therefore, this solution can be lifted to the compactified three-dimensional space and has winding number one.
Before we present the numerical results of our curved space texture solution, we give also the basic equations for a second ansatz. Here, the metric is taken to be

$$g = dt^2 - dr^2 - r^2\omega^2(r/t)d\Omega^2. \tag{0.30}$$

Instead of (0.22) we have now

$$(x^2 - 1)(x^2\omega^2 f')' + \sin 2f = 0. \tag{0.31}$$

Combining the Einstein equations $G_{00} = 8\pi G\ T_{00}$ and $G_{11} = 8\pi G\ T_{11}$, we find the following first order constraint equation:

$$(x\omega)'^2 - (\omega' x^2)^2 + \frac{\kappa}{2}(x^2 - 1)(\omega x f')^2 = 1 - \kappa \sin^2 f. \tag{0.32}$$

Together with the second order equation (0.31) for f, this relation also implies the equation $G_{01} = 8\pi G\ T_{01}$ which reads

$$(x\omega)'' = -\frac{\kappa}{2}(x\omega)f'^2. \tag{0.33}$$

From (0.31) we have again $f(0) = n\pi/2$, and thus $f(0) = 0$. Requiring that $\omega(0)$ and $\omega''(0)$ are finite, we obtain $\omega'(0) = 0$ from (0.33) and then $\omega(0) = 1$ from the constraint equation (0.32). Again $f'(0)$ is the only free parameter and determines the expansion at $x = 0$:

$$\begin{aligned} \omega(x) &= 1 + \alpha x^2 + \beta x^4 + \mathcal{O}(x^5), \\ f(x) &= \gamma x + \delta x^3 + \mathcal{O}(x^5), \end{aligned} \tag{0.34}$$

with

$$\begin{aligned} \alpha &= -\frac{\kappa}{12} f'(0)^2, \\ \beta &= -\frac{6\kappa}{25} f'(0)^2[3 + (\frac{19\kappa}{4} - 2)f'(0)^2], \\ \gamma &= f'(0), \\ \delta &= \frac{\kappa}{15} f'(0)[3 + (\kappa - 2)f'(0)^2]. \end{aligned} \tag{0.35}$$

The critical point is always at $x = 1$ and we have $f(1) = \pi/2$. This condition determines the slope $f'(0)$.

For reasons which will become clear in the next subsection, let us also discuss ω–gauge in the variable $t/r = 1/x \equiv u$. In terms of u the evolution equations for f and ω are

$$(1 - u^2)(\omega^2 f')' + \sin 2f = 0, \tag{0.36}$$

$$\omega'' = -(\kappa/2)\omega f'^2, \tag{0.37}$$

and the constraint yields

$$(\omega' u - \omega)^2 - \omega'^2 + (\kappa/2)(1-u^2)f'^2\omega^2 + \kappa \sin^2 f = 1 \ . \tag{0.38}$$

A prime denotes now the derivative with respect to our new variable u. This system does not exhibit a critical point at $u = 0$ so that in this case $f(0)$ and $\omega(0)$ are not fixed. Since in the linearized solution $f(0) = \pi$, let us assume initial values

$$\begin{aligned} f(0) &= \pi + \epsilon \,, \quad -\pi/2 < \epsilon < \pi/2 \\ \omega(0) &= \omega_0 \neq 0 \,. \end{aligned}$$

The point $u = 0$ describes spatial infinity, $(t, r \to \infty)$ and the time of collapse, $(t = 0, r > 0)$. To find the collapsing solution numerically, we start, e.g., at $u = -1$ where (0.36) requires $f(-1) = n\pi/2$ and choose $f(-1) = \pi/2$. An expansion of f and ω at $u = -1$ then yields:

$$\omega(u) = \omega_0 + \omega_1(u+1) + \omega_2(u+1)^2 + O(u+1)^3, \tag{0.39}$$

$$f(u) = \pi/2 + f_1(u+1) + f_2(u+1)^2 + O(u+1)^3 \tag{0.40}$$

with

$$\begin{aligned} \omega_1 &= (1 - \kappa - \omega_0^2)/2\omega_0, \\ \omega_2 &= (\kappa/2)\omega_0 f_1^2, \\ f_2 &= \frac{f_1}{2}(1 + \kappa/\omega_0^2). \end{aligned}$$

The value of $f_1 = f'(-1)$ is determined by the optimal shooting condition. To describe the collapse and unwinding of a texture we shoot to $f(1) = \pi$. The numerical result $\omega(u)$ is presented in Fig. XXX

At this point we note also that the topological 3-form (0.4) is in both gauges given by

$$\Omega = -\frac{1}{2\pi^2} \sin^2 \chi \, \frac{\partial \chi}{\partial r} \, dr \wedge d\vartheta \wedge d\varphi. \tag{0.41}$$

3.2 Equivalence of the two gauges

Before we discuss the solutions of the coupled ordinary nonlinear differential equations, we show that there exists a coordinate transformation connecting the self similar solutions in AB gauge (metric (0.6)) and in ω gauge (metric (0.30)). Let us denote in this subsection the coordinates in AB gauge with capital letters, A=A(X), B=B(X), X=R/T and in ω gauge with small ones, $\omega = \omega(x)$, $x = r/t$. Inserting the ansatz

$$R(r,t) = \omega(x)\ r, \quad T(r,t) = \omega(x)\ t\ g(x) \tag{0.42}$$

in (0.6) and comparing with (0.30) we obtain the following relation for the partial derivatives

$$T_{,t}\ R_{,t} = T_{,r}\ R_{,r}\ . \tag{0.43}$$

Using relations such as $T_{,r} = T_{,x}\, x_{,r} = (g\omega)_{,x}\, t x_{,r} = (g\omega)'$ (with $' = \partial/\partial x$) we can derive from (0.43) a differential equation for $g(x)$,

$$\frac{(g\omega)'}{(g\omega)} = \frac{x^2\omega'}{x^3\omega' - (x\omega)'}\,. \tag{0.44}$$

A solution $g(x)$ of this equation determines the coordinate transformation $X(x) = x\,g^{-1}(x)$. With the following relations between the metric functions

$$exp(-2B(X)) = (x\omega)'^2 - (x^2\omega')^2$$

$$exp(-A(X)) = exp(-B(X))\, \frac{(g\omega) - x(g\omega)'}{(x\omega)'}\,, \tag{0.45}$$

we obtain the field equations in AB gauge from the field equations in ω gauge (without solving the differential equation (0.44) for $g(x)$).
We can apply this, in particular, to the linearized approximation. To first order in κ we find from (0.29) and (0.33) and the boundary condition $\omega \overset{t\to-\infty}{\to} 1$

$$\omega(x) = 1 \;+\; \kappa\,[\frac{\arctan(x)}{x} - 1]. \tag{0.46}$$

A solution of (0.44) is then

$$(g\omega)^{-1} = 1 \;+\; \kappa\,[\ln(1+x^2) - x\,\arctan(x)], \tag{0.47}$$

which yields the linearized solution in AB gauge:

$$B(X) = \kappa\,\frac{X^2}{1+X^2}\;\;,\quad A(X) = \kappa\,[\ln(1+X^2) - \frac{X^2}{1+X^2}]\,, \tag{0.48}$$

where $X = x + 0(\kappa)$.
For ω–gauge in terms of the variable $u = 1/x$ which we shall call $\tilde{\omega}$ in this subsection to avoid confusion, we obtain for $g(u)$ the differential equation

$$\frac{(g\tilde{\omega})'}{(g\tilde{\omega})} = \frac{\tilde{\omega}'}{u^3\tilde{\omega}' - u^2\tilde{\omega} - u\tilde{\omega}'} \tag{0.49}$$

instead of (0.44), where here $'$ denotes the derivative with respect to u. The linearized solution (e.g. from (0.37)) is

$$\tilde{\omega}(u) = 1 - \kappa[u(\arctan u + \pi/2) + 1]\,. \tag{0.50}$$

This solution is identical with ω only for $t < 0$. For $t > 0$ the two are related to each other by a gauge transformation. Denoting the variables of $\tilde{\omega}$–gauge by (τ,ρ) we easily find the following linearized relation:

$$(\tau,\rho) = \begin{cases} (t,r) & , \text{for } t < 0 \\ (t+\kappa\pi r, r+\kappa\pi t) & , \text{for } t > 0\,. \end{cases}$$

Let us now discuss in which part of parameter space our gauges are actually regular. As we found numerically (see Fig. 3) and as can be inferred from the basic equations, $A \overset{x\to\infty}{\to} \infty$ and thus (AB)–gauge becomes singular for $x \to \infty$, or $t \to 0$.

It is a little less obvious that also ω–gauge becomes singular at $t = 0$. Not the metric itself diverges but the second fundamental form is not continous at $t = 0$. To see this, we first note that ω is an even function of x. Thus

$$\lim_{t\nearrow 0^-} \frac{d\omega}{dt} = \frac{1}{r} \lim_{x\to-\infty} (x^2\omega') = \frac{-1}{r} \lim_{x\to\infty} (x^2\omega') = -\lim_{t\searrow 0^+} \frac{d\omega}{dt}\,,$$

If the second fundamental form is to be continous at $t = 0$ all this limits thus have to vanish. But from Fig. 2 we see directly

$$\lim_{x\to\infty}(x^2\omega') = -(\frac{d\omega}{du})_{u=0} \neq 0$$

This is also true in the linear approximation where one obtains by direct calculation

$$\lim_{t\nearrow 0^-}\frac{d\omega}{dt} = -\pi/2r \quad \text{and} \quad -\lim_{t\searrow 0^+}\frac{d\omega}{dt} = \pi/2r\ ,$$

The only gauge discussed in this paper which is regular in a vicinity of $t = 0$ is $\tilde{\omega}$–gauge. This gauge is thus an extension of ω–gauge to positive times. In linearized approximation it breaks down at $\kappa\pi t \approx r$. For small values of κ we can thus expect this gauge to be regular until $t \gg r$.

When we want to discuss the behavior of matter and radiation in Section 4, we shall be most interested in the induced perturbations due to a collapsing texture. At very early times $t \ll 0$ we start with a homogenous and isotropic distribution of matter and radiation and we want to calculate the resulting distribution long after texture collapse. To do this safely we need a gauge which is regular at $t = 0$ and we shall thus mostly work in $\tilde{\omega}$–gauge in the next section. (Although we shall call it ω–gauge where there is no danger of confusion.)

3.3 The numerical solutions for the two gauges

We discuss first the results in the second gauge. In Fig. 1 we show the amplitude f of the $\vec{\phi}$-field (eq.(0.18)) for various values for the coupling constant κ. The smallest value $\kappa = 10^{-4}$ corresponds to a typical GUT scale, while the others are chosen in order to illustrate the role of gravity. In all cases, f is close to the flat space solution (0.29). The asymptotic value $f(\infty)$ always overshoots π (very slightly for $\kappa = 10^{-4}$), which means that the solution cannot be lifted to the compactified three dimensional space. The metric coefficient $\omega(x)$ is plotted in Fig. 2 for the same values of κ. Its monotonic decrease from $\omega(0) = 1$ to an asymptotic value ω_∞ for $\mid x \mid\to\infty$ can be deduced rigorously from the basic equations. It is instructive to compare ω_∞ with the result of the linearized theory (0.46) which is equal to $1-\kappa$. Even for unreasonably large κ the difference is less than 10%. As a result, the energy density of the $\vec{\phi}$-field is also changed very little.
In the first gauge the function $f(x)$ is almost the same. Fig. 3 shows plots for the metric functions $A(\mid x\mid)$, $B(\mid x\mid)$. Their monotonic increase follows immediately from (0.21) and (0.23). The asymptotic divergence of A is related to the relatively slow decrease of the energy-momentum tensor, which implies a diverging total mass at large r.

The physical interpretation of our solutions is as follows: For $t < 0$ and $x = r/(-t) > 0$ the solution describes a shrinking texture which will collapse and unwind at $t = 0$. Clearly the event of collapse and unwinding at $(r, \mid t\mid) \leq (1/\eta, 1/\eta)$ cannot be described within the σ-model approximation since in this regime the scalar field $\vec{\phi}$ leaves the 3-sphere and is "pulled over the potential barrier". A full description of this process is very complicated. Nevertheless, from physical arguments we expect the unwinding process to be very short and very concentrated so that for $r \gg 1/\eta$ $\vec{\phi}$ never leaves the 3–sphere and thus the σ-model approximation should be very reliable for all times including $t = 0$. We can approximate the solution after the collapse ($t > 0$) by appropriately patching together the σ-model solutions in such a way that the winding number vanishes, as it was done for flat spacetime

[3]. The result is shown in Fig.xx. For comparison, we recall the analytic solution for $\kappa = 0$, which is also plotted in Fig. xx:

$$\begin{aligned} f &= 2\ \arctan(\frac{r}{t}) + \pi \quad , r < t \\ &= 2\ \arctan(\frac{t}{r}) + \pi \quad , r > t. \end{aligned} \tag{0.51}$$

Note that the first derivative of this solution is discontinuous at $r = t > 0$. This unphysical feature is shared by our exact solution. For the reason we have just discussed, this should, however, not be disturbing, because it reflects the failure of the σ-model in describing the unwinding process, (within the σ-model the winding number is conserved !).

3.4. Non self-similar solutions

We have investigated also non-self similar spherically symmetric solutions in AB gauge. For the numerical study it is convenient to write the coupled system of partial differential equations (corresponding to eq.(0.12) and the Einstein equations) in the form

$$\dot{\chi} = e^{a-b}\pi, \tag{0.52}$$

$$\dot{\pi} = \frac{(e^{a-b}r^2\chi')'}{r^2} - \frac{e^{a+b}}{r^2}\sin(2\chi), \tag{0.53}$$

$$\dot{b} = \frac{\kappa}{2} r e^{a-b}\pi\chi', \tag{0.54}$$

$$a' = -\frac{1}{2r} + \frac{e^{2b}}{2r} + \frac{\kappa}{4} r[\pi^2 + \chi'^2 - \frac{2e^{2b}}{r^2}\sin^2\chi], \tag{0.55}$$

$$b' = +\frac{1}{2r} - \frac{e^{2b}}{2r} + \frac{\kappa}{4} r[\pi^2 + \chi'^2 + \frac{2e^{2b}}{r^2}\sin^2\chi], \tag{0.56}$$

where: $' = \partial/\partial r$ and $\dot{} = \partial/\partial t$. As in the self-similar case, the second order equation for $\chi(r,t)$ (0.52,0.53) is still a consequence of the field equations (0.54-0.56). The requirement of regularity leads to the following expansion at the origin:

$$\begin{aligned} \chi(r,t) &= r\chi'(0,t) + \frac{r^2}{2}\chi''(0,t) + \mathcal{O}(r^3), \\ \pi(r,t) &= \pi(0,t) + r\pi'(0,t) + \mathcal{O}(r^2), \\ a(r,t) &= r^2\frac{\kappa}{6}\pi^2(0,t) + \mathcal{O}(r^3), \\ b(r,t) &= r^2\frac{\kappa}{12}[\pi^2(0,t) + 3\chi'^2(0,t)] + \mathcal{O}(r^3). \end{aligned} \tag{0.57}$$

The strategy for solving this hyperbolic, mixed initial-, boundary-value problem is the following: We give $\chi(r,0)$ and $\pi(r,0)$ on N spacial lattice points at $t = 0$. Having numerically computed the first and second derivative of χ with respect to r, we solve the $2N$ coupled ordinary differential equations (0.55) and (0.56) and obtain $a(r,0)$ and $b(r,0)$. Next we propagate the system one time step Δt, by solving the $3N$ evolution equations (0.52)-(0.54) for χ, π and b, taking the boundary conditions at $r = 0$ and $r = r_{max}$ into account. After a further differentiation of $\chi(r, \Delta t)$ with respect to r we obtain $a(r, \Delta t)$ from b, χ, χ' and π by integrating eq. (0.55). Finally we check the constraint equation (0.56). Now we evolve χ, π and b again with (0.52)-(0.54), compute the spacial derivative of χ and integrate (0.55) to obtain a, b, χ and π at $2\Delta t$, and so on.

After each step we make sure that the relation $\Delta t/\Delta r$ respects the hyperbolic characteristics of the system. We wrote two codes in order to perform the time steps, one of which implemented a modified MacCormack predictor-corrector scheme [15, 16, 17]. For a short description of this method we also refer to Ref.[18]. The results of both methods coincide. The main numerical instabilities arise from the numerically computed derivatives of χ with respect to r at the boundaries.

4. INTERACTION WITH MATTER AND RADIATION

The viability of the texture scenario depends on the interaction of the texture field with matter and radiation. This is the subject of the present section.

4.1. Light deflection

First, we discuss the light deflection in the gravitational field of the collapsing texture and give some details for the second gauge. The self-similarity is responsible for the fact, that it is possible to derive a differential equation for null geodesics, which involves only the variable u as a function of the angle φ (we take $\vartheta = \pi/2$). Writing the metric in terms of $v(r,t) = \ln(t)$ and $u(r,t) = t/r$, the Lagrangian reads (now $\dot{} = \partial/\partial\tau$, where τ is the proper time)

$$2\mathcal{L} = \frac{e^{2v}}{u^2}\,[\dot{v}^2(u^2-1) - (\frac{\dot{u}}{u})^2 + 2\dot{v}\frac{\dot{u}}{u} - \omega^2(u)\dot{\varphi}^2]\,. \tag{0.58}$$

Since $\mathcal{L}$ depends on v only via the factor in front of the bracket, it is possible to obtain in addition to the 'angular momentum conservation law' also an 'energy conservation' equation for null geodesics:

$$\dot{\varphi} = L\,e^{-2v}\,u^2\,\omega^{-2} = L\,(\omega r)^{-2}\,, \tag{0.59}$$

$$\dot{v}(u^2-1) + \dot{u}/u = E\,e^{-2v}\,u^2 = E\,r^{-2}\,. \tag{0.60}$$

Using $\dot{u} = (du/d\varphi)\dot{\varphi} \equiv u'\dot{\varphi}$ we immediately obtain the first order geodesic differential equation from (0.58-0.60):

$$u'^{\,2} = \omega^2(u)[1 - u^2 + k^2\omega^2(u)]\,. \tag{0.61}$$

The relation between the constant $k^2 = (E/L)^2$ and the impact parameters r_o and t_o will be discussed in the next section. The total light deflection angle δ is given by

$$\delta = \int_{-1}^{1} \frac{du}{\omega(u)\sqrt{1 - u^2 + k^2\omega^2(u)}} - \pi. \tag{0.62}$$

For $r \to \infty$, $t \to \pm\infty$ we obtain $u \to \pm 1$, as will become clear from the geodesic equation for the function $r = r(u)$, see eqn.(0.70) below. Using the fact that $\omega(u)$ decreases with increasing u and the inequality $0 \le u\omega_{,u}/\omega \le \kappa/2$ for all u, we can establish the following upper bound for δ

$$\delta < \pi(\frac{1+\kappa/2}{\omega_0} - 1) \cong \kappa\frac{3}{2}\pi\,, \tag{0.63}$$

where $\omega_0 := \omega(u=0)$. Let us also evaluate the integral (0.62) to first order in κ. Using

$$\omega(u) = 1 + \kappa\, \omega_1(u), \quad \omega_1(u) = -u\,(\arctan u + \pi/2) - 1, \tag{0.64}$$

we obtain the following expression for $\varphi(u)$:

$$\varphi(u) = \int^u \frac{du}{\sqrt{1+k^2-u^2}} \; - \; \kappa \int^u \frac{(1+2k^2-u^2)\,\omega_1(u)}{\sqrt{1+k^2-u^2}^{\,3}} \; + \; \mathcal{O}(\kappa^2).$$

$= I_o + \kappa I_1$. Both integrals can be performed explicitly. The first integral, when evaluated between the limits $u=-1$ and $u=+1$, is equal to π, and thus only the second integral contributes to light deflection After some manipulations, we find the following expression for the light deflection:

$$\begin{aligned}\Delta\varphi\Big|_1^2 \;&=\; \kappa\,[\frac{u^2-1}{\sqrt{1+k^2-u^2}}\,(\arctan(u)+\pi/2 \;+\; \frac{k^2}{1+k^2}\frac{u}{\sqrt{1+k^2-u^2}} \\ &+\; \frac{2}{\sqrt{2+k^2}}\;\arctan(\sqrt{2+k^2}\frac{uk+\sqrt{1+k^2-u^2}}{k\sqrt{1+k^2-u^2}-u(2+k^2)})]\Big|_1^2 .\end{aligned} \tag{0.65}$$

This equation describes the deviation of the trajectory from the zero'th order geodesic $\varphi_o = \arcsin(u)/\sqrt{1+k^2}$, $u = t/\sqrt{r_o^2+(t-t_o)^2}$. To first order in κ, the total deflection angle for null geodesics becomes

$$\delta_{(1)} = \kappa\,\pi\, \frac{2}{\sqrt{2+k^2}} \;=\; \kappa\,\pi\, \frac{2r_o}{\sqrt{2r_o^2+t_o^2}}\,, \tag{0.66}$$

where $k = t_o/r_o$ as we shall argue in the next section. For $k\to\infty$, corresponding to $r_o = 0$ the effect vanishes, whereas the first order light deflection takes its maximal value at $k=0$ ($r_0\to\infty$ or $t_o\to 0$): $\Delta\varphi = \kappa\sqrt{2}\pi \;<\; \kappa\, 3\pi/2$, in agreement with the upper bound given in (0.63).

4.2. Photon red shift

As it was the case for the light deflection, it is also possible to express the red shift in terms of the self similar coordinate u. In ω gauge, the relevant quantity $\dot{t}$ (a dot denotes the derivative with respect to the proper time) can be written with the help of (0.59) as

$$\dot{t}(u) = \frac{t'}{t}\; L\; \frac{u}{r(u)\,\omega(u)^2}\,. \tag{0.67}$$

Eliminating $t = ur$ from $t'^2 - r^2 = r^2\omega^2$, we obtain the equation

$$(u' + u\,\frac{r'}{r})^2 \;+\; (\frac{r'}{r})^2 \;=\; \omega^2(u)\,, \tag{0.68}$$

which can be solved for r'/r, using the differential equation (0.61) for $u'(\varphi)$:

$$\frac{r'}{r} = \frac{k\,\omega(u)^2 \;-\; uu'}{u^2-1}\,. \tag{0.69}$$

Integration of this equation yields r as a function of u:

$$\frac{r(u)^2}{r(u_o)^2} = \frac{u_o^2-1}{u^2-1}\; exp\left[2k\int_{u_o}^{u}\frac{\omega(u)\,du}{(u^2-1)\sqrt{1+k^2\omega^2-u^2}}\right]\,. \tag{0.70}$$

From equation (0.69) and $t'/t = r'/r + u'/u$ we also obtain t'/t in terms of the self similar coordinate and thus

$$\dot{t}(u) = L\,\frac{k\omega u - \sqrt{1+k^2\omega^2-u^2}}{u}\,r^{-1}(u)\,, \tag{0.71}$$

from which the red shift can be computed as

$$\frac{\delta E}{E}\Big|_1^2 \;=\; \frac{\dot{t}(2)\;-\;\dot{t}(1)}{\dot{t}(1)}\,. \tag{0.72}$$

Let us now evaluate (0.71) to first order in κ. Using (0.64) the geodesic equation (0.70) becomes

$$r(u) = r(u_o)\,\frac{uk - \sqrt{1+k^2-u^2}}{u^2-1}\,[1\;+\;\kappa\,k\,I(u)\;+\;\mathcal{O}(\kappa^2)]\,, \tag{0.73}$$

with

$$I(u) = \int^u \frac{\omega_1(u)\,du}{\sqrt{1+k^2-u^2}^{\,3}}\,.$$

For $\kappa = 0$ this reduces to the flat geodesic equation

$$r(t) = \sqrt{r_o^2\;+\;(t-t_o)^2}\,, \tag{0.74}$$

if we identify the constant k with the parameter u_o

$$k = u_o = t_o/r_o\,, \tag{0.75}$$

where t_o is the time when the distance r_o of the photon to the texture becomes minimal ($t = 0$ is the time at which the texture collapses.). An elementary integration gives the following expression for the photon red shift to first order in κ:

$$\begin{aligned}\frac{\delta E}{E}\Big|_1^2 \;&=\; -\,\kappa\,[\,(1-\frac{k(1-u^2)}{u(uk-\sqrt{1+k^2-u^2})})\;u\,(\arctan(u)+\pi/2)\\ &-\;\frac{k}{\sqrt{2+k^2}}\;\arctan(\sqrt{2+k^2}\,\frac{uk+\sqrt{1+k^2-u^2}}{k\sqrt{1+k^2-u^2}-u(2+k^2)})]\Big|_1^2\,.\end{aligned} \tag{0.76}$$

Since the effect is proportional to κ, we can use the zero'th order part of equation (0.73) to eliminate $\sqrt{1+k^2-u^2}$ and then $u = t/r(t)$ with $r(t)$ from the zero'th order geodesic equation (0.74). This yields the result in terms of t:

$$\begin{aligned}\frac{\delta E}{E}\Big|_1^2 \;&=\; -\,\kappa\,[\frac{t_o-t}{\sqrt{r_o^2+(t-t_o)^2}}\,arcctg(\frac{-t}{\sqrt{r_o^2+(t-t_o)^2}})\\ &\qquad +\frac{-t_o}{\sqrt{2r_o^2+t_o^2}}\,arcctg(\frac{t_o-2t}{\sqrt{2r_o^2+t_o^2}})]\Big|_1^2\,,\end{aligned} \tag{0.77}$$

in agreement with [4]. The total redshift to first order in κ becomes

$$\frac{\delta E}{E} = \kappa\,\pi\,\frac{k}{\sqrt{2+k^2}}\;=\kappa\,\pi\,\frac{t_o}{\sqrt{2r_o^2+t_o^2}}\,. \tag{0.78}$$

For $t_o = 0$ the fractional red and blue shifts compensate and the effect vanishes, whereas the photon is red shifted if it passes the texture knot before the collapse time and blue shifted if it passes afterwards, giving thus rise to a clear signature of hot-spot - cold-spot signal in the microwave background wherever a texture has collapsed. The maximum values of the red shift ($\kappa\pi$) is obtained for $r_0 \to 0$. For cosmological applications one has to take into account also the expansion of the Universe. If we denote with R_H the horizon size at the time of the collapse, $t = 0$, clearly photons for which $\mid t_0 \mid > R_H$ or $r_0 > R_H$ are not yet in causal contact with the texture and thus do not feel its gravitational field. We expect $\delta E = 0$ for such photons. The damping of the signal due to expansion and especially due to photon diffusion as long as photons are coupled to matter is calcutated in [12].

4.3. Density fluctuations

In this subsection we discuss the evolution of texture seeded density fluctuations in the gravitational field of our self-similar solution. For the special case of CDM or dust it will turn out that we can solve the highly nonlinear coupled equations exactly. First we set up the basic hydrodynamic equations for an ideal fluid in both gauges. For the (A,B)-gauge these can be obtained directly from the form (0.9) and (0.10)for the energy-momentum balance. Relative to the orthonormal tetrad (0.8) the four-velocity for a spherically symmetric flow has the components

$$u^0 = \gamma, \quad \gamma^1 = \gamma v, \quad u^2 = u^3 = 0, \tag{0.79}$$

with $\gamma = (1 - v^2)^{-1/2}$. The non-vanishing components of the energy-momentum tensor

$$T^{\mu\nu} = (p + \rho)u^\mu u^\nu - pg^{\mu\nu} \tag{0.80}$$

are

$$\begin{aligned} T^{00} &\equiv \epsilon = \gamma^2(\rho + pv^2), \\ T^{01} &\equiv s = \gamma^2(\rho + p)v, \\ T^{11} &\equiv \tau + p, \quad \tau = \gamma^2(\rho + p)v^2, \\ T^{22} &= T^{33} = p. \end{aligned} \tag{0.81}$$

Inserting these expressions in (0.9) and (0.10) gives

$$e^{-a}\dot{\epsilon} + e^{-a}\dot{b}(\epsilon + \tau + p) + e^{-b}s' + 2e^{-b}a's + 2\frac{e^{-b}}{r}s = 0, \tag{0.82}$$

$$e^{-b}a'(\epsilon + \tau + p) + e^{-b}(\tau + p)' + e^{-a}\dot{s} + 2e^{-a}\dot{b}\dot{s} + 2\frac{e^{-b}}{r}\tau = 0. \tag{0.83}$$

These equations hold for any metric of the form (0.6). For the special case of the self-similar texture field we look for solutions of the matter variables ρ, p and v, which depend also only on $x = r/t$. In this case eqns.(0.82) and (0.83) reduce to (a prime denotes again the derivative with respect to x)

$$\epsilon' = \frac{e^{A-B}}{x}(s' + \frac{2}{x}s + 2A's) - B'(\epsilon + \tau + p), \tag{0.84}$$

$$s' = -2sB' + \frac{e^{A-B}}{x}\left[A'(\epsilon + \tau + p) + (p + \tau)' + \frac{2}{x}\tau\right]. \tag{0.85}$$

Similarly, one finds for the ω-gauge the following equations with $' = \frac{d}{du}$

$$\epsilon' = us' + 2(-1 + \frac{u\omega'}{\omega})s - 2\frac{\omega'}{\omega}\epsilon - 2\frac{\omega'}{\omega}p\,, \tag{0.86}$$

$$s' = -2\frac{\omega'}{\omega}s + u\tau' - 2(1 - u\frac{\omega'}{\omega})\tau + up'. \tag{0.87}$$

From here on we consider only CDM or dust. In this case we have

$$p = 0, \quad \epsilon = \rho\gamma^2, \quad s = \epsilon v, \quad \tau = \epsilon v^2. \tag{0.88}$$

Eqns.(0.84), (0.85) reduce to

$$\epsilon' = \frac{e^{A-B}}{x}\left[(\epsilon v)' + \frac{2}{x}\epsilon v + 2A'\epsilon v\right] - B'\epsilon(1 + v^2), \tag{0.89}$$

$$(\epsilon v)' = -2B'\epsilon v + \frac{e^{A-B}}{x}\left[(\epsilon v^2)' + \epsilon(v^2 + 1)A' + \frac{2\epsilon v^2}{x}\right]. \tag{0.90}$$

Multiplying (0.89) with v and substracting the result from (0.90) leads to the relation

$$(1 - v^2)(A'e^A - xvB'e^B) = v'(xe^B - ve^A). \tag{0.91}$$

Here we can eliminate A' with the help of the field equation (0.22) and find

$$x(1 - v^2)B'e^{B-A}(xe^B - ve^A) = v'(xe^B - ve^A). \tag{0.92}$$

The appearance of a common factor on both sides implies that we have either

$$v(x) = xe^{B-A} \tag{0.93}$$

or

$$\frac{v'}{1 - v^2} = xB'e^{B-A}. \tag{0.94}$$

Using again the field eqn.(0.22) we can write the last equation in the form

$$\frac{v'}{1 - v^2} = \frac{A'}{x}e^{A-B}. \tag{0.95}$$

The first alternative (0.93) is physically uninteresting (e.g. since it does not satisfy our boundary condition $v \overset{t\to-\infty}{\to} 0$). In ω-gauge the equations for CDM or dust can be solved similarly and one finds instead of (0.93) and (0.94),(0.95) the alternative: either $v(u) = 1/u$ or $v' \equiv 0$. Only the solution $v \equiv 0$ is physically interesting. Then we have $\epsilon = \rho$, in which case (0.86) reduces to

$$\frac{\rho'}{\rho} + 2\frac{\omega'}{\omega} = 0. \tag{0.96}$$

This implies $\rho\omega^2 = const$, a result which we have already presented in Ref.[13]. The gauge transform of this solution satisfies (0.94), (0.95).

In the linearized approximation we find for δ in $\rho = \rho_0(1+\delta)$ for both gauges

$$\delta = 2\kappa[1 + u(\arctan u + \pi/2)]. \qquad (0.97)$$

For the velocity field we have

$$v \equiv 0 \qquad (\omega - gauge),$$

resp.

$$v = -\kappa(\frac{x}{1+x^2} + \arctan\frac{1}{x} + \frac{\pi}{2}) \qquad (A, B - gauge). \qquad (0.98)$$

4.4 Collisionless particles

Collisionless particles are described by their 1–particle distribution function F which is defined on the mass bundle, $P_m = \{(x,p) \in T\mathcal{M} | g(x)(p,p) = -m^2\}$ and obeys Liouville's equation, $\mathcal{L}_{X_g} F = 0$ [19, 20]. If we choose coordinates (x^μ, p^i) on P_m with respect to some basis of vector fields (e_μ) on M, we have

$$\mathcal{L}_{X_g} F = p^\mu e_\mu(F) - \omega^i{}_\nu(p) p^\nu \frac{\partial F}{\partial p^i} = 0 \ . \qquad (0.99)$$

We are most interested in the cosmological situation, where at very early times $t \to -\infty$ the distribution function F is homogeneous and isotropic. At $t = -\infty$, F is thus only a function of $p = \sqrt{p^i p_i}$, independent e.g. of p^r and r. We want to calculate how the distribution function looks long after the texture has collapsed, $t \gg 0$. Let us therefore discuss (0.99) in ω–gauge, the gauge , $u = t/r$. We choose the tetrad

$$(e_\mu) = (\partial_t, \partial_r, \frac{1}{r\omega}\partial_\vartheta, \frac{1}{r\omega\sin\vartheta}\partial_\varphi)$$

and look for spherically symmetric solutions, $F(t, r, p^r, p)$, where
$p^2 = (p^r)^2 + (p^2)^2 + (p^3)^2 = (p^r)^2 + (p^-)^2$. In these variables (0.99) becomes

$$p^0 \partial_t F + p^r \partial_r F + \frac{(p^-)^2}{r}[(1 - \frac{u\omega'}{\omega})\frac{\partial F}{\partial p^r} - \frac{\omega'}{\omega}\frac{p^0}{p}\frac{\partial F}{\partial p}] = 0 \ . \qquad (0.100)$$

For cold particles (i.e. particles which move nonrelativistically, $p \ll p^0 \approx m$), using that $|u\omega'/\omega| \leq \mathcal{O}(\kappa)$ also for large negative u, we may neglect the $\partial_r F$ and $\partial_{p^r} F$ terms and obtain

$$\partial_t F = \frac{p^-}{r}\frac{\omega'}{\omega}\frac{\partial F}{\partial p^-} \ . \qquad (0.101)$$

This equation is solved by

$$F = F_o((p^r)^2 + (\omega p^-)^2) \ . \qquad (0.102)$$

Since $\lim_{t\to-\infty}\omega = 1$, this solution has the correct initial conditions, $lim_{t\to-\infty} F = F_o(p)$. An intergration of (0.102) using

$$n = \int F d\pi_m \quad \text{and} \quad T^\mu_\nu = \int p^\mu p_\nu F d\pi_m \quad \text{with} \quad d\pi_m = \frac{d^3p}{p_0}$$

yields

$$n = \omega^{-2} n_o \ , \quad T^{0i} = T^{0i}_{(o)} = 0 \ , \quad T^{00} = \omega^{-2} T^{00}_{(o)} \ .$$

This confirms our results for dust, (0.96). In ω-gauge no velocities are induced. The density enhancment is only due to the shrinking of 3–volume in the missing solid angle geometry.

To obtain qualitative results for hot particles and radiation let us discuss (0.100) in first order perturbation theory. We set

$$F = F_o(p) + \kappa\phi \ , \quad \omega = 1 + \kappa\omega_1 \ .$$

where ω_1 is given by (0.50). Up to first order we then have

$$p^0\partial_t\phi + p^i\partial_i\phi = \frac{\omega'}{r}\frac{(p^-)^2p^0}{p}\frac{dF_o}{dp} \ . \tag{0.103}$$

The general solution of this equation is of the form

$$\phi = \phi_{hom} + \frac{1}{p}\frac{dF_o}{dp}\int_{t_o}^{t}\mathcal{S}(t',\vec{x'},\vec{p})dt' \ , \tag{0.104}$$

with

$$\mathcal{S} = \omega_1'\frac{(p^-)^2}{r} \quad \text{and} \quad \vec{x'} = \vec{x} - \frac{\vec{p}}{p^0}(t-t')$$

$$\text{and} \quad \phi_{hom}(t,\vec{x},\vec{p}) = \phi(t_o, \vec{x} - \frac{\vec{p}}{p^0}(t-t_o), \vec{p})$$

We are of course most interested in $\phi(t=\infty)$ with the boundary condition $\phi(t=-\infty)=0$. To estimate the integral in (0.104) we note that according to (0.50) $\omega_1'(t=-\infty)=0$, $\omega_1'(t=\infty)=-\pi$ and $\omega''<0$ so that $0\le\omega_1'\le-\pi$. If we then replace ω_1' by its mean value $-\pi/2$ in in the integral of $\mathcal{S}$ and use

$$\int_{-\infty}^{t}\frac{(p^-)^2}{r(\vec{x'})}dt' = (p-p^r)p^0$$

we obtain the following approximation for the change in the distribution function due to the texture:

$$\delta F(\vec{x},\vec{p}) \approx -\frac{\kappa\pi}{2}\frac{p^0}{p}(p-p^r)\frac{dF_o}{dp} \ . \tag{0.105}$$

From this result we can now calculate the induced net radial velocity density and the change in number density, energy density and pressure:

$$\delta\rho \approx \kappa\pi[4\pi\int_0^\infty(p(p^0)^2+p^3)F_o dp] \tag{0.106}$$

$$\delta\rho \approx \frac{2\kappa\pi}{3}[4\pi\int_0^\infty p^3F_o dp] \tag{0.107}$$

$$v_r \approx -\frac{\kappa\pi}{2}[4\pi\int_0^\infty\frac{p^2}{p^0}(1-p(p^0)^2/3)F_o dp] \tag{0.108}$$

$$\delta n \approx \kappa\pi[4\pi\int_0^\infty pF_o dp] \ . \tag{0.109}$$

For relativistic particles we can set $p^0\approx p$ in the above integrals and obtain

$$\delta\rho/\rho_o = \delta p/p_o \approx 2\kappa\pi \ , \qquad v_r \approx \kappa\pi n_o \ , \qquad \delta n/n_o \approx \kappa\pi \ .$$

We thus see that hot particles do not accumulate around a texture to arbitrary high densities (as long as $\kappa \ll 1$). The change in the energy momentum variables of hot particles is of the order of ω' and thus remains small for all times in contrary to cold particles or dust, where the density perturbation is proportional to $\omega^{-2} - 1$ and thus can become very large.

The above results for $\delta\rho/\rho_o$ and $\delta n/n_o$ tell us that each relativistic particle aquires an energy shift on the order of $\kappa\pi$ in the vicinity of a collapsing texture. This agrees with our exact result (0.78) on photon redshift.

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Appendix A

In this appendix we briefly discuss the light deflection in the gravitational field of a collapsing texture in the AB gauge. The Lagrangian is then

$$2\mathcal{L} = e^{2a}\dot{t}^2 - e^{2b}\dot{r}^2 - r^2(\dot{\vartheta}^2 + sin^2\theta\dot{\varphi}^2), \tag{A.1}$$

where $t = t(\tau)$, $r = r(\tau)$ and τ is the proper time ($\dot{} = \partial/\partial\tau$). Due to the symmetry we can restrict ourself to the plane with $\vartheta = \frac{\pi}{2}$. Moreover, for photons we have $2\mathcal{L} = 0$, and the Euler-Lagrange equation for φ leads to

$$\dot{\varphi} = \frac{L}{r^2}, \tag{A.2}$$

which expresses the angular momentum conservation ($L = const$). We use $\frac{\partial}{\partial\tau} = \frac{L}{r^2}\frac{\partial}{\partial\varphi}$ (where $' = \partial/\partial\varphi$) and $u = x^{-1}$ in order to eliminate the derivative with respect to the proper time. By suitably combining the Euler-Lagrange equations for t and r as well as eq.(A.1) (with $2\mathcal{L} = 0$) one finds the following equation for u

$$u'' + u'^2(\frac{da}{du} + \frac{db}{du}) + e^{-2b}u + (e^{-2b}\frac{da}{du}u^2 - \frac{db}{du}e^{-2a}) = 0 \tag{A.3}$$

(a, b are functions of $u = t/r$ and $u = u(\varphi)$). After multiplying eq.(A.3) with e^{2a+2b} we can easily find the following first integral (which corresponds to eq.(0.61) in ω gauge)

$$u'^2 = (k^2e^{-2a-2b} + e^{-2a} - e^{-2b}u^2) \tag{A.4}$$

($k^2 = (E/L)^2$ is an integration constant). Alternatively, this equation can be directly derived from the corresponding energy conservation equation as it was done in section 4.1 for ω gauge. The total light deflection angle δ is given by

$$\delta = \int_{u_-}^{u_+} \frac{du}{\sqrt{k^2e^{-2a-2b} + e^{-2a} - e^{-2b}u^2}} - \pi, \tag{A.5}$$

where $u_\pm$ are the solutions of the transcendental equations $u = \pm e^{(b-a)(u)}$.

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Figure Captions

- Amplitude f of the $\vec{\phi}$-field in the second gauge (eq.(0.30)) for various values of the coupling constant κ. In the inteval $1 <\mid x \mid< \infty$ f is plotted as a function of $y = 2 - 1/ \mid x \mid$, in order to show how much the asymptotic value overshoots π.
- Metric coefficient $\omega(x)$ (eq.(0.30)) for various values of κ. For $\kappa = 10^{-6}$ the curve would be almost horizontal.
- Metric functions $A(\mid x \mid)$ and $B(\mid x \mid)$ for the first gauge for $\kappa = 10^{-6}$. (eqs.(0.6) and (0.18)).