

Clarifying perturbations in the ekpyrotic universe

A web-note

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In this note I try to clarify the problem of perturbations in the ekpyrotic universe. I write down the most general matching conditions and specify the choices taken by the two debating sides. I also bring up the problem of surface stresses which always have to be present when a transition from a collapsing to an expanding phase is made.

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I. INTRODUCTION

Lately, the idea that a hot big bang universe might emerge from the collision of two BPS 3-branes in $4 + 1$ dimensions has been put forward [1–3]. For an observer on a brane, the universe is first contracting and then goes through a big bang into a hot, thermal expanding phase. During the contracting phase, the matter content is dominated by a scalar field, the amplitude of which is physically related to the distance to the second brane. It has been argued that, under certain conditions for the scalar field potential, this scenario can lead to a scale invariant spectrum of scalar perturbations. Several papers have objected to this result [4–6], mainly claiming that the scale invariant growing mode before the collision has to be matched to the decaying mode after the collision. The decaying mode before the collision, which then is matched to the growing mode after the collision is blue and will therefore not lead to a scale invariant spectrum of scalar fluctuations.

These objections have come from the usual matching conditions after inflation [7], where the extrinsic curvature on surfaces of constant density in longitudinal gauge is matched. However, the matching surfaces might be chosen differently, and I will show that by choosing to match the surfaces of constant energy in comoving gauge, the growing mode of the contraction phase is in general matched to the 'growing' mode of the expansion phase.

Furthermore, in the ekpyrotic universe the matching has to be done from a collapsing phase into an expanding phase. Here, the extrinsic curvature of the background changes sign. Having a jump in the background extrinsic curvature, it does not seem reasonable to require that its perturbation be continuous. Clearly, within the four dimensional picture, for the extrinsic curvature to jump, we have to require a non-vanishing surface stress density. The perturbations of which will enter the matching conditions.

In this note, I discuss the matching conditions applied in [3] and [4–6]. It appears to me that the matching conditions of [3] are more natural than those applied in [4–6], where the surfaces of constant energy density in longitudinal gauge are matched and the perturbations of

the surface stress density is set to zero. The truly correct matching conditions can only be worked out in the five dimensional picture which has to determine this surface stress density and its fluctuations.

Here I shall remain fully within the four dimensional framework and just discuss the generic matching conditions which I then exemplify in the two cases studied in the literature. One of them matches the constant energy slices in longitudinal gauge, there the scale invariant growing mode of the collapsing phase is matched to the decaying mode in the expanding phase. The second case, where the constant energy slices are matched in comoving gauge, leads to a growing mode in the expanding phase which is inherited from the growing mode in the collapsing phase.

Since this is a web-note, mainly addressed to those who have studied the ekpyrotic scenario and have been puzzled about the ongoing perturbation debate, we shall not repeat the basics of the scenario, but immediately jump in 'medias res'.

II. MATCHING PERTURBATIONS FROM A COLLAPSING TO AN EXPANDING UNIVERSE

Let us first note some generic results about matching of perturbations, which are especially subtle when matching from a contracting to an expanding phase.

The extrinsic curvature in a Friedman universe is

$$K_j^i = - \left(\frac{\dot{a}}{a^2} \right) \delta_j^i = - \frac{\mathcal{H}}{a} \delta_j^i, \quad (1)$$

where a denotes the scale factor and an over-dot is the derivative w.r.t. conformal time η and $\mathcal{H} = \dot{a}/a$. \dot{a} changes sign in the transition from a contracting to an expanding phase. Hence the extrinsic curvature is discontinuous in the four dimensional picture, if we simply 'glue' the contracting phase at very small scale factor (high curvature) to the expanding phase with the same scale factor, but opposite sign for \dot{a} and conformal time η . Clearly, this is the simplest way of connecting a contracting phase to an expanding phase, but it is relatively close to the approach motivated from the 5 dimensional

picture, where the singularity at $a = 0$ becomes a narrow 'throat' [2]. Here we replace this throat by a stiff 'collar' (see also [3]).

If the extrinsic curvature is discontinuous, the Israel junction conditions require [8]

$$[K_j^i]_{\pm} = \kappa^2 S_j^i, \quad (2)$$

where S_j^i is a surface stress-energy tensor, and

$$[h]_{\pm} \equiv \lim_{\epsilon \searrow 0} [h(\eta_1 + \epsilon) - h(\eta_1 - \epsilon)] = h_+ - h_-$$

for an arbitrary function $h(\eta)$. The time η_1 is the time of matching. Let us consider an arbitrary hyper-surface, linearly perturbed from $\eta = \text{const.}$ and defined by $f = f_0(\eta) + \delta f = \text{const.}$, where f is an arbitrary (perturbed) function, e.g. the energy density. We use longitudinal gauge,

$$ds^2 = a^2(\eta) [-(1 + 2\Psi)d\eta^2 - (1 + 2\Phi)\delta_{ij}dx^i dx^j] \quad (3)$$

(we ignore a possible spatial 3-curvature which is unimportant at early time). The perturbation variables Ψ and Φ are the Bardeen potentials. We assume that there are no anisotropic stresses, so that $\Phi = -\Psi$. We want to determine the extrinsic curvature in the coordinate system $(\tilde{\eta}, \tilde{x}^i)$ where the surfaces $\{\tilde{\eta} = \text{const.}\}$ are parallel to $\{f = \text{const.}\}$. The perturbed conformal time is given by $\tilde{\eta} = \eta + \delta f/f$. The normal vector to $f = \text{const.}$ in this coordinate system is [7]

$$\tilde{n}_0 = -a(1 + \Psi + \mathcal{H}\delta f/f), \quad \tilde{n}_i = -a\partial_i \delta f/f. \quad (4)$$

The induced metric and extrinsic curvature in the coordinate system $(\tilde{\eta}, \tilde{x}^i)$ are given by

$$\tilde{q}_{\mu\nu} \equiv \tilde{g}_{\mu\nu} + \tilde{n}_{\mu}\tilde{n}_{\nu}, \quad \tilde{K}_{\mu\nu} \equiv \tilde{\nabla}_{\mu}\tilde{n}_{\nu} \quad (5)$$

where $\tilde{g}_{\mu\nu}$ is the metric and $\tilde{\nabla}$ the covariant derivative. The continuity of the induced metric implies (for the background) that

$$[a(\eta)]_{\pm} = 0. \quad (6)$$

The background extrinsic curvature is

$$K_j^i = -\frac{\mathcal{H}}{a}\delta_j^i. \quad (7)$$

Since \mathcal{H} changes sign at the transition, K_j^i is not continuous and we have

$$[K_j^i]_{\pm} = -2\frac{\mathcal{H}_+}{a}\delta_j^i = \kappa^2 p_s \delta_j^i, \quad (8)$$

where p_s is a negative surface pressure. Within the four dimensional picture we have no explanation for this surface pressure, which has to be present in order for the extrinsic curvature to jump, if we require Einstein's equation to hold during the transition. (If they don't hold, we

have no means to find the matching conditions in a four dimensional analysis.) Eq. (8) is a possibility to 'escape' the violation of the weak energy condition, $\rho + p < 0$, which is needed for a smooth transition from collapse to expansion. This has been one of the objections to the ekpyrotic scenario [9].

At the perturbed level, the continuity of the induced metric leads to [7]

$$\left[\Psi + \mathcal{H}\frac{\delta f}{f}\right]_{\pm} = 0. \quad (9)$$

The extrinsic curvature is given by [7]

$$\delta K_j^i = \frac{1}{a} \left[\dot{\Psi} + \mathcal{H}\Psi + (\dot{\mathcal{H}} - \mathcal{H}^2)\frac{\delta f}{f} \right] \delta_j^i + (\delta f/f)_{,j}^i \quad (10)$$

For simplicity, we assume that also the surface stress tensor has no anisotropic stresses, which implies

$$[(\delta f/f)]_{\pm} = 0. \quad (11)$$

Then

$$\delta K_j^i \equiv (\delta K)\delta_j^i \quad (12)$$

so that the matching conditions for the perturbations become (11) and

$$[\delta K]_{\pm} = \kappa^2 \delta p_s, \quad (13)$$

where δp_s is the perturbation of the surface pressure.

Let us first consider the constant energy hyper-surfaces in longitudinal gauge, which are usually used for matching after inflation, and on which all attention of [4-6] has been concentrated. In this case $f = \rho$ and $\delta f = \delta\rho$ in longitudinal gauge. The perturbed Einstein equations give (see e.g. [10] Eqs. (2.45), (2.46) and use $\delta\rho = \rho D_s$ in longitudinal gauge)

$$\frac{\delta\rho}{\rho} = \frac{-2}{\mathcal{H}^2} \left[(3k^2 + \mathcal{H}^2)\Psi + \mathcal{H}\dot{\Psi} \right] \simeq -2 \left[\Psi + \mathcal{H}^{-1}\dot{\Psi} \right] \quad (14)$$

on super horizon scales. With $\dot{\rho} = 2\frac{\dot{\mathcal{H}} - \mathcal{H}^2}{\mathcal{H}}\rho$ we have

$$\delta\rho/\dot{\rho} \simeq \frac{-1}{\mathcal{H} - \mathcal{H}^2} (\mathcal{H}\Psi + \dot{\Psi}). \quad (15)$$

Eq. (9) then leads to

$$\left[\Psi - \frac{\mathcal{H}}{\mathcal{H} - \mathcal{H}^2} [\mathcal{H}\Psi + \dot{\Psi}] \right]_{\pm} = [\zeta]_{\pm} = 0, \quad (16)$$

where ζ is the curvature perturbation introduced by Bardeen. Note that in general Ψ will not be continuous at the transition since \mathcal{H} jumps and $\frac{1}{\mathcal{H} - \mathcal{H}^2} (\mathcal{H}\Psi + \dot{\Psi})$ is continuous according to Eqs. (11) and (15).

For fluids with adiabatic perturbations and single scalar fields, the general solution for the Bardeen potential on super horizon scales is given by (see e.g. [11])

$$\Psi = A \frac{\mathcal{H}}{a^2} + B. \quad (17)$$

Here A and B are constants for each mode, depending only on the mode k . During the collapse phase in the ekpyrotic universe, the growing mode proportional to A acquires a scale invariant spectrum, $|A|^2 k^3 = \text{const.}$, while the spectrum of the constant mode B is blue, $|B|^2 k^3 \propto k^2$ [3]. Inserting ansatz (17) in the continuity condition (16) for the metric, yields

$$B_+ \left[\frac{\dot{\mathcal{H}}_+ - 2\mathcal{H}_+^2}{\dot{\mathcal{H}}_+ - \mathcal{H}_+^2} \right] = B_- \left[\frac{\dot{\mathcal{H}}_+ - 2\mathcal{H}_-^2}{\dot{\mathcal{H}}_- - \mathcal{H}_-^2} \right]. \quad (18)$$

Clearly, since B_+ couples only to B_- it inherits the blue spectrum of B_- . This is the main argument of Refs. [4–6]. However, in this case with $\delta f = \rho D_s$, Eqs. (13) leads to $\delta K \equiv 0$ and does hence not allow for perturbations of the surface tension. A possible way out would be to admit anisotropic stresses, so that $[\delta f/f]_{\pm} \neq 0$. But we want to clarify the attempt taken in Ref [3]:

Another natural coordinate choice is to set $f = \rho$ in *comoving coordinates*, i.e., $\delta g_{0i} = v^i = 0$. In this case (see e.g. [10] Eq. (2.45), and use $\delta \rho = \rho D$ in comoving gauge)

$$\delta f/f = \frac{k^2}{3(\mathcal{H}\dot{\mathcal{H}} - \mathcal{H}^3)} \Psi. \quad (19)$$

The junction conditions on super horizon scales (i.e. to lowest order k/\mathcal{H}) then become

$$[\Psi]_{\pm} = 0 \quad (20)$$

$$\left[\mathcal{H}\Psi + \dot{\Psi} \right]_{\pm} = a\kappa^2 \delta p_s \quad (21)$$

For our general solution (17) this gives

$$\begin{aligned} A_+/a^2 &= (A_-/a^2)\mathcal{H}_-/\mathcal{H}_+ + (B_- - B_+)/\mathcal{H}_+ \\ B_+ &= \frac{A_-}{a^2} \left[\frac{\mathcal{H}_+(\dot{\mathcal{H}}_- - \mathcal{H}_-^2) - \mathcal{H}_-(\dot{\mathcal{H}}_+ - \mathcal{H}_+^2)}{2\mathcal{H}_+^2 - \dot{\mathcal{H}}_+} \right] \\ &+ B_- \left[1 + \frac{\mathcal{H}_-\mathcal{H}_+ - \mathcal{H}_+^2}{2\mathcal{H}_+^2 - \dot{\mathcal{H}}_+} \right] \\ &+ \kappa^2 a \delta p_s \frac{\mathcal{H}_+}{2\mathcal{H}_+^2 - \dot{\mathcal{H}}_+}. \end{aligned}$$

Even if $B_- = \delta p_s = 0$, B_+ will in general not vanish and will inherit the flat spectrum of A_- . Since the spectra are normalized on small scales, the scale invariant contribution from A_- will dominate also if $B_- \neq 0$ (unless if its pre-factor $A_-[\dots]$ vanishes). This is the matching condition which has been adopted in [3].

III. CONCLUSIONS

We consider the matching conditions which match the energy perturbation as seen by an observer comoving

with the fluid at least as natural as matching the energy in longitudinal gauge. Therefore, from the four dimensional point of view, one cannot decide which one, if any, of these two matching conditions is to be preferred. This issue has to be resolved in the five dimensional picture, which is attempted in Refs. [3,2]. It is also not clear to us how a full, five dimensional treatment of perturbations would alter this result. Clearly, since five dimensional perturbations can be re-written as four dimensional ones with 'seeds' [12], the scale invariant mode remains as a homogeneous solution. Whether it will generically dominate over the 'inhomogeneous modes' is not clear at this stage. The inhomogeneous modes reflect themselves in this treatment in the perturbation of the surface tension δp_s .

The scenario deserves further study.

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