Tachyonic perturbations in AdS_5 orbifolds

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We show that scalar as well as vector and tensor metric perturbations in the Randall Sundrum II braneworld allow normalizable tachyonic modes, *i.e.,* instabilities. These instabilities require non vanishing initial anisotropic stresses on the brane. We show with a specific example that within the Randall Sundrum II model, even though the tachyonic modes are excited, no instability develops. We argue, however, that in the cosmological context instabilities might in principle be present. We conjecture that the tachyonic modes are due to the singularity of the orbifold construction. We illustrate this with a simple but explicit toy model.

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I. INTRODUCTION

Already at the beginning of the last century, the idea that our universe may have more than three spatial dimensions has been explored by Nordström [1], Kaluza [2] and Klein [3]. Since superstring theory, the most promising candidate for a theory of quantum gravity, is consistent only in ten space-time dimensions (11 dimensions for M-theory) these ideas have been revived in recent years [4–6]. It has also been found that string theories naturally predict lower dimensional "branes" to which fermions and gauge particles are confined, while gravitons (and the dilaton) propagate in the bulk [7–9].

Recently it has been emphasized that relatively large extra-dimensions (with typical length $L \simeq \mu$m) can "solve" the hierarchy problem : The effective four-dimensional Newton constant given by $G_4 \approx G/L^n$ can become very small even if the fundamental gravitational constant $G \simeq m_{\mathrm{P}\ell}^{-(2+n)}$ is of the order of the electro-weak scale [10–13]. Here n denotes the number of extra-dimensions. It has also been shown that extra-dimensions may even be infinite if the geometry contains a so-called "warp factor". An especially attractive model of this type, where the bulk is a 5-dimensional anti-de Sitter (AdS_5) space has been developed by Randall and Sundrum [14]. This is the model which we discuss in this work, we shall call it RSII in what follows.

The size of the extra-dimensions is constrained by the requirement of recovering usual four-dimensional Newton's law on the brane, at least on scales tested by experiments [15–17].

Models with finite extra-dimensions always have to invoke some non-gravitational interaction in order to stabilize the gravi-scalar (which is equivalent to the radion) [18]. However, in the case of non-compact warped extra-dimensions, it can happen that this mode is not normalizable and therefore cannot be excited. This is precisely what happens in the RSII model.

Therefore, there is justified hope that, for suitable parameters, this model can reproduce four-dimensional gravity without invoking ad-hoc additional interactions. However, we show in this paper that the gravitational sector coupled to a brane with non-vanishing anisotropic stresses does have negative mass modes. We argue that these instabilities are not relevant for the Randall Sundrum model, but they may be devastating in the cosmological context, where the brane is moving, as has been indicated recently for the vector mode [19].

The tachyonic modes are absent if there are no anisotropic stresses. Furthermore, if anisotropic stresses remain small, they cannot develop an instability. As we shall show, this is the case for the RSII model since there, to first order, anisotropic stresses evolve like in Minkowski spacetime and hence remain small (if there Minkowski evolution is not already unstable). In the cosmological context, however, this is no longer true and large deviations from homogeneity and isotropy may in principle develop.

The outline of the paper is as follows : In the next section, the perturbation theory on RSII is briefly introduced and the relevant perturbation equations are given. We present the solutions to the bulk perturbation equations and the junction conditions for tensor, vector and scalar modes. We pay particular attention to the tachyonic modes which are new and represent a possible instability. In Section III we discuss the simple case of free-streaming, relativistic particles and show that they induce negative mass modes. We also explicitly solve the equations for the RSII background and see that no instability is induced in this case. We then argue that, in principle, this behavior may change in a cosmological setting, although in the example of tensor perturbations discussed in some details, this is not the case.

In Section IV, we present a simple 3 + 1 dimensional Minkowski–orbifold who's bulk modes exhibit the same instability as the AdS_5–orbifold. We explicitly reconstruct the instability from the retarded Green's function, showing that it is causal. In this toy model, instabilities develop due to non–linear couplings. A final

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section is devoted to some conclusions.

II. PERTURBATIONS OF THE RSII MODEL

Our universe is considered to be a 3-brane embedded in five-dimensional anti-de Sitter space-time,

$$\mathrm{d}s^2 = g_{AB}\mathrm{d}x^A\mathrm{d}x^B = \frac{L^2}{y^2}\left[-\mathrm{d}t^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j + \mathrm{d}y^2\right] . \tag{1}$$

Capital Latin indices A, B run from 0 to 4 and lower case Latin indices i, j from 1 to 3. Four-dimensional indices running from 0 to 3 will be denoted by lower case Greek letters. Anti-de Sitter space-time is a solution of Einstein's equations with a negative cosmological constant Λ,

$$G_{AB} + \Lambda g_{AB} = 0 . \tag{2}$$

The curvature radius L is given by

$$L^2 = -\frac{6}{\Lambda} . \tag{3}$$

Another coordinate system for anti-de Sitter space can be defined by the transformation $L^2/y^2 = \exp(-2\varrho/L)$. Then, the metric takes the form

$$\mathrm{d}s^2 = g_{AB}\mathrm{d}x^A\mathrm{d}x^B = e^{-2\rho/L}\left(-\mathrm{d}t^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right) + \mathrm{d}\rho^2 , \tag{4}$$

which is often used in braneworld models.

We now introduce a brane at $y = y_\mathrm{b} = L$ (or equivalently $\rho = 0$) and replace the "left hand side", $0 < y < L$, of AdS_5 by a second copy of the "right hand side". We use the superscripts "$>$" and "$<$" for the bulk sides with $y > y_\mathrm{b}$ and $y < y_\mathrm{b}$, respectively. In terms of the coordinate y, the value of y decreases continuously from ∞ to L and then jumps to $-L$ over the brane whereafter it continues to decrease. At the brane position, $y_\mathrm{b}^> = L$, $y_\mathrm{b}^< = -L$, the metric function $(L/y)^2$ has a kink. The advantage of the coordinate ρ introduced in Eq. (4) is that the variable ρ does not jump, but the metric function in the presence of a brane becomes $\exp(-2|\rho|/L)$.

The Einstein equations at the brane position are singular, they contain a Dirac–delta function. To avoid this, one can integrate them over the brane which leads to the so-called junction conditions [20–23] at the brane position. These read [24]

$$K^>_{\mu\nu} - K^<_{\mu\nu} = \kappa_5\left(S_{\mu\nu} - \frac{1}{3}Sq_{\mu\nu}\right) \equiv \kappa_5\widehat{S}_{\mu\nu} , \tag{5}$$

where $S_{\mu\nu}$ is the energy-momentum tensor on the brane with trace S, and

$$\kappa_5 \equiv 6\pi^2 G_5 = \frac{1}{M_5^3} . \tag{6}$$

M_5 and G_5 are the five-dimensional (fundamental) reduced Planck mass and Newton constant, respectively. $K_{\mu\nu}$ is the extrinsic curvature of the brane and $q_{\mu\nu}$ is the induced metric on the brane. Equation (5) is usually referred to as the second junction condition. The first junction condition simply states that the induced metric, the first fundamental form,

$$q_{\mu\nu} = e^A_\mu e^B_\nu g_{AB} , \tag{7}$$

be continuous across the brane. Here the vectors e_ν are tangent to the brane. In other words, if we parametrize the brane by coordinates (z^μ) and its position in the bulk is given by functions $X^A_\mathrm{b}(z^\mu)$, the vectors e_ν are defined by

$$e^A_\mu = \partial_\mu X^A_\mathrm{b}(z) . \tag{8}$$

Denoting the brane normal by n, we have $g_{AB}e^A_\mu n^B = 0$. The extrinsic curvature can be expressed purely in terms of the internal brane coordinates [25, 26], $K = K_{\mu\nu}dz^\nu dz^\mu$, with

$$K_{\mu\nu} = -\frac{1}{2}\left[g_{AB}\left(e^A_\mu\partial_\nu n^B + e^A_\nu\partial_\mu n^B\right) + e^A_\mu e^B_\nu n^C g_{AB,C}\right] . \tag{9}$$

In the case we are interested in, the background space-time consists of two copies of the part of AdS_5 with $y \geqslant y_\mathrm{b} = L$. We actually let the coordinate y jump from $y = L$ to $y = -L$ across the brane. Since the metric is symmetric in y, the first junction condition is trivially fulfilled. The second fundamental form is proportional to the induced metric, $K_{\mu\nu} = \pm L^{-1}q_{\mu\nu}$, hence the energy-momentum tensor on the brane is a pure brane tension $\mathcal{T}$, $S_{\mu\nu} = -\mathcal{T}q_{\mu\nu}$. The second junction condition becomes

$$K^>_{\mu\nu} - K^<_{\mu\nu} = [K_{\mu\nu}] = 2K^>_{\mu\nu} , \tag{10}$$

with

$$[K_{00}]|_{y_\mathrm{b}} = -\frac{2}{L} = -\frac{1}{3}\kappa_5\mathcal{T} , \tag{11}$$

$$[K_{ii}]|_{y_\mathrm{b}} = \frac{2}{L} = \frac{1}{3}\kappa_5\mathcal{T} . \tag{12}$$

This leads to the well-known RS-fine tuning condition,

$$-\Lambda = \frac{6}{L^2} = \frac{1}{6}\kappa_5^2\mathcal{T}^2 . \tag{13}$$

The most general perturbation of the AdS_5 metric (1) is of the form

$$\begin{aligned}\mathrm{d}s^2 &= g_{AB}\mathrm{d}x^A\mathrm{d}x^B \\ &= \frac{L^2}{y^2}\left[-(1+2\Psi)\mathrm{d}t^2 - 4\Sigma_i\mathrm{d}t\mathrm{d}x^i - 4\mathcal{B}\mathrm{d}t\mathrm{d}y + \right. \\ &\quad \left((1-2\Phi)\delta_{ij} + 2H_{ij}\right)\mathrm{d}x^i\mathrm{d}x^j + 4\Xi_i\mathrm{d}x^i\mathrm{d}y \\ &\quad \left. +(1+2\mathcal{C})\mathrm{d}y^2\right] .\end{aligned} \tag{14}$$

Here Σ_i and Ξ_i are divergence-less vectors and H_{ij} is a divergence-less, traceless tensor. It is easy to show that there exists one fully specified gauge in which the perturbation variables take this form, vectors have no "scalar component" and tensors have neither a vector nor a scalar component. We call this the generalized longitudinal gauge (see also [27, 28]). We shall use it in the following. Within linear perturbation theory, the variables with different spin, the tensor H_{ij}, the vectors Σ_i and Ξ_i, and the scalars $\Psi, \Phi, \mathcal{B}, \mathcal{C}$ do not couple. We can therefore study the perturbations of each type separately. We shall do so in the next sub-sections. There we write down the perturbed Einstein equations for a fixed Fourier-mode $\mathbf{k}$ for which we have $\mathbf{k}\cdot\mathbf{\Sigma} = \mathbf{k}\cdot\mathbf{\Xi} = k^i H_{ij} = 0$. We do not perform a Fourier decomposition in time.

We want to study the perturbations in an empty bulk with possible perturbations on the brane. The five dimensional Einstein equation implies the perturbation equations in the bulk,

$$\delta G_{AB} = -\Lambda \delta g_{AB} \ , \tag{15}$$

and the junction conditions at the brane,

$$2\delta K_{\mu\nu} = \kappa_5 \delta \widehat{S}_{\mu\nu} \ . \tag{16}$$

We first discuss tensor perturbations. As we shall see, the homogeneous equations reduce to the same Bessel equations for all three types of perturbations (see also [25]).

A. Tensor perturbations

In this paragraph we first discuss the simplest case, the tensor perturbation equations. We write them down for a fixed Fourier-mode $\mathbf{k}$ and determine their solutions. We consider only $H_{ij} \neq 0$. For this case, Eq. (15) reduces to

$$\left(\partial_t^2 + k^2 - \partial_y^2 + \frac{3}{y}\partial_y\right) H_{ij} = 0 \ . \tag{17}$$

For a given polarization, $H_\bullet = H_+$ or $H_\bullet = H_\times$, we make the ansatz $H_\bullet = f(t)g(y)$ leading to

$$\frac{\partial_t^2 f}{f} + k^2 = \frac{(\partial_y^2 - \frac{3}{y}\partial_y)g}{g} = Z \ , \tag{18}$$

where Z is an arbitrary separation constant. The behavior of the solutions to these equations depends strongly on the sign of Z. If $Z = -m^2$ is negative, we obtain

$$f = \exp(\pm it\sqrt{m^2+k^2}) \equiv \exp(\pm i\omega t) \ , \tag{19}$$

$$g = N(my)^2 \times \begin{cases} J_2(my) \ , \\ Y_2(my) \ . \end{cases} \tag{20}$$

Here J_ν and Y_ν denote the Bessel functions of order ν. They are oscillating and decaying. They are "δ–function normalizable" perturbations like harmonic waves in flat space, in the sense that [29, 30]

$$\int_0^\infty H_m H_{m'} \frac{dy}{m^2 y^3} \propto m\delta(m-m') \ , \tag{21}$$

These are just the ordinary gravity modes of 4-dimensional mass m without a mass gap which are discussed on the original RS paper [14]. However, if $Z = -m^2$ is positive, the solutions take the form

$$f = \exp(\pm t\sqrt{Z-k^2}) = \exp(\pm t\omega) \ , \tag{22}$$

$$g = N(|m|y)^2 \times \begin{cases} K_2(|m|y) \ , \\ I_2(|m|y) \ . \end{cases} \tag{23}$$

Here K_ν and I_ν are the modified Bessel functions of order ν. The second case, $g \propto I_2$ grows exponentially in y. This is not normalizable and therefore cannot represent a physical, small perturbation. However, the mode K_2 decays exponentially and is normalizable and small for sufficiently small initial amplitudes. However, even with arbitrary small initial data this mode grows exponentially in time for sufficiently small wave numbers, $k^2 < -m^2$; it is a tachyonic instability.

To have a complete solution to the perturbation equations we need to discuss the boundary conditions at the brane, *i.e.*, the junction conditions.

A short computation shows that the non-vanishing components of the extrinsic curvature tensor perturbations are in our case

$$\delta K_{ij}|_{y_b} = \left(\frac{2}{L}H_{ij} - \partial_y H_{ij}\right)\bigg|_{y_b} \ , \quad \text{hence}$$

$$-\,2(\partial_y H_{ij})|_{y_b} = \kappa_5 \Pi_{ij}^{(T)} \ , \tag{24}$$

where $\Pi^{(T)}$ are tensor–type anisotropic stresses on the brane.

Let us first consider the homogeneous case $\Pi^{(T)} \equiv 0$. For $m^2 > 0$, the solutions are of the form

$$H = \exp(\pm i\omega t)(my)^2 \left[AJ_2(my) + BY_2(my)\right] \ . \tag{25}$$

The junction condition (24) then requires

$$B = -A\frac{J_1(mL)}{Y_1(mL)} \simeq \frac{\pi}{4}(mL)^2 A \ , \tag{26}$$

where the last expression is a good approximation for $mL \ll 1$. This is precisely the result of Randall and Sundrum [14]. This is not modified even if we allow for the negative mass modes, $Z = -m^2 > 0$, because a physical solution has to be of the form

$$H = C\exp(\pm t\sqrt{Z-k^2})(|m|y)^2 K_2(|m|y) \ , \tag{27}$$

and since K_1 has no zero, the junction condition (24) requires $C = 0$.

But in a realistic brane universe, $\Pi^{(T)}$ is not exactly zero. In cosmology, it is just typically a factor of 10 smaller than other perturbations of the energy-momentum tensor on the brane. We therefore may not expect $C \equiv 0$. However, as long as $\Pi^{(T)}$ remains small, we do not expect the unstable modes to be present, hence we expect $C(k,m) = 0$ for $k^2 < -m^2$. In the next section, we shall show in a specific example that this is indeed the case in RSII, where the brane is Minkowski space