Gauge invariant cosmological perturbation theory for braneworlds

Alain Riazuelo,[1,2,*] Filippo Vernizzi,[2,†] Danièle Steer,[3,2,‡] and Ruth Durrer[2,§]

[1] *Service de Physique Théorique, CEA/DSM/SPhT, Unité de recherche associée au CNRS, CEA/Saclay, F–91191 Gif-sur-Yvette cédex, France*

[2] *Département de Physique Théorique, Université de Genève, 24, Quai Ernest Ansermet, 1211 Genève 4, Switzerland*

[3] *Laboratoire de Physique Théorique, Bât. 210, Université Paris XI, F–91405 Orsay Cedex, France*

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We derive the gauge invariant perturbation equations for a 5-dimensional bulk spacetime in the presence of a brane. The equations are derived in full generality, without specifying a particular energy content of the bulk or the brane. We do not assume Z_2 symmetry, and show that the degree of freedom associated with brane motion plays a crucial role. Our formalism may also be used in the Z_2 symmetric case where it simplifies considerably.

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*Electronic address: `riazuelo@spht.saclay.cea.fr`
†Electronic address: `vernizzi@amorgos.unige.ch`
‡Electronic address: `steer@th.u-psud.fr`
§Electronic address: `ruth.durrer@physics.unige.ch`

I. INTRODUCTION

The idea that our 4-dimensional observed universe may be a hypersurface or "brane" in a higher dimensional spacetime is motivated by string- and M-theory [1–3]. In particular, 5-dimensional braneworld scenarios, in which our universe represents the boundary of a 5-dimensional spacetime, have recently received considerable attention [4–58]. The case in which the bulk spacetime is Anti de Sitter space and orbifold compactification is realized with the brane as fixed point has been particularly studied. In this situation, it has been shown that 4-dimensional gravity is recovered on the brane [11, 20, 22, 23] on energy scales much lower than the brane tension and/or bulk curvature, and late time cosmology is not changed if the brane tension is sufficiently high [12, 13, 15–18]. In an attempt to solve the fine-tuning problem between the bulk cosmological constant and the brane tension, more complex models in which the bulk or the brane are filled with several species (such as scalar fields) have recently been proposed (see for example [24] and references therein).

In these models Z_2 symmetry is often assumed, and this is particularly convenient when considering boundary conditions on the brane. If Z_2 symmetry is dropped, brane motion in the bulk must be taken into account and involved calculations are required in order to determine the boundary conditions on the brane. Whilst Z_2 symmetry is motivated by M-theory and is required for a supersymmetric brane configuration, such as a BPS state [1], there exist situations in which Z_2 symmetry is broken. This occurs, for example, when the brane is charged and couples to a 4-form field in the bulk [30]. Cosmological asymmetric brane models have been studied in [21, 26–32].

These developments have prompted us to derive gauge invariant perturbation theory for brane cosmology with one codimension. Our aim is develop a formalism which may then be applied to any situation of cosmological interest. Previously, perturbations in braneworld cosmology have been extensively studied in the literature mostly for the case of Z_2 symmetry [35–51]. Here we consider the most general situation in which the spatial background geometry on the brane has maximal symmetry and thus represents a space of constant curvature k. We do not assume Z_2 symmetry, and the boundary conditions on the brane are discussed. Also, no particular gauge choice for the metric component g_{44} is made. The perturbation equations in the bulk and on the brane are derived for general bulk and brane stress-energy tensors. This makes our formalism particularly convenient when analyzing situations in which different bulk components (such as several scalar fields) are also considered. The formalism can be used to study phenomena which have important observational consequences, the most important of them being the calculation of the anisotropies of the cosmic microwave background [47, 51]. Since one must in general first determine the behaviour of perturbations in the bulk before being able to determine their behaviour on the brane [34], we pay particular attention to the relation between bulk and brane gauge invariant perturbation variables. These become more subtle when the position of the

brane is displaced. Indeed we define a set of gauge invariant variables in which the perturbation equations on the brane become similar to the usual 4-dimensional equations. We then study the new terms arising in braneworlds.

Since we assume very general background spacetimes and no Z_2 symmetry, some of our equations are extremely cumbersome. In order to guide the reader through the rest of the paper, we now give a general overview of the methods we use, the variables we introduce, and the equations we derive in this paper.

The basic setup is one of a $3+1$-dimensional brane where the 3-space of constant time is maximally symmetric (a space of constant curvature), embedded in a $4+1$-dimensional bulk. As Z_2 symmetry is not assumed, the bulk spacetimes on each side of the brane will generally differ. Both the brane and the bulk may contain arbitrary matter. Our notation is as follows:

- $x^\alpha,\ \alpha = 0,\ 1,\ 2,\ 3,\ 4$: spacetime coordinates (Greek indices), with metric $g_{\alpha\beta}$ and covariant derivative D_α,
- $x^i,\ i = 1,\ 2,\ 3$: coordinates on the maximally symmetric 3-space (second part of Latin alphabet) with metric γ_{ij} and covariant derivative ∇_i,
- $\sigma^a,\ a = 0,\ 1,\ 2,\ 3$: brane-worldsheet coordinates (first part of Latin alphabet),
- $X^\alpha(\sigma^a)$: brane position in target-space.
- A Roman subscript b indicates "brane" whilst B denotes "bulk".
- Certain variables such as the brane matter content ($\underline{P}(\sigma^a)$, $\underline{\rho}(\sigma^a)$, etc) are only defined *on* the brane. Other variables such as the normal vector to the brane $\underline{\perp}^\alpha$ or the extrinsic curvature $\underline{\mathcal{K}}_{\alpha\beta}$ are also defined at the brane position, but since they describe the embedding of the brane in the bulk, they may take different values *on either side* of the brane. All these brane-related variables are underlined.

The action for the system is

$$\begin{aligned} S &= S_{\rm EH} + S^m_{\rm B} + \underline{S}^m_{\rm b} + S_{\rm GH} \\ &= \int {\rm d}^5x\sqrt{|g|}\left(\frac{1}{2\kappa_5}R + \mathcal{L}^m_{\rm B}\right) + \int {\rm d}^4\sigma\sqrt{|\underline{\chi}|}\,\underline{\mathcal{L}}^m_{\rm b} + S_{\rm GH}. \end{aligned} \tag{1.1}$$

Here $S_{\rm GH}$ is the Gibbons-Hawking boundary term required to consistently derive the Israel junction conditions [54], and κ_5 is the fundamental 5-dimensional Newton constant (related to the 5-dimensional Planck mass M_5 by $\kappa_5 = 6\pi^2 M_5^{-3}$). Furthermore, R is the bulk scalar curvature, $g_{\alpha\beta}$ and $\underline{\chi}_{ab}$ are the bulk metric and the induced metric on the brane respectively, and $\mathcal{L}^m_{\rm B}$ and $\underline{\mathcal{L}}^m_{\rm b}$ are respectively the Lagrangians for arbitrary matter in the bulk and matter confined on the brane. They may also contain a cosmological constant or brane tension. The induced metric on the brane is (see for example [67])

$$\underline{\chi}_{ab}(\sigma) = g_{\mu\nu}(X)\frac{\partial X^\mu}{\partial\sigma^a}\frac{\partial X^\nu}{\partial\sigma^b}, \tag{1.2}$$

and the Einstein equations resulting from action (1.1) are

$$G_{\alpha\beta} = \kappa_5\left(T_{\alpha\beta} + {\rm D}\underline{T}_{\alpha\beta}\right), \tag{1.3}$$

where D is a covariant Dirac δ-function specifying the position of the brane (see Section III B), and

$$T_{\alpha\beta} = \frac{2}{\sqrt{|g|}}\left(\frac{\delta S^m_{\rm B}}{\delta g^{\alpha\beta}}\right), \tag{1.4}$$

$$\underline{T}^{\alpha\beta} = \frac{2}{\sqrt{|\underline{\chi}|}}\frac{\partial X^\alpha}{\partial\sigma^a}\frac{\partial X^\beta}{\partial\sigma^b}\left(\frac{\delta\underline{S}^m_{\rm b}}{\delta\underline{\chi}_{ab}}\right). \tag{1.5}$$

As noted above, we underline $\underline{T}^{\alpha\beta}$ and $\underline{\chi}_{ab}$ to emphasize that they are only defined on the brane (see Section III).

We consider these Einstein equations (1.3) for a homogeneous and isotropic brane and bulk background with first order perturbations. As is summarised schematically in the left hand panels of Fig. 1, these equations contain three parts: one is continuous; the second is discontinuous across the brane; and the third part is singular at the brane position (proportional to D). The coefficients of each of the individual parts must be equated, leading to a number of different equations. The continuous part gives the Einstein equations in the bulk and, via the Gauss-Codacci equation, they also determine the 4-dimensional Einstein tensor on the brane (see Section IV). The discontinuous (but non

singular) part is only non-trivial when Z_2 symmetry is not assumed. It then gives equations for the continuous part of the extrinsic curvature, and it describes the energy and momentum exchange between the brane and bulk, leading to the equation of motion for the brane — the so-called "sail equation" [30–32]. Finally, the singular part represents the second junction condition which relates the bulk geometries on each side of the brane through the brane geometry and matter content.

We also discuss the so-called conservation equations for the stress-energy tensors given in Eqns (1.4,1.5). Again, these contain a discontinuous, continuous and singular part. As is summarized schematically in the right hand panels of Fig. 1, the continuous part gives the bulk energy momentum conservation, the discontinuity simply describes the conservation of the jump of the bulk stress-energy on the brane, and the singular part leads to energy momentum conservation of the brane with a possible contribution from the bulk, and to the sail equation.

When discussing the perturbations of these equations, we will make use of the maximal symmetry of the 3-dimensional subspaces parallel to the brane. Our geometrical quantities will be decomposed into scalar, vector and tensor degrees of freedom (with respect to these 3-spaces). This decomposition is not identical to the more physical one containing density modes, vorticity modes, and 5-dimensional gravitational waves. The relationship between these two approaches is given in Section V A. Finally, in order to set up a consistent gauge-invariant formalism for the evolution of these perturbations, we will see that it is crucial to take fully into account the perturbed brane motion (which can be written in a gauge invariant manner). This degree of freedom will be central to our analysis.

The outline of the paper is the following. In the next section (Section II) we discuss the unperturbed (or background) 5-dimensional bulk: we allow a foliation (with two codimensions) into maximally symmetric 3-spaces, and do not specify the presence of the brane. The Einstein and conservation equations for the bulk background are derived. In Section III we introduce the brane and we discuss the boundary conditions at the brane position for the unperturbed spacetime without imposing Z_2 symmetry. In Section IV we derive the background equations for an observer on the brane. In Section V, we perturb the background. We introduce gauge invariant variables and derive the perturbed Einstein and conservation equations in terms of these variables. The perturbed brane including the perturbation of the brane position is discussed in Section VI. In Section VII we reformulate the perturbation theory from the point of view of an observer confined to the brane, and in the last section we draw some conclusions.

Finally, we also provide an extensive and highly technical Appendix where we present all the relevant intermediate steps required to obtain the results presented in the text. (Examples are, for instance, the perturbed Christoffel symbols and the components of the perturbed Riemann and Weyl tensors.) The Appendix is, in fact, more general than the main text since there we consider an $N+1$-dimensional brane (with an N-dimensional maximally symmetric subspace) embedded in a $N+2$-dimensional bulk: in the text we have set $N=3$. Furthermore, whilst the text presents the perturbation equations in full generality, some specific examples such as a bulk scalar field are discussed briefly in the Appendix.

II. BULK BACKGROUND

In this section we describe the bulk background geometry and energy content without introducing a brane. We assume that the space orthogonal to the fifth dimension is maximally symmetric so that a homogeneous and isotropic brane can be accommodated, as discussed in the next section. We consider the most general stress-energy tensor which satisfies these symmetry conditions, and then derive the Einstein equations and the conservation equations.

A. Metric and notation

We consider a 5-dimensional spacetime with one timelike coordinate $x^0 \equiv \eta$, and four spacelike coordinates $\{x^1, x^2, x^3, x^4\}$, where $x^4 \equiv y$. We assume that the constant time hypersurfaces are locally of the form $\mathcal{M} \times \mathbf{R}$, where $\mathcal{M}$ is a 3-dimensional maximally symmetric space, i.e., a 3-space of constant curvature, parameterized by the coordinates $\{x^1, x^2, x^3\}$, with spatial metric $a^2(\eta, y)\gamma_{ij}$. The curvature of this space will be denoted by k. For example, we may choose the coordinates $\{x^1, x^2, x^3\}$ such that

$$\gamma_{ij} = \delta_{ij} + \frac{\mathrm{k} x^i x^j}{1 - \mathrm{k}\delta_{pq} x^p x^q}, \tag{2.1}$$

where δ_{ij} is the Kronecker symbol. The last spacelike coordinate y (the "extra dimension") is orthogonal to the maximally symmetric space. The metric has the signature $+----$. The line element of the metric can therefore be written as

$$\mathrm{d}s^2 = n^2 \mathrm{d}\eta^2 - a^2 \gamma_{ij} \mathrm{d}x^i \mathrm{d}x^j - b^2 \mathrm{d}y^2. \tag{2.2}$$

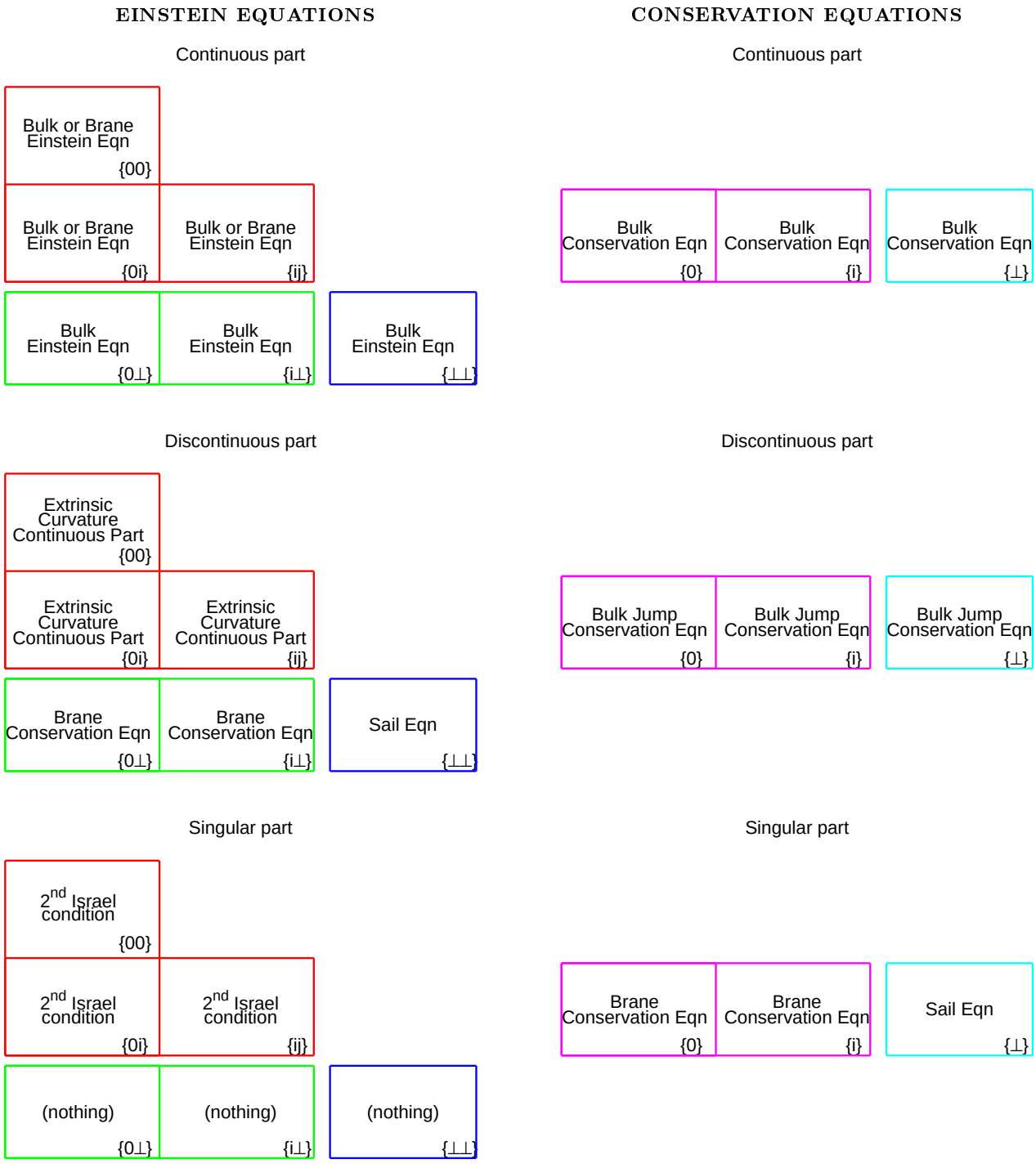

FIG. 1: Structure of the Einstein equations and of the energy momentum conservation equations in the coordinate system (2.2), where coordinates 0, i are also brane coordinates and 4 represents the direction orthogonal to the brane. In components, the Einstein equations can be split into three parts: $\{\mu\nu\}$, $\{\mu 4\}$, and $\{44\}$, where μ, ν run on indices 0, i. These three parts possess a continuous part (defined everywhere in the bulk) and a jump at the brane position. Part $\{\mu\nu\}$ also exhibits a singular term at the brane position. The role played by all these terms is shown in the above diagrams.

An overdot will denote derivation with respect to η, and a prime derivation with respect to y. In addition, we shall define

$$\partial_u \equiv \frac{1}{n}\partial_\eta, \tag{2.3}$$

$$\partial_n \equiv \frac{1}{b}\partial_y. \tag{2.4}$$

Covariant derivatives with respect to the full metric will be denoted by D_α, and those with respect to γ_{ij} by ∇_i. For convenience we also define

$$\mathcal{H} \equiv \frac{1}{n}\frac{\dot{a}}{a} \quad , \qquad \mathcal{I} \equiv \frac{1}{n}\frac{\dot{n}}{n} \quad , \qquad \mathcal{U} \equiv \frac{1}{n}\frac{\dot{b}}{b}, \tag{2.5}$$

$$H \equiv \frac{1}{b}\frac{a'}{a} \quad , \qquad I \equiv \frac{1}{b}\frac{n'}{n} \quad , \qquad U \equiv \frac{1}{b}\frac{b'}{b}, \tag{2.6}$$

as well as

$$\nabla^2 \equiv \nabla_i\nabla^i \quad , \qquad \Delta \equiv \frac{\nabla^2}{a^2} \quad , \qquad \nabla_{ij} \equiv \nabla_i\nabla_j \quad , \qquad K \equiv \frac{\mathrm{k}}{a^2}. \tag{2.7}$$

Notice that ∂_u and ∂_n do not commute since $\partial_u\partial_n - \partial_n\partial_u = I\partial_u - \mathcal{U}\partial_n$.

B. Stress-energy tensor

We now consider the bulk stress-energy tensor $T_{\alpha\beta}$ whose energy flux need not be at rest with respect to our (η, y) coordinates. Let U^α be the normalized timelike eigenvector of T^β_α with eigenvalue ρ_0. Correspondingly let N^α be the normalized eigenvector orthogonal to both U^α and to the maximally symmetric 3-spaces, with eigenvalue Y_0. Finally, let P_0 be the eigenvalue of the three eigenvectors parallel to the maximally symmetric 3-dimensional slices. Note that any symmetric tensor can be decomposed in this way, and that the symmetry requires that the eigenvectors parallel to the symmetric 3-spaces are degenerate. In terms of these variables, the bulk stress-energy tensor can be written as

$$T_{\alpha\beta} = (P_0 + \rho_0)U_\alpha U_\beta - (P_0 - Y_0)N_\alpha N_\beta - P_0 g_{\alpha\beta}, \tag{2.8}$$

where the vectors U^α and N^α are given by

$$U^\alpha = \left(\frac{1}{n}\gamma, \mathbf{0}, \frac{1}{b}\beta\gamma\right) \quad , \qquad U_\mu U^\mu = 1, \tag{2.9}$$

$$N^\alpha = \left(-\frac{1}{n}\beta\gamma, \mathbf{0}, -\frac{1}{b}\gamma\right) \quad , \qquad N_\mu N^\mu = -1. \tag{2.10}$$

Here β represents the Lorentz boost which must be performed along the y axis in order to be in the fluid's rest frame. When one is not in the rest frame of the fluid, both its energy density and its pressure along the extra dimension are modified, and the fluid exhibits a flux through an $y =$ constant hypersurface. As usual, $\gamma = 1/\sqrt{1-\beta^2}$.

Below it will be more convenient to use a different definition for the stress-energy tensor components — a definition which is less adapted to the fluid, but better adapted to our coordinates. To derive it, let us denote by u^α the 5-velocity of a bulk observer who is at rest with respect to our coordinate system,

$$u^\alpha = \left(\frac{1}{n}, \mathbf{0}, 0\right) \quad , \qquad u_\mu u^\mu = 1, \tag{2.11}$$

and by n^α the spacelike unit vector orthogonal to both u^α and $\mathcal{M}$,

$$n^\alpha = \left(0, \mathbf{0}, -\frac{1}{b}\right) \quad , \qquad n_\mu n^\mu = -1 \quad , \qquad n_\mu u^\mu = 0. \tag{2.12}$$

Note that neither u^α nor n^α are geodesic vector fields, but their orthogonality and normalization is conserved. The most general form of the bulk stress-energy tensor satisfying the required symmetry with respect to translations and rotations in $\mathcal{M}$ can be written as

$$T_{\alpha\beta} = (P + \rho)u_\alpha u_\beta - (P - Y)n_\alpha n_\beta - P g_{\alpha\beta} - 2F u_{(\alpha}n_{\beta)}, \tag{2.13}$$

where $f_{(\alpha}g_{\beta)} \equiv \frac{1}{2}(f_\alpha g_\beta + g_\alpha f_\beta)$ denotes symmetrization. In components this gives

$$T_{00} = n^2\rho, \tag{2.14}$$
$$T_{ij} = a^2 P\gamma_{ij}, \tag{2.15}$$
$$T_{04} = -nbF, \tag{2.16}$$
$$T_{44} = b^2 Y. \tag{2.17}$$

Thus $\rho = T_{\mu\nu}u^\mu u^\nu$ is the bulk energy density as measured by an observer with 5-velocity u^μ, $F = T_{\mu\nu}u^\mu n^\nu$ is the energy flux transverse to $\mathcal{M}$, and P, Y are the pressure along the directions x^i, y, respectively. The new variables ρ, Y, P and F are related to the old ones by

$$\rho = \gamma^2(\rho_0 + \beta^2 Y_0), \tag{2.18}$$
$$Y = \gamma^2(Y_0 + \beta^2\rho_0), \tag{2.19}$$
$$P = P_0, \tag{2.20}$$
$$F = \beta\gamma^2(\rho_0 + Y_0). \tag{2.21}$$

The last relation again shows that F represents the energy flux in y direction. This flux, as well as the energy density ρ and pressure Y along the y direction measured by an observer at rest with respect to the coordinate system, are obtained from the components of the stress-energy tensor in a frame at rest with respect to the fluid simply by a Lorentz transformation. Somewhat more complicated but equally straightforward expressions express the old variables in terms of the new ones (see Appendix C2). Note that in (2.9) the bulk velocity of the fluid is $\beta = n_\mu U^\mu/u_\nu U^\nu$. When $\beta = 0$, $\gamma = 1$, we recover the case in which $U^\alpha = u^\alpha$ and $N^\alpha = n^\alpha$, so that the rest frame of the bulk matter and the coordinate system coincide.

C. Einstein equations

The Christoffel symbols, the Riemann, Ricci, Einstein and Weyl tensors for the metric (2.2) are given in Appendices B3, B4, B6, B7 and B8 respectively, and the background bulk Einstein equations are

$$G_{\alpha\beta} = \kappa_5 T_{\alpha\beta}. \tag{2.22}$$

With the stress-energy tensor (2.13) and the Einstein tensor from Appendix B6, Eq. (2.22) becomes

$$3K + 3\mathcal{H}\left(\mathcal{H} + \mathcal{U}\right) - 3\left(\partial_n + 2H\right)H = \kappa_5\rho \qquad \{00\}\,, \tag{2.23}$$
$$-K - 3\left(\mathcal{H}^2 - H^2\right) - \left(\partial_u + \mathcal{U}\right)\left(\mathcal{U} + 2\mathcal{H}\right) + \left(\partial_n + I\right)\left(I + 2H\right) = \kappa_5 P \qquad \{ij\}\,, \tag{2.24}$$
$$3\left(\partial_u H + \mathcal{H}H - \mathcal{H}I\right) = \kappa_5 F \qquad \{04\}\,, \tag{2.25}$$
$$-3K - 3\left(\partial_u + 2\mathcal{H}\right)\mathcal{H} + 3\left(H + I\right)H = \kappa_5 Y \qquad \{44\}\,, \tag{2.26}$$

where we have indicated in braces on the right hand side from which component of the Einstein tensor these bulk Einstein equations are derived. Equations (2.23,2.26) were first discussed in [12], and integrated with respect to the fifth dimension in [13], for the case of a negative bulk cosmological constant, $P = Y = -\rho = \Lambda$.

The first and the third of these equations (2.23,2.25) are constraints (i.e., they do not involve second derivatives with respect to time). The other two are dynamical equations. In fact, there are only two independent dynamical variables which can be written as a combination of the scale factors n, a, and b. One can choose coordinates to remove this ambiguity: for example, in Gaussian coordinates $b = 1$ as in [13], and in conformal coordinates $b = n$ [19, 36]. Of course other choices of coordinates are also allowed. We shall, however, keep b undetermined, so that any useful choice for b can be made at the end.

D. Conservation equations

The Bianchi identities lead to the so-called conservation equations for the stress-energy tensor,

$$D_\mu T^{\mu\alpha} = 0. \tag{2.27}$$

Only for $\alpha = 0$ and $\alpha = 4$ are there non-trivial relations,

$$\partial_u \rho + 3\mathcal{H}(P+\rho) + \mathcal{U}(Y+\rho) + (\partial_n + 3H + 2I)\, F = 0 \qquad \{0\}\,, \tag{2.28}$$
$$(\partial_u + 3\mathcal{H} + 2\mathcal{U})\, F + \partial_n Y + 3H(Y-P) + I(Y+\rho) = 0 \qquad \{4\}\,. \tag{2.29}$$

These are the conservation equations for the energy density and the energy flux of the bulk components, respectively. The generalisation to several components is straightforward (see Appendix C4). Written in term of the intrinsic fluid quantities, they give an equation of evolution for the energy density ρ_0 and for fluid bulk velocity β.

III. BULK BACKGROUND WITH A BRANE

We now consider a homogeneous and isotropic 3-brane orthogonal to y (lying in the space of maximal symmetry) as a singular source, with intrinsic stress-energy tensor $\underline{T}_{\alpha\beta}$.

A. Brane position, induced metric and first fundamental form

Let us choose the intrinsic brane coordinates $(\sigma^0, \sigma^i) = (\eta, x^i)$, and embed the brane according to

$$X^0 = \eta, \tag{3.1}$$
$$X^i = x^i, \tag{3.2}$$
$$X^4 = \mathrm{y_b} = \text{constant}. \tag{3.3}$$

Note that it is always possible to choose the background coordinate y such that the unperturbed brane is at rest: this is the only coordinate choice made in this paper.

As we shall see, the presence of the brane will introduce discontinuities at $y = \mathrm{y_b}$ in several variables. For that reason, it is useful to decompose a given function f as

$$f = [f]\left(\theta(y - \mathrm{y_b}) - \tfrac{1}{2}\right) + \langle f \rangle\,(y), \tag{3.4}$$

where θ is the Heaviside function. This equation defines the continuous function $\langle f \rangle\,(y)$, whilst the discontinuity or jump of f when going from one side to the other side of the brane is given by

$$[f] = \lim_{\varepsilon \to 0^+} \left(f(\mathrm{y_b} + \varepsilon) - f(\mathrm{y_b} - \varepsilon)\right) \equiv f^+ - f^-. \tag{3.5}$$

Notice that we have the two product relations

$$\langle fg \rangle = \langle f \rangle \langle g \rangle + \frac{1}{4}\,[f]\,[g]\,, \tag{3.6}$$
$$[fg] = \langle f \rangle [g] + [f] \langle g \rangle\,. \tag{3.7}$$

For later convenience, and when considering a continuous function $\langle f \rangle$, we will also define the continuous part and the jump of its derivative by

$$\langle \partial_n \rangle \langle f \rangle \equiv \langle \partial_n \langle f \rangle \rangle\,, \tag{3.8}$$
$$[\partial_n] \langle f \rangle \equiv [\partial_n \langle f \rangle]\,. \tag{3.9}$$

Sometimes we shall also need $\big[\underline{f}\big]$ for variables $\underline{f}$ describing the embedding of the brane, and thus which may take different values, $\underline{f}^+$, $\underline{f}^-$, on either side of the brane. The quantities $\big[\underline{f}\big]$ and $\big\langle \underline{f} \big\rangle$ are defined by

$$\big[\underline{f}\big] \equiv \underline{f}^+ - \underline{f}^-, \tag{3.10}$$
$$\big\langle \underline{f} \big\rangle \equiv \tfrac{1}{2}\left(\underline{f}^+ + \underline{f}^-\right). \tag{3.11}$$

The normal unit vector to the brane, $\underline{\perp}_\alpha$, is given by

$$\underline{\perp}_\mu \frac{\partial X^\mu}{\partial \sigma^a} = 0 \quad , \qquad \underline{\perp}_\mu \underline{\perp}^\mu = -1. \tag{3.12}$$

One obtains (up to an overall sign)

$$\underline{\perp}^{\alpha} = \left(0, \mathbf{0}, -\frac{1}{b}\right). \tag{3.13}$$

As we shall see, b can be discontinuous on the brane and $\underline{\perp}^{\alpha}$ can have different values *on either side* of the brane.

From the induced metric one can define the first fundamental form [69] $\underline{q}_{\alpha\beta} = g_{\alpha\beta} + \underline{\perp}_{\alpha}\underline{\perp}_{\beta}$, where $g_{\alpha\beta}$ is again evaluated on (either side of) the brane, and we have $\underline{q}_{\alpha\mu}\underline{\perp}^{\mu} = 0$. On the brane, $\underline{q}_{\alpha\beta}$ is related to $\underline{\chi}_{ab}(\sigma)$ by

$$\underline{q}^{\alpha\beta}(X) = \frac{\partial X^{\alpha}}{\partial \sigma^{a}} \frac{\partial X^{\beta}}{\partial \sigma^{b}} \underline{\chi}^{a\,b}(\sigma). \tag{3.14}$$

We can decompose the stress-energy tensor *on* the brane, $\underline{T}_{\alpha\beta}(X)$, as

$$\underline{T}_{\alpha\beta} = (\underline{P} + \underline{\rho})\underline{u}_{\alpha}\underline{u}_{\beta} - \underline{P}\underline{q}_{\alpha\beta}, \tag{3.15}$$

where $\underline{u}_{\alpha}$ is the 4-vector of the energy flux on the brane matter,

$$\underline{u}^{\alpha} = \left(\frac{1}{n}, \mathbf{0}, 0\right) \quad , \qquad \underline{u}_{\mu}\underline{u}^{\mu} = 1. \tag{3.16}$$

Note that $\underline{T}_{\alpha\mu}\underline{\perp}^{\mu} = \underline{u}_{\mu}\underline{\perp}^{\mu} = 0$. This is the most generic stress-energy tensor compatible with a homogeneous and isotropic brane.

B. Einstein equation

In the presence of the brane, the 5-dimensional Einstein equations become

$$G_{\alpha\beta} = \kappa_5 \left(T_{\alpha\beta} + \mathrm{D}\underline{T}_{\alpha\beta}\right), \tag{3.17}$$

where, from Eqn (1.1), the "covariant Dirac function" D is

$$\mathrm{D} = \frac{\sqrt{|\underline{q}|}}{\sqrt{|g|}} \delta(y - \mathrm{y_b}). \tag{3.18}$$

Here g and $\underline{q}$ are the determinants of the metric $g_{\alpha\beta}$ and first fundamental form $\underline{q}_{\alpha\beta}$ respectively, evaluated at the brane position. Written in components, the Einstein equations with the brane, Eq. (3.17), become

$$3K + 3\mathcal{H}\left(\mathcal{H} + \mathcal{U}\right) - 3\left(\partial_n + 2H\right)H = \kappa_5\left(\rho + \mathrm{D}\underline{\rho}\right) \quad \{00\}\,, \tag{3.19}$$

$$-K - 3\left(\mathcal{H}^2 - H^2\right) - \left(\partial_u + \mathcal{U}\right)\left(\mathcal{U} + 2\mathcal{H}\right) + \left(\partial_n + I\right)\left(I + 2H\right) = \kappa_5\left(P + \mathrm{D}\underline{P}\right) \quad \{ij\}\,, \tag{3.20}$$

$$3\left(\partial_u H + \mathcal{H}H - \mathcal{H}I\right) = \kappa_5 F \quad \{04\}\,, \tag{3.21}$$

$$-3K - 3\left(\partial_u + 2\mathcal{H}\right)\mathcal{H} + 3\left(H + I\right)H = \kappa_5 Y \quad \{44\}\,. \tag{3.22}$$

A global solution to these equations has been derived in [12, 14] with the assumption of a pure negative cosmological constant in the bulk, and using Gaussian coordinates. The right hand sides of Eqns (3.19,3.20) contain a singular term proportional to D due to the presence of the brane. As we will see below, although the first fundamental form is continuous on the brane, its first derivative with respect to the fifth dimension y (i.e., the terms H and I) may jump and its second derivative ($\partial_n H$ and $\partial_n I$) can be singular. Thus the Einstein tensor contains a singular part which must be matched with the singular part of the stress-energy tensor. We now turn to the problem of relating these terms to the brane matter content.

C. Israel junction conditions

The extrinsic curvature formalism is a useful tool in the analysis of junction conditions on a singular surface [70]. The *first Israel condition* [65] imposes the continuity of the first fundamental form,

$$\left[\underline{q}_{\alpha\beta}\right] = 0. \tag{3.23}$$

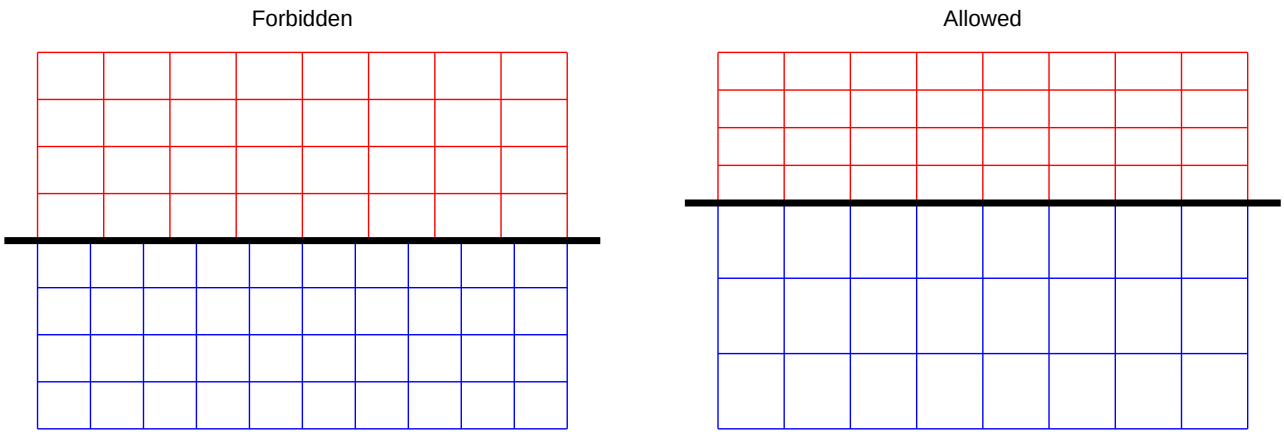

FIG. 2: Schematic illustration of the first Israel condition. We have embedded in a Minkowskian space a 2-dimensional spacelike bulk of metric $\mathrm{d}s^2 = a^2\mathrm{d}x^2 + b^2\mathrm{d}y^2$. The brane is the thick horizontal line in the middle of both panels, and the grids represent the metric coefficients so that the grid spacing is proportional to b and a along the vertical and the horizontal directions respectively. In the left panel, $[a] \neq 0$, $[b] = 0$, in the right one, $[b] \neq 0$, $[a] = 0$. The first Israel condition states that when considering a line of constant x, there must not be any discontinuity when crossing the brane: this is obviously not the case in the left panel. On the contrary, nothing is said about how the spacing of the horizontal lines evolves across the brane. This translates into the fact that b is allowed to be discontinuous (right panel).

Hence $\underline{q}_{\alpha\beta}$ is well-defined *on* the brane. Since we have $\underline{q}_{00} = n^2(X)$ and $\underline{q}_{ij} = -a^2(X)\gamma_{ij}$, this condition implies the continuity of the scale factors n and a: $[n] = [a] = 0$ (see Figure 2). Note that the continuity of the metric function b is not required by the junction conditions and will not be assumed in what follows[1] (see also Appendix F).

Nevertheless, the first derivative with respect to y of a and n (which are proportional to I and H), are allowed to jump. In order to study the behaviour of these quantities on the brane we consider the extrinsic curvature tensor (or second fundamental form) with respect to the brane, namely

$$\underline{\mathcal{K}}_{\alpha\beta} = \underline{q}^{\mu}_{(\alpha} D_{\mu} \underline{\perp}_{\beta)}. \tag{3.24}$$

For the background metric, the components of the extrinsic curvature are

$$\underline{\mathcal{K}}_{00} = -n^2 I, \tag{3.25}$$
$$\underline{\mathcal{K}}_{ij} = a^2 H \gamma_{ij}. \tag{3.26}$$

Let us define the surface "stress tensor" $\underline{\mathcal{S}}_{\alpha\beta}$ on the brane by

$$\underline{\mathcal{S}}_{\alpha\beta} = \underline{T}_{\alpha\beta} - \frac{1}{3}\underline{T}\underline{q}_{\alpha\beta}. \tag{3.27}$$

Then the *second Israel condition* [65] relates the jump in the extrinsic curvature with the energy content on the brane and requires that

$$[\underline{\mathcal{K}}_{\alpha\beta}] = -\kappa_5 \underline{\mathcal{S}}_{\alpha\beta}. \tag{3.28}$$

(Note that the choice of the sign here is consistent with our choice for the sign of $\underline{\perp}_{\alpha}$ in Eq. (3.13).) For our background this condition can be written as (see Appendix D7)

$$[I] = \kappa_5 \left(\frac{2}{3}\underline{\rho} + \underline{P} \right), \tag{3.29}$$
$$[H] = -\kappa_5 \frac{1}{3}\underline{\rho}. \tag{3.30}$$

[1] Allowing b to be discontinuous makes the covariant Dirac function D ill-defined. This is not a serious problem, as all the terms involving this function can be grouped together to give the second Israel condition. Therefore, we shall continue to use the notation D and suppose that when b is not continuous, it corresponds to a regularized and mathematically consistent expression.

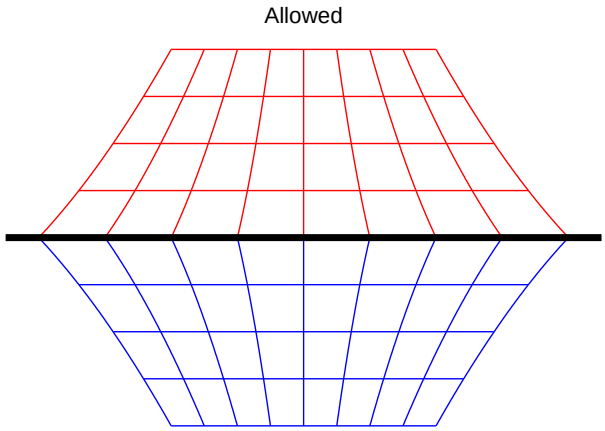

FIG. 3: Illustration of the second Israel condition. With the same conventions as in fig. 2, we show an example where $[H] \neq 0$. This is possible if the energy density on the brane is non zero (it is positive in this illustration).

(See also Figure 3.) Alternatively, Eqns (3.29,3.30) can be obtained directly from the singular part of the Einstein equations (3.19,3.20).

D. Boundary conditions in the bulk

We now comment briefly on the question of boundary conditions at the brane. Consider first the bulk Einstein equations (3.19–3.22). They form a system of second order partial differential equations in η and y, and in order to solve them we must specify initial conditions on a spacelike Cauchy hypersurface, boundary conditions far from our braneworld (at infinity in a one brane scenario, or on another brane), and boundary conditions at our brane. The Israel junction conditions impose the continuity of a and n, and fix the jump in their normal derivatives at the brane. Since the Einstein equations represent a set of second order partial differential equations, these junction conditions are sufficient to allow us to solve the Einstein equations everywhere in the bulk.

We now turn to the Einstein equations on the brane.

IV. THE BRANE POINT OF VIEW

An observer on the brane will not see 4-dimensional Einstein gravity. This may, however, be recovered in particular situations at low energy. The 4-dimensional Einstein tensor in general depends on bulk quantities and is quadratic in the brane stress-energy tensor.

Here we discuss the 4-dimensional "Einstein equations" which lead to the modified Friedmann equations, and also the conservation equation on the brane. Finally we interpret the deviation from the 4-dimensional theory in terms of the 5-dimensional one.

A. Einstein gravity on the brane

Through the Gauss-Codacci equations, we can write the 4-dimensional Einstein tensor ${}^{(4)}\underline{G}_{\alpha\beta}$ in terms of the bulk stress-energy tensor $T_{\alpha\beta}$, the extrinsic curvature $\underline{\mathcal{K}}_{\alpha\beta}$ and the projected Weyl tensor $\underline{\mathcal{E}}_{\alpha\beta}$. The details of the calculation can be found in [34], and the result is

$$\begin{aligned} {}^{(4)}\underline{G}_{\alpha\beta} &= \frac{2}{3}\left(G_{\mu\nu}\underline{q}^{\mu}_{\alpha}\underline{q}^{\nu}_{\beta} - \left(G_{\mu\nu}\underline{\perp}^{\mu}\underline{\perp}^{\nu} + \frac{1}{4}G\right)\underline{q}_{\alpha\beta}\right) \\ &\quad -\underline{\mathcal{K}}\underline{\mathcal{K}}_{\alpha\beta} + \underline{\mathcal{K}}^{\mu}_{\alpha}\underline{\mathcal{K}}_{\mu\beta} + \frac{1}{2}\underline{q}_{\alpha\beta}(\underline{\mathcal{K}}^2 - \underline{\mathcal{K}}^{\mu\nu}\underline{\mathcal{K}}_{\mu\nu}) + \underline{\mathcal{E}}_{\alpha\beta}. \end{aligned} \tag{4.1}$$

The projected Weyl tensor $\underline{\mathcal{E}}_{\alpha\beta}$ on the brane is obtained from the bulk Weyl tensor $C_{\alpha\beta\gamma\delta}$ as follows

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + \frac{2}{3}(g_{\alpha[\delta}R_{\gamma]\beta} + g_{\beta[\gamma}R_{\delta]\alpha}) + \frac{1}{6}Rg_{\alpha[\gamma}g_{\delta]\beta}, \tag{4.2}$$

$$\underline{\mathcal{E}}_{\alpha\beta} = C_{\alpha\mu\beta\nu}\underline{\perp}^{\mu}\,\underline{\perp}^{\nu}. \tag{4.3}$$

Here $f_{[\alpha}g_{\beta]} \equiv \frac{1}{2}(f_\alpha g_\beta - g_\alpha f_\beta)$ denotes antisymmetrization. This projection represents the contribution of the free gravity in the bulk to the gravity on the brane. In components, we have

$$\underline{\mathcal{E}}_{00} = \frac{1}{2}n^2\mathcal{Z}, \tag{4.4}$$

$$\underline{\mathcal{E}}_{ij} = \frac{1}{6}a^2\mathcal{Z}\gamma_{ij}, \tag{4.5}$$

$$\mathcal{Z} = K + (\partial_u + \mathcal{U})(\mathcal{U} - \mathcal{H}) - (\partial_n + I)(I - H). \tag{4.6}$$

There is only one independent component in the Weyl tensor (as well as in its projection on the brane). This is related to the fact that this spacetime is the 5-dimensional analog of a 4-dimensional type-D spacetime in Petrov's classification [68].

B. Friedmann equations on the brane

Since the tensor ${}^{(4)}\underline{G}_{\alpha\beta}$ contains only derivatives of the continuous first fundamental form with respect to η and x^i, it is continuous. Hence, on taking the continuous part of the right hand side of Eq. (4.1) (and applying the product relation (3.6)), we find the projected 4-dimensional Einstein equation on the brane,

$$\begin{aligned}{}^{(4)}\underline{G}_{\alpha\beta} = {} & \frac{2}{3}\kappa_5\left(\langle T_{\mu\nu}\rangle\,\underline{q}^{\mu}_{\alpha}\underline{q}^{\nu}_{\beta} - \left(\langle T_{\mu\nu}\underline{\perp}^{\mu}\,\underline{\perp}^{\nu}\rangle + \frac{1}{4}\langle T\rangle\right)\underline{q}_{\alpha\beta}\right) \\ & - \langle\underline{\mathcal{K}}\rangle\langle\underline{\mathcal{K}}_{\alpha\beta}\rangle + \langle\underline{\mathcal{K}}^{\mu}_{\alpha}\rangle\langle\underline{\mathcal{K}}_{\mu\beta}\rangle + \frac{1}{2}\underline{q}_{\alpha\beta}(\langle\underline{\mathcal{K}}\rangle^2 - \langle\underline{\mathcal{K}}^{\mu\nu}\rangle\langle\underline{\mathcal{K}}_{\mu\nu}\rangle) \\ & + \frac{1}{4}\left(-[\underline{\mathcal{K}}][\underline{\mathcal{K}}_{\alpha\beta}] + [\underline{\mathcal{K}}^{\mu}_{\alpha}][\underline{\mathcal{K}}_{\mu\beta}] + \frac{1}{2}\underline{q}_{\alpha\beta}([\underline{\mathcal{K}}]^2 - [\underline{\mathcal{K}}^{\mu\nu}][\underline{\mathcal{K}}_{\mu\nu}])\right) + \langle\underline{\mathcal{E}}_{\alpha\beta}\rangle\,.\end{aligned} \tag{4.7}$$

The right hand side of equation (4.7) can be split into four parts. The first depends on the average of the bulk stress-energy tensor, $\langle T_{\alpha\beta}\rangle$. The second is given by four terms quadratic in the average of the extrinsic curvature $\langle\underline{\mathcal{K}}_{\alpha\beta}\rangle$. These terms are known once the bulk Einstein equations have been solved. They vanish when Z_2 symmetry is assumed — we return to this point below. Then there is a third part which contains four terms quadratic in the jump of the extrinsic curvature $[\underline{\mathcal{K}}_{\alpha\beta}]$. We have already determined these through the second junction condition (3.29,3.30), and they are related to the brane stress-energy tensor: these terms will be responsible for the non-standard ρ^2 contribution in the brane Friedmann equations. Finally there is a fourth part, the average of the projected bulk Weyl tensor on the brane, describing the effect from the free gravity in the bulk.

In components, we obtain the modified Friedmann equations which contain a dynamical equation and a constraint:

$$3\left(\mathcal{H}^2 + K\right) = \frac{1}{2}\kappa_5\langle\rho + P - Y\rangle + \frac{\kappa_5^2}{12}\underline{\rho}^2 + 3\langle H\rangle^2 + \frac{1}{2}\langle\mathcal{Z}\rangle\,, \tag{4.8}$$

$$-2\partial_u\mathcal{H} - 3\mathcal{H}^2 - K = \frac{1}{6}\kappa_5\langle\rho + P + 3Y\rangle + \frac{\kappa_5^2}{12}(\underline{\rho} + 2\underline{P})\underline{\rho} - \langle H\rangle\langle 2I + H\rangle + \frac{1}{6}\langle\mathcal{Z}\rangle\,, \tag{4.9}$$

where we have isolated the continuous part of the projection of the bulk Weyl tensor,

$$\langle\mathcal{Z}\rangle = K - \partial_u\mathcal{H} + \langle(\partial_u + \mathcal{U} - \mathcal{H})\mathcal{U}\rangle - \langle\partial_n + I\rangle\langle I - H\rangle - \frac{\kappa_5^2}{4}\left(\frac{2}{3}\underline{\rho} + \underline{P}\right)(\underline{\rho} + \underline{P}). \tag{4.10}$$

(These equations could alternatively have been obtained from the continuous part of the Einstein equations, Eqns (3.19–3.22).)

The cosmological consequences of these equations have been studied in [13, 15–18] with assumption of Z_2 symmetry, in which case $\langle\underline{\mathcal{K}}_{\alpha\beta}\rangle = \langle H\rangle = \langle I\rangle = 0$. These authors considered a negative cosmological constant in the bulk and assumed that the brane stress-energy tensor consists of a rigid part — the brane tension — and a fluid,

$$\underline{T}_{\alpha\beta} = \lambda\underline{q}_{\alpha\beta} + \underline{T}^{f}_{\alpha\beta}. \tag{4.11}$$

In this case the bulk stress-energy tensor can be tuned to the brane tension in such a way that deviations from standard Friedmann equations are effective only at energies of order λ and higher.

If we assume $F = \langle H \rangle = \dot{Y} = 0$, integrate the sum of Eq. (3.21) and (3.22) once with respect to time, and compare the result with (4.8), one obtains an expression for the sum of the continuous part of the projected bulk Weyl tensor and the continuous part of the bulk energy density and pressure: this behaves as a radiation term,

$$\langle \mathcal{Z} \rangle + \kappa_5 \langle \rho + P \rangle = \mathcal{C}/a^4, \tag{4.12}$$

where $\mathcal{C}$ is an integration constant (see Appendix E2). In [13, 15–18] this is discussed for the case $\rho + P = 0$ so that $\langle \mathcal{Z} \rangle \propto a^{-4}$.

Furthermore, notice that Eq. (4.11) is not a necessary requirement in order for the 5-dimensional Friedmann equations on the brane to reduce to the standard 4-dimensional Friedmann equations at low energy. It suffices that the brane stress-energy tensor is dominated by a term which is almost a cosmological constant today (i.e., it can be a slow-rolling scalar field, see for example [25]). In this case, one has

$$\underline{\rho} = \underline{\rho}_\phi + \underline{\rho}^f, \tag{4.13}$$

$$\underline{P} = \underline{P}_\phi + \underline{P}^f, \tag{4.14}$$

$$\underline{\rho}_\phi \simeq -\underline{P}_\phi \simeq \lambda, \tag{4.15}$$

where $\underline{\rho}^f$, $\underline{P}^f$ are the energy density and pressure of the ordinary matter content on the brane. Now, if in addition

$$|\rho + P| \ll |Y|, \tag{4.16}$$

$$Y \simeq \Lambda, \tag{4.17}$$

$$\Lambda \simeq \frac{1}{6}\kappa_5 \lambda^2, \tag{4.18}$$

the first two terms of the right hand side of the above Friedmann equations on the brane are proportional to $\kappa_4 \underline{\rho}^f$, $\kappa_4 \underline{P}^f$, with

$$\kappa_4 = \frac{1}{6}\kappa_5^2 \lambda = \kappa_5 \frac{\Lambda}{\lambda}, \tag{4.19}$$

and one re-obtains a term linear in the brane matter content. Note also that there are no extra λ^2 appearing in the projected Weyl tensor (4.10) since with these conditions $\underline{P} + \underline{\rho}$ is λ-independent. This was also noted in Ref. [52].

C. Closing the system when Z_2 symmetry is broken

When solving the Einstein equations on the brane (4.7), we need the continuous part of the extrinsic curvature $\langle \underline{\mathcal{K}}_{\alpha\beta} \rangle$. If Z_2 symmetry is assumed — as motivated by M-theory — the evolution of the bulk is the same on both the sides of the brane. In this case the Israel conditions determine $\underline{\mathcal{K}}_{\alpha\beta}$ entirely: it is always possible to choose a coordinate system in which $n(\mathrm{y_b} + y) = n(\mathrm{y_b} - y)$ and $a(\mathrm{y_b} + y) = a(\mathrm{y_b} - y)$, i.e., where n and a are even. Thus I and H are odd and the continuous value of $\underline{\mathcal{K}}_{\alpha\beta}$ across the brane vanishes,

$$\langle \underline{\mathcal{K}}_{\alpha\beta} \rangle = 0 \qquad (Z_2 \text{ symmetry}). \tag{4.20}$$

This implies

$$\begin{aligned} \langle I \rangle &= \langle H \rangle = 0, \qquad (Z_2 \text{ symmetry}), \\ I^+ &= -I^- = \tfrac{1}{2}[I], \\ H^+ &= -H^- = \tfrac{1}{2}[H]. \end{aligned} \tag{4.21}$$

If Z_2 symmetry is not assumed, as in this paper, the evolution on either side will in general be different and $\langle \underline{\mathcal{K}}_{\alpha\beta} \rangle$ no longer vanishes,

$$\langle \underline{\mathcal{K}}_{\alpha\beta} \rangle \neq 0 \qquad (Z_2 \text{ symmetry broken}). \tag{4.22}$$

One can, however, obtain a condition for the continuous part of the extrinsic curvature by considering the jump of Eq. (4.1). We obtain

$$\begin{aligned} 0 &= \frac{2}{3}\kappa_5 \left([T_{\mu\nu}]\, \underline{q}^{\mu}_{\alpha} \underline{q}^{\nu}_{\beta} - \left([T_{\mu\nu} \underline{\perp}^{\mu} \underline{\perp}^{\nu}] + \frac{1}{4}[T] \right) \underline{q}_{\alpha\beta} \right) \\ &\quad - [\underline{\mathcal{K}}]\, \langle \underline{\mathcal{K}}_{\alpha\beta} \rangle + [\underline{\mathcal{K}}^{\mu}_{\alpha}]\, \langle \underline{\mathcal{K}}_{\mu\beta} \rangle + \underline{q}_{\alpha\beta} \left([\underline{\mathcal{K}}]\, \langle \underline{\mathcal{K}} \rangle - [\underline{\mathcal{K}}^{\mu\nu}]\, \langle \underline{\mathcal{K}}_{\mu\nu} \rangle \right) \\ &\quad - \langle \underline{\mathcal{K}} \rangle\, [\underline{\mathcal{K}}_{\alpha\beta}] + \langle \underline{\mathcal{K}}^{\mu}_{\alpha} \rangle\, [\underline{\mathcal{K}}_{\mu\beta}] + [\underline{\mathcal{E}}_{\alpha\beta}] \,. \end{aligned} \tag{4.23}$$

(Notice that $\left[{}^{(4)}\underline{G}_{\alpha\beta}\right] = 0$.) This becomes, in components,

$$\underline{\rho}\, \langle H \rangle = \frac{1}{4} \left([P + \rho - Y] + \frac{1}{\kappa_5} [\mathcal{Z}] \right), \tag{4.24}$$

$$(\underline{\rho} + 3\underline{P})\, \langle H \rangle - \underline{\rho}\, \langle I \rangle = \frac{1}{4} \left([P + \rho + 3Y] + \frac{1}{\kappa_5} [\mathcal{Z}] \right), \tag{4.25}$$

where, using Eq. (4.6) and the junction conditions (3.29) and (3.30), the jump of $\mathcal{Z}$ on the brane can be expressed as

$$[\mathcal{Z}] = (\partial_u + 2\langle \mathcal{U} \rangle - \mathcal{H})\, [\mathcal{U}] - [\partial_n]\, \langle I - H \rangle - \kappa_5 \langle I \rangle \left(2\underline{P} + \frac{5}{3}\underline{\rho} \right) + \kappa_5 \langle H \rangle \left(\underline{P} + \frac{2}{3}\underline{\rho} \right). \tag{4.26}$$

(Note that Eqns (4.24,4.25) could alternatively have been obtained from the discontinuous part of the Einstein equations (3.19–3.22).)

Equations (4.24,4.25) allow one to fix the unknown quantities $\langle H \rangle$, $\langle I \rangle$, *provided* the jumps of both the bulk matter content and the Weyl tensor are known. Thus the continuous part of the extrinsic curvature depends not only on the brane matter content but also on the discontinuity of the bulk stress-energy and the projected Weyl tensors. If both vanish, Eqns (4.24,4.25) allow in particular the trivial solution $\langle H \rangle = \langle I \rangle = 0$, which holds with Z_2 symmetry. The jump of the Weyl tensor, Eq. (4.26), contains first derivatives of the extrinsic curvature with respect to y, and so it is not possible in general to determine $\langle I \rangle$ and $\langle H \rangle$ without first solving the Einstein equations in the bulk. Nonetheless, if the projection (4.26) of the 5-dimensional bulk Weyl tensor on the brane is known *a priori* (as in the case of a Schwarzschild-Anti de Sitter bulk with a known black hole mass on both side of the brane), then $\langle I \rangle$ and $\langle H \rangle$ can be determined *directly* from Eqns (4.24,4.25) (see [30–32] for a more detailed discussion).

D. Brane motion

On contracting Eq. (4.23) with the first fundamental form $\underline{q}^{\alpha\beta}$ and using the second junction condition (3.28), one obtains

$$\underline{T}^{\mu\nu}\, \langle \underline{\mathcal{K}}_{\mu\nu} \rangle = [\underline{\perp}_{\mu} \underline{\perp}_{\nu} T^{\mu\nu}]\,. \tag{4.27}$$

This equation is known as the "sail equation" [30–32]. The right hand side is an external force density on the brane due to the asymmetry of the bulk stress-energy tensor on the two sides. In analogy with Newton's second law (here the force is due to a pressure difference between the two sides of the brane), $\underline{T}^{\alpha\beta}$, $\langle \underline{\mathcal{K}}_{\alpha\beta} \rangle$, and $[\underline{\perp}_{\mu} \underline{\perp}_{\nu} T^{\mu\nu}]$ play the role of mass, acceleration, and force, respectively. Notice from (4.20) that when Z_2 symmetry is assumed, this equation vanishes identically. When Z_2 symmetry is broken, the "acceleration" $\langle \underline{\mathcal{K}}_{\mu\nu} \rangle$ is non-zero. In this paper we do not assume Z_2 symmetry, but recall that we have chosen a coordinate system in which the brane is at rest: Eq. (4.28) must therefore be understood as dictating the condition that must be satisfied by $\langle H \rangle$ and $\langle I \rangle$ (and therefore by the coordinate system itself) for the brane to remain at a fixed position y_b. Later, however, we will see that Eq. (4.27) does indeed give a more intuitive equation of motion for the perturbed brane position or brane displacement (see Section VI F).

In components the sail equation leads to

$$-\langle I \rangle\, \underline{\rho} + 3\langle H \rangle\, \underline{P} = [Y]\,, \tag{4.28}$$

which can also be obtained by taking the discontinuous part of the $\{44\}$ component of the Einstein equation, Eq. (3.22) or, of course, by a linear combination of Eqns (4.24,4.25). This is the only combination of Equations (4.24,4.25) which does not involve the Weyl tensor.

E. Conservation equations

The singular part of the 5-dimensional energy conservation equation (2.28) yields the stress-energy conservation equation on the brane: we find

$$\partial_u \underline{\rho} + 3\mathcal{H}(\underline{P} + \underline{\rho}) = -[F]\,. \tag{4.29}$$

(Again the generalisation to several interacting components may be found in Appendix D5.) Notice that the jump in the bulk energy flux transverse to the brane enters in the conservation equation, meaning that the brane matter content can act as a source or a sink to the energy flux along the fifth dimension. When this energy flux is continuous, the conservation equation on the brane reduces to the usual one, as discussed in [34]. Another conservation equation appears in brane cosmology: by considering the singular part of Eq. (2.29) we obtain again the sail equation (4.28). Both equations were first found in [12] for the case a bulk cosmological constant.

V. BULK PERTURBATIONS

We now turn to perturbed quantities, and begin in this section by analysing the properties of the perturbed bulk: the perturbed brane itself will be introduced in Section VI. We work throughout with gauge independent perturbation variables, which are inspired from a generalisation of the Newtonian (or longitudinal) gauge to the 5-dimensional case. First we introduce the bulk metric perturbation variables using the standard scalar, vector, tensor decomposition. We study their gauge transformation properties and define gauge invariant combinations. Then, in Section V D, the perturbations of the bulk stress-energy tensor are considered, leading, in Section V E, to the gauge invariant perturbed bulk Einstein equations. Finally we write down the perturbed conservation equations (Bianchi identities).

A. Classification of the perturbations

Let us consider the perturbations of a spacetime with one timelike and n spacelike coordinates. The perturbed metric of this spacetime possesses $\frac{1}{2}(n+1)(n+2)$ different components. Amongst these, a coordinate transformation allows $n+1$ of them to be fixed, so that there are $\frac{1}{2}n(n+1)$ independent metric coefficients. For example, in synchronous gauge, the $\delta g_{0\alpha}$ are set to zero.

When solving perturbation equations about a given spacetime, one is naturally led to classify perturbations. Two classifications are of particular relevance. Firstly, the perturbations may be classified according to their physical meaning, and this is done by looking at the spin of the perturbation in a locally Minkowskian frame. The different perturbations are density (spin 0) modes, vorticity (spin 1) modes, and gravitational (spin 2) waves. Secondly, the perturbations may be classified more geometrically in terms of irreducible components under the group of isometries of the unperturbed spacetime. This leads to scalar, vector and tensor perturbations. Under some circumstances, these two classifications are identical. In particular, this is true for a Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime, which can be foliated by a set of maximally symmetric spacelike hypersurfaces. In brane cosmology, however, the bulk is not as symmetric as in the FLRW case, and the two classifications are different.

Components which transform irreducibly under symmetries of the background spacetime evolve independently (to linear order) while the physical spin components mix.

1. Physical splitting

As explained above, metric perturbations can be decomposed according to their spin with respect to a local rotation of the coordinate system. This leads to density modes, vorticity modes, and gravitational waves. Gravitational (spin 2) waves are "true" degrees of freedom of the gravitational field in the sense that they can exist even in vacuum. The number of gravitational wave modes is given by the dimension of the vector space spanned by symmetric, transverse, traceless rank 2 tensors in an n-dimensional space: this is $\frac{1}{2}(n-2)(n+1)$. In addition, when there is a non trivial matter content, there may be vorticity (or spin 1) modes arising from rotational velocity fields, which have $n-1$ independent components. Finally, there remain $\frac{1}{2}n(n+1) - \frac{1}{2}(n-2)(n+1) - (n-1) = 2$ possible density (spin 0) modes, which are usually represented by the two Bardeen potentials Φ and Ψ [60, 61].

More schematically, let us consider the metric perturbation around a locally inertial frame, written in synchronous gauge and in Fourier space considering the wave vector $k^i = k\delta^i_1$:

$$\delta g_{\alpha\beta} = \left(\begin{array}{c|c|ccccc} 0 & 0 & 0 & 0 & \dots & 0 \\ \hline 0 & 2k^2E - 2C & ikV_2 & ikV_3 & \dots & ikV_n \\ \hline 0 & ikV_2 & -2C + \sum_{i=3}^{n} h_i^+ & h_{23}^\times & \dots & h_{2n}^\times \\ 0 & ikV_3 & h_{23}^\times & -2C - h_3^+ & & h_{3n}^\times \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & ikV_n & h_{2n}^\times & h_{3n}^\times & \dots & -2C - h_n^+ \end{array}\right). \tag{5.1}$$

The quantities E and C describe the density modes (with the standard definition of the Bardeen potentials, one has $\Phi = -C$ and $\Psi = -\partial_t^2 E$), the V_i $(i = 2, \dots, n)$ represent the vorticity modes, and the h_i^+ $(i = 3, \dots, n)$ and $h_{jk}^\times$ $(2 \leq j < k \leq n)$ represent the gravitational waves (when $n = 3$, these notations agree with the standard definition of the h^+ and $h^\times$ modes).

2. Geometrical splitting

The three above types of perturbation generally do not evolve independently: even at linear order, they are coupled if the unperturbed spacetime does not possess any symmetries. However, for most cosmological models (including the ones considered in this paper), spacetime possesses some symmetries, being invariant under a certain group of global transformations. We consider the symmetry group $\mathrm{SO}(N)$ with $N < n$, which is of course relevant when there exists a coordinate system in which N coordinates span a maximally symmetric space.

When this is the case, perturbations may be decomposed into components which transform irreducibly under $\mathrm{SO}(N)$-rotations of the coordinate system. This leads to what we call scalar, vector and tensor perturbations which are perturbations whose spin with respect to $\mathrm{SO}(N)$ is 0, 1 and 2 respectively. The main advantage of this decomposition is that the three new types of perturbation are now decoupled from each other, and hence are convenient when studying the evolution of cosmological perturbations. For example, consider an n-dimensional space with N coordinates (labelled by i, j, etc) spanning an N-dimensional, maximally symmetric sub-space, with metric γ_{ij}, and associated covariant derivative ∇_i. The $n - N$ remaining coordinates will be labelled by A, B, etc. In this case, the metric perturbations can be decomposed as

$$\delta g_{ij} = -2C\gamma_{ij} - 2\nabla_{ij}E - 2\nabla_{(i}\bar{E}_{j)} - 2\bar{\bar{E}}_{ij}, \tag{5.2}$$

$$\delta g_{iA} = \nabla_i E_{(A)} + \bar{E}_{(A)i}, \tag{5.3}$$

$$\delta g_{AB} = E_{(AB)}, \tag{5.4}$$

where barred quantities are divergenceless N-vectors, and double barred quantities are divergenceless, traceless N-tensors of rank 2 (with respect to the covariant derivative ∇_i and metric γ_{ij} respectively). With our definitions, it is clear that C, E, $E_{(A)}$, $E_{(AB)}$ are scalars, $\bar{E}_i$, $\bar{E}_{(A)i}$ are vectors, and $\bar{\bar{E}}_{ij}$ are tensors under $\mathrm{SO}(N)$ rotations. The perturbed metric components can then be written as

$$\delta g_{\alpha\beta} = \left(\begin{array}{c|c|ccccc|c} 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \hline 0 & 2k^2E - 2C & -2ik\bar{E}_2 & -2ik\bar{E}_3 & \dots & -2ik\bar{E}_N & ikE_{(A)} \\ \hline 0 & -2ik\bar{E}_2 & -2C + \sum_{i=3}^{N} h_i^+ & h_{23}^\times & \dots & h_{2N}^\times & \bar{E}_{(A)2} \\ 0 & ik\bar{E}_3 & h_{23}^\times & -2C - h_3^+ & & h_{3N}^\times & \bar{E}_{(A)3} \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & ik\bar{E}_N & h_{2N}^\times & h_{3N}^\times & \dots & -2C - h_N^+ & \bar{E}_{(A)N} \\ \hline 0 & ikE_{(A)} & \bar{E}_{(A)2} & \bar{E}_{(A)3} & \dots & \bar{E}_{(A)N} & E_{(AB)} \end{array}\right), \tag{5.5}$$

with the $h_k^+, h_{lm}^\times$ describing $\bar{\bar{E}}_{ij}$. Obviously, one has

- $2 + (n - N) + \frac{1}{2}(n - N)(n - N + 1)$ scalar degrees of freedom,

- $(N-1)(n-N+1)$ vector degrees of freedom and,
- $\frac{1}{2}(N-2)(N+1)$ tensor degrees of freedom.

By definition, the tensor components are spin 2 quantities and represent gravitational waves. It is clear that when $N \neq n$, not all the gravitational waves are tensor perturbations (with respect to SO(N)): $\frac{1}{2}(n-N)(n+N-1)$ of them are actually scalar or vector perturbations. In fact, the spin of the second decomposition can be understood as the projection of the spin of the first decomposition on the maximally symmetric space. Therefore, density modes are always scalar perturbations, vorticity modes can be either scalar of vector perturbations, and gravitational waves can be any of the three. By comparing Eqns (5.1) and (5.5), it is clear that:

- the $2+(n-N)+\frac{1}{2}(n-N)(n-N+1)$ scalars decompose as 2 density modes, $n-N$ vorticity modes, and $\frac{1}{2}(n-N)(n-N+1)$ gravitational waves,
- the $(N-1)(n-N+1)$ vectors represent $(N-1)$ vorticity modes and $(N-1)(n-N)$ gravitational waves,
- the $\frac{1}{2}(N-2)(N+1)$ tensors all represent gravitational waves.

For our purpose ($n = N+1 = 4$), this reduces to

- 4 scalar degrees of freedom which split into the 2 density modes, 1 vorticity mode, and 1 gravitational wave,
- 4 vector degrees of freedom which go into 2 vorticity modes and 2 gravitational waves,
- 2 tensor degrees of freedom which all represent gravitational waves.

As expected, we have 10 degrees of freedom 5 of which are gravitational waves. This decomposition ensures that even in the vacuum, the scalar and vector parts of the Einstein equation will allow non trivial solutions. These are usually called "graviscalars" and "graviphotons" [63, 64]. This effect represents the most striking change to the physics of brane cosmological perturbations as compared to that of the standard FLRW case since it can occur at arbitrary low energy as long as the corresponding gravitational waves exist in the bulk.

3. The brane point of view

The brane is, by definition, described by $N+1$ coordinates: one timelike and the N spacelike coordinates spanning an N-dimensional maximally symmetric space. For the case of one codimension, we have $N = n-1$. The perturbed induced metric of the maximally symmetric space then has $\frac{1}{2}n(n-1)$ independent components. An important question is how these perturbations can interact with the bulk perturbations. It is clear that whatever the bulk matter content, there are at least $\frac{1}{2}(n-2)(n+1) = \frac{1}{2}n(n-1)-1$ degrees of freedom which arise from the gravitational waves in the bulk. Therefore, one can expect that $\frac{1}{2}(n-2)(n+1)$ of the brane perturbations can interact with the bulk. We will see that this is indeed the case: the second Israel condition essentially states that the discontinuity of some bulk perturbations which can exist even in the vacuum describe the matter content of the brane. But this also suggests that one additional scalar degree of freedom of the brane is likely not to be directly related with the bulk perturbations. It happens, indeed, that this extra degree of freedom physically corresponds to the perturbation of the brane position in the bulk, which is independent of the gravitational waves. For example, if the bulk is pure Minkowski space, one can consider a fixed coordinate system (as, e.g., Newtonian gauge, which is unambiguously fixed). The position of the brane in this coordinate system is defined independently of the metric perturbations. This extra degree of freedom ensures that in any situation all the brane perturbations can be related to bulk perturbations (see also the discussion in Ref. [19]). One of the aims of this paper is to make the link between these two sets of perturbations.

B. Geometrical perturbation variables

We now make use of maximal symmetry on $\mathcal{M}$. Due to rotational invariance, we can split the perturbations into components which transform irreducibly under rotations, i.e., into different SO(3)-spin components, which evolve independently to first order perturbation theory. One could then go on and split these into irreducible components under translations, corresponding to the expansion in terms of eigenvectors of the Laplacian on $\mathcal{M}$ (which is the Fourier transform in the case k = 0) [61]. Following the discussion of the last paragraph, the perturbed line element can be generally written as

$$\mathrm{d}s^2 = n^2(1+2A)\mathrm{d}\eta^2 + 2anB_i\mathrm{d}\eta\mathrm{d}x^i - a^2(\gamma_{ij}+h_{ij})\mathrm{d}x^i\mathrm{d}x^j + 2nbB_\perp\mathrm{d}\eta\mathrm{d}y + 2baE_{\perp i}\mathrm{d}x^i\mathrm{d}y - b^2(1-2E_{\perp\perp})\mathrm{d}y^2. \quad (5.6)$$

Here, the $\perp$ indices of $E_{\perp i}$, $E_{\perp\perp}$ are labels. The quantities B_i and $E_{\perp i}$ are vectors on $\mathcal{M}$ which can be respectively decomposed into scalar (spin 0) components B, $E_\perp$, and divergenceless vector (spin 1) components $\bar{B}_i$, $\bar{E}_{(\perp)i}$, such that $\gamma^{ij}\nabla_i\bar{B}_j = \gamma^{ij}\nabla_i\bar{E}_{(\perp)j} = 0$. Equivalently, the tensor on $\mathcal{M}$, h_{ij}, can be decomposed into two scalars, C and E, a divergenceless vector, $\bar{E}_i$, and divergenceless, traceless, tensor (spin 2) component, $\bar{\bar{E}}_{ij}$, such that $\gamma^{ij}\nabla_i\bar{E}_j = \gamma^{ij}\nabla_i\bar{\bar{E}}_{ij} = \bar{\bar{E}}^i_i = 0$. This decomposition is

$$B_i = \nabla_i B + \bar{B}_i, \tag{5.7}$$
$$E_{\perp i} = \nabla_i E_\perp + \bar{E}_{(\perp)i}, \tag{5.8}$$
$$h_{ij} = 2C\gamma_{ij} + 2E_{ij}, \tag{5.9}$$
$$E_{ij} = \nabla_{(i}E_{j)} + \bar{\bar{E}}_{ij}, \tag{5.10}$$
$$E_i = \nabla_i E + \bar{E}_i. \tag{5.11}$$

The indices of these $\mathcal{M}$-quantities are raised and lowered with the metric γ_{ij}. The symmetries of the metric ensure that the scalar (A, B, C, E, $B_\perp$, $E_\perp$, $E_{\perp\perp}$), vector ($\bar{B}_i$, $\bar{E}_i$, $\bar{E}_{(\perp)i}$) and tensor ($\bar{\bar{E}}_{ij}$) quantities evolve independently.

C. Gauge invariant metric perturbations

Let us consider an infinitesimal coordinate transformation

$$x^\alpha \to x^\alpha + \xi^\alpha, \tag{5.12}$$

with

$$\xi^\alpha = (T, L^i, L^\perp), \tag{5.13}$$
$$L^i = \nabla^i L + \bar{L}^i. \tag{5.14}$$

Under this coordinate change the geometrical perturbations transform in the following way:

$$A \to A + \partial_u(nT) + IbL^\perp, \tag{5.15}$$
$$B_i \to B_i - \frac{a}{n}\dot{L}_i + \frac{n}{a}\nabla_i T, \tag{5.16}$$
$$C \to C + \mathcal{H}nT + HbL^\perp, \tag{5.17}$$
$$E_i \to E_i + L_i, \tag{5.18}$$
$$\bar{\bar{E}}_{ij} \to \bar{\bar{E}}_{ij}, \tag{5.19}$$
$$B_\perp \to B_\perp - \frac{b}{n}\dot{L}^\perp + \frac{n}{b}T', \tag{5.20}$$
$$E_{\perp i} \to E_{\perp i} - \frac{a}{b}L'_i - \frac{b}{a}\nabla_i L^\perp, \tag{5.21}$$
$$E_{\perp\perp} \to E_{\perp\perp} - \mathcal{U}nT - \partial_n(bL^\perp), \tag{5.22}$$
$$\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) \to \left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) + T, \tag{5.23}$$
$$\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right) \to \left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right) - L^\perp. \tag{5.24}$$

(Recall that $\dot{} \equiv \partial/\partial\eta$ and that $' \equiv \partial/\partial y$. In this section we will use both this notation and the $\partial_{u,n}$ defined in Eqns (2.3,2.4): we aim to do so in such a way as to keep the equations as simple as possible.) We can therefore define the following four scalar and two vector perturbation variables, which are invariant under infinitesimal coordinate transformations, also called gauge transformations in this context:

$$\Psi = A - \partial_u\left(aB + \frac{a^2}{n}\dot{E}\right) + I\left(aE_\perp + \frac{a^2}{b}E'\right), \tag{5.25}$$
$$\Phi = -C + \mathcal{H}\left(aB + \frac{a^2}{n}\dot{E}\right) - H\left(aE_\perp + \frac{a^2}{b}E'\right), \tag{5.26}$$

$$\Sigma = B_\perp - n\partial_n\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) - b\partial_u\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right), \tag{5.27}$$

$$h = E_{\perp\perp} + \mathcal{U}\left(aB + \frac{a^2}{n}\dot{E}\right) - \partial_n\left(aE_\perp + \frac{a^2}{b}E'\right), \tag{5.28}$$

$$\bar{\Sigma}_i = \bar{B}_i + \frac{a}{n}\dot{\bar{E}}_i, \tag{5.29}$$

$$\bar{h}_i = \bar{E}_{(\perp)i} + \frac{a}{b}\bar{E}'_i. \tag{5.30}$$

The two vector variables possess two independent components (hence four degrees of freedom). The tensor variable $\bar{\bar{E}}_{ij}$ is gauge invariant since there are no tensor type gauge transformations, and possesses two independent components. All these quantities represent a generalisation of the Newtonian gauge often used in FLRW cosmologies (we can no longer call it a "longitudinal gauge", as $\delta g_{04} \neq 0$). It is completely fixed by setting $\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right)$, E^i, $\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right)$ to 0 and in this case one has

$$\delta g_{00} = 2n^2\Psi, \tag{5.31}$$

$$\delta g_{0i} = an\bar{\Sigma}_i, \tag{5.32}$$

$$\delta g_{ij} = 2a^2(\Phi\gamma_{ij} - \bar{\bar{E}}_{ij}), \tag{5.33}$$

$$\delta g_{04} = nb\Sigma, \tag{5.34}$$

$$\delta g_{i4} = ba\bar{h}_i, \tag{5.35}$$

$$\delta g_{44} = 2b^2 h. \tag{5.36}$$

This gauge is perfectly well-suited for describing the bulk perturbation *without* a brane. In the presence of a brane, however, things are more complicated since some of these quantities involve first or second derivatives with respect to the fifth dimension, and hence they are not always regular at the brane position (see Section VI C). For this reason, other gauge choices are often preferred, but not essential[2].

D. Perturbed stress-energy tensor

We now perturb the unit vectors U^α and N^α, defined in Eqns (2.8–2.10), which are the timelike and spacelike eigenvectors normal to the maximally symmetric 3-spaces. It follows from the normalization conditions, Eqns (2.9,2.10), that each vector has only four independent components. Furthermore, as U^α and N^α are eigenvectors of a symmetric tensor, they are normal to each other, $N_\mu U^\mu = 0$. Hence at perturbed order there are only seven independent components which we denote by v_0^i, f_0^i, w. They are defined by

$$\delta U^\alpha = \left(\frac{1}{n}\gamma(\beta w - A - \beta B_\perp), \frac{1}{a}v_0^i, \frac{1}{b}\gamma(w + \beta E_{\perp\perp})\right), \tag{5.37}$$

$$\delta N^\alpha = \left(\frac{1}{n}\gamma(-w + \beta A + B_\perp), \frac{1}{a}f_0^i, -\frac{1}{b}\gamma(\beta w + E_{\perp\perp})\right). \tag{5.38}$$

Neglecting the metric perturbations, the quantity w represents the perturbation of the Lorentz boost β,

$$w = \frac{\delta\gamma}{\beta\gamma} = \frac{\delta(\beta\gamma)}{\gamma} = \gamma^2\delta\beta. \tag{5.39}$$

As usual, we will decompose v_0^i, f_0^i, into scalar and vector components,

$$v_i^0 = \nabla_i v_0 + \bar{v}_i^0, \tag{5.40}$$

$$f_i^0 = \nabla_i f_0 + \bar{f}_i^0. \tag{5.41}$$

---

[2] In any case, there is no particular reason why the brane and bulk metric perturbations should be the same as the brane perturbation depends explicitly on the brane position, which is not a quantity that can be defined everywhere in the bulk.

Finally, in order to write down the stress-energy tensor, it is also useful to introduce the variables

$$v^i = \gamma(v_0^i + \beta f_0^i), \tag{5.42}$$
$$f^i = \gamma(f_0^i + \beta v_0^i), \tag{5.43}$$

which have a decomposition into scalar and vector components similar to (5.40,5.41). With these definitions, general perturbations of the bulk stress-energy tensor, $\delta T_{\alpha\beta}$, are

$$\delta T_{00} = n^2\left(\delta\rho + 2\rho A\right), \tag{5.44}$$
$$\delta T_{0i} = -an\left((\rho+P)v_i - \rho B_i - F(f_i + E_{\perp i})\right), \tag{5.45}$$
$$\delta T_{04} = -nb\left(\delta F + F(A - E_{\perp\perp}) - \rho B_\perp\right), \tag{5.46}$$
$$\delta T_{ij} = a^2\left(\delta P\gamma_{ij} + \Pi_{ij} + 2P(C\gamma_{ij} + E_{ij})\right), \tag{5.47}$$
$$\delta T_{i4} = ba\left((P-Y)f_i + F(v_i - B_i) - YE_{\perp i}\right), \tag{5.48}$$
$$\delta T_{44} = b^2\left(\delta Y - 2YE_{\perp\perp} - 2FB_\perp\right). \tag{5.49}$$

Here we have defined, according to (2.18–2.21,5.39):

$$\delta\rho = \gamma^2(\delta\rho_0 + \beta^2\delta Y_0) + 2Fw, \tag{5.50}$$
$$\delta Y = \gamma^2(\delta Y_0 + \beta^2\delta\rho_0) + 2Fw, \tag{5.51}$$
$$\delta P = \delta P_0, \tag{5.52}$$
$$\delta F = \beta\gamma^2(\delta\rho_0 + \delta Y_0) + (\rho+Y)w, \tag{5.53}$$

and we have introduced the anisotropic stress tensor Π_{ij}, which again may be decomposed into a scalar, (divergenceless) vector, and (divergenceless, traceless) tensor components according to

$$\Pi_{ij} = \left(\nabla_{ij} - \frac{1}{3}\nabla^2\gamma_{ij}\right)\Pi + \nabla_{(i}\bar{\Pi}_{j)} + \bar{\bar{\Pi}}_{ij}. \tag{5.54}$$

On investigation of the behaviour of these variables under the infinitesimal coordinate transformations (5.12) (see Appendix G), we find the following scalar gauge invariant variables

$$v^\sharp = v + \frac{a}{n}\dot{E}, \tag{5.55}$$
$$\bar{v}_i^\sharp = \bar{v}_i + \frac{a}{n}\dot{\bar{E}}_i, \tag{5.56}$$
$$f^\sharp = f - \frac{a}{b}E', \tag{5.57}$$
$$\bar{f}_i^\sharp = \bar{f}_i - \frac{a}{b}\bar{E}_i', \tag{5.58}$$
$$w^\sharp = w - \frac{\dot{\gamma}}{\beta\gamma}\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) + \frac{\gamma'}{\beta\gamma}\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right) - \frac{b}{n}\partial_\eta\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right), \tag{5.59}$$
$$\delta X_0^\sharp = \delta X_0 - \dot{X}_0\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) + X_0'\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right), \tag{5.60}$$

where X_0 is any scalar quantity (density ρ_0, pressure P_0, etc). The anisotropic stress tensor Π_{ij} is gauge invariant by itself due to the Stewart-Walker lemma [62]. Notice that ρ, Y, F and w are not scalars (since they depend explicitly on the choice of the coordinate system via the vector fields u^α and n^α), but we can, however, define the following gauge invariant variables,

$$\delta\rho^\sharp = \gamma^2(\delta\rho_0^\sharp + \beta^2\delta Y_0^\sharp) + 2Fw^\sharp, \tag{5.61}$$
$$\delta Y^\sharp = \gamma^2(\delta Y_0^\sharp + \beta^2\delta\rho_0^\sharp) + 2Fw^\sharp, \tag{5.62}$$
$$\delta P^\sharp = \delta P_0^\sharp, \tag{5.63}$$
$$\delta F^\sharp = \beta\gamma^2(\delta\rho_0^\sharp + \delta Y_0^\sharp) + (\rho+Y)w^\sharp. \tag{5.64}$$

As an example, the perturbed stress-energy tensor for a scalar field is given in Appendix G5.

E. The perturbed Einstein equations

The explicit forms of the perturbed Christoffel symbols, the perturbed Riemann, Ricci and Einstein tensors are all given in Appendices F4, F5, F8, F7, where they are expressed in terms of the gauge invariant variables introduced above. We now write down the full perturbed bulk Einstein equations also in terms of gauge invariant variables. They split into seven scalar, three vector (each with two independent components), and one tensor (with two independent components) equations, adding up to the required 15 components of a symmetric 5×5 tensor. These equations are given below, where we indicate on the right hand side of each equation from which component of the Einstein equations they were derived and, when necessary, the term to which they are proportional. The seven scalar equations are

$$
\begin{aligned}
&\Delta(2\Phi + h) + 6K\Phi \\
&-3\left(2\mathcal{H}^2 + 2\mathcal{H}\mathcal{U}\right)\Psi - 3\mathcal{H}\partial_u h - 3\left(2\mathcal{H} + \mathcal{U}\right)\partial_u\Phi \\
&-3(H\partial_n + 4H^2)h - 6h\partial_n H \\
&+3(\partial_n + 4H)\partial_n\Phi \\
&+3\left(\partial_n + 3H + I\right)(\mathcal{H}\Sigma) \;=\; \kappa_5\left(\delta\rho^\sharp - F\Sigma\right) \qquad \{00\}\,, \qquad (5.65)\\
&\frac{1}{2}\left(\partial_n + H + 2I\right)\Sigma \\
&-(\mathcal{U} + 2\mathcal{H})\Psi - (\partial_u + \mathcal{U} - \mathcal{H})h - 2\partial_u\Phi \;=\; \kappa_5\left((P+\rho)av^\sharp - Faf^\sharp\right) \qquad \{0i\}\,, \qquad (5.66)\\
&-2K\Phi \\
&+2\left((\mathcal{U} + 2\mathcal{H})(\partial_u + \mathcal{U}) + 3\mathcal{H}^2\right)\Psi + 2\Psi\partial_u(\mathcal{U} + 2\mathcal{H}) \\
&+2\left((I + 2H)(\partial_n + I) + 3H^2\right)h + 2h\partial_n(I + 2H) \\
&+(\partial_u + \mathcal{U})\partial_u(h + 2\Phi) \\
&-(\mathcal{U} + 2\mathcal{H})\partial_u(\Psi - h - 3\Phi) \\
&+(\partial_n + I)\partial_n(\Psi - 2\Phi) \\
&+(I + 2H)\partial_n(\Psi - h - 3\Phi) \\
&-\tfrac{1}{2}(\partial_n\partial_u + \partial_u\partial_n + I\partial_u + \mathcal{U}\partial_n)\Sigma \\
&-\left((\mathcal{U} + 2\mathcal{H})\partial_n + (I + 2H)\partial_u\right)\Sigma \\
&-\Sigma\left((\partial_n + I)(\mathcal{U} + 2\mathcal{H}) + (\partial_u + \mathcal{U})(I + 2H)\right) \\
&-3\left(\mathcal{U}\partial_u - I\partial_n\right)\Phi - 6\mathcal{H}H\Sigma \;=\; \kappa_5\left(\delta P^\sharp + \tfrac{2}{3}\Delta\Pi\right) \qquad \{ij\}\propto\gamma_{ij}\,, \qquad (5.67)\\
&\Phi - \Psi + h \;=\; \kappa_5 a^2\Pi \qquad \{ij\}\propto\nabla_{ij}\,, \qquad (5.68)\\
&-3\left((\partial_u\partial_n + (H - I)\partial_u + \mathcal{H}\partial_n)\Phi + \mathcal{H}\partial_n\Psi - H\partial_u h\right) \\
&-3\Sigma\partial_u\mathcal{H} - \left(\frac{1}{2}\Delta + 3(\mathcal{H}^2 - \mathcal{H}\mathcal{U})\right)\Sigma \;=\; \kappa_5\left(\delta F^\sharp + F(\Psi - h)\right) \qquad \{04\}\,, \qquad (5.69)\\
&\frac{1}{2}\left(\partial_u + \mathcal{H} + 2\mathcal{U}\right)\Sigma \\
&-(\partial_n + I - H)\Psi - (2H + I)h + 2\partial_n\Phi \;=\; \kappa_5\left(Fav^\sharp + (P - Y)af^\sharp\right) \qquad \{i4\}\,, \qquad (5.70)\\
&-\Delta(2\Phi - \Psi) - 6K\Phi \\
&+3\mathcal{H}\left(\partial_u + 4\mathcal{H}\right)\Psi + 6\Psi\partial_u\mathcal{H} \\
&+3\left(\partial_u + 4\mathcal{H}\right)\partial_u\Phi \\
&+3\left(H\partial_n\right)\Psi + 3\left(2H^2 + 2HI\right)h - 3\left(2H\partial_n + I\partial_n\right)\Phi \\
&-3\left(\partial_u + 3\mathcal{H} + \mathcal{U}\right)(H\Sigma) \;=\; \kappa_5\left(\delta Y^\sharp - F\Sigma\right) \qquad \{44\}\,. \qquad (5.71)
\end{aligned}
$$

The three vector equations are:

$$
\begin{aligned}
&-\frac{1}{2}(\Delta + 2K)\bar{\Sigma}_i \\
&-\frac{1}{2}\left(\partial_n + 4H\right)\left((\partial_n + I - H)\bar{\Sigma}_i\right) \\
&+\frac{1}{2}\left(\partial_n + 4H\right)\left((\partial_u + \mathcal{U} - \mathcal{H})\bar{h}_i\right) \;=\; \kappa_5\left((P+\rho)(\bar{v}_i^\sharp - \bar{\Sigma}_i) - F(\bar{f}_i^\sharp + \bar{h}_i)\right) \qquad \{0i\}\,, \qquad (5.72)
\end{aligned}
$$

$$
\begin{aligned}
(\partial_u + 2\mathcal{H} + \mathcal{U})\,\bar{\Sigma}_i - (\partial_n + 2H + I)\,\bar{h}_i &= \kappa_5 a \bar{\Pi}_i && \{ij\}\,, \qquad (5.73)\\
&\quad \frac{1}{2}(\Delta + 2K)\bar{h}_i \\
+\frac{1}{2}\,(\partial_u + 4\mathcal{H})\,\big((\partial_n + I - H)\bar{\Sigma}_i\big) & \\
-\frac{1}{2}\,(\partial_u + 4\mathcal{H})\,\big((\partial_u + \mathcal{U} - \mathcal{H})\bar{h}_i\big) &= \kappa_5 \Big(F(\bar{v}_i^\sharp - \bar{\Sigma}_i) + (P-Y)(\bar{f}_i^\sharp + \bar{h}_i)\Big) && \{i4\}\,, \qquad (5.74)
\end{aligned}
$$

and the tensor equation is

$$
-(\Delta - 2K)\bar{\bar{E}}_{ij} + (\partial_u + 3\mathcal{H} + \mathcal{U})\,\partial_u \bar{\bar{E}}_{ij} - (\partial_n + 3H + I)\,\partial_n \bar{\bar{E}}_{ij} = \kappa_5 \bar{\bar{\Pi}}_{ij} \qquad \{ij\}\,. \qquad (5.75)
$$

As a small aside, it is interesting to check our analysis of Section V A. We shall take for simplicity an empty, Minkowski bulk, so that the terms proportional to K, $\mathcal{H}$, $\mathcal{U}$, H and I vanish. Then the above equations reduce to

$$
\begin{aligned}
\Delta(h + 2\Phi) &= -3\partial_n^2 \Phi, && (5.76)\\
\Delta(\Psi - 2\Phi) &= -3\partial_u^2 \Phi, && (5.77)\\
\Delta\Sigma &= -6\partial_u \partial_n \Phi, && (5.78)\\
\left(\partial_u^2 - \partial_n^2 - \Delta\right)\Phi &= 0, && (5.79)\\
\partial_n \bar{h}_i &= \partial_u \bar{\Sigma}_i, && (5.80)\\
\left(\partial_u^2 - \partial_n^2 - \Delta\right)\bar{\Sigma}_i &= 0, && (5.81)\\
\left(\partial_u^2 - \partial_n^2 - \Delta\right)\bar{\bar{E}}_{ij} &= 0. && (5.82)
\end{aligned}
$$

In the vacuum, in addition to the usual two tensor modes, there are one scalar and two vector degrees of freedom which satisfy wave equations and represent the graviscalar and graviphoton (for a total of five gravitons, as expected). The remaining degrees of freedom can only exist if matter is present. They describe either density or vorticity modes.

F. Perturbed conservation equations

We now compute the perturbed energy momentum conservation equations. Even though they do not contain new information, they can provide useful evolution equations for the matter content of the bulk. Here we write them down just for the total bulk matter. The generalisation to several components is straightforward and is given in Appendix I3. Written in terms of gauge invariant variables there are three scalar conservation equations,

$$
\begin{aligned}
(\partial_u + 3\mathcal{H} + 2\mathcal{U})(\delta\rho^\sharp - F\Sigma) + 3\mathcal{H}\delta P^\sharp + \mathcal{U}(\delta Y^\sharp - \delta\rho^\sharp) & \\
+(\partial_n + 3H + 2I)\left(\delta F^\sharp + F(\Psi + h)\right) & \\
+\Delta\left((P+\rho)av^\sharp - Faf^\sharp\right) & \\
-3(P+\rho)\partial_u\Phi - (\rho + Y)\partial_u h - F\partial_u\Sigma + F\partial_n(\Psi - h - 3\Phi) &= 0 \qquad \{0\}\,, \qquad (5.83)\\
(\partial_u + 3\mathcal{H} + \mathcal{U})\left((P+\rho)av^\sharp - Faf^\sharp\right) & \\
+(\partial_n + 3H + I)\left(Fav^\sharp + (P-Y)af^\sharp\right) & \\
+\delta P^\sharp + \frac{2}{3}(\Delta + 3K)a^2\Pi + (P+\rho)\Psi + (Y-P)h + F\Sigma &= 0 \qquad \{i\}\,, \qquad (5.84)\\
(\partial_u + 3\mathcal{H} + 2\mathcal{U})\left(\delta F^\sharp - F(\Psi + h) - (\rho + Y)\Sigma\right) & \\
+(\partial_n + 3H + 2I)(\delta Y^\sharp - F\Sigma) - 3H\delta P^\sharp - I(\delta Y^\sharp - \delta\rho^\sharp) & \\
+\Delta\left(Fav^\sharp + (P-Y)af^\sharp\right) & \\
+F\partial_u(\Psi - h - 3\Phi) + 3(P-Y)\partial_n\Phi + (\rho + Y)\partial_n\Psi + F\partial_n\Sigma &= 0 \qquad \{4\}\,, \qquad (5.85)
\end{aligned}
$$

and one vector conservation equation,

$$
\begin{aligned}
(\partial_u + 4\mathcal{H} + \mathcal{U})\left((P+\rho)(\bar{v}_i^\sharp - \bar{\Sigma}_i) - F(\bar{f}_i^\sharp + \bar{h}_i)\right) & \\
+(\partial_n + 4H + I)\left(F(\bar{v}_i^\sharp - \bar{\Sigma}_i) + (P-Y)(\bar{f}_i^\sharp + \bar{h}_i)\right) & \\
+\frac{1}{2}\,(\Delta + 2K)\,a\bar{\Pi}_i &= 0 \qquad \{i\}\,. \qquad (5.86)
\end{aligned}
$$

Finally, in order to close the system, we must specify an equation of state for $\delta P^\sharp$, $\delta Y^\sharp$, $f_i^\sharp$ and Π_{ij}, as functions either of $\delta\rho^\sharp$ or of some other non dynamical variables (such as the entropy). For example, all these quantities vanish for a bulk containing non relativistic matter. For a scalar field, most of them are also set to zero, as is discussed in Appendix G5. We can interpret the three scalar equations (5.83–5.85) as the conservation equations for $\delta\rho^\sharp$, $v^\sharp$, and $\delta F^\sharp$.

Notice that with these Bianchi or conservation equations, three scalar and one vector Einstein equations are redundant and can be dropped. Formally, the seven scalar Einstein equations can be split into four dynamical equations for the four scalar metric perturbations Φ, Ψ, Σ and h, and three constraint equations. It happens, however, that with our choice of variables, the splitting is not completely straightforward. For example, the $\{0\alpha\}$ Einstein equations are the constraint equations for the metric components g_{ij}, g_{i4}, g_{44}, the rest being the evolution equations. However, in terms of our gauge invariant variables, Eq. (5.68) is obviously a constraint equation. This is because Ψ involves first and second time derivatives of the metric perturbation E. Equivalently, the three vector Einstein equations can be split into one constraint equation and two dynamical equations for the variables $\bar\Sigma_i$ and $\bar h_i$.

VI. BULK PERTURBATION WITH A BRANE

In the previous section we have considered the most general perturbed 5-dimensional bulk spacetime for which there is a perturbed maximally symmetric space orthogonal to the fifth direction. We have seen that its dynamics can be described in terms of four geometrical scalar perturbation variables governed by four evolution equations and three constraints, two geometrical vector perturbation variables governed by two evolution equations and a constraint, and one geometrical tensor variable governed by one tensor evolution equation. In this section we add a brane to this system — that is we assume, as was the case for the background, that the bulk contains a perturbed homogeneous and isotropic brane as a singular source. This will introduce one new geometrical degree of freedom, the brane displacement, whose dynamics has to be considered in order to fully describe the perturbations on the brane.

A. Brane position and its displacement

The perturbed brane embedding is given by

$$X^0 = \sigma^0 + \zeta^0(\sigma^a), \tag{6.1}$$
$$X^i = \sigma^i + \zeta^i(\sigma^a), \tag{6.2}$$
$$X^4 = \mathrm{y_b} + \epsilon(\sigma^a). \tag{6.3}$$

Here ϵ is the displacement of the brane from its background position $X^4 = \mathrm{y_b}$, and it is a function on the brane worldsheet. It is a true new degree of freedom which sometimes also called "radion" [19, 59]. On the contrary, as we will soon see, the perturbations in the X^0 and X^i directions can be set to zero without loss of generality, as they do not lead to any physical consequences (e.g., to a physical "deformation" of the brane) [19] .

It was first noticed in [22] that, when studying brane perturbations of a Randall-Sundrum background (of Ref. [11]), using the transverse and traceless gauge in Gaussian normal coordinates, the brane position is no more at constant y in presence of matter sources. The presence of an $\epsilon \neq 0$ can be interpreted as a bending of the brane due to the presence of matter or gravitational waves. The bending ϵ can in principle be set to zero by choosing a convenient set of bulk coordinates such that $y = \mathrm{y_b}$, since by the infinitesimal coordinate transformation defined in the previous section, Eq. (5.13), one has

$$\epsilon \to \epsilon - L^\perp. \tag{6.4}$$

This is not, however, the most general possibility. As was noted in [36, 53], such a gauge choice will also fix the gauge of some other perturbation variables. In fact, if we choose $L^\perp$ such that ϵ is zero, $\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right)$ is fixed (see Eq. (5.24)). In a gauge invariant approach one must keep ϵ arbitrary.

We now can define the perturbed vector orthogonal to the brane, $\underline{\perp}^\alpha + \underline{\delta\perp}^\alpha$. One easily obtains (See Appendix H2)

$$\underline{\delta\perp}_\alpha = (-b\dot\epsilon, -b\nabla_i\epsilon, -bE_{\perp\perp}). \tag{6.5}$$

Note that no vector perturbations enter in the above formula. This is just a consequence of Frobenius theorem [66]. Also, the fact that the perturbations ζ^a do not enter in the above expression illustrates that they do not correspond to any physical deformation of the brane (see further comments below).

B. Induced metric and first fundamental form

In this subsection we calculate the perturbed first fundamental form which will be used in the perturbed first Israel junction condition in the following subsections.

We shall first look at the perturbation of the induced metric $\underline{\chi}_{ab}(\sigma^a)$. In doing so, it is important to recall that the brane embedding in the unperturbed and perturbed bulks (respectively given by (3.1–3.3), (6.1–6.3)) are different. Therefore, each brane variable has two contribution to its perturbation: one coming from the perturbation of the variable and a second contribution due to the fact that we have to evaluate it at the perturbed brane position. We obtain

$$\underline{\delta\chi}_{00}(\sigma^a) = 2n^2\left(A + \dot{\zeta}^0 + \mathcal{I}n\zeta^0 + Ib\epsilon\right), \tag{6.6}$$

$$\underline{\delta\chi}_{0i}(\sigma^a) = an\left(B_i - \frac{a}{n}\dot{\zeta}_i + \frac{n}{a}\nabla_i\zeta^0\right), \tag{6.7}$$

$$\underline{\delta\chi}_{ij}(\sigma^a) = -2a^2\left(\left(C + \mathcal{H}n\zeta^0 + Hb\epsilon\right)\gamma_{ij} + E_{ij} + \nabla_{(i}\zeta_{j)}\right), \tag{6.8}$$

where, to first order in perturbation theory, the right hand side of these equations are evaluated at $x^\alpha = X^\alpha$ (corresponding to the unperturbed embedding). As usual, we can decompose the perturbation ζ^i into its scalar and vector parts using the metric γ_{ij} evaluated at the unperturbed brane position: $\zeta_i = \nabla_i\zeta + \bar{\zeta}_i$. This is possible since, for perturbations, the time derivative ∂_η and spatial derivatives ∂_i are equivalent to derivation with respect to σ^0 and σ^i respectively.

A word of caution is in order here. Recall that the quantities A, C, I, H, etc., are defined in the bulk. However, due to the presence of the brane, they may be (and in fact, are — see below) discontinuous at the brane position. For this reason, they may only be evaluated on each side of the brane. Therefore we must check that the above expressions are consistent in the sense that they have the same value when evaluated *on both sides* of the brane — only if that is the case, they can be considered well defined *on* the brane. As we shall soon see, this consistency in fact results from the first Israel condition. Anticipating this result, the above equations allow us to define in the standard way the brane perturbations $\underline{A}$, $\underline{B}_i$, etc.,

$$\underline{A} \equiv A + \dot{\zeta}^0 + \mathcal{I}n\zeta^0 + Ib\epsilon, \tag{6.9}$$

$$\underline{B} \equiv B - \frac{a}{n}\dot{\zeta} + \frac{n}{a}\zeta^0, \tag{6.10}$$

$$\underline{\bar{B}}_i \equiv \bar{B}_i - \frac{a}{n}\dot{\bar{\zeta}}_i, \tag{6.11}$$

$$\underline{C} \equiv C + \mathcal{H}n\zeta^0 + Hb\epsilon, \tag{6.12}$$

$$\underline{E} \equiv E + \zeta, \tag{6.13}$$

$$\underline{\bar{E}}_i \equiv \bar{E}_i + \bar{\zeta}_i, \tag{6.14}$$

$$\underline{\bar{\bar{E}}}_{ij} \equiv \bar{\bar{E}}_{ij}. \tag{6.15}$$

Using the standard 4-dimensional perturbation theory, we construct the two Bardeen potentials, as well as the brane vector and tensor metric perturbations,

$$\underline{\Psi} \equiv \underline{A} - \partial_u\left(a\underline{B} + \frac{a^2}{n}\dot{\underline{E}}\right) = \Psi + I\left(b\epsilon - \left(aE_\perp + \frac{a^2}{b}E'\right)\right), \tag{6.16}$$

$$\underline{\Phi} \equiv -\underline{C} + \mathcal{H}\left(a\underline{B} + \frac{a^2}{n}\dot{\underline{E}}\right) = \Phi - H\left(b\epsilon - \left(aE_\perp + \frac{a^2}{b}E'\right)\right), \tag{6.17}$$

$$\underline{\bar{\Sigma}}_i \equiv \underline{\bar{B}}_i + \frac{a}{n}\dot{\underline{\bar{E}}}_i = \bar{\Sigma}_i, \tag{6.18}$$

$$\underline{\bar{\bar{E}}}_{ij} \equiv \bar{\bar{E}}_{ij}. \tag{6.19}$$

(The two first equations are equivalent to Eqns (5.21) of Ref. [45].) Finally, using Eq. (3.14), the perturbed first fundamental form is

$$\underline{\delta q}_{00} = 2n^2(A + Ib\epsilon), \tag{6.20}$$

$$\underline{\delta q}_{0i} = anB_i, \tag{6.21}$$

$$\underline{\delta q}_{ij} = -2a^2(C + Hb\epsilon)\gamma_{ij} - 2a^2E_{ij}, \tag{6.22}$$

$$\underline{\delta q}_{04} = -nb\left(B_\perp - \frac{b}{n}\dot{\epsilon}\right), \tag{6.23}$$

$$\underline{\delta q}_{i4} = ba\left(E_{\perp\, i} - \frac{b}{a}\nabla_i\epsilon\right), \tag{6.24}$$

$$\underline{\delta q}_{44} = 0. \tag{6.25}$$

Notice that the ζ^i do not appear in equations (6.16–6.25): this is again related to the fact that they do not represent physical degrees of freedom. These above expressions can also be obtained by starting from the definition

$$\underline{g}_{\alpha\beta} = g_{\alpha\beta} + \underline{\perp}_\alpha \underline{\perp}_\beta, \tag{6.26}$$

paying attention that in the perturbed and unperturbed cases, the bulk metric is not evaluated at the same position ($y = \mathrm{y_b} + \epsilon$ and $y = \mathrm{y_b}$ respectively) [19].

C. First Israel condition for the standard 4-dimensional perturbation variables

Using Eqns (5.15–5.24,6.4), the coordinate transformations of the following variables are obviously continuous as they do not involve derivatives with respect to y:

$$A + Ib\epsilon \to A + Ib\epsilon + \partial_u(nT), \tag{6.27}$$

$$B_i \to B_i - \frac{a}{n}\dot{L}_i + \frac{n}{a}\nabla_i T, \tag{6.28}$$

$$C + Hb\epsilon \to C + Hb\epsilon + \mathcal{H}nT, \tag{6.29}$$

$$E_i \to E_i + L_i, \tag{6.30}$$

$$\bar{\bar{E}}_{ij} \to \bar{\bar{E}}_{ij}, \tag{6.31}$$

$$\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) \to \left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) + T. \tag{6.32}$$

Given the first fundamental form (6.20–6.25), the first Israel conditions imply that these quantities, which are linear combinations of the components (6.20) to (6.23) of the perturbed first fundamental form $\underline{\delta g}_{\alpha\beta}$, are continuous,

$$[A] + [Ib]\,\epsilon = 0, \tag{6.33}$$

$$[B_i] = 0, \tag{6.34}$$

$$[C] + [Hb]\,\epsilon = 0, \tag{6.35}$$

$$[E_i] = 0, \tag{6.36}$$

$$\left[\bar{\bar{E}}_{ij}\right] = 0, \tag{6.37}$$

$$\left[\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right)\right] = 0, \tag{6.38}$$

or, equivalently

$$[\Psi] = -\left[Ib\epsilon - I\left(aE_\perp + \frac{a^2}{b}E'\right)\right], \tag{6.39}$$

$$[\Phi] = \left[Hb\epsilon - H\left(aE_\perp + \frac{a^2}{b}E'\right)\right], \tag{6.40}$$

$$\left[\bar{\Sigma}_i\right] = 0, \tag{6.41}$$

$$\left[\bar{\bar{E}}_{ij}\right] = 0. \tag{6.42}$$

Hence tensor perturbations are continuous and only the two scalar quantities Σ, h and the vector quantity $\bar{h}_i$ may jump. The first two equations are equivalent to

$$[\underline{\Phi}] = [\underline{\Psi}] = 0, \tag{6.43}$$

which ensures that the brane Bardeen potentials $\underline{\Phi}$ and $\underline{\Psi}$ are well defined. We also see that the bulk vector and tensor perturbation $\bar{\Sigma}_i$ and $\bar{\bar{E}}_{ij}$ may be defined on the brane where they reduce to the standard vector and tensor metric perturbations of a 4-dimensional spacetime with maximally symmetric spacelike hypersurfaces. Equivalently, the corresponding first fundamental form can be rewritten as

$$\underline{\delta q}_{00} = 2n^2\underline{\Psi} + (\dot{\underline{q}}_{00} + 2\underline{q}_{00}\partial_\eta)\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right), \tag{6.44}$$

$$\underline{\delta q}_{0i} = -\frac{n}{a}\underline{q}_{ij}\bar{\Sigma}^j + \underline{q}_{00}\nabla_i\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) + \underline{q}_{ij}\dot{E}^j, \tag{6.45}$$

$$\underline{\delta q}_{ij} = 2\underline{q}_{k(i}\left(E^k_{j)} - \delta^k_{j)}\underline{\Phi}\right), \tag{6.46}$$

which indeed reduce to $2n^2\underline{\Psi}$, $-\frac{n}{a}\underline{q}_{ij}\bar{\Sigma}^j$ and $2a^2(\underline{\Phi}\gamma_{ij} - \bar{\bar{E}}_{ij})$, in longitudinal gauge ($B = E = 0$) for scalar perturbations and in the gauge $\bar{E}_i = 0$ for vector perturbations.

D. Regularity conditions for coordinate transformations and non-standard 4-dimensional perturbation variables

So far we have given the relationship between the intrinsic brane metric perturbations, the brane displacement ϵ, and some of the bulk metric perturbations. Things become a little bit more involved when we consider the other bulk metric perturbations that appear in $\delta g_{\alpha 4}$.

As we already noticed, the first Israel condition does not constrain $E_{\perp\perp}$ (see Eq. (6.25)). Nevertheless, since the transformation law for $E_{\perp\perp}$ can be written as

$$E_{\perp\perp} \to E_{\perp\perp} - \mathcal{U}nT - \partial_n(bL^\perp), \tag{6.47}$$

and since all the metric coefficients must remain finite, it follows that $\left[bL^\perp\right] = 0$ (see also Appendix F) . Furthermore, as the coordinate transformation $x^\alpha \to x^\alpha + \xi^\alpha$ must be invertible, we also require $\left[L^\perp\right] = 0$. Thus if b is not continuous, then $L^\perp(\mathrm{y_b}) = 0$:

$$[b] \neq 0 \qquad \Rightarrow \qquad L^\perp(y = \mathrm{y_b}) = 0. \tag{6.48}$$

This is the only additional requirement that one must impose on the coordinate system in the vicinity of the brane. Note that if Z_2 symmetry is assumed, $L^\perp(y) = 0$ must be imposed even though b is continuous [36].

As mentioned in Section III C, the only place where discontinuities or singularities are allowed is the brane position. When $\epsilon = 0$, the brane is at $y = \mathrm{y_b}$. However, if the brane position is perturbed, $\epsilon \neq 0$, the above requirement implies

$$[b]\,\epsilon = 0, \tag{6.49}$$

and hence b is not allowed to jump if the brane position is perturbed. If the unperturbed metric has a zeroth order discontinuity in the coefficient b, the brane position must remain at $\epsilon = 0$ to first order perturbation theory (see Figure 4). This statement is in fact valid even in the absence of metric perturbations (see Appendix F).

Let us now consider the coordinate transformations of the variables $\underline{\delta q}_{04}$ and $\underline{\delta q}_{i4}$ given in Eqns (6.23,6.24),

$$bB_\perp - \frac{b^2}{n}\dot{\epsilon} \;\to\; bB_\perp - \frac{b^2}{n}\dot{\epsilon} + nT', \tag{6.50}$$

$$bE_{\perp i} - \frac{b^2}{a}\nabla_i\epsilon \;\to\; bE_{\perp i} - \frac{b^2}{a}\nabla_i\epsilon - aL_i'. \tag{6.51}$$

The first Israel condition also states that

$$[bB_\perp] - \left[\frac{b^2}{n}\right]\dot{\epsilon} = [bB_\perp] = 0, \tag{6.52}$$

$$[bE_{\perp i}] - \left[\frac{b^2}{a}\right]\nabla_i\epsilon = [bE_{\perp i}] = 0. \tag{6.53}$$

Therefore, in order for the transformations (6.50,6.51) to be continuous we have to imply that a valid coordinate change satisfies

$$[T'] = 0, \tag{6.54}$$

$$\left[L^{i\prime}\right] = 0. \tag{6.55}$$

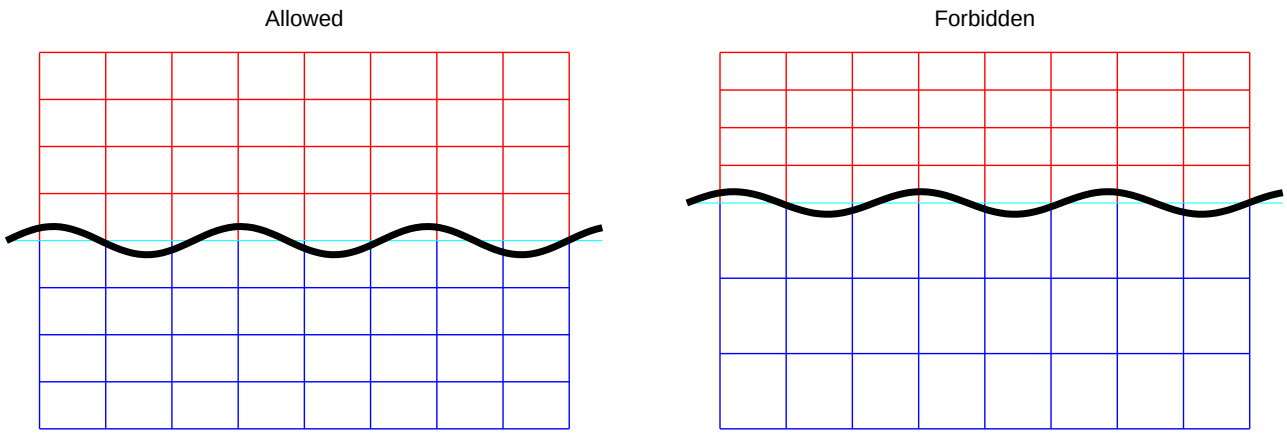

FIG. 4: Illustration of the constraints on the brane position ϵ implied by the unperturbed "scale factor" b. When $[b] = 0$ at zeroth order, then the brane position ϵ can be non zero at first order (left panel). On the contrary, if $[b] \neq 0$ at zeroth order, then the brane position must be zero at first order, in order the perturbation theory to be valid, so that right panel represents a forbidden situation. This is due to the fact that in the situation of the right panel, an infinitesimal, first order coordinate change could change by a large (zeroth order) amount some perturbed metric coefficients in the vicinity of the brane. (Here we have represented a situation corresponding to an unperturbed bulk, but of course it also holds when it is perturbed. In fact the perturbation of the brane position necessarily induces some metric perturbations in the bulk.)

These conditions ensure that T and L^i admit second derivatives. This standard requirement for any valid coordinate transformation is therefore preserved even in the presence of a brane. In particular (see Eqns 5.23,5.18)), this means that *if* $\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right)'$ or $E^{i\prime}$ are discontinuous (which is allowed), then $\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right)$ and E^i *cannot* be transformed identically to zero by coordinate changes. This does not prevent the quantities Φ, Ψ, Σ, h from being well defined. However, it does mean that there may not be a coordinate system in which Eqns (5.31–5.36) are valid (which we never needed to suppose).

The first Israel condition does not require the continuity of Σ and $\bar{h}_i$, see Eqns (6.89,6.97) below. Finally, from

$$\left(aE_\perp + \frac{a^2}{b}E'\right) \to \left(aE_\perp + \frac{a^2}{b}E'\right) - bL^\perp, \tag{6.56}$$

and using the fact that $\left[bL^\perp\right] = 0$, the jump

$$\left[\left(aE_\perp + \frac{a^2}{b}E'\right)\right] \equiv \underline{\Xi} \tag{6.57}$$

is gauge invariant. Therefore the gauge invariant quantity h given in Eqn (5.28) may contain a singular part,

$$h = E_{\perp\perp} + \mathcal{U}\left(aB + \frac{a^2}{n}\dot{E}\right) - \partial_n\left\langle\left(aE_\perp + \frac{a^2}{b}E'\right)\right\rangle - \mathrm{D}\underline{\Xi}, \tag{6.58}$$

which again shows that h cannot always be a component of the perturbed metric tensor in the vicinity of the brane. Of course, this does not invalidate the results found previously, but simply suggests that other variables may be more suitable to describe the metric perturbations in the vicinity of the brane.

After these remarks on the regularity requirements of gauge transformations in a bulk-brane system, we can now define some further gauge invariant scalar variables in terms of which we will express the second Israel condition. They will also be used to write the Einstein equations for an observer on the brane when we want to compare our results with the usual 4-dimensional cosmological perturbation theory.

Let us first define the gauge invariant combination

$$\epsilon^\sharp \equiv \epsilon - \left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right). \tag{6.59}$$

Since $\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right)$ can be discontinuous, $\epsilon^\sharp$ is defined on each side of the brane. Note that we have

$$\left[b\epsilon^\sharp\right] = -\underline{\Xi}. \tag{6.60}$$

Furthermore, we set

$$\langle b\epsilon^\sharp \rangle \equiv \underline{\Upsilon}. \tag{6.61}$$

When b is continuous, $\underline{\Upsilon}/b$ can be interpreted as the "gauge invariant brane position" — that is, the position of the brane unambiguously defined when $\left\langle \left(\frac{a}{b} E_\perp + \frac{a^2}{b^2} E' \right) \right\rangle$ is set to zero by a suitable coordinate change (which always exists *if* b is continuous).

In principle, derivatives normal to the brane are not defined for brane variables. But in what follows we will also use $\partial_n(b\epsilon^\sharp)$ which we simply define as

$$\partial_n(b\epsilon^\sharp) \equiv U\epsilon - \partial_n \left(aE_\perp + \frac{a^2}{b} E' \right). \tag{6.62}$$

In other words, the operator ∂_n acts on every metric perturbation defined in the bulk but not on ϵ (i.e., we define $\partial_n \epsilon \equiv 0$). Along similar lines, one can also define $\partial_n^2(b\epsilon^\sharp)$ and $\partial_n^2 \underline{\Upsilon}$. The quantity $\partial_n(b\epsilon^\sharp)$ can contain a singular term because of the discontinuous part of $\left(aE_\perp + \frac{a^2}{b} E' \right)$. Therefore, it is useful to define $\partial_n \underline{\Upsilon}$ by

$$\partial_n \underline{\Upsilon} \equiv \partial_n(b\epsilon^\sharp) + \mathrm{D}\underline{\Xi}. \tag{6.63}$$

This new quantity can take different values on each side of the brane, so that we can define $[\partial_n \underline{\Upsilon}]$ and $\langle \partial_n \underline{\Upsilon} \rangle$ following Eqns (3.10,3.11).

We may not simply continue ϵ into the bulk as a variable ϵ is independent of y. This is a gauge dependent continuation and the definitions for $\partial_n(b\epsilon^\sharp)$ and $\partial_n^2(b\epsilon^\sharp)$ given above would be valid only in the gauge where ϵ is independent of y. From the above expressions it is also clear that the variables $\partial_n(b\epsilon^\sharp)$ and $\partial_n^2(b\epsilon^\sharp)$ and all brane perturbation variables which contain these derivatives, like e.g. $\underline{h}$ below, are gauge invariant only with respect to gauge transformations parallel to the brane. Therefore, it is important to keep in mind that these quantities are defined only on (each side of) the brane, although the notation $\partial_n(b\epsilon^\sharp)$, $\partial_n \underline{\Upsilon}$ may be slightly misleading. They simply refer to Eqns (6.62,6.63).

Using equation (6.59), one can define several gauge invariant quantities which can also only be evaluated on either side of the brane (that is either at $y = \mathrm{y_b} + \epsilon^+$ or at $y = \mathrm{y_b} + \epsilon^-$) :

$$\underline{\Sigma} \equiv B_\perp - \frac{n}{b} \left(\frac{a}{n} B + \frac{a^2}{n^2} \dot{E} \right)' - \frac{b}{n} \dot{\epsilon} = \Sigma - (\partial_u - \mathcal{U})(b\epsilon^\sharp), \tag{6.64}$$

$$\underline{h} \equiv E_{\perp\perp} + \mathcal{U} \left(aB + \frac{a^2}{n} \dot{E} \right) - Ub\epsilon = h - \partial_n(b\epsilon^\sharp). \tag{6.65}$$

By comparing the last equation to Eq. (6.58), it appears that $\underline{h}$ does not contain a singular term (and hence it is a quantity that has a meaning on each side of the brane).

Using Eqns (6.39,6.40) we have

$$\underline{\Psi} = \Psi + Ib\epsilon^\sharp, \tag{6.66}$$

$$\underline{\Phi} = \Phi - Hb\epsilon^\sharp. \tag{6.67}$$

The derivatives $\partial_n \underline{\Psi}$ and $\partial_n \underline{\Phi}$ are defined via Eq. (6.62,6.63) above. The above equations will become very useful when considering the second Israel conditions and writing down the perturbed Weyl tensor.

The first Israel condition states that $\underline{\Phi}$ and $\underline{\Psi}$ are defined on the brane, and that $\partial_n \underline{\Phi}$, $\partial_n \underline{\Psi}$, $\underline{\Sigma}$, and $\underline{h}$ are well-defined on both sides of the brane, but it does not imply their continuity. In fact, it is their discontinuity which will enter into the perturbed second Israel condition. Using the above definitions we have the following relations for the discontinuous and the continuous parts of the gauge invariant scalar perturbation variables:

$$[\Psi] = -[I]\,\underline{\Upsilon} + \langle I \rangle\, \underline{\Xi}, \tag{6.68}$$

$$[\Phi] = [H]\,\underline{\Upsilon} - \langle H \rangle\, \underline{\Xi}, \tag{6.69}$$

$$[\Sigma] = [\underline{\Sigma}] - (\partial_u - \langle \mathcal{U} \rangle)\, \underline{\Xi} - [\mathcal{U}]\,\underline{\Upsilon}, \tag{6.70}$$

$$[h] = [\underline{h}] + [\partial_n \underline{\Upsilon}]\,, \tag{6.71}$$

and

$$\langle \Psi \rangle = \underline{\Psi} - \langle I \rangle\, \underline{\Upsilon} + \frac{1}{4}\, [I]\, \underline{\Xi}, \tag{6.72}$$

$$\langle\Phi\rangle = \underline{\Phi} + \langle H\rangle\,\underline{\Upsilon} - \frac{1}{4}[H]\,\underline{\Xi}, \tag{6.73}$$

$$\langle\Sigma\rangle = \langle\underline{\Sigma}\rangle + (\partial_u - \langle\mathcal{U}\rangle)\,\underline{\Upsilon} + \frac{1}{4}[\mathcal{U}]\,\underline{\Xi}, \tag{6.74}$$

$$\langle h\rangle = \langle\underline{h}\rangle + \langle\partial_n\underline{\Upsilon}\rangle\,. \tag{6.75}$$

If Z_2 symmetry is assumed, the first of these relations reduce to

$$[\Psi] = -[I]\,\underline{\Upsilon}, \tag{6.76}$$

$$[\Phi] = [H]\,\underline{\Upsilon}, \tag{6.77}$$

and

$$\langle\Psi\rangle = \underline{\Psi} + \frac{1}{4}[I]\,\underline{\Xi}, \tag{6.78}$$

$$\langle\Phi\rangle = \underline{\Phi} - \frac{1}{4}[H]\,\underline{\Xi}. \tag{6.79}$$

These relations are very important. They allow us to move freely between the bulk (non underlined) perturbation variables, which are defined everywhere, and the brane-related (underlined) variables, which are well defined only on the brane, i.e. they are either defined *on* the brane, like $\underline{\Phi}$ and $\underline{\Psi}$, or *on both sides* of the brane, like $b\epsilon^\sharp$, $\partial_n\underline{\Phi}$, $\partial_n\underline{\Psi}$, $\underline{\Sigma}$, or $\underline{h}$, etc.

The main difference between the brane and bulk perturbation variables is $\epsilon^\sharp$ which appears in the former. The brane displacement, however, is not unrelated to the bulk metric perturbations: a displacement of the brane induces metric perturbation in the bulk (as one could have guessed by making an analogy with a charged surface in electromagnetism).

E. Extrinsic curvature and second Israel condition

We define the perturbed stress-energy tensor on the brane as

$$\underline{\delta T}_{\alpha\beta} = (\underline{\delta\rho} + \underline{\delta P})\underline{u}_\alpha\underline{u}_\beta + 2(\underline{\rho} + \underline{P})\underline{u}_{(\alpha}\underline{\delta u}_{\beta)} - \underline{\delta P}\underline{q}_{\alpha\beta} - \underline{P}\underline{\delta q}_{\alpha\beta} + a^2\underline{\Pi}_{\alpha\beta}, \tag{6.80}$$

where $\underline{\delta u}_\alpha$ is the perturbation of the energy velocity on the brane which is given by

$$\underline{\delta u}^\alpha = \left(-\frac{1}{n}A, \frac{1}{a}\underline{v}^i, \frac{b^2}{n}\dot{\epsilon}\right). \tag{6.81}$$

(The $\underline{\delta u}^4$ component is determined by the condition $(\underline{\perp}_\mu + \underline{\delta\perp}_\mu)(\underline{u}^\mu + \underline{\delta u}^\mu) = 0$.) The variable $\underline{\Pi}_{\alpha\beta}$ is the anisotropic stress tensor and it is gauge invariant by itself.

As discussed in Appendix H7, we define the gauge invariant perturbations for the energy density and the pressure on the brane by

$$\underline{\delta\rho}^\sharp = \underline{\delta\rho} - \dot{\underline{\rho}}\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right), \tag{6.82}$$

$$\underline{\delta P}^\sharp = \underline{\delta P} - \dot{\underline{P}}\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right). \tag{6.83}$$

Similarly we define the gauge invariant perturbation variables for the velocity on the brane,

$$\underline{v}^\sharp = \underline{v} + \frac{a}{n}\dot{E}, \tag{6.84}$$

$$\underline{\bar{v}}_i^\sharp = \underline{\bar{v}}_i + \frac{a}{n}\dot{\bar{E}}_i. \tag{6.85}$$

To impose the second Israel junction condition, we need to compute $\underline{\delta\mathcal{K}}_{\alpha\beta}$ which is the difference between the perturbed value of $\underline{\mathcal{K}}_{\alpha\beta}$ at the brane position $y = \mathrm{y_b} + \epsilon$, and the background value of $\underline{\mathcal{K}}_{\alpha\beta}$. The results are given in Appendix H6. The extrinsic curvature has to be compared to the perturbation of the surface stress tensor,

$$\underline{\delta\mathcal{S}}_{\alpha\beta} = \underline{\delta T}_{\alpha\beta} - \frac{1}{3}\underline{\delta T}\underline{q}_{\alpha\beta} - \frac{1}{3}\underline{T}\underline{\delta q}_{\alpha\beta}, \tag{6.86}$$

whose components are given in Appendix H7.

The perturbation of the second Israel condition,

$$[\delta\underline{\mathcal{K}}_{\alpha\beta}] = -\kappa_5\underline{\delta\mathcal{S}}_{\alpha\beta}, \tag{6.87}$$

yields four discontinuity conditions for the scalar perturbation variables $\underline{\Sigma}$, $\underline{h}$, $\underline{\Xi}$, and the first derivatives $\partial_n\underline{\Psi}$ and $\partial_n\underline{\Phi}$,

$$\begin{aligned}
-\Delta\underline{\Xi} + 3\left[\partial_n\underline{\Phi} - H\underline{h}\mathcal{H}\left[\underline{\Sigma}\right]\right] &= \kappa_5\underline{\delta\rho}^\sharp, && (6.88)\\
\frac{1}{2}\left[\underline{\Sigma} - (\partial_u + \mathcal{U} - 2\mathcal{H})(b\epsilon^\sharp)\right] &= \kappa_5(\underline{P} + \underline{\rho})a\underline{v}^\sharp, && (6.89)\\
\left[\partial_n\underline{\Psi} + I\underline{h} - (\partial_u + \mathcal{U})\,\underline{\Sigma}\right] &= \kappa_5\left(\underline{\delta P}^\sharp + \frac{2}{3}\underline{\delta\rho}^\sharp\right), && (6.90)\\
-\underline{\Xi} &= \kappa_5 a^2\underline{\Pi}. && (6.91)
\end{aligned}$$

In terms of the gauge invariant bulk variables these conditions read

$$\begin{aligned}
&-\Delta\underline{\Xi} + 3\left[\partial_n\Phi - Hh + \mathcal{H}\Sigma\right]\\
&\quad +3\underline{\Upsilon}\left(\mathcal{H}\left[\mathcal{U}\right] - \left[\partial_n\right]\langle H\rangle\right)\\
&-3\underline{\Xi}\left(\mathcal{H}\langle\mathcal{U}\rangle - \langle\partial_n\rangle\langle H\rangle\right) + 3\partial_u\underline{\Xi} &= \kappa_5\underline{\delta\rho}^\sharp, && (6.92)\\
&\frac{1}{2}\left[\Sigma\right] + (\partial_u - \mathcal{H})\underline{\Xi} &= \kappa_5(\underline{P} + \underline{\rho})a\underline{v}^\sharp, && (6.93)\\
&\left[\partial_n\Psi + Ih - (\partial_u + \mathcal{U})\,\Sigma\right]\\
&\quad +\underline{\Upsilon}\left(\left[\partial_n\right]\langle I\rangle - \left[(\partial_u + \mathcal{U})\mathcal{U}\right]\right)\\
&-\underline{\Xi}\left(\langle\partial_n\rangle\langle I\rangle - \langle(\partial_u + \mathcal{U})\mathcal{U}\rangle\right) - \partial_u^2\underline{\Xi} &= \kappa_5\left(\underline{\delta P}^\sharp + \frac{2}{3}\underline{\delta\rho}^\sharp\right), && (6.94)\\
&-\underline{\Xi} &= \kappa_5 a^2\underline{\Pi}. && (6.95)
\end{aligned}$$

Notice that when Z_2 symmetry is imposed $\underline{\Upsilon}$ never appears in these equations.

For the vector perturbation variables we obtain two discontinuity conditions for $\bar{h}_i$ and the first derivative $\partial_n\bar{\Sigma}_i$,

$$\begin{aligned}
-\frac{1}{2}\left[\partial_n + I - H\right]\bar{\Sigma}_i + \frac{1}{2}\left[(\partial_u + \mathcal{U} - \mathcal{H})\bar{h}_i\right] &= \kappa_5(\underline{P} + \underline{\rho})(\underline{v}_i^\sharp - \bar{\Sigma}_i), && (6.96)\\
-\left[\bar{h}_i\right] &= \kappa_5 a\underline{\bar{\Pi}}_i. && (6.97)
\end{aligned}$$

Finally, there is also a discontinuity condition for the normal derivative of the tensor perturbation variable $\bar{\bar{E}}_{ij}$:

$$-\left[\partial_n\bar{\bar{E}}_{ij}\right] = \kappa_5\underline{\bar{\bar{\Pi}}}_{ij}. \tag{6.98}$$

Note that the Israel conditions do not give any constraint on the $\{\alpha 4\}$ components of Eq. (6.87). The above constraints can also be found directly from the singular part of Einstein's equations (5.65–5.75). This is relatively straightforward for the vector and tensor modes, but much more involved for the scalar part, as one must rewrite the equations using the underlined quantities defined above, and also because one has to consider the perturbation of the covariant Dirac function D. However, for completeness, this has been undertaken in Appendix I1, and we have checked that both approaches lead to the same result.

F. Sail equation

As we have seen, the junction conditions are conveniently written using the underlined variables $\underline{\Phi}$, etc. In order to use them, we must know $b\epsilon^\sharp$. The jump of this quantity, $\underline{\Xi}$, is given by Eq. (6.91). As its continuous part $\underline{\Upsilon}$ represents the brane displacement, it is natural to seek an equation describing the brane motion. As for the unperturbed case, such an equation is found by taking the discontinuous part of the $\{44\}$ component of Einstein's equations. This yields

$$\begin{aligned}
&3\langle H\rangle\,\underline{\delta P}^\sharp - \langle I\rangle\,\underline{\delta\rho}^\sharp\\
&-3\underline{P}\left\langle\partial_n\underline{\Phi} - H\underline{h} + \mathcal{H}\underline{\Sigma} + \frac{1}{3}\Delta\underline{\Upsilon}\right\rangle\\
&\quad -\underline{\rho}\left\langle\partial_n\underline{\Psi} + I\underline{h} - (\partial_u + \mathcal{U})\,\underline{\Sigma}\right\rangle &= \left[\overline{\delta Y}^\sharp\right], && (6.99)
\end{aligned}$$

where $\overline{\delta Y^\sharp}$ corresponds to the pressure perturbation along the extra dimension as measured by an observer at rest with respect to the brane. The relationship between $\delta Y^\sharp$ and $\overline{\delta Y^\sharp}$ is given in Eq. (7.26) below. Equation (6.99) is the typical equation for the displacement of a membrane (it involves the Laplacian of the displacement $\underline{\Upsilon}$). When going back to the bulk (non underlined) perturbations, Eq. (6.99) becomes, as expected, a wave equation for $\underline{\Upsilon}$:

$$\begin{aligned}
-\partial_u^2(\underline{\rho}\underline{\Upsilon}) - 3\mathcal{H}\partial_u\left(2\underline{\rho}\underline{\Upsilon} + \underline{P}\underline{\Upsilon}\right) - \underline{P}\Delta\underline{\Upsilon} + 2K\underline{\rho}\underline{\Upsilon} & \\
-3\left(\underline{P} + \frac{2}{3}\underline{\rho}\right)\underline{\Upsilon}\left(2\partial_u\mathcal{H} + 4\mathcal{H}^2\right) & \\
-(\underline{P}+\underline{\rho})\underline{\Upsilon}\left(3\langle H\rangle\langle H - I\rangle + \frac{\kappa_5}{4}\underline{\rho}(\underline{P}+\underline{\rho})\right) & \\
+\underline{\Upsilon}\left(3\langle H\rangle[Y-P] + \langle I\rangle[Y+\rho]\right) &= \left[\delta Y^\sharp\right] - 3\langle H\rangle\underline{\delta P^\sharp} + \langle I\rangle\underline{\delta\rho^\sharp} \\
&\quad +3\underline{P}\langle\partial_n\Phi - Hh + \mathcal{H}\Sigma\rangle \\
&\quad +\underline{\rho}\langle\partial_n\Psi + Ih - (\partial_u + \mathcal{U})\Sigma\rangle \\
&\quad +2\partial_u\langle F\rangle\underline{\Xi} - \underline{\Xi}(\partial_u - N\mathcal{H})\langle F\rangle \\
&\quad +3\langle H\rangle\underline{\Xi}\left(\frac{\kappa_5}{4}(\underline{P}+\underline{\rho})^2 + \langle Y - P\rangle\right) \\
&\quad +\langle I\rangle\underline{\Xi}\left(-\frac{\kappa_5}{4}(\underline{P}+\underline{\rho})\underline{\rho} + \langle Y+\rho\rangle\right). \qquad (6.100)
\end{aligned}$$

Note that there is nothing which guarantees *a priori* that the motion of the brane is stable. Even in the simplest case (Z_2 symmetry, k = 0, no bulk perturbation contributing to the right hand side of the above equation and brane stress-energy tensor dominated by a constant tension term), this equation becomes

$$\left(\partial_\eta^2 + 2\frac{\dot{a}}{a}\partial_\eta - \nabla^2 - 2\frac{\ddot{a}}{a}\right)\underline{\Upsilon} = 0, \qquad (6.101)$$

and the mass term becomes negative for sufficiently fast expansion rate of the brane!

VII. THE BRANE POINT OF VIEW

In the previous sections we have derived the bulk perturbation equations and their boundary conditions on the brane. This allows us in principle to solve the full system of perturbation equations in the bulk for given initial conditions. From these one can determine also the perturbed Weyl tensor and the second fundamental form.

In order to make contact with 4-dimensional cosmology in this section, we want to write the perturbed version of the 4-dimensional Einstein equations on the brane. As for the background, this can either be done directly from the perturbed bulk Einstein equations (5.65–5.75), or using the Gauss-Codacci equation.

A. Projected Weyl tensor on the brane

The full expression of the perturbed Weyl tensor $\delta C_{\alpha\beta\gamma\delta}$ is given in Appendix F9. Here we write only the components of the perturbed projected Weyl tensor, $\underline{\delta\mathcal{E}}_{\alpha\beta}$, on the spacelike direction $\underline{\perp}^\alpha + \underline{\delta\perp}^\alpha$, written in terms of the underlined gauge invariant variables. We have

$$\begin{aligned}
\underline{\delta\mathcal{E}}_{00} &= \frac{1}{2}n^2\underline{\delta\mathcal{Z}^\sharp} + 2\underline{\mathcal{E}}_{00}\underline{\Psi} \\
&\quad +(\dot{\underline{\mathcal{E}}}_{00} + 2\underline{\mathcal{E}}_{00}\partial_\eta)\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right), \qquad (7.1) \\
\underline{\delta\mathcal{E}}_{0i} &= -n\nabla_i\underline{\delta\mathcal{E}}^v - an\underline{\bar{\delta}\mathcal{E}}_i^v \\
&\quad +\underline{\mathcal{E}}_{00}\nabla_i\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) + \underline{\mathcal{E}}_{ij}\dot{E}^j, \qquad (7.2) \\
\underline{\delta\mathcal{E}}_{ij} &= \frac{1}{6}a^2\gamma_{ij}\underline{\delta\mathcal{Z}^\sharp} \\
&\quad +\left(\nabla_{ij} - \frac{1}{3}\nabla^2\gamma_{ij}\right)\underline{\delta\mathcal{E}}^\Pi + a\nabla_{(i}\underline{\bar{\delta}\mathcal{E}}_{j)}^\Pi + a^2\underline{\bar{\bar{\delta}}\mathcal{E}}_{ij}^\Pi
\end{aligned}$$

$$+\underline{\dot{\mathcal{E}}}_{ij}\left(\frac{a}{n}B+\frac{a^2}{n^2}\dot{E}\right)+2\underline{\mathcal{E}}_{k(i}(E^k_{j)}-\delta^k_{j)}\underline{\Phi}), \tag{7.3}$$

$$\underline{\delta\mathcal{E}}_{04} = \underline{\mathcal{E}}^0_0\underline{\delta q}_{04}, \tag{7.4}$$

$$\underline{\delta\mathcal{E}}_{i4} = \underline{\mathcal{E}}^j_i\underline{\delta q}_{j4}, \tag{7.5}$$

$$\underline{\delta\mathcal{E}}_{44} = 0, \tag{7.6}$$

where we have set

$$\begin{aligned}\underline{\delta\mathcal{Z}}^\sharp \equiv\ & \frac{2}{3}(\Delta+3K)\underline{\Phi}+\frac{1}{3}\Delta(\underline{\Psi}-\underline{h})\\ &+(\partial_u+\mathcal{U})\left(\partial_u\underline{\Phi}+\mathcal{H}\underline{\Psi}\right)-(\partial_u+2\mathcal{U}-\mathcal{H})\left(\partial_u\underline{h}+\mathcal{U}\underline{\Psi}\right)\\ &-(\partial_n+I)\left(\partial_n\underline{\Phi}-H\underline{h}+\mathcal{H}\underline{\Sigma}+\frac{1}{3}\Delta(b\epsilon^\sharp)\right)\\ &-(\partial_n+2I-H)\left(\partial_n\underline{\Psi}+I\underline{h}-(\partial_u+\mathcal{U})\,\underline{\Sigma}\right)\\ &-\underline{\Psi}\partial_u\left(\mathcal{U}-\mathcal{H}\right)-\left(\underline{h}\partial_n-\underline{\Sigma}\partial_u\right)(I-H)\,,\end{aligned} \tag{7.7}$$

$$\begin{aligned}\underline{\delta\mathcal{E}}^v \equiv\ & \frac{2}{3}\left(-\left(\mathcal{H}\underline{\Psi}+\partial_u\underline{\Phi}\right)+\left(\partial_u\underline{h}+\mathcal{U}\underline{\Psi}\right)+(\mathcal{U}-\mathcal{H})\,\underline{h}+(H-I)\,\underline{\Sigma}\right.\\ &\left.-\tfrac{1}{2}\partial_n\left(\underline{\Sigma}-(\partial_u+\mathcal{U}-2\mathcal{H})\,(b\epsilon^\sharp)\right)\right),\end{aligned} \tag{7.8}$$

$$\underline{\delta\mathcal{E}}^\Pi \equiv \frac{1}{3}\left(\underline{\Phi}-\underline{\Psi}-2\underline{h}+(H+I-2\partial_n)(b\epsilon^\sharp)\right). \tag{7.9}$$

Since, by solving the bulk equations, we in principle obtain the non underlined variables, it is useful to express the above components also in terms of these,

$$\begin{aligned}\underline{\delta\mathcal{Z}}^\sharp =\ & \frac{2}{3}(\Delta+3K)\Phi+\frac{1}{3}\Delta(\Psi-h)\\ &+(\partial_u+\mathcal{U})\left(\partial_u\Phi+\mathcal{H}\Psi\right)-(\partial_u+2\mathcal{U}-\mathcal{H})\left(\partial_u h+\mathcal{U}\Psi\right)\\ &-(\partial_n+I)\left(\partial_n\Phi-Hh+\mathcal{H}\Sigma\right)\\ &-(\partial_n+2I-H)\left(\partial_n\Psi+Ih-(\partial_u+\mathcal{U})\,\Sigma\right)\\ &-\Psi\partial_u\left(\mathcal{U}-\mathcal{H}\right)-\left(h\partial_n-\Sigma\partial_u\right)(I-H)\\ &+\mathcal{Z}'\epsilon^\sharp,\end{aligned} \tag{7.10}$$

$$\underline{\delta\mathcal{E}}^v = \frac{2}{3}\left(-\left(\mathcal{H}\Psi+\partial_u\Phi\right)+\left(\partial_u h+\mathcal{U}\Psi\right)+(\mathcal{U}-\mathcal{H})\,h-\left(\tfrac{1}{2}\partial_n+I-H\right)\Sigma\right), \tag{7.11}$$

$$\underline{\delta\mathcal{E}}^\Pi = \frac{1}{3}\left(\Phi-\Psi-2h\right). \tag{7.12}$$

For the vector and tensor part of the projected Weyl tensor we have defined

$$\underline{\delta\bar{\mathcal{E}}}^v_i \equiv -\frac{1}{6}(\Delta+2K)\bar{\Sigma}_i+\frac{1}{3}\left(\partial_n+H\right)\left(\left(\partial_n+I-H\right)\bar{\Sigma}_i-\left(\partial_u+\mathcal{U}-\mathcal{H}\right)\bar{h}_i\right), \tag{7.13}$$

$$\underline{\delta\bar{\mathcal{E}}}^\Pi_i \equiv \frac{1}{3}\left(\left(\partial_u+2\left(\mathcal{H}-\mathcal{U}\right)\right)\bar{\Sigma}_i+\left(2\partial_n+(H-I)\right)\bar{h}_i\right), \tag{7.14}$$

$$\underline{\delta\bar{\bar{\mathcal{E}}}}^\Pi_{ij} \equiv \frac{1}{3}\left(\left(\partial_u+3\mathcal{H}-2\mathcal{U}\right)\partial_u-(\Delta-2K)+\left(2\partial_n+3H-I\right)\partial_n\right)\bar{\bar{E}}_{ij}. \tag{7.15}$$

B. Perturbed Einstein equations on the brane

With these definitions, we can now write the projected perturbed Einstein equations on the brane. They split into four scalar equations,

$$\begin{aligned}&2\left(\Delta+3K\right)\underline{\Phi}\\ -6\mathcal{H}\left(\mathcal{H}\underline{\Psi}+\partial_u\underline{\Phi}\right) &= \frac{1}{6}\kappa_5^2\left(\sum_{\rm b}\underline{\rho}_{\rm b}\right)\sum_{\rm b}\underline{\delta\rho}^\sharp_{\rm b}\end{aligned}$$

$$
\begin{aligned}
& -2\left\langle H\right\rangle \left\langle 3\partial_n\underline{\Phi}-3H\underline{h}+3\mathcal{H}\underline{\Sigma}+\Delta b\epsilon^\sharp\right\rangle \\
& +\frac{1}{2}\kappa_5\sum_{\mathrm{B}}\left\langle \overline{\delta P}^\sharp_{\mathrm{B}}+\overline{\delta\rho}^\sharp_{\mathrm{B}}-\overline{\delta Y}^\sharp_{\mathrm{B}}\right\rangle +\frac{1}{2}\left\langle \underline{\delta\mathcal{Z}}^\sharp\right\rangle , \qquad (7.16)
\end{aligned}
$$

$$
\begin{aligned}
-2\left(\mathcal{H}\underline{\Psi}+\partial_u\underline{\Phi}\right) &= \frac{1}{6}\kappa_5^2\left(\sum_{\mathrm{b}}\underline{\rho}_{\mathrm{b}}\right)\sum_{\mathrm{b}}\left((\underline{P}_{\mathrm{b}}+\underline{\rho}_{\mathrm{b}})av^\sharp_{\mathrm{b}}\right) \\
& -\frac{3}{2}\left\langle H\right\rangle \left\langle \underline{\Sigma}-(\partial_u+\mathcal{U}-2\mathcal{H})(b\epsilon^\sharp)\right\rangle \\
& +\frac{2}{3}\kappa_5\sum_{\mathrm{B}}\left\langle (P_{\mathrm{B}}+\rho_{\mathrm{B}})v^\sharp_{\mathrm{B}}-F_{\mathrm{B}}\overline{f}^\sharp_{\mathrm{B}}\right\rangle +\left\langle \underline{\delta\mathcal{E}}^v\right\rangle , \qquad (7.17)
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{2}{3}\left(\Delta\underline{\Psi}-(\Delta+3K)\underline{\Phi}\right) \\
& +2\left(\partial_u+3\mathcal{H}\right)\left(\mathcal{H}\underline{\Psi}+\partial_u\underline{\Phi}\right) \\
+2\underline{\Psi}\partial_u\mathcal{H} &= \frac{1}{6}\kappa_5^2\left(\sum_{\mathrm{b}}\underline{\rho}_{\mathrm{b}}\right)\sum_{\mathrm{b}}\underline{\delta P}^\sharp_{\mathrm{b}} \\
& -2\left\langle H\right\rangle \left\langle \partial_n\underline{\Psi}+I\underline{h}-(\partial_u+\mathcal{U})\underline{\Sigma}\right\rangle \\
& +2\left\langle H+I\right\rangle \left\langle \partial_n\underline{\Phi}-H\underline{h}+\mathcal{H}\underline{\Sigma}+\frac{1}{3}\Delta b\epsilon^\sharp\right\rangle \\
& +\frac{1}{6}\kappa_5\sum_{\mathrm{B}}\left\langle \overline{\delta P}^\sharp_{\mathrm{B}}+\overline{\delta\rho}^\sharp_{\mathrm{B}}+3\overline{\delta Y}^\sharp_{\mathrm{B}}\right\rangle \\
& +\frac{1}{6}\kappa_5^2\left(\sum_{\mathrm{b}}(\underline{P}_{\mathrm{b}}+\underline{\rho}_{\mathrm{b}})\right)\sum_{\mathrm{b}}\underline{\delta\rho}^\sharp_{\mathrm{b}}+\frac{1}{6}\left\langle \underline{\delta\mathcal{Z}}^\sharp\right\rangle , \qquad (7.18)
\end{aligned}
$$

$$
\begin{aligned}
\underline{\Phi}-\underline{\Psi} &= \frac{1}{6}\kappa_5^2\left(\sum_{\mathrm{b}}\underline{\rho}_{\mathrm{b}}\right)a^2\sum_{\mathrm{b}}\underline{\Pi}_{\mathrm{b}} \\
& -\left\langle H+I\right\rangle \underline{\Upsilon} \\
& -\frac{1}{4}\kappa_5^2\left(\sum_{\mathrm{b}}(\underline{P}_{\mathrm{b}}+\underline{\rho}_{\mathrm{b}})\right)a^2\sum_{\mathrm{b}}\underline{\Pi}_{\mathrm{b}} \\
& +\frac{2}{3}\kappa_5 a^2\sum_{\mathrm{B}}\left\langle \Pi_{\mathrm{B}}\right\rangle +\left\langle \underline{\delta\mathcal{E}}^\Pi\right\rangle , \qquad (7.19)
\end{aligned}
$$

two vector equations,

$$
\begin{aligned}
-\frac{1}{2}\left(\Delta+2K\right)\bar{\Sigma}_i &= \frac{1}{6}\kappa_5^2\left(\sum_{\mathrm{b}}\underline{\rho}_{\mathrm{b}}\right)\sum_{\mathrm{b}}\left((\underline{P}_{\mathrm{b}}+\underline{\rho}_{\mathrm{b}})(\bar{v}_i^{\mathrm{b}\,\sharp}-\bar{\Sigma}_i)\right) \\
& +\frac{2}{2}\left\langle H\right\rangle \left\langle (\partial_n+I-H)\bar{\Sigma}_i-(\partial_u+\mathcal{U}-\mathcal{H})\bar{h}_i\right\rangle \\
& +\frac{2}{3}\kappa_5\sum_{\mathrm{B}}\left\langle (P_{\mathrm{B}}+\rho_{\mathrm{B}})(\bar{v}_i^{\mathrm{B}\,\sharp}-\bar{\Sigma}_i)-F_{\mathrm{B}}(\bar{f}_i^{\mathrm{B}\,\sharp}+\bar{h}_i)\right\rangle +\left\langle \underline{\delta\bar{\mathcal{E}}}^v_i\right\rangle , \qquad (7.20)
\end{aligned}
$$

$$
\begin{aligned}
\left(\partial_u+2\mathcal{H}\right)\bar{\Sigma}_i &= \frac{1}{6}\kappa_5^2\left(\sum_{\mathrm{b}}\underline{\rho}_{\mathrm{b}}\right)a\sum_{\mathrm{b}}\underline{\bar{\Pi}}^{\mathrm{b}}_i \\
& +\left\langle H+I\right\rangle \left\langle \bar{h}_i\right\rangle \\
& -\frac{1}{4}\kappa_5^2\left(\sum_{\mathrm{b}}(\underline{P}_{\mathrm{b}}+\underline{\rho}_{\mathrm{b}})\right)a\sum_{\mathrm{b}}\underline{\bar{\Pi}}^{\mathrm{b}}_i \\
& +\frac{2}{3}\kappa_5 a\sum_{\mathrm{B}}\left\langle \bar{\Pi}^{\mathrm{B}}_i\right\rangle +\left\langle \underline{\delta\bar{\mathcal{E}}}^\Pi_i\right\rangle , \qquad (7.21)
\end{aligned}
$$

and one tensor equation

$$
\begin{aligned}
(\partial_u + 3\mathcal{H})\,\partial_u \bar{\bar{E}}_{ij} - (\Delta - 2K)\,\bar{\bar{E}}_{ij} \;=\; & \frac{1}{6}\kappa_5^2 \left(\sum_{\rm b} \underline{\rho}_{\rm b}\right) \sum_{\rm b} \underline{\bar{\bar{\Pi}}}^{\rm b}_{ij} \\
& + \langle H + I \rangle \left\langle \partial_n \bar{\bar{E}}_{ij} \right\rangle \\
& - \frac{1}{4}\kappa_5^2 \left(\sum_{\rm b} (\underline{P}_{\rm b} + \underline{\rho}_{\rm b})\right) \sum_{\rm b} \underline{\bar{\bar{\Pi}}}^{\rm b}_{ij} \\
& + \frac{2}{3}\kappa_5 \sum_{\rm B} \left\langle \bar{\bar{\Pi}}^{\rm B}_{ij} \right\rangle + \left\langle \underline{\bar{\bar{\mathcal{E}}}}^{\Pi}_{ij} \right\rangle , \qquad (7.22)
\end{aligned}
$$

where we have set, for the bulk matter quantities evaluated at the brane position,

$$\overline{\delta\rho}^{\sharp} = \delta\rho^{\sharp} + \rho'\epsilon^{\sharp} - 2F\frac{b}{n}\dot{\epsilon}^{\sharp}, \qquad (7.23)$$

$$\overline{\delta P}^{\sharp} = \delta P^{\sharp} + P'\epsilon^{\sharp}, \qquad (7.24)$$

$$\overline{\delta F}^{\sharp} = \delta F^{\sharp} + F'\epsilon^{\sharp} - (\rho + Y)\frac{b}{n}\dot{\epsilon}^{\sharp}, \qquad (7.25)$$

$$\overline{\delta Y}^{\sharp} = \delta Y^{\sharp} + Y'\epsilon^{\sharp} - 2F\frac{b}{n}\dot{\epsilon}^{\sharp}, \qquad (7.26)$$

$$a\overline{f}^{\sharp} = af^{\sharp} - b\epsilon^{\sharp}. \qquad (7.27)$$

These corrections follow from the fact that we have to go in a coordinate system which follows the brane. The terms proportional to $X'\epsilon^{\sharp}$ are here because we consider the bulk matter content at $y = \mathrm{y_b} + \epsilon$ rather than at $y = \mathrm{y_b}$, the terms proportional to $\dot{\epsilon}^{\sharp}$ come from the fact that we also perform a Lorentz boost in order to follow the brane motion, and the term $b\epsilon^{\sharp}$ in the last equation comes from the fact that the brane is bent.

As for the unperturbed case, the continuous parts of the bulk stress-energy tensor and of the projected Weyl tensor appear on the right hand side of these equations, as well as the components of the continuous part of the perturbed extrinsic curvature. These are related through the discontinuous part of the Einstein equations to the discontinuity of the perturbed Weyl tensor and of the bulk perturbed matter content. The corresponding equations can be found in Appendix I5.

C. Perturbed conservation equation

The brane matter conservation equations follow from the singular part of $D_\mu T^{\mu\alpha} = 0$ or from the Bianchi identities. One obtains (see Appendix I3)

$$
\begin{aligned}
& \partial_u \underline{\delta\rho}^{\sharp} + 3\mathcal{H}(\underline{\delta\rho}^{\sharp} + \underline{\delta P}^{\sharp}) \\
& + (\underline{P} + \underline{\rho})\Delta a\underline{v}^{\sharp} - 3(\underline{P} + \underline{\rho})\partial_u \underline{\Phi} \;=\; -\left[\overline{\delta F}^{\sharp} + F\underline{\Psi}\right], \qquad (7.28) \\
& (\partial_u + 3\mathcal{H})\left((\underline{P} + \underline{\rho})a\underline{v}^{\sharp}\right) + \underline{\delta P}^{\sharp} \\
& + \frac{2}{3}(\Delta + 3K)\,a^2\underline{\Pi} + (\underline{P} + \underline{\rho})\underline{\Psi} \;=\; -\left[Fav^{\sharp} + (P - Y)a\overline{f}^{\sharp}\right], \qquad (7.29) \\
& (\partial_u + 4\mathcal{H})\left((\underline{P} + \underline{\rho})(\underline{\bar{v}}_i^{\sharp} - \bar{\Sigma}_i)\right) \\
& \qquad + \frac{1}{2}(\Delta + 2K)\,a\underline{\bar{\Pi}}_i \;=\; -\left[F(\bar{v}_i^{\sharp} - \bar{\Sigma}_i) + (P - Y)(\bar{f}_i^{\sharp} + \bar{h}_i)\right]. \qquad (7.30)
\end{aligned}
$$

Again, when there are no discontinuities in the bulk matter perturbations, one obtains the usual conservation equations.

VIII. CONCLUSION

In this paper we have derived gauge invariant cosmological perturbation theory in braneworld scenarios with one codimension. The unperturbed background system we considered (Sections II–IV) consists of a 5-dimensional bulk

spacetime with a maximally symmetric 3-dimensional subspace of curvature k, containing arbitrary (possibly interacting) matter with energy-momentum tensor $T_{\alpha\beta}$, and a homogeneous and isotropic 3-brane again with arbitrary stress energy tensor $\underline{T}_{\alpha\beta}$. We have not assumed Z_2 symmetry across the brane. As such, our work generalises that of previous authors who have considered perturbation theory mainly in the Z_2-symmetric case, and with specific bulk (and brane) matter (e.g., a bulk cosmological constant [45] or scalar field [24]). We believe that the general setup considered here is a necessary component of any serious attempt which may be made to tackle such important questions as the cosmic microwave background anisotropies in braneworlds.

The only coordinate choice we have made is to fix the unperturbed brane to be at a given position y_b in the extra dimension. The bulk metric is explicitly time-dependent. When the bulk contains only a cosmological constant, this is not the most natural coordinate system: there one would work with (static) Schwarzschild-AdS$_5$ and a dynamical brane [14, 55–57]. However, in the case of arbitrary bulk matter and especially for the study of perturbations, we have found it more convenient to work in a coordinate system in which the brane is at rest.

In Sections II–IV we derived all the relevant background equations, ending with the brane Friedmann equation (4.8–4.9). As discussed in Section IV, when Z_2 symmetry is not assumed, one has additional contributions to the 4-dimensional Einstein tensor on the brane. In order to study these terms one has to include equations for the extrinsic curvature.

In the remainder of the paper we studied perturbations of this system, setting up a completely gauge invariant formalism. Section V contains a general discussion of the classification of perturbations in an $n+1$-dimensional space time, as well as the interplay between bulk and brane perturbations. An important point which we note there is the existence of one extra scalar degree of freedom on the brane which is not a metric perturbation (although it interacts with some bulk metric perturbations): this is the brane displacement. In Section VI we have derived an equation of motion for the gauge invariant perturbation variable describing this quantity.

In Section V we introduced the perturbed 5-dimensional bulk spacetime. This led to the definition of four scalar, two vector and one tensor gauge invariant bulk perturbation variables given in equations (5.25–5.30). Following the definition of gauge invariant variables for the perturbations of the bulk matter, we were able to write down the perturbed bulk Einstein equations in a gauge invariant manner. The perturbed brane was then introduced in Section VI. In analogy with usual 4-dimensional cosmological perturbation theory, our aim was to introduce two scalar gauge invariant brane perturbation variables (the Bardeen potentials), one vector and one tensor metric perturbation. The correct definition of these variables can only be given once the perturbed brane metric and Israel junction conditions are used determine the brane variables in terms of the continuous part and the jump of the bulk perturbations. The brane variables are defined in equations (6.16–6.19). The gauge invariant brane displacement also enters in these definitions. Finally the perturbed Einstein equations on the brane were derived in Section VII. As in the unperturbed case, they contain a contribution from the projection of the perturbed bulk Weyl tensor which in general have to be determined by solving the bulk equations.

Despite the fact that we have tried to present our results as clearly as possible, the formalism presented in this paper is technically rather complicated. This reflects the fact that we have considered a very general scenario. The corollary is, however, that our results should be applicable to a whole variety of different (and possibly simpler) situations of interest in braneworld scenarios. In a forthcoming paper, we plan to apply this formalism to a specific model and solve some of the perturbation equations presented here.

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[1] J. Polchinski, *String Theory* (Cambridge Universty Press, Cambridge, England, 1998), volumes 1 and 2.
[2] P. Horava and E. Witten, Nucl. Phys. **B640**, 506 (1996).
[3] P. Horava and E. Witten, Nucl. Phys. **B675**, 94 (1996).
[4] K. Akama, *Pregeometry*, Lecture Notes in Physics **176**, edited by K. Kikkawa, N. Nakanishi, and H. Nariai (Springer-Verlag, Berlin, 1982), p. 267.
[5] V.A. Rubakov and M.E. Shaposhnikov, Phys. Lett. B **125**, 136 (1983).

[6] M. Visser, Phys. Lett. B **159**, 22 (1985).
[7] E.J. Squires, Phys. Lett. B **167**, 286 (1986).
[8] N. Arkani-Hamed, S. Dimopolous, and G. Dvali, Phys. Lett. B **429**, 263 (1998).
[9] I. Antoniadis, N. Arkani-Hamed, S. Dimopolous, and G. Dvali, Phys. Lett. B **436**, 257 (1998).
[10] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999).
[11] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4670 (1999).
[12] P. Binétruy, C. Deffayet, and D. Langlois, Nucl. Phys. **B565**, 269 (2000).
[13] P. Binétruy, C. Deffayet, U. Ellwanger, and D. Langlois, Phys. Lett. B **477**, 285 (1999).
[14] S. Mukohyama, Phys. Lett. B **473**, 241-245 (2000).
[15] J.M. Cline, C. Grojean, and G. Servant, Phys. Rev. Lett. **83**, 4245 (1999).
[16] C. Csaki, M. Graesser, C. Kolda, and J. Terning, Phys. Lett. B **462**, 34 (1999).
[17] E.E. Flannagan, S.-H. Tye, and I. Wasserman, Phys. Rev. D **62**, 044039 (2000).
[18] J. Lesgourgues, S. Pastor, M. Peloso, and L. Sorbo, Phys. Lett. B **489**, 411 (2000).
[19] N. Deruelle, T. Doležel, and J. Katz, Phys. Rev. D **63**, 083513 (2001).
[20] N. Deruelle and J. Katz, Phys. Rev. D **64**, 083515 (2001).
[21] N. Deruelle and T. Doležel, Phys. Rev. D **64**, 103506 (2001).
[22] J. Garriga and T. Tanaka, Phys. Rev. Lett. **84**, 2778-2781 (2000).
[23] S.B. Giddings, E. Katz, and L. Randall, JHEP **0003**, 023 (2000).
[24] S. Kachru, M. Schulz, E. Silverstein, Phys. Rev. D **62**, 045021 (2000).
[25] P.F. Gonzalez-Diaz Phys. Lett. B **481**, 353 (2000).
[26] A.-C. Davis, I.R. Vernon, S. Davis, and W.B. Perkins, Phys. Lett. B **504**, 254 (2001).
[27] W.B. Perkins, Phys. Lett. B **504**, 28 (2001).
[28] B. Carter, Int. J. Theor. Phys. **40**, 2099 (2001).
[29] R.A. Battye and B. Carter, Phys. Lett. B **B50**, 331 (2001).
[30] B. Carter and J.-P. Uzan, Nucl. Phys. **B606**, 45 (2001).
[31] R.A. Battye, B. Carter, A. Mennim, and J.-P. Uzan, Phys. Rev. D **64**, 124007 (2001).
[32] B. Carter, J.-P. Uzan, R.A. Battye, and A. Mennim, Class. Quantum Grav. **18**, 4871 (2001).
[33] C. van de Bruck, M. Dorca, C.J.A.P. Martins, and M. Parry, Phys. Lett. B **495**, 183 (2000).
[34] T. Shiromizu, K. Maeda, and M. Sasaki, Phys. Rev. D **62**, 024012 (2000).
[35] R. Maartens, Phys. Rev. D **62**, 084023 (2000).
[36] C. van de Bruck, M. Dorca, R. Brandenberger, and A. Lukas, Phys. Rev. D **62**, 123515 (2000).
[37] H. Kodama, A. Ishibashi, and O. Seto, Phys. Rev. D **62**, 064022 (2000).
[38] S. Mukohyama, Phys. Rev. D **62**, 084015 (2000).
[39] S. Mukohyama, Class. Quantum Grav. **17**, 4777 (2000).
[40] S. Mukohyama, Phys. Rev. D **64**, 064006 (2001).
[41] D. Langlois, Phys. Rev. D **62**, 126012 (2000).
[42] D. Langlois, Phys. Rev. Lett. **86**, 2212 (2001).
[43] A. Neronov and I. Sachs, Phys. Lett. B **513**, 173 (2001).
[44] C. van de Bruck and M. Dorca, hep-th/0012073.
[45] H. Bridgman, K. Malik, and D. Wands, Phys. Rev. D **65**, 043502 (2002).
[46] K. Koyama and J. Soda, Phys. Rev. D **62**, 123502 (2000).
[47] K. Koyama and J. Soda, Phys. Rev. D **65**, 023514 (2002).
[48] V. Bozza, M. Gasperini, and G. Veneziano, Nucl. Phys. **B619**, 191 (2001).
[49] O. Seto and H. Kodama, Phys. Rev. D **63**, 123506 (2001).
[50] M. Giovannini, Phys. Rev. D **65**, 064008 (2002).
[51] J.D. Barrow and R. Maartens, gr-qc/0108073.
[52] J. Khoury and R.-J. Zhang, hep-th/0203274.
[53] M. Dorca and C. van de Bruck, Nucl. Phys. **B605**, 215 (2001).
[54] R. Dick, Phys. Lett. B **491**, 333 (2000).
[55] P. Bowcock, C. Charmousis, and R. Gregory, Class. Quantum Grav. **17**, 4745 (2000).
[56] P. Kraus, JHEP **9912**, 011 (1999).
[57] D. Ida, JHEP **0009**, 014 (2000).
[58] S. Mukohyama, T. Shiromizu, and K. Maeda, Phys. Rev. D **62**, 024028 (2000).
[59] C. Charmousis, R. Gregory, and V. Rubakov, Phys. Rev. D **62**, 067505 (2000).
[60] J.M. Bardeen, Phys. Rev. D **22**, 1882 (1980).
[61] H. Kodama and M. Sasaki, Prog. Theor. Phys. **78**, 1 (1984).
[62] J.M. Stewart and M. Walker, Proc. Roy. Soc. London, **A341** (1974).
[63] H.A. Bridgman, K.A. Malik, and D. Wands, Phys. Rev. D **63**, 084012 (2001).
[64] Y.S. Myung, hep-th/0009117
[65] W. Israel, Nuovo Cimento **B44**, 1 (1966); Erratum: *ibid.* **B48**, 463 (1967).
[66] R.M. Wald, *General Relativity*, (The University of Chicago Press, Chicago, 1984).
[67] B. Carter and P. Peter, Phys. Lett. B 466, 41 (1999).
[68] A.Z. Petrov, *Einstein Spaces*, (Pergamon Press, Oxford, 1969).
[69] B. Carter, Phys. Lett. B **228**, 446 (1989).

[70] W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation*, (Freeman, New York, 1970).

APPENDIX

In this Appendix, we give all the necessary formulae that were used to obtain the results presented in the text.

Here we will consider an $N+1+1$-dimensional bulk with, N-dimensional maximally symmetric, spacelike hypersurfaces of constant curvature k. Therefore, here $\alpha = 0, 1, \ldots, M$, where $M = N+1$, and $i = 1, 2, \ldots, N$. We will consider both the unperturbed (Section B–E) and perturbed cases (Section F–I). Both the bulk and the brane matter content are arbitrary as well as the global geometry of the bulk. Furthermore, we do not assume Z_2 symmetry.

APPENDIX A: SOME USEFUL FORMULAE

1. Some ensor definitions and sign conventions

Following the definitions of [70], we use the sign convention $(-++)$, that is the signature of the metric is $(+-\ldots-)$, and the Riemann and Ricci tensors are respectively defined by

$$R^{\alpha}{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^{\alpha}_{\beta\delta} - \partial_\delta \Gamma^{\alpha}_{\beta\gamma} + \Gamma^{\alpha}_{\gamma\mu}\Gamma^{\mu}_{\beta\delta} - \Gamma^{\alpha}_{\delta\nu}\Gamma^{\nu}_{\beta\gamma}, \tag{A1}$$

$$R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta}. \tag{A2}$$

The Weyl tensor is defined by

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - \frac{1}{N}\left(R_{\alpha\gamma}g_{\beta\delta} - R_{\alpha\delta}g_{\beta\gamma} + R_{\beta\delta}g_{\alpha\gamma} - R_{\beta\gamma}g_{\alpha\delta}\right) + \frac{1}{N(N+1)}R(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}). \tag{A3}$$

2. Brane-related metric quantities

The induced metric is defined by

$$\underline{\chi}_{ab} \equiv g_{\mu\nu}\ \partial_a X^\mu\ \partial_b X^\nu. \tag{A4}$$

The metric can be projected back in the bulk to give the first fundamental form

$$\underline{q}^{\alpha\beta} \equiv \underline{\chi}^{pq}\partial_p X^\alpha\ \partial_q X^\beta. \tag{A5}$$

More generally, any tensor $\underline{X}_{a_1\ldots a_n}$ defined for the brane can be projected back in the bulk using

$$\underline{X}^{\alpha_1\ldots\alpha_n} = \underline{X}^{p_1\ldots p_n}\ \partial_{p_1}X^{\alpha_1}\ \ldots\ \partial_{p_n}X^{\alpha_n}. \tag{A6}$$

In particular, for the brane Riemann tensor, one has

$$\underline{R}^{\alpha\beta\gamma\delta} = \partial_p X^\alpha\ \partial_q X^\beta\ \partial_r X^\gamma\ \partial_s X^\delta\ \underline{R}_{pqrs}. \tag{A7}$$

One can also define the normal spacelike unit vector $\underline{\perp}_\alpha$ to the brane according to

$$\underline{\perp}_\mu \partial_a X^\mu = 0 \quad , \qquad \underline{\perp}_\mu \underline{\perp}^\mu = -1, \tag{A8}$$

and the bulk metric evaluated at the brane position can be split into

$$g_{\alpha\beta} = \underline{q}_{\alpha\beta} - \underline{\perp}_{\alpha\beta}, \tag{A9}$$

with

$$\underline{\perp}_{\alpha\beta} \equiv \underline{\perp}_\alpha \underline{\perp}_\beta. \tag{A10}$$

Note that Eq. (A8) implies

$$\underline{\perp}_\sigma \partial^2_{ab} X^\sigma = -\partial_a X^\mu \, \partial_b X^\nu (\underline{\mathcal{K}}_{\mu\nu} + \Gamma^\rho_{\mu\nu} \underline{\perp}_\rho). \tag{A11}$$

One can also define the extrinsic curvature according to

$$\underline{\mathcal{K}}_{\alpha\beta} \equiv \underline{q}^\mu_{(\alpha} D_\mu \underline{\perp}_{\beta)}, \tag{A12}$$

which obeys the following relations

$$\underline{q}^\mu_\alpha \underline{\mathcal{K}}_{\mu\beta} = \underline{\mathcal{K}}_{\alpha\beta}, \tag{A13}$$
$$\underline{\perp}^\mu_\alpha \underline{\mathcal{K}}_{\mu\beta} = 0. \tag{A14}$$

With these definition, the brane Riemann and Ricci tensors, the brane scalar curvature and the brane Einstein tensor can be rewritten

$$\underline{R}_{abcd} = \partial_a X^\mu \, \partial_b X^\nu \, \partial_c X^\rho \, \partial_d X^\sigma \, \left(R_{\mu\nu\rho\sigma} - D_{(\mu} \underline{\perp}_{\rho)} D_{(\nu} \underline{\perp}_{\sigma)} - D_{(\mu} \underline{\perp}_{\sigma)} D_{(\nu} \underline{\perp}_{\rho)} \right), \tag{A15}$$
$$\underline{R}_{\alpha\beta\gamma\delta} = \underline{q}^\mu_\alpha \underline{q}^\nu_\beta \underline{q}^\rho_\gamma \underline{q}^\sigma_\delta R_{\mu\nu\rho\sigma} - \underline{\mathcal{K}}_{\alpha\gamma} \underline{\mathcal{K}}_{\beta\delta} + \underline{\mathcal{K}}_{\alpha\delta} \underline{\mathcal{K}}_{\beta\gamma}, \tag{A16}$$
$$\underline{R}_{\alpha\beta} = \frac{N-1}{N} \left(\underline{q}^\mu_\alpha \underline{q}^\nu_\beta R_{\mu\nu} - \frac{1}{N-1} \underline{q}_{\alpha\beta} \left(G_{\mu\nu} \underline{\perp}^{\mu\nu} + \frac{1}{N+1} G \right) \right)$$
$$- \underline{\mathcal{K}} \underline{\mathcal{K}}_{\alpha\beta} + \underline{\mathcal{K}}^\mu_\alpha \underline{\mathcal{K}}_{\mu\beta} + \underline{\mathcal{E}}_{\alpha\beta}, \tag{A17}$$
$$\underline{R} = R + R_{\mu\nu} \underline{\perp}^{\mu\nu} + \underline{\mathcal{K}}_{\mu\nu} \underline{\mathcal{K}}^{\mu\nu} - \underline{\mathcal{K}}^2, \tag{A18}$$
$$\underline{G}_{\alpha\beta} = \frac{N-1}{N} \left(\underline{q}^\mu_\alpha \underline{q}^\nu_\beta G_{\mu\nu} - \underline{q}_{\alpha\beta} \left(G_{\mu\nu} \underline{\perp}^{\mu\nu} + \frac{1}{N+1} G \right) \right)$$
$$- \underline{\mathcal{K}} \underline{\mathcal{K}}_{\alpha\beta} + \underline{\mathcal{K}}^\mu_\alpha \underline{\mathcal{K}}_{\mu\beta} + \frac{1}{2} \underline{q}_{\alpha\beta} \left(\underline{\mathcal{K}}^2 - \underline{\mathcal{K}}_{\mu\nu} \underline{\mathcal{K}}^{\mu\nu} \right) + \underline{\mathcal{E}}_{\alpha\beta}. \tag{A19}$$

APPENDIX B: BACKGROUND GEOMETRIC QUANTITIES

1. Metric

$$g_{00} = n^2 \quad , \quad g^{00} = \frac{1}{n^2}, \tag{B1}$$
$$g_{ij} = -a^2 \gamma_{ij} \quad , \quad g^{ij} = -\frac{1}{a^2} \gamma^{ij}, \tag{B2}$$
$$g_{MM} = -b^2 \quad , \quad g^{MM} = -\frac{1}{b^2}. \tag{B3}$$

2. Notations

$$\tilde{\mathcal{H}} = \frac{\dot{a}}{a} \quad , \quad \tilde{\mathcal{I}} = \frac{\dot{n}}{n} \quad , \quad \tilde{\mathcal{U}} = \frac{\dot{b}}{b}, \tag{B4}$$
$$\tilde{H} = \frac{a'}{a} \quad , \quad \tilde{I} = \frac{n'}{n} \quad , \quad \tilde{U} = \frac{b'}{b}. \tag{B5}$$

3. Christoffel symbols

$$\Gamma^0_{ij} = \frac{a^2}{n^2}\tilde{\mathcal{H}}\gamma_{ij} \quad , \qquad \Gamma^0_{MM} = \frac{b^2}{n^2}\tilde{\mathcal{U}}, \tag{B6}$$

$$\Gamma^0_{00} = \tilde{\mathcal{I}} \quad , \qquad \Gamma^i_{j0} = \tilde{\mathcal{H}}\delta^i_j \quad , \qquad \Gamma^M_{M0} = \tilde{\mathcal{U}}, \tag{B7}$$

$$\Gamma^0_{0M} = \tilde{I} \quad , \qquad \Gamma^i_{jM} = \tilde{H}\delta^i_j \quad , \qquad \Gamma^M_{MM} = \tilde{U}, \tag{B8}$$

$$\Gamma^M_{00} = \frac{n^2}{b^2}\tilde{I} \quad , \qquad \Gamma^M_{ij} = -\frac{a^2}{b^2}\tilde{H}\gamma_{ij}, \tag{B9}$$

$$\Gamma^k_{ij} = {}^{(N)}\Gamma^k_{ij}, \tag{B10}$$

$$\Gamma^\mu_{\mu 0} = \tilde{\mathcal{I}} + N\tilde{\mathcal{H}} + \tilde{\mathcal{U}} \quad , \qquad \Gamma^\mu_{\mu M} = \tilde{I} + N\tilde{H} + \tilde{U}. \tag{B11}$$

The superscript N means that the corresponding quantity is evaluated using the metric γ_{ij}.

4. Ricci tensor

$$R_{00} = -(\partial_\eta + \tilde{\mathcal{U}} - \tilde{\mathcal{I}})\tilde{\mathcal{U}} - N(\partial_\eta + \tilde{\mathcal{H}} - \tilde{\mathcal{I}})\tilde{\mathcal{H}} + \frac{n^2}{b^2}(\partial_y + \tilde{I} - \tilde{U})\tilde{I} + N\frac{n^2}{b^2}\tilde{H}\tilde{I}, \tag{B12}$$

$$R_{ij} = (N-1)\gamma_{ij}\mathrm{k} + \frac{a^2}{n^2}\gamma_{ij}(\partial_\eta + N\tilde{\mathcal{H}} + \tilde{\mathcal{U}} - \tilde{\mathcal{I}})\tilde{\mathcal{H}} - \frac{a^2}{b^2}\gamma_{ij}(\partial_y + N\tilde{H} + \tilde{I} - \tilde{U})\tilde{H}, \tag{B13}$$

$$R_{0M} = N(-\dot{\tilde{H}} - \tilde{\mathcal{H}}\tilde{H} + \tilde{\mathcal{H}}\tilde{I} + \tilde{\mathcal{U}}\tilde{H}), \tag{B14}$$

$$R_{MM} = -(\partial_y + \tilde{I} - \tilde{U})\tilde{I} - N(\partial_y + \tilde{H} - \tilde{U})\tilde{H} + \frac{b^2}{n^2}(\partial_\eta + \tilde{\mathcal{U}} - \tilde{\mathcal{I}})\tilde{\mathcal{U}} + N\frac{b^2}{n^2}\tilde{\mathcal{H}}\tilde{\mathcal{U}}. \tag{B15}$$

5. Scalar curvature

$$R = -N(N+1)\left(\frac{\tilde{\mathcal{H}}^2}{n^2} + \frac{\mathrm{k}}{a^2} - \frac{\tilde{H}}{b^2}\right) + 2N\frac{\mathrm{k}}{a^2} - \frac{2}{n^2}(\partial_\eta + \tilde{\mathcal{U}} - \tilde{\mathcal{I}})(\tilde{\mathcal{U}} + N\tilde{\mathcal{H}}) + \frac{2}{b^2}(\partial_y + \tilde{I} - \tilde{U})(\tilde{I} + N\tilde{H}). \tag{B16}$$

6. Einstein tensor

$$G_{00} = \frac{N(N-1)}{2}\left(\tilde{\mathcal{H}}^2 + \frac{n^2}{a^2}\mathrm{k} - \frac{n^2}{b^2}\tilde{H}^2\right) + N\tilde{\mathcal{H}}\tilde{\mathcal{U}} - N\frac{n^2}{b^2}(\tilde{H}' + \tilde{H}^2 - \tilde{H}\tilde{U}), \tag{B17}$$

$$\begin{aligned} G_{ij} = {} & -\frac{N(N-1)}{2}\left(\frac{a^2}{n^2}\tilde{\mathcal{H}}^2 + \mathrm{k} - \frac{a^2}{b^2}\tilde{H}^2\right)\gamma_{ij} + (N-1)\mathrm{k}\gamma_{ij} \\ & -\frac{a^2}{n^2}\gamma_{ij}(\partial_\eta + \tilde{\mathcal{U}} - \tilde{\mathcal{I}})\left(\tilde{\mathcal{U}} + (N-1)\tilde{\mathcal{H}}\right) + \frac{a^2}{b^2}\gamma_{ij}(\partial_y + \tilde{I} - \tilde{U})\left(\tilde{I} + (N-1)\tilde{H}\right), \end{aligned} \tag{B18}$$

$$G_{0M} = N(-\dot{\tilde{H}} - \tilde{\mathcal{H}}\tilde{H} + \tilde{\mathcal{H}}\tilde{I} + \tilde{\mathcal{U}}\tilde{H}), \tag{B19}$$

$$G_{MM} = -\frac{N(N-1)}{2}\left(\frac{b^2}{n^2}\tilde{\mathcal{H}}^2 + \frac{b^2}{a^2}\mathrm{k} - \tilde{H}^2\right) - N\frac{b^2}{n^2}(\dot{\tilde{\mathcal{H}}} + \tilde{\mathcal{H}}^2 - \tilde{\mathcal{H}}\tilde{\mathcal{I}}) + N\tilde{H}\tilde{I}. \tag{B20}$$

7. Riemann tensor

$$R_{0M0M} = b^2(\partial_\eta + \tilde{\mathcal{U}} - \tilde{\mathcal{I}})\tilde{\mathcal{U}} - n^2(\partial_y + \tilde{I} - \tilde{U})\tilde{I}, \tag{B21}$$

$$R_{0i0j} = a^2\gamma_{ij}(\partial_\eta + \tilde{\mathcal{H}} - \tilde{\mathcal{I}})\tilde{\mathcal{H}} - \frac{a^2n^2}{b^2}\gamma_{ij}\tilde{H}\tilde{I}, \tag{B22}$$

$$R_{0iMj} = a^2\gamma_{ij}(\dot{\tilde{H}} + \tilde{\mathcal{H}}\tilde{H} - \tilde{\mathcal{H}}\tilde{I} - \tilde{\mathcal{U}}\tilde{H}), \tag{B23}$$

$$R_{MiMj} = a^2\gamma_{ij}(\partial_y + \tilde{H} - \tilde{U})\tilde{H} - \frac{b^2a^2}{n^2}\gamma_{ij}\tilde{\mathcal{H}}\tilde{\mathcal{U}}, \tag{B24}$$

$$R_{ijkl} = -a^4(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk})\left(\frac{\tilde{\mathcal{H}}^2}{n^2} + \frac{\mathrm{k}}{a^2} - \frac{\tilde{H}^2}{b^2}\right). \tag{B25}$$

8. Weyl tensor

$$C_{0M0M} = \frac{N-1}{N+1} n^2 b^2 \mathcal{Z}, \tag{B26}$$

$$C_{0i0j} = -\frac{1}{N}\frac{N-1}{N+1} a^2 n^2 \mathcal{Z} \gamma_{ij}, \tag{B27}$$

$$C_{MiMj} = \frac{1}{N}\frac{N-1}{N+1} b^2 a^2 \mathcal{Z} \gamma_{ij}, \tag{B28}$$

$$C_{ijkl} = -\frac{2}{N(N+1)} (\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}) a^4 \mathcal{Z}, \tag{B29}$$

with

$$\mathcal{Z} = \frac{\mathrm{k}}{a^2} + \frac{1}{n^2}(\partial_\eta + \tilde{\mathcal{U}} - \tilde{\mathcal{I}})(\tilde{\mathcal{U}} - \tilde{\mathcal{H}}) - \frac{1}{b^2}(\partial_y + \tilde{I} - \tilde{U})(\tilde{I} - \tilde{H}). \tag{B30}$$

APPENDIX C: BACKGROUND MATTER CONTENT

1. Unit vectors

From the fields $X^{(\eta)} = \eta$ and $X^{(\perp)} = y$, one can build two unit vectors u^α and n^α:

$$u_\alpha = \frac{D_\alpha X^{(\eta)}}{\sqrt{D_\mu X^{(\eta)} D^\mu X^{(\eta)}}}, \tag{C1}$$

$$u_\alpha = (n, \mathbf{0}, 0), \tag{C2}$$

$$u^\alpha = \left(\frac{1}{n}, \mathbf{0}, 0\right), \tag{C3}$$

$$u_\mu u^\mu = 1, \tag{C4}$$

$$n_\alpha = \frac{D_\alpha X^{(\perp)}}{\sqrt{-D_\mu X^{(\perp)} D^\mu X^{(\perp)}}}, \tag{C5}$$

$$n_\alpha = (0, \mathbf{0}, b), \tag{C6}$$

$$n^\alpha = \left(0, \mathbf{0}, -\frac{1}{b}\right), \tag{C7}$$

$$n_\mu n^\mu = -1. \tag{C8}$$

One can define the following operators:

$$\partial_u \equiv u^\mu \partial_\mu = \frac{1}{n}\partial_\eta, \tag{C9}$$

$$\partial_n \equiv -n^\mu \partial_\mu = \frac{1}{b}\partial_y. \tag{C10}$$

As in the main text, we also use

$$\mathcal{H} = \frac{\partial_u a}{a} \quad , \qquad \mathcal{I} = \frac{\partial_u n}{n} \quad , \qquad \mathcal{U} = \frac{\partial_u b}{b}, \tag{C11}$$

$$H = \frac{\partial_n a}{a} \quad , \qquad I = \frac{\partial_n n}{n} \quad , \qquad U = \frac{\partial_n b}{b}. \tag{C12}$$

2. Stress-energy tensor

For the bulk matter, it is more convenient to introduce the unit vector U_α which represents the bulk $N+2$-velocity of the fluid:

$$\begin{aligned} U_\alpha &= (n\gamma, \mathbf{0}, -b\beta\gamma), && \text{(C13)} \\ U^\alpha &= \left(\frac{1}{n}\gamma, \mathbf{0}, \frac{1}{b}\beta\gamma\right), && \text{(C14)} \\ U_\mu U^\mu &= 1, && \text{(C15)} \\ \gamma^2(1-\beta^2) &= 1. && \text{(C16)} \end{aligned}$$

Here β represents the Lorentz boost which must be performed along the y axis in order to be in the rest frame of the bulk matter. As usual $\gamma = 1/\sqrt{1-\beta^2}$. Due to the symmetries of spacetime, the stress energy tensor of any component possesses N identical eigenvalues P_0. The other eigenvalues are ρ_0 (associated to the timelike eigenvector U^α) and Y_0 (associated to the spacelike eigenvector N^α). One has

$$\begin{aligned} N^\alpha &= \left(-\frac{1}{n}\beta\gamma, \mathbf{0}, -\frac{1}{b}\gamma\right), && \text{(C17)} \\ N_\alpha &= (-n\beta\gamma, \mathbf{0}, b\gamma). && \text{(C18)} \end{aligned}$$

$$\begin{aligned} T_{\alpha\beta} &= (P_0+\rho_0)U_\alpha U_\beta - (P_0 - Y_0)N_\alpha N_\beta - P_0 g_{\alpha\beta}, && \text{(C19)} \\ &= (P+\rho)u_\alpha u_\beta - (P-Y)n_\alpha n_\beta - P g_{\alpha\beta} - 2F u_{(\alpha} n_{\beta)}. && \text{(C20)} \end{aligned}$$

$$T_{00} = n^2\gamma^2(\rho_0 + \beta^2 Y_0) \equiv n^2\rho \quad , \quad T^{00} = \frac{1}{n^2}\rho, \qquad \text{(C21)}$$

$$T_{ij} = a^2 P_0 \gamma_{ij} \equiv a^2 P \gamma_{ij} \quad , \quad T^{ij} = \frac{1}{a^2} P \gamma^{ij}, \qquad \text{(C22)}$$

$$T_{0M} = -nb\beta\gamma^2(Y_0+\rho_0) \equiv -nbF \quad , \quad T^{0M} = \frac{1}{nb}F, \qquad \text{(C23)}$$

$$T_{MM} = b^2\gamma^2(Y_0 + \beta^2\rho_0) \equiv b^2 Y \quad , \quad T^{MM} = \frac{1}{b^2}Y. \qquad \text{(C24)}$$

$$\begin{aligned} \rho &= \gamma^2(\rho_0 + \beta^2 Y_0), && \text{(C25)} \\ Y &= \gamma^2(Y_0 + \beta^2\rho_0), && \text{(C26)} \\ P &= P_0, && \text{(C27)} \\ F &= \beta\gamma^2(\rho_0 + Y_0), && \text{(C28)} \end{aligned}$$

$$\begin{aligned} \frac{\beta}{1+\beta^2} &= \frac{F}{\rho+Y}, && \text{(C29)} \\ \beta &= \frac{\rho+Y}{2F}\left(1 - \sqrt{1 - \left(\frac{2F}{\rho+Y}\right)^2}\right), && \text{(C30)} \\ \rho_0 &= \frac{\rho - \beta^2 Y}{1+\beta^2} \\ &= \frac{\rho - Y}{2} + \sqrt{\frac{(\rho+Y)^2}{4} - F^2}, && \text{(C31)} \\ Y_0 &= \frac{Y - \beta^2\rho}{1+\beta^2} \\ &= \frac{Y-\rho}{2} + \sqrt{\frac{(\rho+Y)^2}{4} - F^2}, && \text{(C32)} \\ P_0 &= P. && \text{(C33)} \end{aligned}$$

3. Einstein equations

$$\frac{N(N-1)}{2}\left(\mathcal{H}^2+K-H^2\right)+N\mathcal{H}\mathcal{U}-N(\partial_n H+H^2) = \kappa_{N+2}\rho, \tag{C34}$$
$$-\frac{N(N-1)}{2}\left(\mathcal{H}^2+K-H^2\right)+(N-1)K$$
$$-(\partial_u+\mathcal{U})\left(\mathcal{U}+(N-1)\mathcal{H}\right)+(\partial_n+I)\left(I+(N-1)H\right) = \kappa_{N+2}P, \tag{C35}$$
$$N(\partial_u H+\mathcal{H}H-\mathcal{H}I)=N(\partial_n\mathcal{H}+\mathcal{H}H-H\mathcal{U}) = \kappa_{N+2}F, \tag{C36}$$
$$-\frac{N(N-1)}{2}\left(\mathcal{H}^2+K-H^2\right)-N(\partial_u\mathcal{H}+\mathcal{H}^2)+NHI = \kappa_{N+2}Y. \tag{C37}$$

4. Conservation equations

For any species,

$$D_\mu T_f^{\mu\alpha} = Q_f^\alpha, \tag{C38}$$
$$Q_f^\alpha = \Gamma_0^f U_f^\mu - D_0^f N_f^\mu, \tag{C39}$$
$$Q_\alpha^f = (n\gamma_f(\Gamma_0^f+\beta_f D_0^f),\mathbf{0},-b\gamma_f(D_0^f+\beta_f\Gamma_0^f)) \equiv (n\Gamma_f,\mathbf{0},-bD_f), \tag{C40}$$
$$\Gamma_f = \gamma(\Gamma_0^f+\beta D_0^f), \tag{C41}$$
$$D_f = \gamma(D_0^f+\beta\Gamma_0^f), \tag{C42}$$
$$\Gamma_0^f = \gamma(\Gamma_f-\beta D_f), \tag{C43}$$
$$D_0^f = \gamma(D_f-\beta\Gamma_f), \tag{C44}$$
$$Q_f^\alpha = \left(\frac{1}{n}\Gamma_f,\mathbf{0},\frac{1}{b}D_f\right), \tag{C45}$$
$$\sum_f \Gamma_f = \sum_f D_f = 0, \tag{C46}$$

$$\partial_u\rho_f+N\mathcal{H}(P_f+\rho_f)+\mathcal{U}(Y_f+\rho_f)+(\partial_n+NH+2I)F_f = \Gamma_f, \tag{C47}$$
$$(\partial_u+N\mathcal{H}+2\mathcal{U})F_f+\partial_n Y_f+I(Y_f+\rho_f)+NH(Y_f-P_f) = D_f. \tag{C48}$$

APPENDIX D: BACKGROUND BRANE-RELATED QUANTITIES

1. Brane position

In general, one has the brane position X^α as a function of $N+1$ variables σ^a. We choose

$$X^0 = \sigma^0, \tag{D1}$$
$$X^i = \sigma^i, \tag{D2}$$
$$X^M = \mathrm{y_b}. \tag{D3}$$

2. Induced metric

One first builds the unit vector orthogonal to the brane:

$$\underline{\perp}_\mu\frac{\partial X^\mu}{\partial\sigma^a} = 0, \tag{D4}$$
$$\underline{\perp}_\alpha = (0,\mathbf{0},b), \tag{D5}$$
$$\underline{\perp}^\alpha = \left(0,\mathbf{0},-\frac{1}{b}\right). \tag{D6}$$

The components of this vector have the same functional form but possibly different numerical values when evaluated at $y = \mathrm{y_b}^+$ and $y = \mathrm{y_b}^-$. Then the induced metric is given by:

$$\underline{q}_{\alpha\beta} = g_{\alpha\beta} + \underline{\perp}_\alpha \underline{\perp}_\beta, \tag{D7}$$
$$\underline{q}_{\alpha\mu}\underline{\perp}^\mu = 0. \tag{D8}$$

$$\underline{q}_{00} = n^2, \tag{D9}$$
$$\underline{q}_{ij} = -a^2\gamma_{ij}. \tag{D10}$$

3. First Israel conditions

For any quantity f, we define

$$f = [f]\left(\theta(y - \mathrm{y_b}) - \tfrac{1}{2}\right) + \langle f \rangle, \tag{D11}$$

where $[f]$ is the discontinuity of f, $\langle f \rangle$ is the continuous part of f and θ is the Heaviside function. The first Israel condition states that $\underline{q}_{\alpha\beta}(\mathrm{y_b}^+) = \underline{q}_{\alpha\beta}(\mathrm{y_b}^-)$:

$$[a] = 0, \tag{D12}$$
$$[n] = 0. \tag{D13}$$

In particular, this means that b is allowed to be discontinuous at the brane position. Also, the derivatives of a and n with respect to y can be discontinuous.

4. Extrinsic curvature

$$\underline{\mathcal{K}}_{\alpha\beta} = \underline{q}^\mu_{(\alpha} D_\mu \underline{\perp}_{\beta)}, \tag{D14}$$
$$\underline{\mathcal{K}}_{\alpha\mu}\underline{\perp}^\mu = 0. \tag{D15}$$

$$\underline{\mathcal{K}}_{00} = -\frac{n^2}{b}\tilde{I}, \tag{D16}$$
$$\underline{\mathcal{K}}_{ij} = \frac{a^2}{b}\tilde{H}\gamma_{ij}, \tag{D17}$$
$$\underline{\mathcal{K}} \equiv g^{\mu\nu}\underline{\mathcal{K}}_{\mu\nu} = \underline{q}^{\mu\nu}\underline{\mathcal{K}}_{\mu\nu} = -\frac{1}{b}(\tilde{I} + N\tilde{H}), \tag{D18}$$
$$\underline{\mathcal{K}}^{\mu\nu}\underline{\mathcal{K}}_{\mu\nu} = \frac{1}{b^2}(\tilde{I}^2 + N\tilde{H}^2), \tag{D19}$$
$$\underline{\mathcal{K}}^2 - \underline{\mathcal{K}}^{\mu\nu}\underline{\mathcal{K}}_{\mu\nu} = \frac{N}{b^2}((N-1)\tilde{H}^2 + 2\tilde{H}\tilde{I}). \tag{D20}$$

5. Stress-energy tensor

Formally, one can take a stress-energy tensor of the above form to describe the brane content, provided that we have

$$\rho_0 \equiv \mathrm{D}\underline{\rho}, \tag{D21}$$
$$P_0 \equiv \mathrm{D}\underline{P}, \tag{D22}$$
$$Y_0 \equiv 0, \tag{D23}$$
$$\gamma = 1, \tag{D24}$$
$$\beta = 0, \tag{D25}$$

with

$$\mathrm{D} = \frac{\sqrt{|\underline{q}|}}{\sqrt{|g|}}\delta(y - \mathrm{y_b}). \tag{D26}$$

The condition $\gamma = 1$ and $\beta = 0$ has not necessarily to be satisfied but is a consequence of the coordinate choice to put the brane at rest with respect to the coordinate system. Because of the Dirac term (D26), $\underline{P}$ and $\underline{\rho}$ depend on η, x^i only. Since the stress energy tensor of the brane is strictly zero elsewhere, its eigenvectors are not defined outside this hypersurface. The vector $\underline{\perp}^\alpha$ appears as the analog of the vector N^α as an eigenvector associated to the eigenvalue $Y_0 = 0$. Equivalently one can define an $N+2$-velocity $\underline{u}_\alpha$ which corresponds to the eigenvector associated with the eigenvalue ρ_0. Therefore,

$$
\begin{aligned}
T^{\rm b}_{\alpha\beta} &= {\rm D}\underline{T}_{\alpha\beta}, && \text{(D27)}\\
\underline{T}_{\alpha\mu}\ \perp^\mu &= 0, && \text{(D28)}\\
\underline{T}_{\alpha\beta} &= (\underline{P}+\underline{\rho})\underline{u}_\alpha\underline{u}_\beta - \underline{P}\underline{q}_{\alpha\beta}, && \text{(D29)}\\
\underline{u}_\alpha &\equiv (n, \mathbf{0}, 0). && \text{(D30)}
\end{aligned}
$$

6. $\underline{\mathcal{S}}_{\alpha\beta}$ tensor

$$
\underline{\mathcal{S}}_{\alpha\beta} = \underline{T}_{\alpha\beta} - \frac{1}{N}\underline{T}\underline{q}_{\alpha\beta}. \qquad \text{(D31)}
$$

$$
\begin{aligned}
\underline{\mathcal{S}}_{00} &= n^2\left(\frac{N-1}{N}\underline{\rho} + \underline{P}\right), && \text{(D32)}\\
\underline{\mathcal{S}}_{ij} &= \frac{1}{N}a^2\underline{\rho}\gamma_{ij}. && \text{(D33)}
\end{aligned}
$$

7. Second Israel condition

$$
[\underline{\mathcal{K}}_{\alpha\beta}] = -\kappa_{N+2}\underline{\mathcal{S}}_{\alpha\beta}. \qquad \text{(D34)}
$$

$$
\begin{aligned}
\left[\frac{\tilde{I}}{b}\right] &= \kappa_{N+2}\left(\frac{N-1}{N}\underline{\rho} + \underline{P}\right), && \text{(D35)}\\
\left[\frac{\tilde{H}}{b}\right] &= -\kappa_{N+2}\frac{1}{N}\underline{\rho}, && \text{(D36)}\\
-N\left[\frac{\tilde{H}}{b}\right] &= \kappa_{N+2}\underline{\rho}, && \text{(D37)}\\
\left[\frac{\tilde{I}}{b}\right] + (N-1)\left[\frac{\tilde{H}}{b}\right] &= \kappa_{N+2}\underline{P}, && \text{(D38)}\\
\left[\frac{\tilde{I}}{b}\right] - \left[\frac{\tilde{H}}{b}\right] &= \kappa_{N+2}(\underline{P}+\underline{\rho}). && \text{(D39)}
\end{aligned}
$$

8. Projected Weyl tensor

$$
\begin{aligned}
\underline{\mathcal{E}}_{\alpha\beta} &= C_{\alpha\mu\beta\nu}\underline{\perp}^\mu\underline{\perp}^\nu, && \text{(D40)}\\
g^{\mu\nu}\underline{\mathcal{E}}_{\mu\nu} = \underline{q}^{\mu\nu}\underline{\mathcal{E}}_{\mu\nu} = \underline{\perp}^\mu\underline{\mathcal{E}}_{\mu\alpha} &= 0. && \text{(D41)}
\end{aligned}
$$

$$
\begin{aligned}
\underline{\mathcal{E}}_{00} &= \frac{N-1}{N+1}n^2\mathcal{Z}, && \text{(D42)}\\
\underline{\mathcal{E}}_{ij} &= \frac{N-1}{N(N+1)}a^2\mathcal{Z}\gamma_{ij}. && \text{(D43)}
\end{aligned}
$$

APPENDIX E: BRANE POINT OF VIEW, UNPERTURBED CASE

Unless otherwise noted, all the quantities are evaluated at the brane position. The quantity $\langle\partial_n\rangle X$ stands for the continuous part of $\partial_n X$ at the brane position.

1. Friedmann equation

Taking the continuous part of the Einstein equation at the brane position, we get

$$\begin{aligned}\frac{N(N-1)}{2}\left(\mathcal{H}^2+K\right) &= \frac{\kappa_{N+2}^2}{8}\frac{N+1}{N}\underline{\rho}^2+\kappa_{N+2}\langle\rho_{\rm B}\rangle \\ &\quad -N\mathcal{H}\langle\mathcal{U}\rangle+N\langle\partial_n\rangle\langle H\rangle+\frac{1}{2}N(N+1)\langle H\rangle^2\,, \qquad \text{(E1)}\\ -\frac{1}{2}(N-1)\left(\mathcal{H}^{-1}\partial_u+N\right)\left(\mathcal{H}^2+K\right) &= -\frac{\kappa_{N+2}^2}{8}\left(2\underline{P}^2+2\frac{N-1}{N}\underline{P}\underline{\rho}+\frac{N-1}{N}\underline{\rho}^2\right)+\kappa_{N+2}\langle P_{\rm B}\rangle \\ &\quad +\langle(\partial_u+\mathcal{U}+(N-1)\mathcal{H})\mathcal{U}\rangle \\ &\quad -\langle\partial_n+I\rangle\langle I+(N-1)H\rangle-\frac{1}{2}N(N-1)\langle H\rangle^2\,. \qquad \text{(E2)}\end{aligned}$$

As such, these equations are not yet very useful because they involve many terms which are not explicit 'brane variables'.

2. New Friedmann equation

Consider the combination $\langle\mathcal{H}\{MM\}+H\{0M\}\rangle$ of the Einstein equations. It yields

$$(\partial_u+(N+1)\mathcal{H})\left(\frac{N(N-1)}{2}\left(\mathcal{H}^2+K-\langle H\rangle^2\right)-\frac{N-1}{8N}\kappa_{N+2}^2\underline{\rho}^2\right)=-(N-1)\kappa_{N+2}\langle\mathcal{H}Y+HF\rangle\,. \qquad \text{(E3)}$$

In the case $\langle HF\rangle=\partial_\eta\langle Y\rangle=0$, they can be integrated exactly and we find

$$\frac{N(N-1)}{2}\left(\mathcal{H}^2+K\right) = \frac{N-1}{8N}\kappa_{N+2}^2\underline{\rho}^2-\frac{N-1}{N+1}\kappa_{N+2}\langle Y\rangle+\frac{\mathcal{C}}{a^{N+1}}. \qquad \text{(E4)}$$

3. Friedmann equations using the Weyl tensor

In general, the Friedmann equation can conveniently be rewritten using the Weyl tensor. One has

$$\begin{aligned}\frac{N(N-1)}{2}\left(\mathcal{H}^2+K\right) &= \frac{N-1}{8N}\kappa_{N+2}^2\left(\sum_{\rm b}\underline{\rho}_{\rm b}\right)^2 \\ &\quad +\frac{N(N-1)}{2}\langle H\rangle^2 \\ &\quad +\frac{N-1}{N+1}\kappa_{N+2}\sum_{\rm B}\langle P_{\rm B}+\rho_{\rm B}-Y_{\rm B}\rangle+\frac{N-1}{N+1}\langle\mathcal{Z}\rangle\,, \qquad \text{(E5)}\\ -\frac{N-1}{2}\left(N+\mathcal{H}^{-1}\partial_u\right)\left(\mathcal{H}^2+K\right) &= \frac{N-1}{8N}\kappa_{N+2}^2\left(\sum_{\rm b}\underline{\rho}_{\rm b}\right)\left(\sum_{\rm b}(\underline{\rho}_{\rm b}+2\underline{P}_{\rm b})\right) \\ &\quad -\frac{N(N-1)}{2}\langle H\rangle^2-(N-1)\langle H\rangle\langle I-H\rangle \\ &\quad +\frac{N-1}{N(N+1)}\kappa_{N+2}\sum_{\rm B}\langle P_{\rm B}+\rho_{\rm B}+NY_{\rm B}\rangle+\frac{N-1}{N(N+1)}\langle\mathcal{Z}\rangle\,. \qquad \text{(E6)}\end{aligned}$$

4. Relationship between $\langle\underline{\mathcal{K}}_{\alpha\beta}\rangle$ and $\left[\underline{\mathcal{E}}_{\alpha\beta}\right]$

$$\underline{\rho}\,\langle H\rangle \;=\; \frac{1}{N+1}\left([P+\rho-Y]+\frac{1}{\kappa_{N+2}}\,[\mathcal{Z}]\right), \tag{E7}$$

$$(N\underline{P}+\underline{\rho})\,\langle H\rangle-\underline{\rho}\,\langle I\rangle \;=\; \frac{1}{N+1}\left([P+\rho+NY]+\frac{1}{\kappa_{N+2}}\,[\mathcal{Z}]\right). \tag{E8}$$

5. Conservation equation

These can be found either by taking the singular part of (C47), or by considering the discontinuity of (B19). A bulk energy exchange term $\Gamma_{\rm B}$ can have a singular component $\Gamma_{\rm B}^{(\rm D)}$ so that $\Gamma_{\rm B}=\mathrm{D}\Gamma_{\rm B}^{(\rm D)}+[\Gamma_{\rm B}]\left(\theta(y-y_{\rm b})-\frac{1}{2}\right)+\langle\Gamma_{\rm B}\rangle$.

$$\partial_u\underline{\rho}_{\rm b}+N\mathcal{H}(\underline{P}_{\rm b}+\underline{\rho}_{\rm b}) \;=\; \underline{\Gamma}_{\rm b}, \tag{E9}$$

$$[F_{\rm B}] \;=\; \Gamma_{\rm B}^{(\rm D)}, \tag{E10}$$

$$\sum_{\rm b}\underline{\Gamma}_{\rm b} \;=\; -\sum_{\rm B}[F_{\rm B}]\,. \tag{E11}$$

One also has the sail equation

$$-\langle I\rangle\sum_{\rm b}\underline{\rho}_{\rm b}+N\langle H\rangle\sum_{\rm b}\underline{P}_{\rm b} \;=\; \sum_{\rm B}[Y_{\rm B}]\,. \tag{E12}$$

APPENDIX F: PERTURBED GEOMETRIC VARIABLES

1. Metric

$$\delta g_{00}=2n^2A \quad,\quad \delta g^{00}=-\frac{1}{n^2}2A, \tag{F1}$$

$$\delta g_{0i}=anB_i \quad,\quad \delta g^{0i}=\frac{1}{an}B^i, \tag{F2}$$

$$\delta g_{ij}=-a^2h_{ij} \quad,\quad \delta g^{ij}=\frac{1}{a^2}h^{ij}, \tag{F3}$$

$$\delta g_{0M}=nbB_\perp \quad,\quad \delta g^{0M}=\frac{1}{nb}B_\perp, \tag{F4}$$

$$\delta g_{iM}=baE_{\perp i} \quad,\quad \delta g^{iM}=-\frac{1}{ba}E_\perp^i, \tag{F5}$$

$$\delta g_{MM}=2b^2E_{\perp\perp} \quad,\quad \delta g^{MM}=-\frac{1}{b^2}2E_{\perp\perp}. \tag{F6}$$

$$B_i \;=\; \nabla_iB+\bar{B}_i, \tag{F7}$$

$$h_{ij} \;=\; 2C\gamma_{ij}+2E_{ij}, \tag{F8}$$

$$E_{ij} \;=\; \nabla_{(i}E_{j)}+\bar{\bar{E}}_{ij}, \tag{F9}$$

$$E_i \;=\; \nabla_iE+\bar{E}_i, \tag{F10}$$

$$E_{\perp i} \;=\; \nabla_iE_\perp+\bar{E}_{(\perp)\,i}. \tag{F11}$$

All 3-vectors indices are raised and lowered using metric γ_{ij}. ∇_i represents its associated covariant derivative and $\nabla^2=\nabla_i\nabla^i$. Barred vectors are divergenceless, double barred tensors are divergenceless and traceless with respect to γ_{ij} and ∇_i.

2. Infinitesimal coordinate transformation

Under an infinitesimal coordinate transformation $x^\alpha \to x^\alpha + \xi^\alpha$, the perturbed part of a tensor transforms as

$$\delta T^{\alpha_1\ldots\alpha_u}_{\beta_1\ldots\beta_d} \to \delta T^{\alpha_1\ldots\alpha_u}_{\beta_1\ldots\beta_d} + \xi^\mu \partial_\mu T^{\alpha_1\ldots\alpha_u}_{\beta_1\ldots\beta_d} - \partial_{\mu_i}\xi^{\alpha_i} T^{\alpha_1\ldots\mu_i\ldots\alpha_u}_{\beta_1\ldots\beta_d} + \partial_{\beta_j}\xi^{\nu_j} T^{\alpha_1\ldots\alpha_u}_{\beta_1\ldots\nu_j\ldots\beta_d}. \tag{F12}$$

Setting

$$\xi^\alpha = (T, L^i, L^\perp), \tag{F13}$$
$$L^i = \nabla^i L + \bar{L}^i, \tag{F14}$$

the metric perturbations transform into

$$A \to A + \dot{T} + \tilde{\mathcal{I}} T + \tilde{I} L^\perp, \tag{F15}$$
$$B_i \to B_i - \frac{a}{n}\dot{L}_i + \frac{n}{a}\nabla_i T, \tag{F16}$$
$$C \to C + \tilde{\mathcal{H}} T + \tilde{H} L^\perp, \tag{F17}$$
$$E_i \to E_i + L_i, \tag{F18}$$
$$\bar{\bar{E}}_{ij} \to \bar{\bar{E}}_{ij}, \tag{F19}$$
$$B_\perp \to B_\perp - \frac{b}{n}\dot{L}^\perp + \frac{n}{b}T', \tag{F20}$$
$$E_{\perp i} \to E_{\perp i} - \frac{a}{b}L_i' - \frac{b}{a}\nabla_i L^\perp, \tag{F21}$$
$$E_{\perp\perp} \to E_{\perp\perp} - \tilde{\mathcal{U}} T - L^{\perp\prime} - \tilde{U} L^\perp, \tag{F22}$$
$$\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) \to \left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) + T, \tag{F23}$$
$$\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right) \to \left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right) - L^\perp. \tag{F24}$$

There is one subtlety due to the fact that b may be discontinuous. We consider the above infinitesimal coordinate transformation. For the $\{MM\}$ component, we have

$$g_{MM} \to g_{MM} - T\partial_\eta(b^2) - 2b\partial_y(bL^\perp). \tag{F25}$$

For the coordinate change to be valid, the metric components must remain finite, therefore one *must* have

$$[bL^\perp] = 0. \tag{F26}$$

If b is continuous, then $L^\perp$ can be an arbitrary (continuous) coordinate transformation, but if b is discontinuous, then $L^\perp(\eta, x^i, \mathrm{y_b}) = 0$. Geometrically, this is related to the fact that the coordinate system is allowed to exhibit some pathologies only at the brane position, but not in its vicinity.

3. Gauge invariant metric perturbations

Using the transformation laws for $\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right)$, E^i, $-\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right)$, it is possible to construct the following gauge invariant quantities:

$$\Psi = A - (\partial_\eta + \tilde{\mathcal{I}})\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) + \tilde{I}\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right), \tag{F27}$$
$$\Phi = -C + \tilde{\mathcal{H}}\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) - \tilde{H}\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right), \tag{F28}$$
$$\Sigma = B_\perp - \frac{n}{b}\partial_y\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) - \frac{b}{n}\partial_\eta\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right), \tag{F29}$$
$$h = E_{\perp\perp} + \tilde{\mathcal{U}}\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) - (\partial_y + \tilde{U})\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right), \tag{F30}$$
$$\bar{\Sigma}_i = \bar{B}_i + \frac{a}{n}\dot{\bar{E}}_i, \tag{F31}$$
$$\bar{h}_i = \bar{E}_{(\perp)i} + \frac{a}{b}\bar{E}_i'. \tag{F32}$$

$$\delta g_{00} = 2n^2\Psi + (2g_{00}\partial_\eta + \dot{g}_{00})\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) - g'_{00}\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right), \tag{F33}$$

$$\delta g_{0i} = -\frac{n}{a}g_{ij}\bar{\Sigma}^j + g_{00}\nabla_i\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) + g_{ij}\dot{E}^j, \tag{F34}$$

$$\delta g_{ij} = \dot{g}_{ij}\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) - g'_{ij}\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right) + 2g_{k(i}(E^k_{j)} - \delta^k_{j)}\Phi), \tag{F35}$$

$$\delta g_{0M} = nb\Sigma + g_{00}\partial_y\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) - g_{MM}\partial_\eta\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right), \tag{F36}$$

$$\delta g_{iM} = -\frac{b}{a}g_{ij}\bar{h}^j - g_{MM}\nabla_i\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) + g_{ij}E^{j\prime}, \tag{F37}$$

$$\delta g_{MM} = 2b^2h + \dot{g}_{MM}\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) - (2g_{MM}\partial_y + g'_{MM})\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right). \tag{F38}$$

4. Christoffel symbols

$$\delta\Gamma^0_{00} = \dot{A} - \frac{n}{b}\tilde{I}B_\perp, \tag{F39}$$

$$\delta\Gamma^0_{0i} = \nabla_i A - \frac{a}{n}\tilde{\mathcal{H}}B_i, \tag{F40}$$

$$\delta\Gamma^0_{ij} = \frac{a^2}{n^2}\left(-2A\tilde{\mathcal{H}}\gamma_{ij} + \tilde{\mathcal{H}}h_{ij} + \frac{1}{2}\dot{h}_{ij}\right) + \frac{a}{n}\nabla_{(i}B_{j)} + \frac{a^2}{nb}\tilde{H}B_\perp\gamma_{ij}, \tag{F41}$$

$$\delta\Gamma^0_{0M} = A' - \frac{b}{n}\tilde{\mathcal{U}}B_\perp, \tag{F42}$$

$$\delta\Gamma^0_{iM} = \frac{1}{2}\frac{a}{n}(\partial_y + \tilde{I} - \tilde{H})B_i - \frac{1}{2}\frac{ba}{n^2}(\partial_\eta + \tilde{\mathcal{H}} + \tilde{\mathcal{U}})E_{\perp i} + \frac{1}{2}\frac{b}{n}\nabla_i B_\perp, \tag{F43}$$

$$\delta\Gamma^0_{MM} = \frac{b}{n}(\partial_y + \tilde{I})B_\perp - \frac{b^2}{n^2}(\partial_\eta + 2\tilde{\mathcal{U}})E_{\perp\perp} - 2\frac{b^2}{n^2}\tilde{\mathcal{U}}A, \tag{F44}$$

$$\delta\Gamma^i_{00} = \frac{n^2}{a^2}\nabla^i A - \frac{n}{a}(\partial_\eta + \tilde{\mathcal{H}})B^i + \frac{n^2}{ba}\tilde{I}E^i_\perp, \tag{F45}$$

$$\delta\Gamma^i_{0j} = \frac{1}{2}\dot{h}^i_j + \frac{1}{2}\frac{n}{a}(\nabla^i B_j - \nabla_j B^j), \tag{F46}$$

$$\delta\Gamma^i_{jk} = \frac{1}{2}(\nabla_j h^i_k + \nabla_k h^i_j - \nabla^k h_{ij}) + \gamma_{jk}\left(\frac{a}{n}\tilde{\mathcal{H}}B^i - \frac{a}{b}\tilde{H}E^i_\perp\right), \tag{F47}$$

$$\delta\Gamma^i_{0M} = -\frac{1}{2}\frac{b}{a}(\partial_\eta + \tilde{\mathcal{H}} - \tilde{\mathcal{U}})E^i_\perp - \frac{1}{2}\frac{n}{a}(\partial_y + \tilde{H} - \tilde{I})B^i + \frac{1}{2}\frac{nb}{a^2}\nabla^i B_\perp, \tag{F48}$$

$$\delta\Gamma^i_{jM} = \frac{1}{2}h^{i\prime}_j + \frac{1}{2}\frac{b}{a}(\nabla^i E_{\perp j} - \nabla_j E^i_\perp), \tag{F49}$$

$$\delta\Gamma^i_{MM} = -\frac{b}{a}(\partial_y + \tilde{H})E^i_\perp + \frac{b^2}{an}\tilde{\mathcal{U}}B^i + \frac{b^2}{a^2}\nabla^i E_{\perp\perp}, \tag{F50}$$

$$\delta\Gamma^M_{00} = -\frac{n}{b}(\partial_\eta + \tilde{\mathcal{U}})B_\perp + \frac{n^2}{b^2}(\partial_y + 2\tilde{I})A + 2\frac{n^2}{b^2}\tilde{I}E_{\perp\perp}, \tag{F51}$$

$$\delta\Gamma^M_{0i} = \frac{1}{2}\frac{an}{b^2}(\partial_y + \tilde{I} + \tilde{H})B_i - \frac{1}{2}\frac{a}{b}(\partial_\eta + \tilde{\mathcal{U}} - \tilde{\mathcal{H}})E_{\perp i} - \frac{1}{2}\frac{n}{b}\nabla_i B_\perp, \tag{F52}$$

$$\delta\Gamma^M_{ij} = -\frac{a^2}{b^2}\left(2\tilde{H}E_{\perp\perp}\gamma_{ij} + \tilde{H}h_{ij} + \frac{1}{2}h'_{ij}\right) - \frac{a}{b}\nabla_{(i}E_{\perp j)} + \frac{a^2}{nb}\tilde{\mathcal{H}}B_\perp\gamma_{ij}, \tag{F53}$$

$$\delta\Gamma^M_{0M} = -\dot{E}_{\perp\perp} + \frac{n}{b}\tilde{I}B_\perp, \tag{F54}$$

$$\delta\Gamma^M_{iM} = -\nabla_i E_{\perp\perp} + \frac{a}{b}\tilde{H}E_{\perp i}, \tag{F55}$$

$$\delta\Gamma^M_{MM} = -E_{\perp\perp}{}' + \frac{b}{n}\tilde{\mathcal{U}}B_\perp, \tag{F56}$$

$$\delta\Gamma^{\ \mu}_{\mu 0} = \partial_\eta(A - E_{\perp\perp} + NC + \nabla^2 E), \tag{F57}$$

$$\delta\Gamma^{\ \mu}_{\mu i} = \nabla_i(A - E_{\perp\perp} + NC + \nabla^2 E), \tag{F58}$$

$$\delta\Gamma^{\ \mu}_{\mu M} = \partial_y(A - E_{\perp\perp} + NC + \nabla^2 E). \tag{F59}$$

5. Ricci tensor

From now on, we shall write any perturbation variables mostly with the gauge invariant quantities defined above. The non-gauge invariant terms will be put under the form of some factors involving components of the corresponding unperturbed tensor multiplied by $\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right)$, E^i and $\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right)$.

$$\begin{aligned}
\delta R_{00} &= \frac{n^2}{a^2}\nabla^2\Psi \\
&+(N\tilde{\mathcal{H}} + \tilde{\mathcal{U}})\partial_\eta\Psi + (\partial_\eta^2 + (2\tilde{\mathcal{U}} - \tilde{\mathcal{I}})\partial_\eta)h + N(\partial_\eta^2 + (2\tilde{\mathcal{H}} - \tilde{\mathcal{I}})\partial_\eta)\Phi \\
&+\frac{n^2}{b^2}(\partial_y + N\tilde{H} + \tilde{I} - \tilde{U})(\partial_y\Psi + 2\tilde{I}\Psi + 2\tilde{I}h) - \frac{n^2}{b^2}\tilde{I}\partial_y(\Psi + h + N\Phi) \\
&-\frac{n}{b}(\partial_\eta\partial_y + \tilde{\mathcal{U}}\partial_y + (N\tilde{H} + \tilde{I})\partial_\eta + N\tilde{\mathcal{H}}\tilde{I} + N\tilde{\mathcal{U}}\tilde{H} + \dot{\tilde{I}} + \tilde{\mathcal{U}}')\Sigma \\
&+(\dot{R}_{00} + 2R_{00}\partial_\eta)\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) - (R'_{00} + 2R_{0M}\partial_\eta)\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right),
\end{aligned} \tag{F60}$$

$$\begin{aligned}
\delta R_{0i} &= ((N-1)\tilde{\mathcal{H}} + \tilde{\mathcal{U}})\nabla_i\Psi + (\partial_\eta - \tilde{\mathcal{H}} + \tilde{\mathcal{U}})\nabla_i h + (N-1)\partial_\eta\nabla_i\Phi \\
&-\frac{1}{2}\frac{n}{b}(\partial_y + (N-2)\tilde{H} + 2\tilde{I})\Sigma \\
&+\frac{1}{2}\frac{n}{a}(\nabla^2 + K(N-1))\bar{\Sigma}_i - \frac{n}{a}R_{ij}\bar{\Sigma}^j \\
&+\frac{1}{2}\frac{an}{b^2}(\partial_y + (N+1)\tilde{H} - \tilde{U})\left((\partial_y + \tilde{I} - \tilde{H})\bar{\Sigma}_i\right) - \frac{1}{2}\frac{a}{b}(\partial_y + (N+1)\tilde{H} - \tilde{I})\left((\partial_\eta + \tilde{\mathcal{U}} - \tilde{\mathcal{H}})\bar{h}_i\right) \\
&+R_{00}\nabla_i\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) - R_{0M}\nabla_i\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right) + R_{ij}\dot{E}^j,
\end{aligned} \tag{F61}$$

$$\begin{aligned}
\delta R_{ij} &= (\gamma_{ij}\nabla^2 + N\nabla_{ij} + 2K(N-1)\gamma_{ij})\Phi - \nabla_{ij}(\Psi + 2\Phi - h) \\
&-\frac{a^2}{n^2}\gamma_{ij}\left(2\dot{\tilde{\mathcal{H}}} + 2\tilde{\mathcal{H}}(N\tilde{\mathcal{H}} + \tilde{\mathcal{U}} - \tilde{\mathcal{I}})\Psi + \tilde{\mathcal{H}}\partial_\eta(\Psi + h + N\Phi)\right) \\
&-\frac{a^2}{b^2}\gamma_{ij}\left(2\tilde{H}' + 2\tilde{H}(N\tilde{H} + \tilde{I} - \tilde{U})h + \tilde{H}\partial_y(\Psi + h + N\Phi)\right) \\
&+\frac{a^2}{nb}\gamma_{ij}\left(\tilde{\mathcal{H}}\partial_y + \tilde{H}\partial_\eta + 2N\tilde{\mathcal{H}}\tilde{H} + \tilde{\mathcal{H}}' + \dot{\tilde{H}}\right)\Sigma \\
&+\frac{a}{n}(\partial_\eta + (N-1)\tilde{\mathcal{H}} + \tilde{\mathcal{U}})\nabla_{(i}\bar{\Sigma}_{j)} - \frac{a}{b}(\partial_y + (N-1)\tilde{H} + \tilde{I})\nabla_{(i}\bar{h}_{j)} \\
&+\left(\frac{a^2}{n^2}(\partial_\eta^2 + (N\tilde{\mathcal{H}} + \tilde{\mathcal{U}} - \tilde{\mathcal{I}})\partial_\eta) - \frac{a^2}{b^2}(\partial_y^2 + (N\tilde{H} + \tilde{I} - \tilde{U})\partial_y)\right)(\bar{\bar{E}}_{ij} - \gamma_{ij}\Phi) - (\nabla^2 - 2K)\bar{\bar{E}}_{ij} \\
&+\dot{R}_{ij}\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) - R'_{ij}\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right) + 2R_{k(i}(E^k_{j)} - \delta^k_{j)}\Phi),
\end{aligned} \tag{F62}$$

$$\begin{aligned}
\delta R_{0M} &= N\tilde{\mathcal{H}}\partial_y\Psi - N\tilde{H}\partial_\eta h + N(\partial_\eta\partial_y + (\tilde{H} - \tilde{I})\partial_\eta + (\tilde{\mathcal{H}} - \tilde{\mathcal{U}})\partial_y)\Phi \\
&+\frac{1}{2}\frac{nb}{a^2}\nabla^2\Sigma + \frac{n}{b}(\tilde{I}' + \tilde{I}(N\tilde{H} + \tilde{I} - \tilde{U}))\Sigma - \frac{b}{n}(\dot{\tilde{\mathcal{U}}} + \tilde{\mathcal{U}}(N\tilde{\mathcal{H}} + \tilde{\mathcal{U}} - \tilde{\mathcal{I}}))\Sigma \\
&+(\dot{R}_{0M} + R_{0M}\partial_\eta + R_{00}\partial_y)\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) - (R'_{0M} + R_{0M}\partial_y + R_{MM}\partial_\eta)\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right),
\end{aligned} \tag{F63}$$

$$\begin{aligned}
\delta R_{iM} &= -((N-1)\tilde{H} + \tilde{I})\nabla_i h - (\partial_y - \tilde{H} + \tilde{I})\nabla_i\Psi + (N-1)\partial_y\nabla_i\Phi \\
&+\frac{1}{2}\frac{b}{n}(\partial_\eta + (N-2)\tilde{\mathcal{H}} + 2\tilde{\mathcal{U}})\nabla_i\Sigma \\
&+\frac{1}{2}\frac{b}{a}(\nabla^2 + K(N-1))\bar{h}_i - \frac{b}{a}R_{ij}\bar{h}^j
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\frac{a}{n}(\partial_\eta+(N+1)\tilde{\mathcal{H}}-\tilde{\mathcal{U}})\left((\partial_y+\tilde{I}-\tilde{H})\bar{\Sigma}_i\right)-\frac{1}{2}\frac{ba}{n^2}(\partial_\eta+(N+1)\tilde{\mathcal{H}}-\tilde{\mathcal{I}})\left((\partial_\eta+\tilde{\mathcal{U}}-\tilde{\mathcal{H}})\bar{h}_i\right)\\
&+R_{0M}\nabla_i\left(\frac{a}{n}B+\frac{a^2}{n^2}\dot{E}\right)-R_{MM}\nabla_i\left(\frac{a}{b}E_\perp+\frac{a^2}{b^2}E'\right)+R_{ij}E^{j\prime},
\end{aligned}
\tag{F64}
$$

$$
\begin{aligned}
\delta R_{MM} &= \frac{b^2}{a^2}\nabla^2 h\\
&-\frac{b^2}{n^2}(\partial_\eta+N\tilde{\mathcal{H}}+\tilde{\mathcal{U}}-\tilde{\mathcal{I}})(\partial_\eta h+2\tilde{\mathcal{U}}h+2\tilde{\mathcal{U}}\Psi)+\frac{b^2}{n^2}\tilde{\mathcal{U}}\partial_\eta(\Psi+h-N\Phi)\\
&-(\partial_y^2+(2\tilde{I}-\tilde{U})\partial_y)\Psi-(N\tilde{H}+\tilde{I})\partial_y h+N(\partial_y^2+(2\tilde{H}-\tilde{U})\partial_y)\Phi\\
&+\frac{b}{n}(\partial_\eta\partial_y+(N\tilde{\mathcal{H}}+\tilde{\mathcal{U}})\partial_y+\tilde{I}\partial_\eta+N\tilde{\mathcal{H}}\tilde{I}+N\tilde{\mathcal{U}}\tilde{H}+\dot{\tilde{I}}+\tilde{\mathcal{U}}')\Sigma\\
&+(\dot{R}_{MM}+2R_{MM}\partial_y)\left(\frac{a}{n}B+\frac{a^2}{n^2}\dot{E}\right)-(R'_{MM}+2R_{0M}\partial_y)\left(\frac{a}{b}E_\perp+\frac{a^2}{b^2}E'\right).
\end{aligned}
\tag{F65}
$$

6. Scalar curvature

$$
\begin{aligned}
\delta R &= \frac{2}{a^2}\left(\nabla^2(\Psi-h-(N-1)\Phi-KN(N-1)\Phi\right)\\
&+\frac{2}{n^2}\Big(2(\tilde{\mathcal{U}}+N\tilde{\mathcal{H}})(\partial_\eta+\tilde{\mathcal{U}}-\tilde{\mathcal{I}})\Psi+2(\dot{\tilde{\mathcal{U}}}+N\dot{\tilde{\mathcal{H}}})\Psi\\
&\quad+(\partial_\eta+\tilde{\mathcal{U}}-\tilde{\mathcal{I}})(\partial_\eta h+N\partial_\eta\Phi)\\
&\quad+(N\tilde{\mathcal{H}}+\tilde{\mathcal{U}})\partial_\eta(h-\Psi)+N(N+1)(\tilde{\mathcal{H}}^2\Psi+\tilde{\mathcal{H}}\partial_\eta\Phi)\Big)\\
&+\frac{2}{b^2}\Big(2(\tilde{I}+N\tilde{H})(\partial_y+\tilde{I}-\tilde{U})h+2(\tilde{I}'+N\tilde{H}')h\\
&\quad+(\partial_y+\tilde{I}-\tilde{U})(\partial_y h+N\partial_y\Phi))\\
&\quad+(N\tilde{H}+\tilde{I})\partial_y(\Psi-h)+N(N+1)(\tilde{H}^2h-\tilde{H}\partial_y\Phi)\Big)\\
&-\frac{2}{nb}\Big(\partial_y\partial_\eta+(N\tilde{\mathcal{H}}+\tilde{\mathcal{U}})\partial_y+(N\tilde{H}+\tilde{I})\partial_\eta\\
&\quad+N\tilde{\mathcal{H}}'+\tilde{\mathcal{U}}'+N\dot{\tilde{H}}+\dot{\tilde{I}}+N(N+1)\tilde{\mathcal{H}}\tilde{H}\Big)\Sigma\\
&+\dot{R}\left(\frac{a}{n}B+\frac{a^2}{n^2}\dot{E}\right)-R'\left(\frac{a}{b}E_\perp+\frac{a^2}{b^2}E'\right),
\end{aligned}
\tag{F66}
$$

7. Einstein tensor

$$
\begin{aligned}
\delta G_{00} &= \frac{n^2}{a^2}(\nabla^2-KN(N-1))(h-\Phi)\\
&+N\left(\frac{n^2}{a^2}\nabla^2-(N\tilde{\mathcal{H}}+\tilde{\mathcal{U}})\partial_\eta+\frac{n^2}{b^2}(\partial_y^2+(N\tilde{H}-\tilde{U})\partial_y)\right)\Phi\\
&-N\left(\frac{n^2}{b^2}\tilde{H}\partial_y(h-\Phi)+\tilde{\mathcal{H}}\partial_\eta(h-\Phi)+((N-1)\tilde{\mathcal{H}}^2+2\tilde{\mathcal{H}}\tilde{\mathcal{U}})(\Psi+h)\right)\\
&+N\frac{n}{b}\left(\tilde{\mathcal{H}}\partial_y+N\tilde{\mathcal{H}}\tilde{H}+\tilde{\mathcal{H}}'\right)\Sigma\\
&+2G_{00}(\Psi+h)-\frac{n}{b}G_{0M}\Sigma\\
&+(\dot{G}_{00}+2G_{00}\partial_\eta)\left(\frac{a}{n}B+\frac{a^2}{n^2}\dot{E}\right)-(G'_{00}+2G_{0M}\partial_\eta)\left(\frac{a}{b}E_\perp+\frac{a^2}{b^2}E'\right),
\end{aligned}
\tag{F67}
$$

$$
\delta G_{0i} = \nabla_i((\tilde{\mathcal{U}}-\tilde{\mathcal{H}})(\Psi+h)+\partial_\eta(h-\Phi)+N(\tilde{\mathcal{H}}\Psi+\dot{\Phi}))
$$

$$
\begin{aligned}
&-\frac{1}{2}\frac{n}{b}(\partial_y+(N-2)\tilde{H}+2\tilde{I})\nabla_i\Sigma\\
&+\frac{1}{2}\frac{an}{b^2}(\partial_y+(N+1)\tilde{H}-\tilde{U})\left((\partial_y+\tilde{I}-\tilde{H})\bar{\Sigma}_i\right)-\frac{1}{2}\frac{a}{b}(\partial_y+(N+1)\tilde{H}-\tilde{I})\left((\partial_\eta+\tilde{\mathcal{U}}-\tilde{\mathcal{H}})\bar{h}_i\right)\\
&+\frac{1}{2}\frac{n}{a}(\nabla^2+K(N-1))\bar{\Sigma}_i-\frac{n}{a}G_{ij}\bar{\Sigma}^j\\
&+G_{00}\nabla_i\left(\frac{a}{n}B+\frac{a^2}{n^2}\dot{E}\right)-G_{0M}\nabla_i\left(\frac{a}{b}E_\perp+\frac{a^2}{b^2}E'\right)+G_{ij}\dot{E}^j, \qquad \text{(F68)}\\
\delta G_{ij} &= (\gamma_{ij}\nabla^2-\nabla_{ij})(\Psi+\Phi-(h-\Phi)-N\Phi)\\
&+\frac{a^2}{n^2}\gamma_{ij}\left(2(\tilde{\mathcal{U}}+(N-1)\tilde{\mathcal{H}})(\partial_\eta+\tilde{\mathcal{U}}-\tilde{\mathcal{I}})+N(N-1)\tilde{\mathcal{H}}^2\right)(\Phi+\Psi)\\
&+\frac{a^2}{n^2}\gamma_{ij}\left((\partial_\eta^2+(\tilde{\mathcal{U}}-\tilde{\mathcal{I}})\partial_\eta)(h-(N-1)\Phi)-((N-1)\tilde{\mathcal{H}}+\tilde{\mathcal{U}})\partial_\eta(\Psi+h-N\Phi)\right)\\
&+\frac{a^2}{b^2}\gamma_{ij}\left(2(\tilde{I}+(N-1)\tilde{H})(\partial_y+\tilde{I}-\tilde{U})+N(N-1)\tilde{H}^2\right)(h-\Phi)\\
&+\frac{a^2}{b^2}\gamma_{ij}\left((\partial_y^2+(\tilde{I}-\tilde{U})\partial_y)(\Psi-(N-1)\Phi)-((N-1)\tilde{H}+\tilde{I})\partial_y(\Psi+h+N\Phi)\right)\\
&-\frac{a^2}{nb}\gamma_{ij}\left(\partial_y\partial_\eta+((N-1)\tilde{\mathcal{H}}+\tilde{\mathcal{U}})\partial_y+((N-1)\tilde{H}+\tilde{I})\partial_\eta+N(N-1)\tilde{\mathcal{H}}\tilde{H}\right)\Sigma\\
&+2a^2\gamma_{ij}\left(\frac{1}{n^2}(\Phi+\Psi)\partial_\eta(\tilde{\mathcal{U}}+(N-1)\tilde{\mathcal{H}})+\frac{1}{b^2}(h-\Phi)\partial_y(\tilde{I}+(N-1)\tilde{H})\right)\\
&-\frac{a^2}{nb}\gamma_{ij}\Sigma\left((N-1)\dot{\tilde{\mathcal{H}}}'+\dot{\tilde{\mathcal{U}}}'+(N-1)\dot{\tilde{H}}+\dot{\tilde{I}}\right)\\
&-N\left(\frac{a^2}{n^2}\tilde{\mathcal{U}}\partial_\eta-\frac{a^2}{b^2}\tilde{I}\partial_y\right)\Phi-2G_{ij}\Phi\\
&+\frac{a}{n}\left(\partial_\eta+(N-1)\tilde{\mathcal{H}}+\tilde{\mathcal{U}}\right)\nabla_{(i}\bar{\Sigma}_{j)}-\frac{a}{b}\left(\partial_y+(N-1)\tilde{H}+\tilde{I}\right)\nabla_{(i}\bar{h}_{j)}\\
&+\left(\frac{a^2}{n^2}(\partial_\eta^2+(N\tilde{\mathcal{H}}+\tilde{\mathcal{U}}-\tilde{\mathcal{I}})\partial_\eta)-(\nabla^2-2K)-\frac{a^2}{b^2}(\partial_y^2+(N\tilde{H}+\tilde{I}-\tilde{U})\partial_y)\right)\bar{\bar{E}}_{ij}\\
&+\dot{G}_{ij}\left(\frac{a}{n}B+\frac{a^2}{n^2}\dot{E}\right)-G'_{ij}\left(\frac{a}{b}E_\perp+\frac{a^2}{b^2}E'\right)+2G_{k(i}(E^k_{j)}-\delta^k_{j)}\Phi), \qquad \text{(F69)}\\
\delta G_{0M} &= \frac{1}{2}\left(\frac{nb}{a^2}\nabla^2+N\frac{b}{n}(\dot{\tilde{\mathcal{H}}}+\tilde{\mathcal{H}}^2-\tilde{\mathcal{H}}\tilde{\mathcal{I}}-\tilde{\mathcal{H}}\tilde{\mathcal{U}})-N\frac{n}{b}(\tilde{H}'+\tilde{H}^2-\tilde{H}\tilde{I}-\tilde{H}\tilde{U})\right)\Sigma\\
&+N(\tilde{\mathcal{H}}\partial_y(\Phi+\Psi)-\tilde{H}\partial_\eta(h-\Phi)+(\partial_\eta\partial_y-\tilde{\mathcal{U}}\partial_y-\tilde{I}\partial_\eta)\Phi)\\
&+\frac{1}{2}\left(\frac{b}{n}G_{00}-\frac{n}{b}G_{MM}\right)\Sigma\\
&+(\dot{G}_{0M}+G_{0M}\partial_\eta+G_{00}\partial_y)\left(\frac{a}{n}B+\frac{a^2}{n^2}\dot{E}\right)-(G'_{0M}+G_{0M}\partial_y+G_{MM}\partial_\eta)\left(\frac{a}{b}E_\perp+\frac{a^2}{b^2}E'\right), \qquad \text{(F70)}\\
\delta G_{iM} &= -\nabla_i((\tilde{I}-\tilde{H})(\Psi+h)+\partial_y(\Phi+\Psi)+N(\tilde{H}h+\Phi'))\\
&+\frac{1}{2}\frac{b}{n}(\partial_\eta+(N-2)\tilde{\mathcal{H}}+2\tilde{\mathcal{U}})\nabla_i\Sigma\\
&+\frac{1}{2}\frac{a}{n}(\partial_\eta+(N+1)\tilde{\mathcal{H}}-\tilde{\mathcal{U}})\left((\partial_y+\tilde{I}-\tilde{H})\bar{\Sigma}_i\right)-\frac{1}{2}\frac{ba}{n^2}(\partial_\eta+(N+1)\tilde{\mathcal{H}}-\tilde{\mathcal{I}})\left((\partial_\eta+\tilde{\mathcal{U}}-\tilde{\mathcal{H}})\bar{h}_i\right)\\
&+\frac{1}{2}\frac{b}{a}(\nabla^2+K(N-1))\bar{h}_i-\frac{b}{a}G_{ij}\bar{h}^j\\
&+G_{0M}\nabla_i\left(\frac{a}{n}B+\frac{a^2}{n^2}\dot{E}\right)-G_{MM}\nabla_i\left(\frac{a}{b}E_\perp+\frac{a^2}{b^2}E'\right)+G_{ij}E^{j\prime}, \qquad \text{(F71)}\\
\delta G_{MM} &= \frac{b^2}{a^2}(\nabla^2-KN(N-1))(\Psi+\Phi)\\
&-N\left(\frac{b^2}{a^2}\nabla^2-\frac{b^2}{n^2}(N\tilde{\mathcal{H}}-\tilde{\mathcal{I}})\partial_\eta+(N\tilde{H}+\tilde{I})\partial_y\right)\Phi
\end{aligned}
$$

$$
\begin{aligned}
&+N\left(\frac{b^2}{n^2}\tilde{\mathcal{H}}\partial_\eta(\Psi+\Phi)+\tilde{H}\partial_y(\Phi+\Psi)+((N-1)\tilde{H}^2+2\tilde{H}\tilde{I})(\Phi+\Psi)\right)\\
&-N\frac{b}{n}\left(\tilde{H}\partial_\eta+N\tilde{\mathcal{H}}\tilde{H}+\dot{\tilde{H}}\right)\Sigma\\
&-2G_{M\,M}(\Psi+h)+\frac{b}{n}G_{0M}\Sigma\\
&+(\dot{G}_{M\,M}+2G_{0M}\partial_y)\left(\frac{a}{n}B+\frac{a^2}{n^2}\dot{E}\right)-(G'_{M\,M}+2G_{M\,M}\partial_y)\left(\frac{a}{b}E_\perp+\frac{a^2}{b^2}E'\right). \qquad \text{(F72)}
\end{aligned}
$$

8. Riemann tensor

$$
\begin{aligned}
\delta R_{0i0j} &= -\left(n^2\nabla_{ij}+a^2\tilde{\mathcal{H}}\gamma_{ij}\partial_\eta+\frac{a^2n^2}{b^2}\tilde{H}\gamma_{ij}\partial_y\right)\Psi-2\frac{a^2n^2}{b^2}\tilde{H}\tilde{I}\gamma_{ij}(\Psi+h)\\
&\quad+\frac{a^2n}{b}\gamma_{ij}(\tilde{H}\partial_\eta+\tilde{\mathcal{H}}\tilde{I}+\tilde{\mathcal{U}}\tilde{H})\Sigma+an(\partial_\eta+\tilde{\mathcal{H}})\nabla_{(i}\bar{\Sigma}_{j)}-\frac{an^2}{b}\tilde{I}\nabla_{(i}\bar{h}_{j)}\\
&\quad+\left(a^2(\partial_\eta^2+(2\tilde{\mathcal{H}}-\tilde{\mathcal{I}})\partial_\eta)-\frac{a^2n^2}{b^2}\tilde{I}\partial_y\right)(\bar{\bar{E}}_{ij}-\gamma_{ij}\Phi)\\
&\quad+(\dot{R}_{0i0j}+2R_{0i0j}\partial_\eta)\left(\frac{a}{n}B+\frac{a^2}{n^2}\dot{E}\right)-(R'_{0i0j}+R_{Mi0j}\partial_\eta+R_{0iMj}\partial_\eta)\left(\frac{a}{b}E_\perp+\frac{a^2}{b^2}E'\right)\\
&\quad+R_{0k0j}(E_i^k-\delta_i^k\Phi)+R_{0i0k}(E_j^k-\delta_j^k\Phi), \qquad \text{(F73)}\\
\delta R_{0ijk} &= a^2\nabla_j\partial_\eta(\bar{\bar{E}}_{ik}-\gamma_{ik}\Phi)-a^2\tilde{\mathcal{H}}\gamma_{ik}\nabla_j\Psi+an\nabla_j\nabla_{(i}\bar{\Sigma}_{k)}\\
&\quad+\frac{1}{2}a^2\tilde{H}\gamma_{ik}\left(\frac{n}{b}\nabla_j\Sigma-\frac{an}{b^2}(\partial_y+\tilde{I}-\tilde{H})\bar{\Sigma}_j+\frac{a}{b}(\partial_\eta+\tilde{\mathcal{U}}-\tilde{\mathcal{H}})\bar{h}_j\right)\\
&\quad-[j\leftrightarrow k]\\
&\quad+(R_{0i0k}\nabla_j+R_{0ij0}\nabla_k)\left(\frac{a}{n}B+\frac{a^2}{n^2}\dot{E}\right)-(R_{0iMk}\nabla_j+R_{0ijM}\nabla_k)\left(\frac{a}{b}E_\perp+\frac{a^2}{b^2}E'\right)\\
&\quad+R_{lijk}\left(\nabla^l\dot{E}-\frac{n}{a}\bar{\Sigma}^l\right), \qquad \text{(F74)}\\
\delta R_{0i0M} &= -n^2(\partial_y+\tilde{I}-\tilde{H})\nabla_i\Psi-n^2\tilde{I}\nabla_i h+\frac{1}{2}nb(\partial_\eta+2\tilde{\mathcal{U}}-\tilde{\mathcal{H}})\nabla_i\Sigma\\
&\quad+\frac{1}{2}an(\partial_\eta+2\tilde{\mathcal{H}}-\tilde{\mathcal{U}})\left((\partial_y+\tilde{U}-\tilde{H})\bar{\Sigma}_i\right)-\frac{1}{2}ba(\partial_\eta+2\tilde{\mathcal{H}}-\tilde{\mathcal{I}})\left((\partial_\eta+\tilde{\mathcal{I}}-\tilde{\mathcal{H}})\bar{h}_i\right)\\
&\quad-R_{0M0M}\nabla_i\left(\frac{a}{b}E_\perp+\frac{a^2}{b^2}E'\right)+R_{0ijM}\left(\dot{E}^j-\frac{n}{a}\bar{\Sigma}^j\right)+R_{0i0j}\left(E^{j\prime}-\frac{b}{a}\bar{h}^j\right), \qquad \text{(F75)}\\
\delta R_{0iMj} &= a^2\gamma_{ij}(\tilde{H}\dot{h}-\tilde{\mathcal{H}}\Psi')-\frac{1}{2}\left(nb\nabla_{ij}-\frac{ba^2}{n}\gamma_{ij}\tilde{\mathcal{H}}\tilde{\mathcal{U}}+\frac{a^2n}{b}\gamma_{ij}\tilde{H}\tilde{I}\right)\Sigma\\
&\quad+a^2(\partial_\eta\partial_y+(\tilde{\mathcal{H}}-\tilde{\mathcal{U}})\partial_y+(\tilde{H}-\tilde{I})\partial_\eta)(\bar{\bar{E}}_{ij}-\gamma_{ij}\Phi)\\
&\quad+\frac{1}{2}an(\partial_y+\tilde{H}-\tilde{I})\nabla_{(i}\bar{\Sigma}_{j)}+\frac{1}{2}ba(\partial_\eta+\tilde{\mathcal{H}}-\tilde{\mathcal{U}})\nabla_{(i}\bar{h}_{j)}\\
&\quad+(\dot{R}_{0iMj}+R_{0iMj}\partial_\eta+R_{0i0j}\partial_y)\left(\frac{a}{n}B+\frac{a^2}{n^2}\dot{E}\right)\\
&\quad-(R'_{0iMj}+R_{0iMj}\partial_y+R_{MiMj}\partial_\eta)\left(\frac{a}{b}E_\perp+\frac{a^2}{b^2}E'\right)\\
&\quad+R_{0kMj}(E_i^k-\delta_i^k\Phi)+R_{0iMk}(E_j^k-\delta_j^k\Phi), \qquad \text{(F76)}\\
\delta R_{ijkl} &= \left(\frac{a^4}{n^2}\tilde{\mathcal{H}}^2\Psi+\frac{a^4}{b^2}\tilde{H}^2h-\frac{a^4}{nb}\tilde{\mathcal{H}}\tilde{H}\Sigma\right)\gamma_{ik}\gamma_{jl}-\gamma_{ki}\left(\frac{a^3}{n}\tilde{\mathcal{H}}\nabla_{(j}\bar{\Sigma}_{l)}-\frac{a^3}{b}\tilde{H}\nabla_{(j}\bar{h}_{l)}\right)\\
&\quad+\left(a^2(\nabla_{ki}+K\gamma_{ki})-\frac{a^4}{n^2}\gamma_{ki}\tilde{\mathcal{H}}\partial_\eta+\frac{a^4}{b^2}\gamma_{ki}\tilde{H}\partial_y\right)(\bar{\bar{E}}_{jl}-\gamma_{jl}\Phi)\\
&\quad-[i\leftrightarrow j]+[ik\leftrightarrow jl]-[k\leftrightarrow l]
\end{aligned}
$$

$$
\begin{aligned}
&+R_{mjkl}(E^m_i-\delta^m_i\Phi)+R_{imkl}(E^m_j-\delta^m_j\Phi)+R_{ijml}(E^m_k-\delta^m_k\Phi)+R_{ijkm}(E^m_l-\delta^m_l\Phi)\\
&+\dot{R}_{ijkl}\left(\frac{a}{n}B+\frac{a^2}{n^2}\dot{E}\right)-R'_{ijkl}\left(\frac{a}{b}E_\perp+\frac{a^2}{b^2}E'\right), \qquad \text{(F77)}\\
\delta R_{ij0_M} &= \nabla_i\left(an(\partial_y+\tilde{I}-\tilde{H})\bar{\Sigma}_j-ba(\partial_\eta+\tilde{\mathcal{U}}-\tilde{\mathcal{H}})\bar{h}_j\right)-[i\leftrightarrow j], \qquad \text{(F78)}\\
\delta R_{Mijk} &= a^2\nabla_j\partial_y(\bar{\bar{E}}_{ij}-\gamma_{ij}\Phi)+a^2\tilde{H}\gamma_{ik}\nabla_j h+ba\nabla_j\nabla_{(i}\bar{h}_{k)}\\
&-\frac{1}{2}a^2\tilde{\mathcal{H}}\gamma_{ik}\left(\frac{b}{n}\nabla_j\Sigma+\frac{a}{n}(\partial_y+\tilde{I}-\tilde{H})\bar{\Sigma}_j-\frac{ba}{n^2}(\partial_\eta+\tilde{\mathcal{U}}-\tilde{\mathcal{H}})\bar{h}_j\right)\\
&-[j\leftrightarrow k]\\
&+(R_{Mi0k}\nabla_j+R_{Mij0}\nabla_k)\left(\frac{a}{n}B+\frac{a^2}{n^2}\dot{E}\right)-(R_{MiMk}\nabla_j+R_{MijM}\nabla_k)\left(\frac{a}{b}E_\perp+\frac{a^2}{b^2}E'\right)\\
&+R_{lijk}\left(\nabla^l E'-\frac{b}{a}\bar{h}^l\right), \qquad \text{(F79)}\\
\delta R_{0_M0_M} &= -n^2(\partial_y-\tilde{U})\partial_y\Psi-2n^2(\partial_y+\tilde{I}-\tilde{U})(\tilde{I}\Psi)-n^2\tilde{I}\partial_y h\\
&-b^2(\partial_\eta-\tilde{\mathcal{I}})\partial_\eta h-2b^2(\partial_\eta+\tilde{\mathcal{U}}-\tilde{\mathcal{I}})(\tilde{\mathcal{U}}h)-b^2\tilde{\mathcal{U}}\partial_\eta\Psi\\
&+nb\left(\partial_\eta(\tfrac{1}{2}\partial_y+\tilde{I})+\partial_y(\tfrac{1}{2}\partial_\eta+\tilde{\mathcal{U}})\right)\Sigma\\
&+(\dot{R}_{0_M0_M}+2R_{0_M0_M}\partial_\eta)\left(\frac{a}{n}B+\frac{a^2}{n^2}\dot{E}\right)-(R'_{0_M0_M}+2R_{0_M0_M}\partial_y)\left(\frac{a}{b}E_\perp+\frac{a^2}{b^2}E'\right), \qquad \text{(F80)}\\
\delta R_{M0_Mi} &= -b^2(\partial_\eta+\tilde{\mathcal{U}}-\tilde{\mathcal{H}})\nabla_i h-b^2\tilde{\mathcal{U}}\nabla_i\Psi+\frac{1}{2}nb(\partial_y+2\tilde{I}-\tilde{H})\nabla_i\Sigma\\
&-\frac{1}{2}an(\partial_y+2\tilde{H}-\tilde{U})\left((\partial_y+\tilde{I}-\tilde{H})\bar{\Sigma}_i\right)+\frac{1}{2}ba(\partial_y+2\tilde{H}-\tilde{I})\left((\partial_\eta+\tilde{\mathcal{U}}-\tilde{\mathcal{H}})\bar{h}_i\right)\\
&+R_{M0_M0}\nabla_i\left(\frac{a}{n}B+\frac{a^2}{n^2}\dot{E}\right)+R_{j0_Mi}\left(E^{j\prime}-\frac{b}{a}\bar{h}^j\right)+R_{MjM_i}\left(\dot{E}^j-\frac{n}{a}\bar{\Sigma}^j\right), \qquad \text{(F81)}\\
\delta R_{MiMj} &= \left(-b^2\nabla_{ij}+a^2\tilde{H}\gamma_{ij}\partial_y+\frac{b^2a^2}{n^2}\tilde{\mathcal{H}}\gamma_{ij}\partial_\eta\right)h+2\frac{b^2a^2}{n^2}\tilde{\mathcal{H}}\tilde{\mathcal{U}}\gamma_{ij}(\Psi+h)\\
&-\frac{ba^2}{n}\gamma_{ij}(\tilde{\mathcal{H}}\partial_y+\tilde{\mathcal{H}}\tilde{I}+\tilde{\mathcal{U}}\tilde{H})\Sigma-\frac{b^2a}{n}\tilde{\mathcal{U}}\nabla_{(i}\bar{\Sigma}_{j)}+ba(\partial_y+\tilde{H})\nabla_{(i}\bar{h}_{j)}\\
&+\left(a^2(\partial_y^2+(2\tilde{H}-\tilde{U})\partial_y)-\frac{b^2a^2}{n^2}\tilde{\mathcal{U}}\partial_\eta\right)(\bar{\bar{E}}_{ij}-\gamma_{ij}\Phi)\\
&(\dot{R}_{MiMj}+2R_{0iMj}\partial_\eta+2R_{Mi0j}\partial_\eta)\left(\frac{a}{n}B+\frac{a^2}{n^2}\dot{E}\right)\\
&-(R'_{MiMj}+2R_{MiMj}\partial_y)\left(\frac{a}{b}E_\perp+\frac{a^2}{b^2}E'\right)\\
&+R_{MkMj}(E^k_i-\delta^k_i)+R_{MiMk}(E^k_j-\delta^k_j). \qquad \text{(F82)}
\end{aligned}
$$

9. Weyl tensor

Defining

$$
\begin{aligned}
E^{(01)}_{ij} &= (\partial_\eta+\tilde{\mathcal{U}}-\tilde{\mathcal{I}})\partial_\eta\bar{\bar{E}}_{ij}+\frac{n}{a}(\partial_\eta+\tilde{\mathcal{U}}-\tilde{\mathcal{H}})\nabla_{(i}\bar{\Sigma}_{j)}, \qquad \text{(F83)}\\
E^{(02)}_{ij} &= (\tilde{\mathcal{H}}-\tilde{\mathcal{U}})\partial_\eta\bar{\bar{E}}_{ij}+\frac{n}{a}(\tilde{\mathcal{H}}-\tilde{\mathcal{U}})\nabla_{(i}\bar{\Sigma}_{j)}, \qquad \text{(F84)}\\
E^{(\perp 1)}_{ij} &= (\partial_y+\tilde{I}-\tilde{U})\partial_y\bar{\bar{E}}_{ij}+\frac{b}{a}(\partial_y+\tilde{I}-\tilde{H})\nabla_{(i}\bar{h}_{j)}, \qquad \text{(F85)}\\
E^{(\perp 2)}_{ij} &= (\tilde{H}-\tilde{I})\partial_y\bar{\bar{E}}_{ij}+\frac{b}{a}(\tilde{H}-\tilde{I})\nabla_{(i}\bar{h}_{j)}, \qquad \text{(F86)}\\
X^{(0)} &= (\partial_\eta+\tilde{\mathcal{U}}-\tilde{\mathcal{I}})\left((\tilde{\mathcal{H}}-\tilde{\mathcal{U}})(\Psi+h)-\partial_\eta(h-\Phi)\right), \qquad \text{(F87)}
\end{aligned}
$$

$$X^{(\perp)} = (\partial_y + \tilde{I} - \tilde{U})\left((\tilde{H} - \tilde{I})(\Psi + h) - \partial_y(\Phi + \Psi)\right), \tag{F88}$$

$$X^{(*)} = (\partial_\eta\partial_y + (\tilde{I} - \tilde{H})\partial_\eta + (\tilde{\mathcal{U}} - \tilde{\mathcal{H}})\partial_y + \dot{\tilde{I}} - \dot{\tilde{H}} + \tilde{\mathcal{U}}' - \tilde{\mathcal{H}}')\Sigma. \tag{F89}$$

$$\begin{aligned}
\delta C_{0i0j} &= -n^2\left(\nabla_{ij} - \frac{1}{N}\nabla^2\gamma_{ij}\right)\left(\frac{N-1}{N}(\Phi + \Psi) + \frac{1}{N}(h - \Phi)\right) \\
&\quad -n^2\frac{N-1}{N^2(N+1)}\gamma_{ij}\left(\nabla^2 + KN\right)(2\Phi + \Psi - h) \\
&\quad +C_{0i0j}(\Psi + h) + n^2\frac{1}{N}(\nabla^2 - 2K)\bar{\bar{E}}_{ij} \\
&\quad -\frac{N-1}{N(N+1)}\gamma_{ij}\left(a^2X^{(0)} + \frac{a^2n^2}{b^2}X^{(\perp)} + \frac{a^2n}{b}X^{(*)}\right) \\
&\quad +a^2\left(\frac{N-1}{N}E_{ij}^{(01)} + E_{ij}^{(02)}\right) + \frac{a^2n^2}{b^2}\left(\frac{1}{N}E_{ij}^{(\perp 1)} + E_{ij}^{(\perp 2)}\right) \\
&\quad (\dot{C}_{0i0j} + 2C_{0i0j}\partial_\eta)\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) - (C'_{0i0j} + C_{Mi0j}\partial_\eta + C_{0iMj}\partial_\eta)\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right) \\
&\quad +C_{0k0j}(E_i^k - \delta_i^k\Phi) + C_{0i0k}(E_j^k - \delta_j^k\Phi),
\end{aligned} \tag{F90}$$

$$\begin{aligned}
\delta C_{0ijk} &= \frac{1}{N}\gamma_{ik}\nabla_j\left(a^2\partial_\eta(h - \Phi) + a^2(\tilde{\mathcal{U}} - \tilde{\mathcal{H}})(\Psi + h) - \frac{a^2n}{b}(\tfrac{1}{2}\partial_y + \tilde{I} - \tilde{H})\Sigma\right) \\
&\quad +\frac{1}{2}\frac{1}{N}\gamma_{ik}\left(\frac{a^3n}{b^2}(\partial_y + \tilde{H} - \tilde{U})\left((\partial_y + \tilde{I} - \tilde{H})\bar{\Sigma}_j\right) - \frac{a^3}{b}(\partial_y + \tilde{H} - \tilde{I})\left((\partial_\eta + \tilde{\mathcal{U}} - \tilde{\mathcal{H}})\bar{h}_j\right)\right) \\
&\quad +\frac{1}{2}\frac{1}{N}\gamma_{ik}an(\nabla^2 + K(N-1))\bar{\Sigma}_j + an\nabla_j\nabla_{(i}\bar{\Sigma}_{k)} + a^2\nabla_j\dot{\bar{\bar{E}}}_{ik} \\
&\quad -[j \leftrightarrow k] \\
&\quad +(C_{0i0k}\nabla_j + C_{0ij0}\nabla_k)\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) - (C_{0iMk}\nabla_j + C_{0ijM}\nabla_k)\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right) \\
&\quad +C_{lijk}\left(\nabla^l\dot{E} - \frac{n}{a}\bar{\Sigma}^l\right),
\end{aligned} \tag{F91}$$

$$\begin{aligned}
\delta C_{0i0M} &= -\frac{N-1}{N}n^2\nabla_i\left(\partial_y(\Phi + \Psi) + (\tilde{I} - \tilde{H})(h + \Psi)\right) + \frac{N-1}{N}nb\left(\tfrac{1}{2}\partial_u + \tilde{\mathcal{U}} - \tilde{\mathcal{H}}\right)\nabla_i\Sigma \\
&\quad -\frac{1}{2}\frac{1}{N}\frac{n^2b}{a}(\nabla^2 + K(N-1))\bar{h}_i \\
&\quad +\frac{1}{2}\frac{N-1}{N}\left(an(\partial_\eta + \tilde{\mathcal{H}} - \tilde{\mathcal{U}})\left((\partial_y + \tilde{I} - \tilde{H})\bar{\Sigma}_i\right) - ba(\partial_\eta + \tilde{\mathcal{H}} - \tilde{\mathcal{I}})\left((\partial_\eta + \tilde{\mathcal{U}} - \tilde{\mathcal{H}})\bar{h}_i\right)\right) \\
&\quad -C_{0M0M}\nabla_i\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right) + C_{jM0i}\left(\dot{E}^j - \frac{n}{a}\bar{\Sigma}^j\right) + C_{0j0i}\left(E^{j\prime} - \frac{b}{a}\bar{h}^j\right),
\end{aligned} \tag{F92}$$

$$\begin{aligned}
\delta C_{0iMj} &= -\frac{1}{2}nb\left(\nabla_{ij} - \frac{1}{N}\nabla^2\gamma_{ij}\right)\Sigma - \frac{1}{N}\frac{a^2}{nb}C_{0M0M}\gamma_{ij}\Sigma \\
&\quad +\frac{1}{2}an(\partial_y + \tilde{H} - \tilde{I})\nabla_{(i}\bar{\Sigma}_{j)} + \frac{1}{2}ba(\partial_\eta + \tilde{\mathcal{H}} - \tilde{\mathcal{U}})\nabla_{(i}\bar{h}_{j)} \\
&\quad +a^2\left((\tfrac{1}{2}\partial_y + \tilde{H} - \tilde{I})\partial_\eta + (\tfrac{1}{2}\partial_\eta + \tilde{\mathcal{H}} - \tilde{\mathcal{U}})\partial_y\right)\bar{\bar{E}}_{ij} \\
&\quad +(\dot{C}_{0iMj} + C_{0iMj}\partial_\eta + C_{0i0j}\partial_y)\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) - (C'_{0iMj} + C_{0iMj}\partial_y + C_{MiMj}\partial_\eta)\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right) \\
&\quad +C_{0kMj}(E_i^k - \delta_i^k\Phi) + C_{0iMk}(E_j^k - \delta_j^k\Phi),
\end{aligned} \tag{F93}$$

$$\begin{aligned}
\delta C_{ijkl} &= -\frac{1}{N}a^2\left(\nabla_{ik} - \frac{1}{N}\nabla^2\gamma_{ik}\right)\gamma_{jl}(2\Phi + \Psi - h) \\
&\quad -\frac{1}{N^2(N+1)}a^2\gamma_{ik}\gamma_{jl}\left(\nabla^2 + KN\right)(2\Phi + \Psi - h) \\
&\quad +C_{ijkl}(h - \Psi) + a^2\left(\nabla_{ki} + K\gamma_{ki} - \frac{1}{N}\gamma_{ki}(\nabla^2 - 2K)\right)\bar{\bar{E}}_{jl}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{N(N+1)}\gamma_{ik}\gamma_{jl}\left(\frac{a^4}{n^2}X^{(0)}+\frac{a^4}{b^2}X^{(\perp)}+\frac{a^4}{nb}X^{(*)}\right)\\
&+\frac{a^4}{n^2}\frac{1}{N}\gamma_{ik}E^{(01)}_{jl}-\frac{a^4}{b^2}\frac{1}{N}\gamma_{ik}E^{(\perp 1)}_{jl}\\
&-[i\leftrightarrow j]+[ik\leftrightarrow jl]-[k\leftrightarrow l]\\
&+C_{mjkl}(E^m_i-\delta^m_i\Phi)+C_{imkl}(E^m_j-\delta^m_j\Phi)+C_{ijml}(E^m_k-\delta^m_k\Phi)+C_{ijkm}(E^m_l-\delta^m_l\Phi)\\
&+\dot{C}_{ijkl}\left(\frac{a}{n}B+\frac{a^2}{n^2}\dot{E}\right)-C'_{ijkl}\left(\frac{a}{b}E_\perp+\frac{a^2}{b^2}E'\right), \qquad \text{(F94)}
\end{aligned}
$$

$$
\delta C_{ij0_M} = \nabla_i\left(an(\partial_y+\tilde{I}-\tilde{H})\bar{\Sigma}_j-ba(\partial_\eta+\tilde{\mathcal{U}}-\tilde{\mathcal{H}})\bar{h}_j\right)-[i\leftrightarrow j], \qquad \text{(F95)}
$$

$$
\begin{aligned}
\delta C_{Mijk} &= -\frac{1}{N}\gamma_{ik}\nabla_j\left(a^2\partial_y(\Phi+\Psi)+a^2(\tilde{I}-\tilde{H})(\Psi+h)-\frac{ba^2}{n}(\tfrac{1}{2}\partial_\eta+\tilde{\mathcal{U}}-\tilde{\mathcal{H}})\Sigma\right)\\
&+\frac{1}{2}\frac{1}{N}\gamma_{ik}\left(\frac{a^3}{n}(\partial_\eta+\tilde{\mathcal{H}}-\tilde{\mathcal{U}})\left((\partial_y+\tilde{I}-\tilde{H})\bar{\Sigma}_j\right)-\frac{ba^3}{n^2}(\partial_\eta+\tilde{\mathcal{H}}-\tilde{\mathcal{I}})\left((\partial_\eta+\tilde{\mathcal{U}}-\tilde{\mathcal{H}})\bar{h}_j\right)\right)\\
&+\frac{1}{2}\frac{1}{N}\gamma_{ik}ba(\nabla^2+K(N-1))\bar{h}_j+ba\nabla_j\nabla_{(i}\bar{h}_{k)}+a^2\nabla_j\bar{\bar{E}}'_{ik}\\
&-[j\leftrightarrow k]\\
&+(C_{Mi0k}\nabla_j+C_{Mij0}\nabla_k)\left(\frac{a}{n}B+\frac{a^2}{n^2}\dot{E}\right)-(C_{MimK}\nabla_j+C_{MijM}\nabla_k)\left(\frac{a}{b}E_\perp+\frac{a^2}{b^2}E'\right)\\
&+C_{lijk}\left(\nabla^l E'-\frac{b}{a}\bar{h}^l\right), \qquad \text{(F96)}
\end{aligned}
$$

$$
\begin{aligned}
\delta C_{0_M0_M} &= \frac{n^2b^2}{a^2}\frac{N-1}{N(N+1)}\left(\nabla^2+KN\right)(2\Phi+\Psi-h)\\
&+\frac{N-1}{N+1}\left(b^2X^{(0)}+n^2X^{(\perp)}+nbX^{(*)}\right)\\
&+(\dot{C}_{0M0M}+2C_{0M0M}\partial_\eta)\left(\frac{a}{n}B+\frac{a^2}{n^2}\dot{E}\right)-(C'_{0M0M}+2C_{0M0M}\partial_y)\left(\frac{a}{b}E_\perp+\frac{a^2}{b^2}E'\right), \qquad \text{(F97)}
\end{aligned}
$$

$$
\begin{aligned}
\delta C_{M0Mi} &= \frac{N-1}{N}b^2\nabla_i\left(\partial_\eta(\Phi-h)+(\tilde{\mathcal{H}}-\tilde{\mathcal{U}})(\Psi+h)+(\tfrac{1}{2}\partial_y+\tilde{I}-\tilde{H})\Sigma\right)\\
&+\frac{1}{2}\frac{1}{N}\frac{nb^2}{a}(\nabla^2+K(N-1))\bar{\Sigma}_i\\
&-\frac{1}{2}\frac{N-1}{N}\left(an(\partial_y+\tilde{H}-\tilde{U})\left((\partial_y+\tilde{I}-\tilde{H})\bar{\Sigma}_i\right)-ba(\partial_y+\tilde{H}-\tilde{I})\left((\partial_\eta+\tilde{\mathcal{U}}-\tilde{\mathcal{H}})\bar{h}_i\right)\right)\\
&+C_{M0M0}\nabla_i\left(\frac{a}{n}B+\frac{a^2}{n^2}\dot{E}\right)+C_{j0Mi}\left(E^{j\prime}-\frac{b}{a}\bar{h}^j\right)+C_{MjMi}\left(\dot{E}^j-\frac{n}{a}\bar{\Sigma}^j\right), \qquad \text{(F98)}
\end{aligned}
$$

$$
\begin{aligned}
\delta C_{MiMj} &= -b^2\left(\nabla_{ij}-\frac{1}{N}\nabla^2\gamma_{ij}\right)\left(\frac{N-1}{N}(h-\Phi)+\frac{1}{N}(\Phi+\Psi)\right)\\
&+b^2\frac{N-1}{N^2(N+1)}\gamma_{ij}\left(\nabla^2+KN\right)(2\Phi+\Psi-h)\\
&-C_{MiMj}(\Psi+h)-b^2\frac{1}{N}(\nabla^2-2K)\bar{\bar{E}}_{ij}\\
&+\frac{N-1}{N(N+1)}\gamma_{ij}\left(\frac{b^2a^2}{n^2}X^{(0)}+a^2X^{(\perp)}+\frac{ba^2}{n}X^{(*)}\right)\\
&+\frac{b^2a^2}{n^2}\left(\frac{1}{N}E^{(01)}_{ij}+E^{(02)}_{ij}\right)+a^2\left(\frac{N-1}{N}E^{(\perp 1)}_{ij}+E^{(\perp 2)}_{ij}\right)\\
&+(\dot{C}_{MiMj}+2C_{MiMj}\partial_\eta)\left(\frac{a}{n}B+\frac{a^2}{n^2}\dot{E}\right)-(C'_{MiMj}+2C_{MiMj}\partial_y)\left(\frac{a}{b}E_\perp+\frac{a^2}{b^2}E'\right)\\
&+C_{MkMj}(E^k_i-\delta^k_i\Phi)+C_{MiMk}(E^k_j-\delta^k_j\Phi). \qquad \text{(F99)}
\end{aligned}
$$

APPENDIX G: PERTURBED MATTER CONTENT

1. Unit vectors

$$\delta u^\alpha = \left(-\frac{1}{n}A, \mathbf{0}, 0\right), \tag{G1}$$
$$\delta u_\alpha = (nA, aB_i, bB_\perp), \tag{G2}$$
$$\delta n^\alpha = \left(0, \mathbf{0}, -\frac{1}{b}E_{\perp\perp}\right), \tag{G3}$$
$$\delta n_\alpha = (-nB_\perp, -aE_{\perp i}, -bE_{\perp\perp}), \tag{G4}$$
$$\delta U^\alpha = \left(\frac{1}{n}\gamma(\beta w - A - \beta B_\perp), \frac{1}{a}v_0^i, \frac{1}{b}\gamma(w + \beta E_{\perp\perp})\right), \tag{G5}$$
$$\delta U_\alpha = \left(n\gamma(\beta w + A), -a(v_i^0 - \gamma B_i - \beta\gamma E_{\perp i}), -b\gamma(w - B_\perp - \beta E_{\perp\perp})\right), \tag{G6}$$
$$\delta N^\alpha = \left(\frac{1}{n}\gamma(-w + \beta A + B_\perp), \frac{1}{a}f_0^i, -\frac{1}{b}\gamma(E_{\perp\perp} + \beta w)\right), \tag{G7}$$
$$\delta N_\alpha = \left(-n\gamma(w + \beta A), -a(f_i^0 + \beta\gamma B_i + \gamma E_{\perp i}), b\gamma(\beta w - E_{\perp\perp} - \beta B_\perp)\right), \tag{G8}$$

with

$$u_\mu U^\mu = \gamma, \tag{G9}$$
$$\delta(u_\mu U^\mu) = \delta\gamma, \tag{G10}$$
$$w = \frac{\delta\gamma}{\beta\gamma} = \frac{\delta(\beta\gamma)}{\gamma}, \tag{G11}$$
$$\frac{w}{\beta} = \frac{\delta\beta}{\beta} + \frac{\delta\gamma}{\gamma} = \frac{\delta(\beta\gamma)}{\beta\gamma}, \tag{G12}$$
$$\gamma w = \delta(\beta\gamma), \tag{G13}$$
$$v_i^0 = \nabla_i v_0 + \bar{v}_i^0, \tag{G14}$$
$$f_i^0 = \nabla_i f_0 + \bar{f}_i^0. \tag{G15}$$

Since $n^\alpha + \delta n^\alpha$ is not orthogonal to $u^\alpha + \delta u^\alpha$, one does not have $\delta(n_\mu U^\mu) = \delta(\beta\gamma)$, but rather $\delta(n_\mu U^\mu) = \delta(\beta\gamma) + (U^\mu u_\mu)\delta(n^\nu u_\nu)$.

2. Gauge transformation

$$v_0 \to v_0 - \frac{a}{n}\gamma\dot{L} - \frac{a}{b}\beta\gamma L', \tag{G16}$$
$$\bar{v}_i^0 \to \bar{v}_i^0 - \frac{a}{n}\gamma\dot{\bar{L}}_i - \frac{a}{b}\beta\gamma\bar{L}_i', \tag{G17}$$
$$f_0 \to f_0 + \frac{a}{n}\beta\gamma\dot{L} + \frac{a}{b}\gamma L', \tag{G18}$$
$$\bar{f}_i^0 \to \bar{f}_i^0 + \frac{a}{n}\beta\gamma\dot{\bar{L}}_i + \frac{a}{b}\gamma\bar{L}_i', \tag{G19}$$
$$w \to w + \frac{\dot{\gamma}}{\beta\gamma}T + \frac{\gamma'}{\beta\gamma}L^\perp - \frac{b}{n}\dot{L}^\perp, \tag{G20}$$
$$\delta X_0 \to \delta X_0 + \dot{X}_0 T + X_0' L^\perp, \tag{G21}$$

where X_0 is any $(N+2)$-scalar quantity (density ρ_0, pressure P_0, etc). Note: γ, β, ρ, Y, F are *not* scalars as they are defined through the vector fields u^α, n^α which specifically depend on a coordinate choice.

3. Gauge invariant quantities

$$
\begin{aligned}
v_0^\sharp &= v_0 + \frac{a}{n}\gamma\dot{E} + \frac{a}{b}\beta\gamma E', && \text{(G22)}\\
\bar{v}_i^{0\sharp} &= \bar{v}_i^0 + \frac{a}{n}\gamma\dot{\bar{E}}_i + \frac{a}{b}\beta\gamma\bar{E}_i', && \text{(G23)}\\
f_0^\sharp &= f_0 - \frac{a}{n}\beta\gamma\dot{E} - \frac{a}{b}\gamma E', && \text{(G24)}\\
\bar{f}_i^{0\sharp} &= \bar{f}_i^0 - \frac{a}{n}\beta\gamma\dot{\bar{E}}_i - \frac{a}{b}\gamma\bar{E}_i', && \text{(G25)}\\
w^\sharp &= w - \frac{\dot{\gamma}}{\beta\gamma}\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) + \frac{\gamma'}{\beta\gamma}\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right) - \frac{b}{n}\partial_\eta\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right)\\
&= w - \frac{\partial_\eta(\beta\gamma)}{\gamma}\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) + \frac{(\beta\gamma)'}{\gamma}\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right) - \frac{b}{n}\partial_\eta\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right), && \text{(G26)}\\
\delta X_0^\sharp &= \delta X_0 - \dot{X}_0\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) + X_0'\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right). && \text{(G27)}
\end{aligned}
$$

It is useful to define

$$
\begin{aligned}
v^\sharp &\equiv \gamma v_0^\sharp + \beta\gamma f_0^\sharp, && \text{(G28)}\\
\bar{v}_i^\sharp &\equiv \gamma\bar{v}_i^{0\sharp} + \beta\gamma\bar{f}_i^{0\sharp}, && \text{(G29)}\\
f^\sharp &\equiv \beta\gamma v_0^\sharp + \gamma f_0^\sharp, && \text{(G30)}\\
\bar{f}_i^\sharp &\equiv \beta\gamma\bar{v}_i^{0\sharp} + \gamma\bar{f}_i^{0\sharp}. && \text{(G31)}
\end{aligned}
$$

4. Stress-energy tensor

$$
\delta T_{\alpha\beta} = \delta((P_0+\rho_0)U_\alpha U_\beta) - \delta((P_0 - Y_0)N_\alpha N_\beta) - \delta(P_0 g_{\alpha\beta}) + \pi_{\alpha\beta}. \qquad \text{(G32)}
$$

$$
\begin{aligned}
\pi_{\alpha\mu}U^\mu &= 0, && \text{(G33)}\\
\pi_{\alpha\mu}N^\mu &= 0, && \text{(G34)}\\
\pi_{0\alpha} &= 0, && \text{(G35)}\\
\pi_{M\alpha} &= 0, && \text{(G36)}\\
\pi_{ij} &= a^2\Pi_{ij} = a^2\left(\left(\nabla_{ij} - \frac{1}{N}\nabla^2\gamma_{ij}\right)\Pi + \nabla_{(i}\bar{\Pi}_{j)} + \bar{\bar{\Pi}}_{ij}\right), && \text{(G37)}
\end{aligned}
$$

$$
\begin{aligned}
\delta T_{00} &= n^2\gamma^2(\delta\rho_0 + \beta^2\delta Y_0 + 2A(\rho_0 + \beta^2 Y_0) + 2\beta(Y_0+\rho_0)w)\\
&= n^2\left(\delta\rho^\sharp + 2\rho\Psi\right)\\
&\quad + (\dot{T}_{00} + 2T_{00}\partial_\eta)\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) - (T_{00}' + 2T_{0M}\partial_\eta)\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right), && \text{(G38)}\\
\delta T_{0i} &= -an((P_0+\rho_0)\gamma v_i^0 - (Y_0 - P_0)\beta\gamma f_i^0 - \gamma^2(\rho_0 + \beta^2 Y_0)B_i - (\rho_0 + Y_0)E_{\perp i})\\
&= -an\left((P+\rho)v_i^\sharp - F f_i^\sharp\right) + \frac{a}{n}T_{00}\bar{\Sigma}_i - \frac{a}{b}T_{0M}\bar{h}_i\\
&\quad + T_{00}\nabla_i\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) - T_{0M}\nabla_i\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right) + T_{ij}\dot{E}^j, && \text{(G39)}\\
\delta T_{ij} &= a^2(\delta P_0\gamma_{ij} + \Pi_{ij} + 2P_0(C\gamma_{ij} + E_{ij}))\\
&= a^2\gamma_{ij}\delta P^\sharp + a^2\Pi_{ij}\\
&\quad + \dot{T}_{ij}\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) - T_{ij}'\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right) + 2T_{k(i}(E^k_{j)} - \delta^k_{j)}\Phi), && \text{(G40)}\\
\delta T_{0M} &= -nb\gamma^2\left(\beta(\delta Y_0 + \delta\rho_0) + (Y_0+\rho_0)(1+\beta^2)w + \beta(Y_0+\rho_0)(A - E_{\perp\perp}) - (\rho_0 + \beta^2 Y_0)B_\perp\right)
\end{aligned}
$$

$$
\begin{aligned}
&= -nb\left(\delta F^\sharp + F(\Psi - h) - \rho\Sigma\right) \\
&\quad +(\dot{T}_{0M} + T_{0M}\partial_\eta + T_{00}\partial_y)\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) - (T'_{0M} + T_{0M}\partial_y + T_{MM}\partial_\eta)\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right), \qquad \text{(G41)} \\
\delta T_{iM} &= ba\left(\gamma(P_0 - Y_0)f_i^0 + \beta\gamma(P_0 + \rho_0)v_i^0 - \gamma^2(Y_0 + \beta^2\rho_0)E_{\perp i} - \beta\gamma^2(\rho_0 + Y_0)B_i\right) \\
&= ba\left(Fv_i^\sharp + (P - Y)f_i^\sharp\right) + \frac{a}{n}T_{0M}\bar{\Sigma}_i - \frac{a}{b}T_{MM}\bar{h}_i \\
&\quad +T_{0M}\nabla_i\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) - T_{MM}\nabla_i\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right) + T_{ij}E^{j\prime}, \qquad \text{(G42)} \\
\delta T_{MM} &= b^2\gamma^2\left((\delta Y_0 + \beta^2\delta\rho_0) + 2\beta(Y_0 + \rho_0)w - 2(Y_0 + \beta^2\rho_0)E_{\perp\perp} - 2\beta(Y_0 + \rho_0)B_\perp\right) \\
&= b^2\left(\delta Y^\sharp - 2Yh - 2F\Sigma\right) \\
&\quad +(\dot{T}_{MM} + 2T_{0M}\partial_y)\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) - (T'_{MM} + 2T_{MM}\partial_y)\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right). \qquad \text{(G43)}
\end{aligned}
$$

$$
\begin{aligned}
\delta T^{00} &= \frac{1}{n^2}\left(\delta\rho^\sharp - 2\rho\Psi - 2F\Sigma\right) \\
&\quad +(\dot{T}^{00} - 2T^{00}\partial_\eta - 2T^{0M}\partial_y)\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) - T^{00\prime}\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right), \qquad \text{(G44)} \\
\delta T^{0i} &= \frac{1}{an}\left((P + \rho)v_\sharp^i - Ff_\sharp^i\right) - \frac{a}{n}T^{ij}\bar{\Sigma}_j \\
&\quad -T^{ij}\nabla_j\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) - (T^{00}\partial_\eta + T^{0M}\partial_y)E^i, \qquad \text{(G45)} \\
\delta T^{ij} &= \frac{1}{a^2}\left(\delta P^\sharp\gamma^{ij} + \Pi^{ij}\right) \\
&\quad +\dot{T}^{ij}\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) - T^{ij\prime}\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right) - 2T^{k(i}(E_k^{j)} - \delta_k^{j)}\Phi), \qquad \text{(G46)} \\
\delta T^{0M} &= \frac{1}{nb}\left(\delta F^\sharp + F(h - \Psi) - Y\Sigma\right) \\
&\quad +(\dot{T}^{0M} - T^{0M}\partial_\eta - T^{MM}\partial_y)\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) - (T^{0M\prime} - T^{0M}\partial_y - T^{00}\partial_\eta)\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right), \qquad \text{(G47)} \\
\delta T^{iM} &= \frac{1}{ba}\left(Fv_\sharp^i + (P - Y)f_\sharp^i\right) + \frac{a}{b}T^{ij}\bar{h}_j \\
&\quad +T^{ij}\nabla_j\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right) - (T^{0M}\partial_\eta + T^{MM}\partial_y)E^i, \qquad \text{(G48)} \\
\delta T^{MM} &= \frac{1}{b^2}\left(\delta Y^\sharp + 2Yh\right) \\
&\quad +\dot{T}^{MM}\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) - (T^{MM\prime} - 2T^{MM}\partial_y - 2T^{0M}\partial_\eta)\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right). \qquad \text{(G49)}
\end{aligned}
$$

5. An example: a scalar field

$$
\begin{aligned}
\rho_0 &= \frac{1}{2}D_\mu\phi D^\mu\phi + V \\
&= \frac{1}{2}\left(\frac{\dot{\phi}^2}{n^2} - \frac{\phi'^2}{b^2}\right) + V, \qquad \text{(G50)} \\
P_0 &= \frac{1}{2}D_\mu\phi D^\mu\phi - V \\
&= \frac{1}{2}\left(\frac{\dot{\phi}^2}{n^2} - \frac{\phi'^2}{b^2}\right) - V, \qquad \text{(G51)}
\end{aligned}
$$

$$Y_0 = P_0, \tag{G52}$$

$$U_\alpha = \frac{D_\alpha\phi}{\pm\sqrt{D_\mu\phi D^\mu\phi}}, \tag{G53}$$

$$\gamma = \frac{1}{\sqrt{1-\frac{n^2}{b^2}\frac{\phi'^2}{\dot\phi^2}}}, \tag{G54}$$

$$\beta\gamma = -\frac{n}{b}\frac{\phi'}{\dot\phi}\frac{1}{\sqrt{1-\frac{n^2}{b^2}\frac{\phi'^2}{\dot\phi^2}}}, \tag{G55}$$

$$\beta = -\frac{n}{b}\frac{\phi'}{\dot\phi}, \tag{G56}$$

with the $\pm$ sign determined by the condition $U_0 \geq 0$.

$$\rho = \frac{1}{2}\left(\frac{\dot\phi^2}{n^2}+\frac{\phi'^2}{b^2}\right)+V, \tag{G57}$$

$$P = \frac{1}{2}\left(\frac{\dot\phi^2}{n^2}-\frac{\phi'^2}{b^2}\right)-V, \tag{G58}$$

$$Y = \frac{1}{2}\left(\frac{\dot\phi^2}{n^2}+\frac{\phi'^2}{b^2}\right)-V, \tag{G59}$$

$$F = -\frac{\dot\phi}{n}\frac{\phi'}{b}. \tag{G60}$$

$$\delta\rho_0^\sharp = \frac{\dot\phi\dot{\delta\phi}^\sharp}{n^2}-\frac{\phi'\delta\phi^{\sharp\prime}}{b^2}-\frac{\dot\phi^2}{n^2}\Psi-\frac{\phi'^2}{b^2}h+\frac{\dot\phi}{n}\frac{\phi'}{b}\Sigma+\frac{\mathrm{d}V}{\mathrm{d}\phi}\delta\phi^\sharp, \tag{G61}$$

$$\delta P_0^\sharp = \frac{\dot\phi\dot{\delta\phi}^\sharp}{n^2}-\frac{\phi'\delta\phi^{\sharp\prime}}{b^2}-\frac{\dot\phi^2}{n^2}\Psi-\frac{\phi'^2}{b^2}h+\frac{\dot\phi}{n}\frac{\phi'}{b}\Sigma-\frac{\mathrm{d}V}{\mathrm{d}\phi}\delta\phi^\sharp, \tag{G62}$$

$$\delta\rho^\sharp = (P+\rho)\left(\frac{\dot{\delta\phi}^\sharp}{\dot\phi}-\Psi\right)+(Y-P)\left(\frac{\delta\phi^{\sharp\prime}}{\phi'}+h\right)+F\Sigma+\frac{\mathrm{d}V}{\mathrm{d}\phi}\delta\phi^\sharp, \tag{G63}$$

$$\delta Y^\sharp = (P+\rho)\left(\frac{\dot{\delta\phi}^\sharp}{\dot\phi}-\Psi\right)+(Y-P)\left(\frac{\delta\phi^{\sharp\prime}}{\phi'}+h\right)+F\Sigma-\frac{\mathrm{d}V}{\mathrm{d}\phi}\delta\phi^\sharp, \tag{G64}$$

$$\delta P^\sharp = (P+\rho)\left(\frac{\dot{\delta\phi}^\sharp}{\dot\phi}-\Psi\right)-(Y-P)\left(\frac{\delta\phi^{\sharp\prime}}{\phi'}+h\right)-F\Sigma-\frac{\mathrm{d}V}{\mathrm{d}\phi}\delta\phi^\sharp, \tag{G65}$$

$$\delta F^\sharp = (P+\rho)\Sigma+F\left(\frac{\dot{\delta\phi}^\sharp}{\dot\phi}-\Psi+\frac{\delta\phi^{\sharp\prime}}{\phi'}+h\right), \tag{G66}$$

$$av_0^\sharp = -n\frac{\frac{\delta\phi^\sharp}{\dot\phi}}{\sqrt{1-\frac{n^2}{b^2}\frac{\phi'^2}{\dot\phi^2}}}, \tag{G67}$$

$$\bar v_i^{0\sharp} = \gamma\bar\Sigma_i+\beta\gamma\bar h_i, \tag{G68}$$

$$af_0^\sharp = -\beta a v_0^\sharp, \tag{G69}$$

$$\bar f_i^{0\sharp} = -\gamma\bar h_i-\beta\gamma\bar\Sigma_i, \tag{G70}$$

$$w = \beta\gamma^2\left(A+E_{\perp\perp}+\frac{\delta\phi'}{\phi'}-\frac{\dot{\delta\phi}}{\dot\phi}\right)+\gamma^2 B_\perp, \tag{G71}$$

$$w^\sharp = \beta\gamma^2\left(\Psi + h + \frac{\delta\phi^{\sharp\prime}}{\phi'} - \frac{\dot{\delta\phi}^\sharp}{\dot\phi}\right) + \gamma^2\Sigma, \tag{G72}$$

$$\pi_{ij} = 0. \tag{G73}$$

Here, the components f_0^i are arbitrary as the eigenvalues P_0 and Y_0 are degenerate. The expression chosen here is purely for convenience in order to simplify the following untilded components.

$$av^\sharp = -n\frac{\delta\phi^\sharp}{\dot\phi}, \tag{G74}$$

$$\bar v_i^\sharp = \bar\Sigma_i, \tag{G75}$$

$$af^\sharp = 0, \tag{G76}$$

$$\bar f_i^\sharp = -\bar h_i. \tag{G77}$$

6. Interaction term

$$\begin{aligned}\delta Q^0 &= \frac{1}{n}\left(\delta\Gamma^\sharp - \Gamma\Psi - D\Sigma\right)\\ &\quad +(\dot Q^0 - Q^0\partial_\eta - Q^\perp\partial_y)\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot E\right) - Q^{0\prime}\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right),\end{aligned} \tag{G78}$$

$$\delta Q^i = \frac{1}{a}\mathcal{Q}_\sharp^i - (Q^0\partial_\eta + Q^\perp\partial_y)E^i, \tag{G79}$$

$$\begin{aligned}\delta Q^\perp &= \frac{1}{b}\left(\delta D^\sharp + Dh\right)\\ &\quad +\dot Q^\perp\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot E\right) - (Q^{\perp\prime} - Q^\perp\partial_y - Q^0\partial_\eta)\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right),\end{aligned} \tag{G80}$$

with

$$\mathcal{Q}_i^\sharp = \nabla_i\mathcal{Q}^\sharp + \bar{\mathcal{Q}}_i^\sharp. \tag{G81}$$

7. Conservation equations

$$\begin{aligned}&(\partial_u + N\mathcal{H} + 2\mathcal{U})(\delta\rho^\sharp - F\Sigma) + N\mathcal{H}\delta P^\sharp + \mathcal{U}(\delta Y^\sharp - \delta\rho^\sharp)\\ &\quad +(\partial_n + NH + 2I)\left(\delta F^\sharp + F(\Psi + h)\right)\\ &\quad +\Delta\left((P+\rho)av^\sharp - Faf^\sharp\right)\\ &-N(P+\rho)\partial_u\Phi - (\rho+Y)\partial_u h - F\partial_u\Sigma + F\partial_n(\Psi - h - N\Phi) = \delta\Gamma^\sharp + \Gamma\Psi,\\ &\quad (\partial_u + N\mathcal{H} + \mathcal{U})\left((P+\rho)av^\sharp - Faf^\sharp\right)\\ &\quad +(\partial_n + NH + I)\left(Fav^\sharp + (P-Y)af^\sharp\right)\end{aligned} \tag{G82}$$

$$+\delta P^\sharp + \frac{N-1}{N}(\Delta + NK)a^2\Pi + (P+\rho)\Psi + (Y-P)h + F\Sigma = a\mathcal{Q}^\sharp, \tag{G83}$$

$$\begin{aligned}&(\partial_u + N\mathcal{H} + 2\mathcal{U})\left(\delta F^\sharp - F(\Psi+h) - (\rho+Y)\Sigma\right)\\ &+(\partial_n + NH + 2I)(\delta Y^\sharp - F\Sigma) - NH\delta P^\sharp - I(\delta Y^\sharp - \delta\rho^\sharp)\\ &\quad +\Delta\left(Fav^\sharp + (P-Y)af^\sharp\right)\\ &+F\partial_u(\Psi - h - N\Phi) + N(P-Y)\partial_n\Phi + (\rho+Y)\partial_n\Psi + F\partial_n\Sigma = \delta D^\sharp - Dh - \Gamma\Sigma,\end{aligned} \tag{G84}$$

$$\begin{aligned}&(\partial_u + (N+1)\mathcal{H} + \mathcal{U})\left((P+\rho)(\bar v_i^\sharp - \bar\Sigma_i) - F(\bar f_i^\sharp + \bar h_i)\right)\\ &+(\partial_n + (N+1)H + I)\left(F(\bar v_i^\sharp - \bar\Sigma_i) + (P-Y)(\bar f_i^\sharp + \bar h_i)\right)\\ &\quad +\frac{1}{2}\left(\Delta + (N-1)K\right)a\bar\Pi_i = \bar{\mathcal{Q}}_i^\sharp - D\bar h_i - \Gamma\bar\Sigma_i.\end{aligned} \tag{G85}$$

8. Einstein equations

$$
\begin{aligned}
&\Delta((N-1)\Phi+h)+KN(N-1)\Phi \\
&-N\left(\left((N-1)\mathcal{H}^2+2\mathcal{H}\mathcal{U}\right)\Psi+\mathcal{H}\partial_u h+\left((N-1)\mathcal{H}+\mathcal{U}\right)\partial_u\Phi\right) \\
&-N\left(H\partial_n+(N+1)H^2\right)h-2Nh\partial_n H \\
&+N(\partial_n+(N+1)H)\partial_n\Phi \\
&+N\left(\partial_n+NH+I\right)(\mathcal{H}\Sigma) &&= \kappa_{N+2}\left(\delta\rho^\sharp-F\Sigma\right), && \text{(G86)}\\
&\frac{1}{2}\left(\partial_n+(N-2)H+2I\right)\Sigma \\
&-(\mathcal{U}+(N-1)\mathcal{H})\Psi-(\partial_u+\mathcal{U}-\mathcal{H})h-(N-1)\partial_u\Phi &&= \kappa_{N+2}\left((P+\rho)av^\sharp-Faf^\sharp\right), && \text{(G87)}\\
&-K(N-1)(N-2)\Phi \\
&+\left(2(\mathcal{U}+(N-1)\mathcal{H})(\partial_u+\mathcal{U})+N(N-1)\mathcal{H}^2\right)\Psi \\
&+2\Psi\partial_u(\mathcal{U}+(N-1)\mathcal{H}) \\
&+\left(2(I+(N-1)H)(\partial_n+I)+N(N-1)H^2\right)h \\
&+2h\partial_n(I+(N-1)H) \\
&+(\partial_u+\mathcal{U})\partial_u(h+(N-1)\Phi) \\
&-(\mathcal{U}+(N-1)\mathcal{H})\partial_u(\Psi-h-N\Phi) \\
&+(\partial_n+I)\partial_n(\Psi-(N-1)\Phi) \\
&+(I+(N-1)H)\partial_n(\Psi-h-N\Phi) \\
&-\tfrac{1}{2}(\partial_n\partial_u+\partial_u\partial_n+I\partial_u+\mathcal{U}\partial_n)\Sigma \\
&-\left((\mathcal{U}+(N-1)\mathcal{H})\partial_n+(I+(N-1)H)\partial_u\right)\Sigma \\
&-\Sigma\left((\partial_n+I)(\mathcal{U}+(N-1)\mathcal{H})+(\partial_u+\mathcal{U})(I+(N-1)H)\right) \\
&-N\left(\mathcal{U}\partial_u-I\partial_n\right)\Phi-N(N-1)\mathcal{H}H\Sigma &&= \kappa_{N+2}\left(\delta P^\sharp+\frac{N-1}{N}\nabla^2\Pi\right), && \text{(G88)}\\
&N\Phi-(\Phi+\Psi)+(h-\Phi) &&= \kappa_{N+2}a^2\Pi, && \text{(G89)}\\
&-N\left((\partial_u\partial_n+(H-I)\partial_u+\mathcal{H}\partial_n)\Phi+\mathcal{H}\partial_n\Psi-H\partial_u h\right) \\
&-\frac{1}{2}\Delta\Sigma-N\Sigma(\partial_u\mathcal{H}+\mathcal{H}^2-\mathcal{H}\mathcal{U}) &&= \kappa_{N+2}\left(\delta F^\sharp+F(\Psi-h)\right), && \text{(G90)}\\
&\frac{1}{2}\left(\partial_u+(N-2)\mathcal{H}+2\mathcal{U}\right)\Sigma \\
&-(\partial_n+I-H)\Psi-((N-1)H+I)h+(N-1)\partial_n\Phi &&= \kappa_{N+2}\left(Fav^\sharp+(P-Y)af^\sharp\right), && \text{(G91)}\\
&-\left(\Delta((N-1)\Phi-\Psi)+KN(N-1)\Phi\right) \\
&+N\left(\mathcal{H}\partial_u+(N+1)\mathcal{H}^2\right)\Psi+2N\Psi\partial_u\mathcal{H} \\
&+N\left(\partial_u+(N+1)\mathcal{H}\right)\partial_u\Phi \\
&+N\left((H\partial_n)\Psi+\left((N-1)H^2+2HI\right)h-\left((N-1)H\partial_n+I\partial_n\right)\Phi\right) \\
&-N\left(\partial_u+N\mathcal{H}+\mathcal{U}\right)(H\Sigma) &&= \kappa_{N+2}\left(\delta Y^\sharp-F\Sigma\right), && \text{(G92)}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}(\Delta+(N-1)K)\bar{\Sigma}_i \\
&-\frac{1}{2}\left(\partial_n+(N+1)H\right)\left((\partial_n+I-H)\bar{\Sigma}_i\right) \\
&+\frac{1}{2}\left(\partial_n+(N+1)H\right)\left((\partial_u+\mathcal{U}-\mathcal{H})\bar{h}_i\right) &&= \kappa_{N+2}\left((P+\rho)(\bar{v}_i^\sharp-\bar{\Sigma}_i)-F(\bar{f}_i^\sharp+\bar{h}_i)\right), && \text{(G93)}\\
&\left(\partial_u+(N-1)\mathcal{H}+\mathcal{U}\right)\bar{\Sigma}_i-\left(\partial_n+(N-1)H+I\right)\bar{h}_i &&= \kappa_{N+2}a\bar{\Pi}_i, && \text{(G94)}\\
&\frac{1}{2}(\Delta+(N-1)K)\bar{h}_i \\
&+\frac{1}{2}\left(\partial_u+(N+1)\mathcal{H}\right)\left((\partial_n+I-H)\bar{\Sigma}_i\right)
\end{aligned}
$$

$$-\frac{1}{2}\left(\partial_u+(N+1)\mathcal{H}\right)\left((\partial_u+\mathcal{U}-\mathcal{H})\bar{h}_i\right) \;=\; \kappa_{N+2}\left(F(\bar{v}_i^\sharp-\bar{\Sigma}_i)+(P-Y)(\bar{f}_i^\sharp+\bar{h}_i)\right), \tag{G95}$$

$$-(\Delta-2K)\bar{\bar{E}}_{ij}+\left(\partial_u+N\mathcal{H}+\mathcal{U}\right)\partial_u\bar{\bar{E}}_{ij}-\left(\partial_n+NH+I\right)\partial_n\bar{\bar{E}}_{ij} \;=\; \kappa_{N+2}\bar{\bar{\Pi}}_{ij}. \tag{G96}$$

APPENDIX H: PERTURBED BRANE-RELATED QUANTITIES

1. Brane position

$$X^0 \;=\; \sigma^0+\zeta^0(\sigma^a), \tag{H1}$$
$$X^i \;=\; \sigma^i+\zeta^i(\sigma^a), \tag{H2}$$
$$X^M \;=\; \mathrm{y_b}+\epsilon(\sigma^a). \tag{H3}$$

Under an infinitesimal coordinate change, the brane position ϵ transforms into

$$\epsilon\to\epsilon-L^\perp. \tag{H4}$$

2. Normal vector to the brane

$$(\underline{\perp}_\mu+\underline{\delta\perp}_\mu)\frac{\partial X^\mu}{\partial\sigma_a} \;=\; 0, \tag{H5}$$
$$\underline{\delta\perp}^\alpha \;=\; \left(\frac{1}{n}\left(B_\perp-\frac{b}{n}\dot{\epsilon}\right),\frac{1}{a}\left(\frac{b}{a}\nabla^i\epsilon-E_\perp^i\right),-\frac{1}{b}E_{\perp\perp}\right), \tag{H6}$$
$$\underline{\delta\perp}_\alpha \;=\; \left(-n\frac{b}{n}\dot{\epsilon},-a\frac{b}{a}\nabla_i\epsilon,-bE_{\perp\perp}\right). \tag{H7}$$

Since $\underline{\perp}^\alpha$ plays the same role for the brane as N^α for a bulk component, this means that formally, the quantities w, f_i can be defined for the brane (we will note them $\underline{w}$ and $\underline{f}_i$ respectively) at $y=\mathrm{y_b}$,

$$\underline{w} \;=\; \frac{b}{n}\dot{\epsilon}, \tag{H8}$$
$$\underline{f}_i+E_{\perp i} \;=\; \frac{b}{a}\nabla_i\epsilon, \tag{H9}$$
$$\tag{H10}$$

or, equivalently,

$$\underline{w}^\sharp \;=\; \frac{b}{n}\dot{\epsilon}^\sharp, \tag{H11}$$
$$\underline{f}^\sharp \;=\; \frac{b}{a}\epsilon^\sharp, \tag{H12}$$
$$\underline{\bar{f}}_i^\sharp \;=\; -\bar{h}_i. \tag{H13}$$

3. Induced metric

$$\underline{\delta q}_{00} \;=\; 2n^2(A+\tilde{I}\epsilon), \tag{H14}$$
$$\underline{\delta q}_{0i} \;=\; anB_i, \tag{H15}$$
$$\underline{\delta q}_{ij} \;=\; -2a^2(C+\tilde{H}\epsilon)\gamma_{ij}-2a^2E_{ij}, \tag{H16}$$
$$\underline{\delta q}_{0M} \;=\; nb\left(B_\perp-\frac{b}{n}\dot{\epsilon}\right), \tag{H17}$$
$$\underline{\delta q}_{iM} \;=\; ba\left(E_{\perp i}-\frac{b}{a}\nabla_i\epsilon\right), \tag{H18}$$
$$\underline{\delta q}_{MM} \;=\; 0. \tag{H19}$$

4. First Israel condition

$$
\begin{aligned}
[A] + \left[\tilde{I}\right]\epsilon &= 0, && \text{(H20)}\\
[B_i] &= 0, && \text{(H21)}\\
[C] + \left[\tilde{H}\right]\epsilon &= 0, && \text{(H22)}\\
[E_i] &= 0, && \text{(H23)}\\
\left[\bar{\bar{E}}_{ij}\right] &= 0, && \text{(H24)}\\
\left[\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right)\right] &= 0. && \text{(H25)}
\end{aligned}
$$

Or, equivalently

$$
\begin{aligned}
[\Psi] &= -\left[\tilde{I}\epsilon - \tilde{I}\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right)\right], && \text{(H26)}\\
[\Phi] &= \left[\tilde{H}\epsilon - \tilde{H}\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right)\right], && \text{(H27)}\\
\left[\bar{\Sigma}_i\right] &= 0, && \text{(H28)}\\
\left[\bar{\bar{E}}_{ij}\right] &= 0. && \text{(H29)}
\end{aligned}
$$

With,

$$
\begin{aligned}
\underline{\Psi} &= A - (\partial_\eta + \tilde{\mathcal{I}})\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) + \tilde{I}\epsilon = \Psi + \tilde{I}\left(\epsilon - \left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right)\right), && \text{(H30)}\\
\underline{\Phi} &= -C + \tilde{\mathcal{H}}\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) - \tilde{H}\epsilon = \Phi - \tilde{H}\left(\epsilon - \left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right)\right), && \text{(H31)}
\end{aligned}
$$

one has

$$
\begin{aligned}
\underline{\delta q}_{00} &= 2n^2\underline{\Psi} + (\dot{\underline{q}}_{00} + 2\underline{q}_{00}\partial_\eta)\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right), && \text{(H32)}\\
\underline{\delta q}_{0i} &= -\frac{n}{a}\underline{q}_{ij}\bar{\Sigma}^j + \underline{q}_{00}\nabla_i\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) + \underline{q}_{ij}\dot{E}^j, && \text{(H33)}\\
\underline{\delta q}_{ij} &= 2\underline{q}_{k(i}\left(E^k_{j)} - \delta^k_{j)}\underline{\Phi}\right), && \text{(H34)}
\end{aligned}
$$

$$
E_{\perp\perp} \to E_{\perp\perp} - \tilde{\mathcal{U}}T - \frac{1}{b}(bL^\perp)', \qquad \text{(H35)}
$$

therefore

$$
\begin{aligned}
\left[L^\perp\right] &= 0, && \text{(H36)}\\
\left[bL^\perp\right] &= 0. && \text{(H37)}
\end{aligned}
$$

Then,

$$
\begin{aligned}
[b] \neq 0 \Rightarrow L^\perp(y = \mathrm{y_b}) &= 0, && \text{(H38)}\\
[b\epsilon] &= 0. && \text{(H39)}
\end{aligned}
$$

$$
\begin{aligned}
bB_\perp - \frac{b^2}{n}\epsilon &\to bB_\perp - \frac{b^2}{n}\epsilon + nT', && \text{(H40)}\\
bE_{\perp i} - \frac{b^2}{a}\nabla_i\epsilon &\to bE_{\perp i} - \frac{b^2}{a}\nabla_i\epsilon - aL'_i, && \text{(H41)}
\end{aligned}
$$

and

$$[bB_\perp] - \left[\frac{b^2}{n}\right]\dot{\epsilon} = [bB_\perp] \;=\; 0, \tag{H42}$$

$$[bE_{\perp i}] - \left[\frac{b^2}{a}\right]\nabla_i\epsilon = [bE_{\perp i}]\,\epsilon \;=\; 0, \tag{H43}$$

therefore

$$[T'] \;=\; 0, \tag{H44}$$

$$\left[L^{i\prime}\right] \;=\; 0. \tag{H45}$$

5. New brane-related gauge invariant quantities

At the brane position (or on both sides of the brane),

$$\epsilon^\sharp \;\equiv\; \epsilon - \left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right), \tag{H46}$$

$$\underline{\Sigma} \;\equiv\; B_\perp - \frac{n}{b}\partial_y\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) - \frac{b}{n}\partial_\eta\epsilon = \Sigma - \frac{b}{n}\partial_\eta\epsilon^\sharp, \tag{H47}$$

$$\underline{h} \;\equiv\; E_{\perp\perp} + \tilde{\mathcal{U}}\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) - \tilde{U}\epsilon = h - \frac{1}{b}\partial_y(b\epsilon^\sharp), \tag{H48}$$

$$\underline{\Psi'} \;\equiv\; A' - \partial_y(\partial_\eta + \tilde{\mathcal{I}})\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) + \tilde{I}'\epsilon = \Psi' + \tilde{I}'\epsilon^\sharp - \tilde{I}\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right)', \tag{H49}$$

$$\underline{\Phi'} \;\equiv\; -C' + (\tilde{\mathcal{H}}\partial_y + \tilde{\mathcal{H}}')\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) - \tilde{H}'\epsilon = \Psi' - \tilde{H}'\epsilon^\sharp + \tilde{H}\left(\frac{a}{b}E_\perp + \frac{a^2}{b^2}E'\right)', \tag{H50}$$

$$\partial_n\underline{\Psi} \;\equiv\; \frac{1}{b}\underline{\Psi}', \tag{H51}$$

$$\partial_n\underline{\Phi} \;\equiv\; \frac{1}{b}\underline{\Phi}'. \tag{H52}$$

$$\left\langle b\epsilon^\sharp\right\rangle \;\equiv\; \underline{\Upsilon}, \tag{H53}$$

$$\left[b\epsilon^\sharp\right] \;\equiv\; -\underline{\Xi}. \tag{H54}$$

$$[\Psi] \;=\; -\left[\frac{\tilde{I}}{b}\right]\underline{\Upsilon} + \left\langle\frac{\tilde{I}}{b}\right\rangle\underline{\Xi}, \tag{H55}$$

$$[\Phi] \;=\; \left[\frac{\tilde{H}}{b}\right]\underline{\Upsilon} - \left\langle\frac{\tilde{H}}{b}\right\rangle\underline{\Xi}, \tag{H56}$$

$$[\Sigma] \;=\; [\underline{\Sigma}] - \frac{1}{n}\left(\partial_\eta - \left\langle\tilde{\mathcal{U}}\right\rangle\right)\underline{\Xi} - \frac{1}{n}\left[\tilde{\mathcal{U}}\right]\underline{\Upsilon}, \tag{H57}$$

$$[h] \;=\; [\underline{h}] + \left[\frac{\partial_y}{b}\underline{\Upsilon}\right], \tag{H58}$$

$$\langle\Psi\rangle \;=\; \underline{\Psi} - \left\langle\frac{\tilde{I}}{b}\right\rangle\underline{\Upsilon} + \frac{1}{4}\left[\frac{\tilde{I}}{b}\right]\underline{\Xi}, \tag{H59}$$

$$\langle\Phi\rangle \;=\; \underline{\Phi} + \left\langle\frac{\tilde{H}}{b}\right\rangle\underline{\Upsilon} - \frac{1}{4}\left[\frac{\tilde{H}}{b}\right]\underline{\Xi}, \tag{H60}$$

$$\langle\Sigma\rangle \;=\; \langle\underline{\Sigma}\rangle + \frac{1}{n}\left(\partial_\eta - \left\langle\tilde{\mathcal{U}}\right\rangle\right)\underline{\Upsilon} + \frac{1}{4}\frac{1}{n}\left[\tilde{\mathcal{U}}\right]\underline{\Xi}, \tag{H61}$$

$$\langle h\rangle \;=\; \langle\underline{h}\rangle + \left\langle\frac{\partial_y}{b}\underline{\Upsilon}\right\rangle, \tag{H62}$$

where the terms $\partial_y\underline{\Upsilon}$ are defined by setting ϵ constant.

6. Extrinsic curvature

$$\begin{aligned}
\underline{\delta\mathcal{K}}_{00} &= n(\partial_\eta+\tilde{\mathcal{U}})\left(\Sigma-\frac{b}{n}\dot{\epsilon}+\frac{b}{n}\partial_\eta\left(\frac{a}{b}E_\perp+\frac{a^2}{b^2}E'\right)\right)-\frac{n^2}{b}(\Psi'+\tilde{I}h)+2\underline{\mathcal{K}}_{00}\Psi+\underline{\mathcal{K}}'_{00}\epsilon\\
&\quad+(\dot{\underline{\mathcal{K}}}_{00}+2\underline{\mathcal{K}}_{00}\partial_\eta)\left(\frac{a}{n}B+\frac{a^2}{n^2}\dot{E}\right)-\underline{\mathcal{K}}'_{00}\left(\frac{a}{b}E_\perp+\frac{a^2}{b^2}E'\right)\\
&= n(\partial_\eta+\tilde{\mathcal{U}})\underline{\Sigma}-\frac{n^2}{b}(\underline{\Psi}'+\tilde{I}\underline{h})+2\underline{\mathcal{K}}_{00}\underline{\Psi}+(\dot{\underline{\mathcal{K}}}_{00}+2\underline{\mathcal{K}}_{00}\partial_\eta)\left(\frac{a}{n}B+\frac{a^2}{n^2}\dot{E}\right), \qquad \text{(H63)}\\
\underline{\delta\mathcal{K}}_{0i} &= \frac{1}{2}n\Sigma-(\partial_\eta-\tilde{\mathcal{H}})\left(b\epsilon-b\left(\frac{a}{b}E_\perp+\frac{a^2}{b^2}E'\right)\right)\\
&\quad+\frac{1}{2}a(\partial_\eta+\tilde{\mathcal{U}}-\tilde{\mathcal{H}})\bar{h}_i-\frac{1}{2}\frac{an}{b}(\partial_y+\tilde{H}-\tilde{I})\bar{\Sigma}_i+\frac{a}{n}\underline{\mathcal{K}}_{00}\bar{\Sigma}_i\\
&\quad+\underline{\mathcal{K}}_{00}\nabla_i\left(\frac{a}{n}B+\frac{a^2}{n^2}\dot{E}\right)+\underline{\mathcal{K}}_{ij}\dot{E}^j\\
&= \frac{1}{2}n\underline{\Sigma}-(\tfrac{1}{2}\partial_\eta+\tfrac{1}{2}\tilde{\mathcal{U}}-\tilde{\mathcal{H}})\left(b\epsilon^\sharp\right)+\frac{1}{2}a(\partial_\eta+\tilde{\mathcal{U}}-\tilde{\mathcal{H}})\bar{h}_i-\frac{1}{2}\frac{an}{b}(\partial_y+\tilde{H}-\tilde{I})\bar{\Sigma}_i\\
&\quad+\frac{a}{n}\underline{\mathcal{K}}_{00}\bar{\Sigma}_i+\underline{\mathcal{K}}_{00}\nabla_i\left(\frac{a}{n}B+\frac{a^2}{n^2}\dot{E}\right)+\underline{\mathcal{K}}_{ij}\dot{E}^j, \qquad \text{(H64)}\\
\underline{\delta\mathcal{K}}_{ij} &= \frac{a^2}{n}\gamma_{ij}\tilde{\mathcal{H}}\left(\frac{b}{n}\dot{\epsilon}-\frac{b}{n}\partial_\eta\left(\frac{a}{b}E_\perp+\frac{a^2}{b^2}E'\right)-\Sigma\right)-\frac{a^2}{b}\gamma_{ij}(\Phi'-\tilde{H}h)\\
&\quad-\nabla_{ij}\left(b\epsilon-b\left(\frac{a}{b}E_\perp+\frac{a^2}{b^2}E'\right)\right)+a\nabla_{(i}\bar{h}_{j)}+\frac{a^2}{b}\bar{\bar{E}}'_{ij}+\underline{\mathcal{K}}'_{ij}\epsilon\\
&\quad+\dot{\underline{\mathcal{K}}}_{ij}\left(\frac{a}{n}B+\frac{a^2}{n^2}\dot{E}\right)-\underline{\mathcal{K}}'_{ij}\left(\frac{a}{b}E_\perp+\frac{a^2}{b^2}E'\right)+2\underline{\mathcal{K}}_{k(i}(E^k_{j)}-\delta^k_{j)}\Phi)\\
&= -\frac{a^2}{n}\gamma_{ij}\tilde{\mathcal{H}}\underline{\Sigma}-\frac{a^2}{b}\gamma_{ij}(\underline{\Phi}'-\tilde{H}\underline{h})-\nabla_{ij}(b\epsilon^\sharp)+a\nabla_{(i}\bar{h}_{j)}+\frac{a^2}{b}\bar{\bar{E}}'_{ij}\\
&\quad+\dot{\underline{\mathcal{K}}}_{ij}\left(\frac{a}{n}B+\frac{a^2}{n^2}\dot{E}\right)+2\underline{\mathcal{K}}_{k(i}(E^k_{j)}-\delta^k_{j)}\underline{\Phi}), \qquad \text{(H65)}\\
\underline{\delta\mathcal{K}}_{0M} &= \underline{\mathcal{K}}^0_0\underline{\delta q}_{0M}, \qquad \text{(H66)}\\
\underline{\delta\mathcal{K}}_{iM} &= \underline{\mathcal{K}}^j_i\underline{\delta q}_{jM}, \qquad \text{(H67)}\\
\underline{\delta\mathcal{K}}_{M\ finN} &= 0. \qquad \text{(H68)}
\end{aligned}$$

7. $\underline{\mathcal{S}}_{\alpha\beta}$ tensor

The perturbed stress-energy tensor on the brane is given in Eqns (6.80–6.85). Formally, for any quantity $\underline{X}$ defined on the brane, the quantity $\mathrm{D}\underline{X}$ is a bulk scalar. The quantity D is also a scalar quantity of the bulk since one is allowed to consider the case $\underline{X}=$ constant. Its perturbation is

$$\delta\mathrm{D}=\mathrm{D}E_{\perp\perp}-\epsilon(\partial_y+\tilde{U})\mathrm{D}. \qquad \text{(H69)}$$

Note that this derivation is a bit formal: since $E_{\perp\perp}$ and $\underline{h}$ can be discontinuous, this expression and the next one are ill-defined, even in the case where b is continuous. But again, all the pathological terms cancel each other when one writes the Einstein equations, so that we consider that this is not a serious problem. Using the formula (G21), it is possible to build the gauge invariant counterparts of both $\delta\mathrm{D}$ and $\underline{\delta X}$:

$$\begin{aligned}
\delta\mathrm{D}^\sharp &= \mathrm{D}\underline{h}-\mathrm{D}'\epsilon^\sharp, \qquad \text{(H70)}\\
\underline{\delta X}^\sharp &= \underline{\delta X}-\dot{\underline{X}}\left(\frac{a}{n}B+\frac{a^2}{n^2}\dot{E}\right). \qquad \text{(H71)}
\end{aligned}$$

This last quantity is invariant under any infinitesimal reparametrization of the σ^a. Equivalently, one has, using (G22,G23), as well as the fact that $\gamma=1$ for the brane,

$$\underline{v}^\sharp = \underline{v}+\frac{a}{n}\dot{E}, \qquad \text{(H72)}$$

$$\underline{\bar{v}_i^\sharp} = \underline{\bar{v}}_i + \frac{a}{n}\dot{\underline{\bar{E}}}_i. \tag{H73}$$

With these definitions,

$$\begin{aligned}
\underline{\delta\mathcal{S}_{00}} &= n^2\left(\frac{N-1}{N}\underline{\delta\rho}^\sharp + \underline{\delta P}^\sharp\right) \\
&\quad +2\underline{\mathcal{S}_{00}}\underline{\Psi} + (\dot{\underline{\mathcal{S}}}_{00} + 2\underline{\mathcal{S}_{00}}\partial_\eta)\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right), && \text{(H74)}\\
\underline{\delta\mathcal{S}_{0i}} &= -an(\underline{P}+\underline{\rho})\underline{v}_i^\sharp \\
&\quad +\frac{a}{n}\underline{\mathcal{S}_{00}}\bar{\Sigma}_i + \underline{\mathcal{S}_{00}}\nabla_i\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) + \underline{\mathcal{S}}_{ij}\dot{E}^j, && \text{(H75)}\\
\underline{\delta\mathcal{S}_{ij}} &= a^2\underline{\Pi}_{ij} + a^2\frac{1}{N}\underline{\delta\rho}^\sharp\gamma_{ij} \\
&\quad +\dot{\underline{\mathcal{S}}}_{ij}\left(\frac{a}{n}B + \frac{a^2}{n^2}\dot{E}\right) + 2\underline{\mathcal{S}}_{k(i}(E^k_{j)} - \delta^k_{j)}\underline{\Phi}), && \text{(H76)}\\
\underline{\delta\mathcal{S}_{0M}} &= \underline{\mathcal{S}}^0_0\underline{\delta q_{0M}}, && \text{(H77)}\\
\underline{\delta\mathcal{S}_{iM}} &= \underline{\mathcal{S}}^j_i\underline{\delta q_{jM}}, && \text{(H78)}\\
\underline{\delta\mathcal{S}_{MM}} &= 0. && \text{(H79)}
\end{aligned}$$

8. Second Israel condition

$$\begin{aligned}
-\Delta\underline{\Xi} + N\left[\partial_n\underline{\Phi} - H\underline{h}\right] + N\mathcal{H}\left[\underline{\Sigma}\right] &= \kappa_{N+2}\underline{\delta\rho}^\sharp, && \text{(H80)}\\
\frac{1}{2}\left[\underline{\Sigma} - (\partial_u + \mathcal{U} - 2\mathcal{H})(b\epsilon^\sharp)\right] &= \kappa_{N+2}(\underline{P}+\underline{\rho})a\underline{v}^\sharp, && \text{(H81)}\\
\left[\partial_n\underline{\Psi} + I\underline{h}\right] - \left[(\partial_u + \mathcal{U})\,\underline{\Sigma}\right] &= \kappa_{N+2}\left(\underline{\delta P}^\sharp + \frac{N-1}{N}\underline{\delta\rho}^\sharp\right), && \text{(H82)}\\
-\underline{\Xi} &= \kappa_{N+2}a^2\underline{\Pi}, && \text{(H83)}
\end{aligned}$$

$$\begin{aligned}
-\frac{1}{2}\left[\partial_n + I - H\right]\bar{\Sigma}_i + \frac{1}{2}\left[(\partial_u + \mathcal{U} - \mathcal{H})\bar{h}_i\right] &= \kappa_{N+2}(\underline{P}+\underline{\rho})(\underline{v}_i^\sharp - \bar{\Sigma}_i), && \text{(H84)}\\
-\left[\bar{h}_i\right] &= \kappa_{N+2}a\underline{\bar{\Pi}}_i, && \text{(H85)}
\end{aligned}$$

$$-\left[\partial_n\bar{\bar{E}}_{ij}\right] = \kappa_{N+2}\underline{\bar{\bar{\Pi}}}_{ij}. \tag{H86}$$

9. Projected Weyl tensor

As this quantity is defined on the brane, it is more convenient to express it in term of the brane-related (underlined) metric perturbations instead of the bulk-related (non underlined) metric perturbations as it was the case for the Weyl tensor.

$$\begin{aligned}
\underline{\delta\mathcal{Z}}^\sharp &\equiv \frac{2}{N}(\Delta + NK)\underline{\Phi} + \frac{1}{N}\Delta(\underline{\Psi} - \underline{h}) \\
&\quad + (\partial_u + \mathcal{U})\left(\partial_u\underline{\Phi} + \mathcal{H}\underline{\Psi}\right) - (\partial_u + 2\mathcal{U} - \mathcal{H})\left(\partial_u\underline{h} + \mathcal{U}\underline{\Psi}\right) \\
&\quad - (\partial_n + I)\left(\partial_n\underline{\Phi} - H\underline{h} + \mathcal{H}\underline{\Sigma} + \frac{1}{N}\Delta(b\epsilon^\sharp)\right) \\
&\quad - (\partial_n + 2I - H)\left(\partial_n\underline{\Psi} + I\underline{h} - (\partial_u + \mathcal{U})\,\underline{\Sigma}\right) \\
&\quad - \underline{\Psi}\partial_u\left(\mathcal{U} - \mathcal{H}\right) - \left(\underline{h}\partial_n - \underline{\Sigma}\partial_u\right)(I - H), && \text{(H87)}\\
\underline{\delta\mathcal{E}}^v &\equiv \frac{N-1}{N}\left(-\left(\mathcal{H}\underline{\Psi} + \partial_u\underline{\Phi}\right) + \left(\partial_u\underline{h} + \mathcal{U}\underline{\Psi}\right) + (\mathcal{U} - \mathcal{H})\,\underline{h} + (H - I)\,\underline{\Sigma}\right. \\
&\qquad \left. -\tfrac{1}{2}\partial_n\left(\underline{\Sigma} - (\partial_u + \mathcal{U} - 2\mathcal{H})\,(b\epsilon^\sharp)\right)\right), && \text{(H88)}\\
\underline{\delta\mathcal{E}}^\Pi &\equiv \frac{1}{N}\left((N-2)\underline{\Phi} - \underline{\Psi} - (N-1)\underline{h} + ((N-2)H + I - (N-1)\partial_n)(b\epsilon^\sharp)\right), && \text{(H89)}
\end{aligned}$$

$$\underline{\delta\bar{\mathcal{E}}_i^v} \equiv -\frac{1}{2}\frac{1}{N}(\Delta+(N-1)K)\bar{\Sigma}_i+\frac{1}{2}\frac{N-1}{N}(\partial_n+H)\left((\partial_n+I-H)\bar{\Sigma}_i-(\partial_u+\mathcal{U}-\mathcal{H})\bar{h}_i\right), \tag{H90}$$

$$\underline{\delta\bar{\mathcal{E}}_i^\Pi} \equiv \frac{1}{N}\left((\partial_u+(N-1)(\mathcal{H}-\mathcal{U}))\bar{\Sigma}_i+((N-1)\partial_n+(H-I))\bar{h}_i\right), \tag{H91}$$

$$\underline{\delta\bar{\bar{\mathcal{E}}}_{ij}^\Pi} \equiv \frac{1}{N}((\partial_u+N\mathcal{H}-(N-1)\mathcal{U})\partial_u-(\Delta-2K)+((N-1)\partial_n+NH-I)\partial_n)\bar{\bar{E}}_{ij}, \tag{H92}$$

$$\begin{aligned}
\underline{\delta\mathcal{E}_{00}} &= \frac{N-1}{N+1}n^2\underline{\delta\mathcal{Z}}^\sharp+2\underline{\mathcal{E}_{00}\Psi}\\
&\quad+(\dot{\underline{\mathcal{E}}}_{00}+2\underline{\mathcal{E}}_{00}\partial_\eta)\left(\frac{a}{n}B+\frac{a^2}{n^2}\dot{E}\right), &\text{(H93)}\\
\underline{\delta\mathcal{E}_{0i}} &= -n\nabla_i\underline{\delta\mathcal{E}}^v-an\underline{\delta\bar{\mathcal{E}}_i^v}\\
&\quad+\underline{\mathcal{E}}_{00}\nabla_i\left(\frac{a}{n}B+\frac{a^2}{n^2}\dot{E}\right)+\underline{\mathcal{E}}_{ij}\dot{E}^j, &\text{(H94)}\\
\underline{\delta\mathcal{E}_{ij}} &= \frac{N-1}{N(N+1)}a^2\gamma_{ij}\underline{\delta\mathcal{Z}}^\sharp\\
&\quad+\left(\nabla_{ij}-\frac{1}{N}\nabla^2\gamma_{ij}\right)\underline{\delta\mathcal{E}}^\Pi+a\nabla_{(i}\underline{\delta\bar{\mathcal{E}}_{j)}^\Pi}+a^2\underline{\delta\bar{\bar{\mathcal{E}}}_{ij}^\Pi}\\
&\quad+\dot{\underline{\mathcal{E}}}_{ij}\left(\frac{a}{n}B+\frac{a^2}{n^2}\dot{E}\right)+2\underline{\mathcal{E}}_{k(i}(E^k_{j)}-\delta^k_{j)}\underline{\Phi}), &\text{(H95)}\\
\underline{\delta\mathcal{E}_{0M}} &= \underline{\mathcal{E}}^0_0\underline{\delta q}_{0M}, &\text{(H96)}\\
\underline{\delta\mathcal{E}_{iM}} &= \underline{\mathcal{E}}^j_i\underline{\delta q}_{jM}, &\text{(H97)}\\
\underline{\delta\mathcal{E}_{MM}} &= 0. &\text{(H98)}
\end{aligned}$$

APPENDIX I: BRANE POINT OF VIEW, PERTURBED CASE

1. New Einstein equations

We rewrite the perturbed Einstein near the brane, that is near $y=\mathrm{y_b}+\epsilon$. We first define some new bulk matter content perturbations by

$$\begin{aligned}
\overline{\delta\rho}^\sharp &= \delta\rho^\sharp+\rho'\epsilon^\sharp-2F\frac{b}{n}\dot{\epsilon}^\sharp, &\text{(I1)}\\
\overline{\delta P}^\sharp &= \delta P^\sharp+P'\epsilon^\sharp, &\text{(I2)}\\
\overline{\delta F}^\sharp &= \delta F^\sharp+F'\epsilon^\sharp-(\rho+Y)\frac{b}{n}\dot{\epsilon}^\sharp, &\text{(I3)}\\
\overline{\delta Y}^\sharp &= \delta Y^\sharp+Y'\epsilon^\sharp-2F\frac{b}{n}\dot{\epsilon}^\sharp, &\text{(I4)}\\
a\overline{f}^\sharp &= af^\sharp-b\epsilon^\sharp, &\text{(I5)}\\
\overline{\delta\Gamma}^\sharp &= \delta\Gamma^\sharp+\Gamma'\epsilon^\sharp-D\frac{b}{n}\dot{\epsilon}^\sharp+F\frac{\nabla^2}{a^2}b\epsilon^\sharp, &\text{(I6)}\\
a\overline{\mathcal{Q}}^\sharp &= a\mathcal{Q}^\sharp+Db\epsilon^\sharp, &\text{(I7)}\\
\overline{\delta D}^\sharp &= \delta D^\sharp-\Gamma\Sigma+D'\epsilon^\sharp+(Y-P)\frac{\nabla^2}{a^2}b\epsilon^\sharp. &\text{(I8)}
\end{aligned}$$

The terms proportional to $\epsilon^\sharp$ come from the fact that we are considering the bulk perturbations at $y=\mathrm{y_b}+\epsilon$ instead of $y=\mathrm{y_b}$. The other terms come from the fact that the brane is not at rest with respect to the bulk coordinate system, and are a mere consequence of a Lorentz boost with velocity $v^\perp=\frac{b}{n}\dot{\epsilon}$ along the y axis. Going from the non

underlined (bulk) metric perturbation to the underlined (brane-related) metric perturbations, the Einstein equation simplify a little bit to

$$
\begin{aligned}
&(N-1)\left(\Delta+NK\right)\underline{\Phi}-N(N-1)\mathcal{H}\left(\mathcal{H}\underline{\Psi}+\partial_u\underline{\Phi}\right)\\
&-N\mathcal{U}\left(\mathcal{H}\underline{\Psi}+\partial_u\underline{\Phi}\right)-N\mathcal{H}\left(\partial_u\underline{h}+\mathcal{U}\underline{\Psi}\right)\\
&+\Delta\underline{h}-N\underline{h}\partial_n\left\langle H\right\rangle+N\underline{\Sigma}\partial_u H\\
&+N\left(\partial_n+(N+1)H\right)\left(\partial_n\underline{\Phi}-H\underline{h}+\mathcal{H}\underline{\Sigma}+\frac{1}{N}\Delta b\epsilon^\sharp\right) &=\ \kappa_{N+2}\left(\mathrm{D}\underline{\delta\rho}^\sharp_{\mathrm{b}}+\overline{\delta\rho}^\sharp_{\mathrm{B}}\right), &\text{(I9)}\\
&-(N-1)\left(\mathcal{H}\underline{\Psi}+\partial_u\underline{\Phi}\right)\\
&-\left(\partial_u\underline{h}+\mathcal{U}\underline{\Psi}\right)-\left(\mathcal{U}-\mathcal{H}\right)\underline{h}-\left(H-I\right)\underline{\Sigma}\\
&+\frac{1}{2}\left(\partial_n+NH\right)\left(\underline{\Sigma}-\left(\partial_u+\mathcal{U}-2\mathcal{H}\right)\left(b\epsilon^\sharp\right)\right) &=\ \kappa_{N+2}\Big(\mathrm{D}(\underline{P}_{\mathrm{b}}+\underline{\rho}_{\mathrm{b}})a\underline{v}^\sharp_{\mathrm{b}}\\
&&&+(P_{\mathrm{B}}+\rho_{\mathrm{B}})av^\sharp_{\mathrm{B}}-F_{\mathrm{B}}a\overline{f}^\sharp_{\mathrm{B}}\Big), &\text{(I10)}\\
&-(N-2)(N-1)K\underline{\Phi}\\
&+(N-1)\left(\partial_u+N\mathcal{H}\right)\left(\mathcal{H}\underline{\Psi}+\partial_u\underline{\Phi}\right)+(N-1)\underline{\Psi}\partial_u\mathcal{H}\\
&+(N-1)\mathcal{U}\left(\mathcal{H}\underline{\Psi}+\partial_u\underline{\Phi}\right)+\underline{\Psi}\partial_u\mathcal{U}\\
&+\left(\partial_u+2\mathcal{U}+(N-1)\mathcal{H}\right)\left(\partial_u\underline{h}+\mathcal{U}\underline{\Psi}\right)\\
&-(N-1)\left(\partial_n+I+NH\right)\left(\partial_n\underline{\Phi}-H\underline{h}+\mathcal{H}\underline{\Sigma}\right)\\
&+\left(\partial_n+2I+(N-1)H\right)\left(\partial_n\underline{\Psi}+I\underline{h}-\left(\partial_u+\mathcal{U}\right)\underline{\Sigma}\right)\\
&+\underline{h}\partial_n\left\langle I+(N-1)H\right\rangle-\underline{\Sigma}\partial_u\left(I+(N-1)H\right) &=\ \kappa_{N+2}\Big(\mathrm{D}\underline{\delta P}^\sharp_{\mathrm{b}}+\frac{N-1}{N}\mathrm{D}\nabla^2\Pi_{\mathrm{b}} &\text{(I11)}\\
&&&+\overline{\delta P}^\sharp_{\mathrm{B}}+\frac{N-1}{N}\nabla^2\Pi_{\mathrm{B}}\Big), &\text{(I12)}\\
&(N-2)\underline{\Phi}-\underline{\Psi}\\
&+\underline{h}+\left(\partial_n+(N-2)H+I\right)\left(b\epsilon^\sharp\right) &=\ \kappa_{N+2}\left(\mathrm{D}a^2\underline{\Pi}_{\mathrm{b}}+a^2\Pi_{\mathrm{B}}\right), &\text{(I13)}\\
&-N\left(\partial_u+\mathcal{H}\right)\left(\partial_n\underline{\Phi}-H\underline{h}+\mathcal{H}\underline{\Sigma}+\frac{1}{N}\Delta(b\epsilon^\sharp)\right)\\
&-N\mathcal{H}\left(\partial_n\underline{\Psi}+I\underline{h}-\left(\partial_u+\mathcal{U}\right)\underline{\Sigma}\right)\\
&-\frac{1}{2}\Delta\left(\underline{\Sigma}-\left(\partial_u+\mathcal{U}-2\mathcal{H}\right)\left(b\epsilon^\sharp\right)\right)-N\left(H-I\right)\partial_u\underline{\Phi} &=\ \kappa_{N+2}\left(\overline{\delta F}^\sharp_{\mathrm{B}}+F_{\mathrm{B}}\underline{\Psi}\right), &\text{(I14)}\\
&-\frac{1}{2}\left(\partial_u+N\mathcal{H}\right)\left(\underline{\Sigma}-\left(\partial_u+\mathcal{U}-2\mathcal{H}\right)\left(b\epsilon^\sharp\right)\right)\\
&-\left(\frac{1}{b}\underline{\Psi}'+I\underline{h}-\left(\partial_u+\mathcal{U}\right)\underline{\Sigma}\right)\\
&+(N-1)\left(\frac{1}{b}\underline{\Phi}'-H\underline{h}+\mathcal{H}\underline{\Sigma}+\frac{1}{N}\Delta(b\epsilon^\sharp)\right)\\
&+\left(H-I\right)\underline{\Psi}-\frac{N-1}{N}\left(\Delta+NK\right)\left(b\epsilon^\sharp\right) &=\ \kappa_{N+2}\left(F_{\mathrm{B}}av^\sharp_{\mathrm{B}}+(P_{\mathrm{B}}-Y_{\mathrm{B}})a\overline{f}^\sharp_{\mathrm{B}}\right), &\text{(I15)}\\
&-(N-1)\left(\Delta+NK\right)\underline{\Phi}+\Delta\underline{\Psi}\\
&+N\underline{\Psi}\partial_u\mathcal{H}+N\left(\partial_u+(N+1)\mathcal{H}\right)\left(\partial_u\underline{\Phi}+\mathcal{H}\underline{\Psi}\right)\\
&+NH\left(\partial_n\underline{\Psi}+I\underline{h}-\left(\partial_u+\mathcal{U}\right)\underline{\Sigma}\right)\\
&-N\left((N-1)H+I\right)\left(\partial_n\underline{\Phi}-H\underline{h}+\mathcal{H}\underline{\Sigma}+\frac{1}{N}\Delta(b\epsilon^\sharp)\right) &=\ \kappa_{N+2}\overline{\delta Y}^\sharp_{\mathrm{B}}, &\text{(I16)}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\left(\Delta+(N-1)K\right)\bar{\Sigma}_i\\
&-\frac{1}{2}\left(\partial_n+(N+1)H\right)\left(\left(\partial_n+I-H\ \right)\bar{\Sigma}_i\right)\\
&+\frac{1}{2}\left(\partial_n+(N+1)H\right)\left(\left(\partial_u+\mathcal{U}-\mathcal{H}\right)\bar{h}_i\right) &=\ \kappa_{N+2}\Big(\mathrm{D}(\underline{P}_{\mathrm{b}}+\underline{\rho}_{\mathrm{b}})(\underline{\bar{v}}^\sharp_{\mathrm{b}\ i}-\bar{\Sigma}_i)
\end{aligned}
$$

$$+(P_{\rm B}+\rho_{\rm B})(\bar v^\sharp_{{\rm B}\,i}-\bar\Sigma_i)-F_{\rm B}(\bar f^\sharp_{{\rm B}\,i}+\bar h_i)\Big)\,, \tag{I17}$$

$$\begin{aligned}
&(\partial_u+(N-1)\mathcal{H})\,\bar\Sigma_i-(\partial_n+(N-1)H)\,\bar h_i\\
&\qquad+\mathcal{U}\bar\Sigma_i-I\bar h_i \;=\; \kappa_{N+2}\left({\rm D}a\underline{\bar\Pi}^{\rm b}_i+a\bar\Pi^{\rm B}_i\right),
\end{aligned} \tag{I18}$$

$$\begin{aligned}
&\frac{1}{2}\,(\Delta+(N-1)K)\,\bar h_i\\
&+\frac{1}{2}\,(\partial_u+(N+1)\mathcal{H})\left((\partial_n+I-H)\,\bar\Sigma_i\right)\\
&-\frac{1}{2}\,(\partial_u+(N+1)\mathcal{H})\left((\partial_u+\mathcal{U}-\mathcal{H})\,\bar h_i\right) \;=\; \kappa_{N+2}\left(F_{\rm B}(\bar v^\sharp_{{\rm B}\,i}-\bar\Sigma_i)+(P_{\rm B}-Y_{\rm B})(\bar f^\sharp_{{\rm B}\,i}+\bar h_i)\right),
\end{aligned} \tag{I19}$$

$$\begin{aligned}
&(\partial_u+N\mathcal{H})\,\partial_u\bar{\bar E}_{ij}-(\Delta-2K)\,\bar{\bar E}_{ij}\\
&+\mathcal{U}\partial_u\bar{\bar E}_{ij}-(\partial_n+NH+I)\left(\partial_n\bar{\bar E}_{ij}\right) \;=\; \kappa_{N+2}\left({\rm D}\underline{\bar{\bar\Pi}}^{\rm b}_{ij}+\bar{\bar\Pi}^{\rm B}_{ij}\right).
\end{aligned} \tag{I20}$$

(A sum on all the brane and bulk species is implicitly assumed on the right hand side of these equations.) One can easily check that the singular part of the above equation reduces to the second Israel condition.

2. Sail equation

In the following a sum on all the brane and bulk species is implicitly assumed.

$$\begin{aligned}
&N\langle H\rangle\,\underline{\delta P}^\sharp-\langle I\rangle\,\underline{\delta\rho}^\sharp\\
&-N\underline{P}\left\langle\partial_n\underline{\Phi}-H\underline{h}+\mathcal{H}\underline{\Sigma}+\frac{1}{N}\Delta\underline{\Upsilon}\right\rangle\\
&-\underline{\rho}\,\langle\partial_n\underline{\Psi}+I\underline{h}-(\partial_u+\mathcal{U})\,\underline{\Sigma}\rangle \;=\; \left[\overline{\delta Y}^\sharp\right],
\end{aligned} \tag{I21}$$

$$\begin{aligned}
&-\partial_u^2(\underline{\rho}\underline{\Upsilon})-N\mathcal{H}\partial_u\left(2\underline{\rho}\underline{\Upsilon}+\underline{P}\underline{\Upsilon}\right)-\Delta\underline{P}\underline{\Upsilon}+K(N-1)\underline{\rho}\underline{\Upsilon}\\
&\qquad-N\left(\underline{P}+\frac{N-1}{N}\underline{\rho}\right)\underline{\Upsilon}\left(2\partial_u\mathcal{H}+(N+1)\mathcal{H}^2\right)\\
&\qquad-(\underline{P}+\underline{\rho})\underline{\Upsilon}\left(N\langle H\rangle\langle H-I\rangle+\frac{\kappa_{N+2}}{4}\underline{\rho}(\underline{P}+\underline{\rho})\right)\\
&\qquad+\underline{\Upsilon}\left(N\langle H\rangle[Y-P]+\langle I\rangle[Y+\rho]\right) \;=\; \left[\delta Y^\sharp\right]-N\langle H\rangle\,\underline{\delta P}^\sharp+\langle I\rangle\,\underline{\delta\rho}^\sharp\\
&\qquad\qquad+N\underline{P}\,\langle\partial_n\Phi-Hh+\mathcal{H}\Sigma\rangle\\
&\qquad\qquad+\underline{\rho}\,\langle\partial_n\Psi+Ih-(\partial_u+\mathcal{U})\,\Sigma\rangle\\
&\qquad\qquad+(2\partial_u+N\mathcal{H})(\langle F\rangle\,\underline{\Xi})-\underline{\Xi}\partial_u\langle F\rangle\\
&\qquad\qquad+N\langle H\rangle\,\underline{\Xi}\left(\frac{\kappa_{N+2}}{4}(\underline{P}+\underline{\rho})^2+\langle Y-P\rangle\right)\\
&\qquad\qquad+\langle I\rangle\,\underline{\Xi}\left(-\frac{\kappa_{N+2}}{4}(\underline{P}+\underline{\rho})\underline{\rho}+\langle Y+\rho\rangle\right).
\end{aligned} \tag{I22}$$

3. Perturbed conservation equation

They transform into

$$\begin{aligned}
&(\partial_u+N\mathcal{H}+\mathcal{U})\left(\overline{\delta\rho}^\sharp-F\underline{\Sigma}\right)+N\mathcal{H}\overline{\delta P}^\sharp+\mathcal{U}\left(\overline{\delta Y}^\sharp-F\underline{\Sigma}\right)\\
&\qquad+(\partial_n+NH+2I)\left(\overline{\delta F}^\sharp+F(\underline{\Psi}+\underline{h})\right)\\
&\qquad+\Delta\left(a(P+\rho)v^\sharp-aFf^\sharp\right)\\
&-N(P+\rho)\partial_u\underline{\Phi}-(\rho+Y)\partial_u\underline{h}-F\partial_u\underline{\Sigma}+F\partial_n(\underline{\Psi}-\underline{h}-N\underline{\Phi}) \;=\; \overline{\delta\Gamma}^\sharp+\Gamma\underline{\Psi},\\
&\qquad(\partial_u+N\mathcal{H}+\mathcal{U})\left(a(P+\rho)v^\sharp-aF\overline{f}^\sharp\right)
\end{aligned} \tag{I23}$$

$$
\begin{aligned}
&+(\partial_n+NH+I)\left(aFv^\sharp+a(P-Y)\overline{f}^\sharp\right)\\
+\overline{\delta P}^\sharp+\frac{N-1}{N}(\Delta+NK)\,a^2\Pi+(P+\rho)\underline{\Psi}+(Y-P)\underline{h}+F\underline{\Sigma} &= a\overline{\mathcal{Q}}^\sharp, \qquad \text{(I24)}\\
&(\partial_u+N\mathcal{H}+2\mathcal{U})\left(\overline{\delta F}^\sharp-F(\underline{\Psi}+\underline{h})-(\rho+Y)\underline{\Sigma}\right)\\
&+(\partial_n+NH+2I)\left(\overline{\delta Y}^\sharp-F\underline{\Sigma}\right)-NH\overline{\delta P}^\sharp-I(\overline{\delta Y}^\sharp-\overline{\delta\rho}^\sharp)\\
&+\Delta\left(aFv^\sharp+a(P-Y)f^\sharp\right)\\
+F\partial_u(\underline{\Psi}-\underline{h}-N\underline{\Phi})+N(P-Y)\partial_n\underline{\Phi}+(\rho+Y)\partial_n\underline{\Psi}+F\partial_n\underline{\Sigma} &= \overline{\delta D}^\sharp-D\underline{h}, \qquad \text{(I25)}
\end{aligned}
$$

$$
\begin{aligned}
&(\partial_u+(N+1)\mathcal{H}+\mathcal{U})\left((P+\rho)(\bar v_i^\sharp-\bar\Sigma_i)-F(\bar f_i^\sharp+\bar h_i)\right)\\
&+(\partial_n+(N+1)H+I)\left(F(\bar v_i^\sharp-\bar\Sigma_i)+(P-Y)(\bar f_i^\sharp+\bar h_i)\right)\\
&+\frac{1}{2}(\Delta+(N-1)K)\,a\bar\Pi_i &= \bar{\mathcal{Q}}_i^\sharp-D\bar h_i-\Gamma\bar\Sigma_i. \qquad \text{(I26)}
\end{aligned}
$$

For the brane components, they are obtained by considering the discontinuity of the $\{0_M\}$, $\{i_M\}$ components of Einstein equation or by taking the singular part of the above equations.

$$
\begin{aligned}
&\partial_u\underline{\delta\rho}_{\rm b}^\sharp+N\mathcal{H}(\underline{\delta\rho}_{\rm b}^\sharp+\underline{\delta P}_{\rm b}^\sharp)\\
+(\underline{P}_{\rm b}+\underline{\rho}_{\rm b})\Delta a\underline{v}_{\rm b}^\sharp-N(\underline{P}_{\rm b}+\underline{\rho}_{\rm b})\partial_u\underline{\Phi} &= \underline{\delta\Gamma}_{\rm b}^\sharp+\Gamma_{\rm b}\underline{\Psi}, \qquad \text{(I27)}\\
\sum_{\rm b}\underline{\delta\Gamma}_{\rm b}^\sharp+\Gamma_{\rm b}\underline{\Psi} &= -\sum_{\rm B}\left[\overline{\delta F}_{\rm B}^\sharp+F_{\rm B}\underline{\Psi}\right], \qquad \text{(I28)}\\
&(\partial_u+N\mathcal{H})\left((\underline{P}_{\rm b}+\underline{\rho}_{\rm b})a\underline{v}_{\rm b}^\sharp\right)+\underline{\delta P}_{\rm b}^\sharp\\
+\frac{N-1}{N}(\Delta+NK)\,a^2\underline{\Pi}_{\rm b}+(\underline{P}_{\rm b}+\underline{\rho}_{\rm b})\underline{\Psi} &= a\underline{\mathcal{Q}}_{\rm b}^\sharp, \qquad \text{(I29)}\\
\sum_{\rm b}a\underline{\mathcal{Q}}_{\rm b}^\sharp &= -\sum_{\rm B}\left[F_{\rm B}a\underline{v}_{\rm B}^\sharp+(P_{\rm B}-Y_{\rm B})a\overline{f}_{\rm B}^\sharp\right], \qquad \text{(I30)}\\
(\partial_u+(N+1)\mathcal{H})&\left((\underline{P}_{\rm b}+\underline{\rho}_{\rm b})(\underline{\bar v}_i^{\rm b\,\sharp}-\bar\Sigma_i)\right)\\
+\frac{1}{2}(\Delta+(N-1)K)\,a\underline{\bar\Pi}_i^{\rm b} &= \underline{\bar{\mathcal{Q}}}_i^{\rm b\,\sharp}-\underline{\Gamma}_{\rm b}\bar\Sigma_i, \qquad \text{(I31)}\\
\sum_{\rm b}\underline{\bar{\mathcal{Q}}}_i^{\rm b\,\sharp}-\underline{\Gamma}_{\rm b}\bar\Sigma_i &= -\sum_{\rm B}\left[F_{\rm B}(\bar v_i^{\rm B\,\sharp}-\bar\Sigma_i)+(P_{\rm B}-Y_{\rm B})(\bar f_i^{\rm B\,\sharp}+\bar h_i)\right]. \qquad \text{(I32)}
\end{aligned}
$$

4. Einstein equations using the Weyl tensor

$$
\begin{aligned}
&(N-1)(\Delta+NK)\,\underline{\Phi}\\
-N(N-1)\mathcal{H}(\mathcal{H}\underline{\Psi}+\partial_u\underline{\Phi}) &= \frac{1}{4}\frac{N-1}{N}\kappa_{N+2}^2\left(\sum_{\rm b}\underline{\rho}_{\rm b}\right)\sum_{\rm b}\underline{\delta\rho}_{\rm b}^\sharp\\
&\quad-(N-1)\langle H\rangle\left\langle N\partial_n\underline{\Phi}-NH\underline{h}+N\mathcal{H}\underline{\Sigma}+\Delta b\epsilon^\sharp\right\rangle\\
&\quad+\frac{N-1}{N+1}\kappa_{N+2}\sum_{\rm B}\left\langle\overline{\delta P}_{\rm B}^\sharp+\overline{\delta\rho}_{\rm B}^\sharp-\overline{\delta Y}_{\rm B}^\sharp\right\rangle+\frac{N-1}{N+1}\left\langle\underline{\delta\mathcal{Z}}^\sharp\right\rangle, \qquad \text{(I33)}\\
-(N-1)(\mathcal{H}\underline{\Psi}+\partial_u\underline{\Phi}) &= \frac{1}{4}\frac{N-1}{N}\kappa_{N+2}^2\left(\sum_{\rm b}\underline{\rho}_{\rm b}\right)\sum_{\rm b}\left((\underline{P}_{\rm b}+\underline{\rho}_{\rm b})a\underline{v}_{\rm b}^\sharp\right)\\
&\quad-\frac{N-1}{2}\langle H\rangle\left\langle\underline{\Sigma}-(\partial_u+\mathcal{U}-2\mathcal{H})(b\epsilon^\sharp)\right\rangle
\end{aligned}
$$

$$+\frac{N-1}{N}\kappa_{N+2}\sum_{\rm B}\left\langle (P_{\rm B}+\rho_{\rm B})av_{\rm B}^\sharp - F_{\rm B}a\overline{f}_{\rm B}^\sharp\right\rangle + \langle\underline{\delta\mathcal{E}}^v\rangle\,, \tag{I34}$$

$$\begin{aligned}
&\frac{N-1}{N}\left(\Delta\underline{\Psi}-(N-2)(\Delta+NK)\underline{\Phi}\right)\\
&+(N-1)\left(\partial_u+N\mathcal{H}\right)\left(\mathcal{H}\underline{\Psi}+\partial_u\underline{\Phi}\right)\\
+(N-1)\underline{\Psi}\partial_u\mathcal{H} \;&=\; \frac{1}{4}\frac{N-1}{N}\kappa_{N+2}^2\left(\sum_{\rm b}\underline{\rho}_{\rm b}\right)\sum_{\rm b}\underline{\delta P}_{\rm b}^\sharp\\
&\quad -(N-1)\left\langle H\right\rangle\left\langle\partial_n\underline{\Psi}+I\underline{h}-(\partial_u+\mathcal{U})\underline{\Sigma}\right\rangle\\
&\quad +(N-1)\left\langle(N-2)H+I\right\rangle\left\langle\partial_n\underline{\Phi}-H\underline{h}+\mathcal{H}\underline{\Sigma}+\frac{1}{N}\Delta b\epsilon^\sharp\right\rangle\\
&\quad +\frac{N-1}{N(N+1)}\kappa_{N+2}\sum_{\rm B}\left\langle\overline{\delta P}_{\rm B}^\sharp+\overline{\delta\rho}_{\rm B}^\sharp+N\overline{\delta Y}_{\rm B}^\sharp\right\rangle\\
&\quad +\frac{1}{4}\frac{N-1}{N}\kappa_{N+2}^2\left(\sum_{\rm b}(\underline{P}_{\rm b}+\underline{\rho}_{\rm b})\right)\sum_{\rm b}\underline{\delta\rho}_{\rm b}^\sharp+\frac{N-1}{N(N+1)}\left\langle\underline{\delta\mathcal{Z}}^\sharp\right\rangle\,,
\end{aligned} \tag{I35}$$

$$\begin{aligned}
(N-2)\underline{\Phi}-\underline{\Psi} \;&=\; \frac{1}{4}\frac{N-1}{N}\kappa_{N+2}^2\left(\sum_{\rm b}\underline{\rho}_{\rm b}\right)a^2\sum_{\rm b}\underline{\Pi}_{\rm b}\\
&\quad -\left\langle(N-2)H+I\right\rangle\underline{\Upsilon}\\
&\quad -\frac{1}{4}\kappa_{N+2}^2\left(\sum_{\rm b}(\underline{P}_{\rm b}+\underline{\rho}_{\rm b})\right)a^2\sum_{\rm b}\underline{\Pi}_{\rm b}\\
&\quad +\frac{N-1}{N}\kappa_{N+2}a^2\sum_{\rm B}\left\langle\Pi_{\rm B}\right\rangle+\left\langle\underline{\delta\mathcal{E}}^{\Pi}\right\rangle\,,
\end{aligned} \tag{I36}$$

$$\begin{aligned}
-\frac{1}{2}\left(\Delta+(N-1)K\right)\bar{\Sigma}_i \;&=\; \frac{1}{4}\frac{N-1}{N}\kappa_{N+2}^2\left(\sum_{\rm b}\underline{\rho}_{\rm b}\right)\sum_{\rm b}\left((\underline{P}_{\rm b}+\underline{\rho}_{\rm b})(\bar{v}_i^{\rm b\,\sharp}-\bar{\Sigma}_i)\right)\\
&\quad +\frac{N-1}{2}\left\langle H\right\rangle\left\langle(\partial_n+I-H)\bar{\Sigma}_i-(\partial_u+\mathcal{U}-\mathcal{H})\bar{h}_i\right\rangle\\
&\quad +\frac{N-1}{N}\kappa_{N+2}\sum_{\rm B}\left\langle(P_{\rm B}+\rho_{\rm B})(\bar{v}_i^{\rm B\,\sharp}-\bar{\Sigma}_i)-F_{\rm B}(\bar{f}_i^{\rm B\,\sharp}+\bar{h}_i)\right\rangle+\left\langle\underline{\bar{\delta\mathcal{E}}}_i^v\right\rangle\,,
\end{aligned} \tag{I37}$$

$$\begin{aligned}
\left(\partial_u+(N-1)\mathcal{H}\right)\bar{\Sigma}_i \;&=\; \frac{1}{4}\frac{N-1}{N}\kappa_{N+2}^2\left(\sum_{\rm b}\underline{\rho}_{\rm b}\right)a\sum_{\rm b}\underline{\bar{\Pi}}_i^{\rm b}\\
&\quad +\left\langle(N-2)H+I\right\rangle\left\langle\bar{h}_i\right\rangle\\
&\quad -\frac{1}{4}\kappa_{N+2}^2\left(\sum_{\rm b}(\underline{P}_{\rm b}+\underline{\rho}_{\rm b})\right)a\sum_{\rm b}\underline{\bar{\Pi}}_i^{\rm b}\\
&\quad +\frac{N-1}{N}\kappa_{N+2}a\sum_{\rm B}\left\langle\bar{\Pi}_i^{\rm B}\right\rangle+\left\langle\underline{\bar{\delta\mathcal{E}}}_i^{\Pi}\right\rangle\,,
\end{aligned} \tag{I38}$$

$$\begin{aligned}
\left(\partial_u+N\mathcal{H}\right)\partial_u\bar{\bar{E}}_{ij}-\left(\Delta-2K\right)\bar{\bar{E}}_{ij} \;&=\; \frac{1}{4}\frac{N-1}{N}\kappa_{N+2}^2\left(\sum_{\rm b}\underline{\rho}_{\rm b}\right)\sum_{\rm b}\underline{\bar{\bar{\Pi}}}_{ij}^{\rm b}\\
&\quad +\left\langle(N-2)H+I\right\rangle\left\langle\partial_n\bar{\bar{E}}_{ij}\right\rangle\\
&\quad -\frac{1}{4}\kappa_{N+2}^2\left(\sum_{\rm b}(\underline{P}_{\rm b}+\underline{\rho}_{\rm b})\right)\sum_{\rm b}\underline{\bar{\bar{\Pi}}}_{ij}^{\rm b}\\
&\quad +\frac{N-1}{N}\kappa_{N+2}\sum_{\rm B}\left\langle\bar{\bar{\Pi}}_{ij}^{\rm B}\right\rangle+\left\langle\underline{\bar{\bar{\delta\mathcal{E}}}}_{ij}^{\Pi}\right\rangle\,.
\end{aligned} \tag{I39}$$

5. Relationship between $\langle \underline{\delta\mathcal{K}}_{\alpha\beta} \rangle$ and $\left[\underline{\delta\mathcal{E}}_{\alpha\beta}\right]$

$$
\begin{aligned}
-\underline{\rho}_{\mathrm{b}} \left\langle \partial_n \underline{\Phi} - H\underline{h} + \mathcal{H}\underline{\Sigma} + \frac{1}{N}\Delta\underline{\Upsilon} \right\rangle &= -\underline{\delta\rho}^{\sharp}_{\mathrm{b}} \langle H \rangle \\
&\quad + \frac{1}{N+1}\left(\left[\overline{\delta P}^{\sharp}_{\mathrm{B}} + \overline{\delta\rho}^{\sharp}_{\mathrm{B}} - \overline{\delta Y}^{\sharp}_{\mathrm{B}} \right] + \frac{1}{\kappa_{N+2}} \left[\underline{\delta\mathcal{Z}}^{\sharp} \right] \right), \qquad \text{(I40)} \\
-(N\underline{P}_{\mathrm{b}} + \underline{\rho}_{\mathrm{b}}) \left\langle \partial_n \underline{\Phi} - H\underline{h} + \mathcal{H}\underline{\Sigma} + \frac{1}{N}\Delta\underline{\Upsilon} \right\rangle & \\
-\underline{\rho}_{\mathrm{b}} \left\langle \partial_n \underline{\Psi} + I\underline{h} - (\partial_u + \mathcal{U})\,\underline{\Sigma} \right\rangle &= -(N\underline{\delta P}^{\sharp}_{\mathrm{b}} + \underline{\delta\rho}^{\sharp}_{\mathrm{b}}) \langle H \rangle + \underline{\delta\rho}^{\sharp}_{\mathrm{b}} \langle I \rangle \\
&\quad + \frac{1}{N+1}\left[\overline{\delta P}^{\sharp}_{\mathrm{B}} + \overline{\delta\rho}^{\sharp}_{\mathrm{B}} + N\overline{\delta Y}^{\sharp}_{\mathrm{B}} \right] \\
&\quad + \frac{1}{N+1}\frac{1}{\kappa_{N+2}} \left[\underline{\delta\mathcal{Z}}^{\sharp} \right], \qquad \text{(I41)} \\
-\underline{\rho}_{\mathrm{b}} \left\langle \tfrac{1}{2}\underline{\Sigma} - \tfrac{1}{2}(\partial_u + \mathcal{U} - 2\mathcal{H})\, b\epsilon^{\sharp} \right\rangle &= -N \langle H \rangle (\underline{P}_{\mathrm{b}} + \underline{\rho}_{\mathrm{b}}) a \underline{v}^{\sharp}_{\mathrm{b}} \\
&\quad + \left[(P_{\mathrm{B}} + \rho_{\mathrm{B}}) a v^{\sharp}_{\mathrm{B}} - F_{\mathrm{B}} a \overline{f}^{\sharp}_{\mathrm{B}} \right] \\
&\quad + \frac{N}{N-1}\frac{1}{\kappa_{N+2}} \left[\underline{\delta\mathcal{E}}^{v} \right], \qquad \text{(I42)} \\
(N\underline{P}_{\mathrm{b}} + \underline{\rho}_{\mathrm{b}})\underline{\Upsilon} &= -N \langle (N-2)H + I \rangle a^2 \underline{\Pi}_{\mathrm{b}} \\
&\quad + (N-1)a^2 \left[\Pi_{\mathrm{B}} \right] + N\frac{1}{\kappa_{N+2}} \left[\underline{\delta\mathcal{E}}^{\Pi} \right], \qquad \text{(I43)}
\end{aligned}
$$

$$
\begin{aligned}
-\underline{\rho}_{\mathrm{b}} \left\langle \tfrac{1}{2}(\partial_u + \mathcal{U} - \mathcal{H})\,\bar{h}_i - \tfrac{1}{2}(\partial_n + I - H)\,\bar{\Sigma}_i \right\rangle &= -N \langle H \rangle (\underline{P}_{\mathrm{b}} + \underline{\rho}_{\mathrm{b}})(\underline{v}^{\sharp}_{\mathrm{b}\,i} - \bar{\Sigma}_i) \\
&\quad + \left[(P_{\mathrm{B}} + \rho_{\mathrm{B}})(\bar{v}^{\sharp}_{\mathrm{B}\,i} - \bar{\Sigma}_i) - F_{\mathrm{B}}(\bar{f}^{\sharp}_{\mathrm{B}\,i} + \bar{h}_i) \right] \\
&\quad + \frac{N}{N-1}\frac{1}{\kappa_{N+2}} \left[\underline{\delta\bar{\mathcal{E}}}^{v}_{i} \right], \qquad \text{(I44)} \\
-(N\underline{P}_{\mathrm{b}} + \underline{\rho}_{\mathrm{b}}) \langle \bar{h}_i \rangle &= -N \langle (N-2)H + I \rangle a^2 \underline{\bar{\Pi}}^{\mathrm{b}}_{i} \\
&\quad + (N-1)a \left[\bar{\Pi}^{\mathrm{B}}_{i} \right] + N\frac{1}{\kappa_{N+2}} \left[\underline{\delta\bar{\mathcal{E}}}^{\Pi}_{i} \right], \qquad \text{(I45)}
\end{aligned}
$$

$$
\begin{aligned}
-(N\underline{P}_{\mathrm{b}} + \underline{\rho}_{\mathrm{b}}) \left\langle \partial_n \bar{\bar{E}}_{ij} \right\rangle &= -N \langle (N-2)H + I \rangle \underline{\bar{\bar{\Pi}}}^{\mathrm{b}}_{ij} \\
&\quad + (N-1) \left[\bar{\bar{\Pi}}^{\mathrm{B}}_{ij} \right] + N\frac{1}{\kappa_{N+2}} \left[\underline{\delta\bar{\bar{\mathcal{E}}}}^{\Pi}_{ij} \right]. \qquad \text{(I46)}
\end{aligned}
$$