On Adiabatic Renormalization of Inflationary Perturbations

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We discuss the impact of adiabatic renormalization on the power spectrum of scalar and tensor perturbations from inflation. In the range $v \equiv k/(aH) \gtrsim 0.1$, we find that the renormalized tensor-to-scalar ratio strongly depends on v. This means that, at fixed k, the ratio depends on the time at which it is calculated. We argue that in the far infrared regime, $v \ll 1$, the adiabatic expansion is no longer valid, and the unrenormalized spectra are the physical, measurable quantities. These findings cast some doubt on the validity of the adiabatic subtraction at horizon exit, $v = 1$, to determine the perturbation spectra from inflation which has recently advocated in the literature.

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I. INTRODUCTION

Inflation was originally proposed to solve the initial condition problem of standard big bang cosmology. At the same time, it was found that inflation typically leads to a nearly scale invariant spectrum of scalar and tensor fluctuations [1, 2]. It is this finding, which is so well confirmed by the observed anisotropies and polarization in the cosmic microwave background [3], which has led to a wide acceptance of the inflationary paradigm. Using present and future CMB data, in combination with other cosmological data sets, we are now in the position to constrain models.

Typically, an inflationary model predicts the value of three parameters, namely the scalar spectral index n_s, the tensor spectral index n_t, and the tensor-to-scalar ratio r. So far, observations just provide upper limits on tensor fluctuations. These are not independent of the scalar spectral index n_s as it is evident from the 2-dimensional one- and two-σ confidence contours, shown in Fig. 5 of Ref. [4]. These data can be used to constrain inflationary models. For example, in [4] it is noted that a model of inflation with a scalar field potential of the form $\lambda\phi^4$ is ruled out if the number N of e-foldings of inflation after horizon crossing of the scales probed by WMAP, $k \simeq 0.002h/\text{Mpc} \simeq 6H_0$, is of the order of $N \sim 50 - 60$. In this expression, H_0 is the current value of the Hubble parameter and $h = 0.72 \pm 0.08$.

This is a truly breath taking result meaning that CMB data, i.e. cosmological observations on the largest scales, can provide information about the physics at energy scales much higher than those attainable in the laboratory, hence about the physics on the smallest scales. It is therefore of the utmost importance that these results are subjected to the deepest scrutiny. With this point in mind, we have studied the recent works [5]-[7]. In particular, in Ref. [5], the author argues that the inflationary power spectra, as they are usually calculated, are not correct. In fact, since they diverge at coincident points, one should subtract an appropriate adiabatic counterterm (see also [8] for a different point of view). In Ref. [7] the authors perform explicit calculations along these lines, and subtract the adiabatic term at the Hubble exit, namely when $v = k/(aH) = 1$. As a result, the values of the tensor-to-scalar ratio differ significantly from the ones usually adopted to be compared with the data, [4]. The most surprising consequence is that, for example, the chaotic inflationary model $\lambda\phi^4$ is no longer ruled out by the data.

It is well known that the standard power spectra are nearly time-independent on super-Hubble scales, i.e. when $v < 1$. In this paper we show that this is not the case for the adiabatic contribution to the spectra in a realistic model of inflation. The renormalized spectra strongly depend on v. For scalar perturbations, the ratio $P^{(2)}/P^{(IR)}$ becomes negligible for $v \ll 1$. Here, $P^{(IR)}$ denotes the standard inflationary power spectrum in the infrared region, while $P^{(2)}$ denotes the second order adiabatic counterterm. This result is very sound as, for infrared modes, the expansion of the Universe is not slow compared to the oscillation frequency of the mode and thus not adiabatic. Thus, $P^{(IR)}$ should not have an adiabatic contribution.

On the other hand, the adiabatic regularization, when performed at horizon exit or a few Hubble times later, is ambiguous in the sense that it gives a time-dependent result. This leads us to also argue that the correct time at which the adiabatic subtraction has to be performed is the end of inflation, rather than the time of Hubble exit. However, at the end of inflation all the modes relevant for observational cosmology are in the far infrared region where the adiabatic expansion is not appropriate. We

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shall argue that the physical result is not affected by adiabatic regularization.

The paper is organized as follows: in the next section we present approximate expressions for the scalar and tensor power spectra in the framework of slow-roll inflation. In Section III we discuss the adiabatic subtraction for both the scalar and tensor power spectrum, and we argue that the difference with the original power spectra becomes irrelevant in the far infrared regime. In Section IV we draw our conclusions. Some technical results are deferred to appendices.

II. POWER SPECTRA FROM SLOW-ROLL INFLATION

A. Linear perturbations in slow-roll inflation

In this paper, we consider a flat Universe, whose dynamics is driven by a classical minimally coupled scalar field, described by the action

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi G} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] . \quad (1)$$

For a spatially flat Friedmann spacetime, of the form $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$, the background equations of motion for ϕ and for the scale factor $a(t)$ read

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi = 0 , \quad (2)$$

$$H^2 = \frac{1}{3M_{\rm pl}^2}\left[\frac{\dot{\phi}^2}{2} + V\right] , \quad (3)$$

$$\dot{H} = -\frac{1}{2M_{\rm pl}^2}\dot{\phi}^2 , \quad (4)$$

where $M_{\rm pl}^2 = 1/(8\pi G)$ is the reduced Planck mass. The dot denotes a derivative with respect to the cosmic time t. Linear perturbations of the metric in longitudinal gauge are given by

$$ds^2 = -(1+2\Psi)dt^2 + a^2\left[(1-2\Phi)\delta_{ij} + h_{ij}\right]dx^i dx^j , \quad (5)$$

Here Ψ and Φ represent the Bardeen potentials, and h_{ij} describes traceless, transverse tensor degrees of freedom, that is gravitational waves. We do not discuss vector perturbations.

In single-field inflationary models, and to first order in perturbation theory, we have $\Phi = \Psi$. Scalar perturbations have only one degree of freedom, which can be studied by means of a single gauge invariant variable, such as the so-called Mukhanov variable [9], defined, in the longitudinal gauge, by

$$Q = \varphi + \frac{\dot{\phi}}{H}\Psi . \quad (6)$$

In this expression, we assume that the scalar field can be written as a background term plus a linear perturbation, namely as $\phi + \varphi$.

Often, one also uses the curvature variable ζ, which, for $\Phi = \Psi$, is defined as [10]

$$\zeta = \frac{H}{\dot{\phi}}Q = \frac{2(H^{-1}\dot{\Psi} + \Psi)}{3(1+w)} + \Psi . \quad (7)$$

Here, w is the equation of state parameter, which satisfies the equation

$$1 + w = \frac{2\dot{\phi}^2}{\dot{\phi}^2 + 2V(\phi)} . \quad (8)$$

It is important to note that both ζ and Q are related to the Bardeen potential Ψ via a first order equation. They are not independent degrees of freedom and it is therefore not consistent to think of Q as a quantum degree of freedom and of Ψ as a classical variable. When we quantize Q, or rather aQ as below, we also quantize the Bardeen potential. In fact, we do not equate expectation values of some quantum fields to classical first order perturbations of the metric via Einstein's equation, but we do quantize the metric perturbations.

The equation governing Q in Fourier space, is given by

$$\ddot{Q}_k + 3H\dot{Q}_k + \frac{1}{a^2}k^2 Q_k + \left[V_{\phi\phi} + 2\frac{d}{dt}\left(3H + \frac{\dot{H}}{H}\right)\right]Q_k = 0 . \quad (9)$$

A subscript ϕ on V denotes the derivative with respect to ϕ, and $V_{\phi\phi}$ acts as an effective mass, as shown below. During inflation, we assume that the so-called slow-roll parameters

$$\epsilon = \frac{M_{\rm pl}^2}{2}\left(\frac{V_\phi}{V}\right)^2 , \qquad \eta = M_{\rm pl}^2\frac{V_{\phi\phi}}{V} \quad (10)$$

are small, ϵ, $|\eta| \ll 1$. To leading order in the slow-roll parameters each mode Q_k satisfies the equation

$$\ddot{Q}_k + 3H\dot{Q}_k + H^2\left[\frac{k^2}{a^2H^2} + 3\eta - 6\epsilon\right]Q_k = 0 . \quad (11)$$

Analytic solutions for Eq. (11) can be found if the slow roll parameters are constant. More generally, along the lines of [11, 12], we can study this equation in the infrared regime (IR), corresponding to $k/(aH) < c$, and in the ultraviolet regime (UV), corresponding to $k/(aH) > c$, where $1/10 \lesssim c \lesssim 1$. We will shortly see that we can "match" the UV solution to the IR one at $k/(aH) = c$. In the UV, the slow-roll parameters can be considered as constant. The canonically normalized solution to Eq. (11) with adiabatic vacuum initial conditions then reads

$$Q_k^{(UV)} = \frac{1}{a^{3/2}}\sqrt{\frac{\pi(1+\epsilon)}{4H}}H_\nu^{(1)}\left[\frac{k}{aH}(1+\epsilon)\right] , \quad (12)$$

where $H_\nu^{(1)}$ is the Hankel function of the first kind with index $\nu = \frac{3}{2} - \eta + 3\epsilon$. It is instructive to rewrite Eq. (9)

in the form

$$(aQ_k)'' + \left(k^2 - \frac{z''}{z}\right) aQ_k \; = \; 0\,, \qquad (13)$$

$$z = a\frac{\dot\phi}{H} = -aM_{pl}\sqrt{2\epsilon}\ , \qquad (14)$$

where primes denote derivatives with respect to conformal time τ, defined by $ad\tau = dt$. Eq. (13) is simply the equation of a harmonic oscillator with a negative time-dependent mass $-z''/z$. When $k^2 - \frac{z''}{z} < 0$, this leads to amplification on the mode aQ_k. This form of the perturbation equation is completely general and independent of the form of the potential.

In the far IR, $v \ll 1$, one can neglect the term k^2, and the non-decaying mode of the solution Q_k is well approximated by $Q_k \propto \frac{\dot\phi}{H} = -M_{pl}\sqrt{2\epsilon}$. On the other hand, in the far UV, $-k\tau \simeq v \gg 1$ and one can neglect the term z''/z, so that Eq. (13) reduces to the equation for a simple harmonic oscillator.

As mentioned above, by imposing that the UV solution approximately matches the IR solution for $k/(aH) = c$, we obtain the solution valid for $k < caH$, namely [12]

$$Q_k^{(IR)} = \frac{1}{a^{3/2}}\sqrt{\frac{\pi(1+\epsilon)}{4H}}\left(\frac{H_c}{H}\right)^{\gamma} H^{(1)}_{3/2}\left[\frac{k}{aH}(1+\epsilon)\right]\ . \qquad (15)$$

Here, H_c is the value of the Hubble parameter at the time t_c when $k = caH$, and

$$\gamma = 3 + \frac{V_{\phi\phi}}{3\dot H} = 3\left(1 - \frac{\eta}{3\epsilon}\right)\ . \qquad (16)$$

We now turn to tensor perturbations h_{ij}. In Fourier space both tensor polarizations evolve according to

$$\ddot h_k + 3H\dot h_k + \frac{k^2}{a^2}h_k = 0\ . \qquad (17)$$

As before, one can derive approximate solutions in the UV and IR. The mode and the amplitude are chosen such that the canonically normalized variable $(aM_{\rm pl}/\sqrt{2})h$ satisfies adiabatic vacuum initial conditions in the UV. Thus, one finds

$$h_k^{(UV)} = \frac{1}{a^{3/2}M_{\rm pl}^2}\sqrt{\frac{\pi(1+\epsilon)}{2H}}H^{(1)}_{\nu}\left[\frac{k}{aH}(1+\epsilon)\right]\ , \qquad (18)$$

with $\nu = 3/2 + \epsilon$, and

$$h_k^{(IR)} = \frac{1}{a^{3/2}M_{\rm pl}^2}\sqrt{\frac{\pi(1+\epsilon)}{2H}}\left(\frac{H_c}{H}\right)H^{(1)}_{3/2}\left[\frac{k}{aH}(1+\epsilon)\right]. \qquad (19)$$

B. Power Spectra

With the solutions discussed in the previous subsection, we can now compute the scalar and tensor power spectrum, defined by

$$P_\zeta(k) = \frac{k^3}{2\pi^2}\left(\frac{H}{\dot\phi}\right)^2 |Q_k|^2\ , \quad P_t(k) = \frac{2k^3}{\pi^2}|h_k|^2\ . \qquad (20)$$

In terms of the variable $v = k/(aH)$, the expansion of the spectra with respect to the slow-roll parameters, in the UV and IR, respectively yields

$$P_\zeta^{(UV)} \; = \; \frac{1}{2M_{\rm pl}^2}\left(\frac{H_v}{2\pi}\right)^2\left[\frac{1+v^2+f(v)\epsilon_v+g(v)\eta_v}{\epsilon_v}\right], \qquad (21)$$

$$P_t^{(UV)} \; = \; \frac{8}{M_{\rm pl}^2}\left(\frac{H_v}{2\pi}\right)^2\left[1+v^2+f_t(v)\epsilon_v\right]\ , \qquad (22)$$

and

$$P_\zeta^{(IR)} \; = \; \frac{1}{2M_{\rm pl}^2}\left(\frac{H_v}{2\pi}\right)^2\left(\frac{H_c}{H_v}\right)^{2\gamma}\left[\frac{1+v^2-2\epsilon_v}{\epsilon_v}\right] \qquad (23)$$

$$P_t^{(IR)} \; = \; \frac{8}{M_{\rm pl}^2}\left(\frac{H_v}{2\pi}\right)^2\left(\frac{H_c}{H_v}\right)^2\left[1+v^2-2\epsilon_v\right]\ . \qquad (24)$$

These UV and IR spectra have the correct asymptotic form but they do not match exactly at $v = c$ since we have neglected the decaying mode contribution in $P^{(IR)}$. The small discontinuity is of the order of the slow roll parameters. The functions $f(v)$, $f_t(v)$, and $g(v)$ appearing in these expressions are defined in Appendix A and plotted in Fig. 3. H_v, ϵ_v, and η_v are the values of these quantities calculated at the time t_v for which $k/(aH) = v$. The functions $f(v)$, $f_t(v)$, and $g(v)$ always appear multiplied by η or ϵ. Therefore, they are always subdominant for wave numbers k which exit the Hubble scale in the slow-roll regime, i.e. when $\epsilon_v \ll 1$ and $\eta_v \ll 1$.

An important observable parameter is the tensor-to-scalar ratio $r = P_t/P_\zeta$. On considering the particular case when $V = m^2\phi^2/2$, we have $\eta = \epsilon$ and $\gamma = 2$. Therefore, for this particular case and at the leading order in the slow-roll parameters, we find

$$r^{(UV)} = 16\epsilon_v\ , \quad r^{(IR)} = 16\epsilon_c\ . \qquad (25)$$

Note that, during slow-roll evolution, ϵ varies slowly ($\dot\epsilon$ is second order in the slow-roll parameters), so that also $r^{(IR)}$ is nearly constant for scales which reach $k = caH$ during slow-roll.

To compare our findings with the five-year WMAP results, we must write $r^{(UV,IR)}$ in terms of the spectral index $n_s = 1 + \frac{d}{d\ln k}\ln P_\zeta$. In turn, n_s must be expressed as a function of v and N, i.e. the number of e-folds between the epoch when the modes corresponding to the scales probed by WMAP exit the Hubble scale and the end of inflation. In Appendix B, we show that, when $V = m^2\phi^2/2$,

$$n_s^{(UV)} = 1 - 4\epsilon_v\ , \quad n_s^{(IR)} = 1 - 4\epsilon_c\ , \qquad (26)$$

where

$$\epsilon_s = \eta_s = \frac{1}{2(N + \ln s)} . \tag{27}$$

where s is either v or c. It then follows that

$$r^{(UV,IR)} = 4\left(1 - n_s^{(UV,IR)}\right) .$$

The generic slow-roll expression for the scalar spectral index is [10]

$$n_s = 1 - 6\epsilon + 2\eta . \tag{28}$$

One easily verifies that for general chaotic inflation models with $V = \frac{\lambda}{p}\frac{\phi^p}{M_{pl}^{p-4}}$ one has $\epsilon = \frac{p^2}{2}\left(\frac{M_{\rm pl}}{\phi}\right)^2$, while $\eta = p(p-1)\left(\frac{M_{\rm pl}}{\phi}\right)^2$, hence

$$n_s = 1 - \left(2 + \frac{4}{p}\right)\epsilon . \tag{29}$$

III. RENORMALIZED POWER SPECTRA

We now investigate how the power spectrum is modified, when corrected by the subtraction of the adiabatic expansion up to the second order. The adiabatic contribution to the power spectrum, in terms of the Mukhanov variable, can be found in the appendix of [13], by taking in consideration only terms with at most two derivatives with respect to the time for the second order expansion. The adiabatic expansion for the tensor perturbation can be found using the results in Appendix A of [14] with $m^2 = 0$. In this paper, the authors consider a scalar field propagating on an unperturbed space-time. On such a background, the equation of motion of the scalar field with $m^2 = 0$ coincides exactly with the equation of motion of the tensor perturbation.

In summary, the adiabatic expansions for the power spectra P_ζ and P_t are

$$P_\zeta^{(2)} = \frac{1}{2M_{\rm pl}^2}\left(\frac{H_v}{2\pi}\right)^2\frac{1}{\epsilon_v}\left[\frac{v^3}{(v^2+3\eta_v)^{1/2}} + \left(1+\frac{5}{2}\epsilon_v\right)\frac{v^3}{(v^2+3\eta_v)^{3/2}} + \frac{9\eta_v v^3}{4(v^2+3\eta_v)^{5/2}} - \frac{45\eta_v^2 v^3}{8(v^2+3\eta_v)^{7/2}}\right], \tag{30}$$

$$P_t^{(2)} = \frac{8}{M_{\rm pl}^2}\left(\frac{H_v}{2\pi}\right)^2\left[v^2 + 1 - \frac{\epsilon_v}{2}\right]. \tag{31}$$

where $\eta_v = \frac{V_{\phi\phi}}{3H^2}$, and $V_{\phi\phi}$ is the mass of the inflaton. In this expansion, we have kept only the the leading order, in the slow-roll parameters, in each term. It is important to note that, strictly speaking, this expansion is meaningful only if v is sufficiently large. This makes sense since this corresponds to the adiabatic regime. On the contrary, when v is small, the expansion of the Universe is no longer slow compared to the oscillations of the mode (especially in the massless case) and we are not in the adiabatic regime.

With these expressions, we readily find the renormalized power spectra in the two regimes. In particular, in the UV case, the renormalized power spectra are

$$P_\zeta = P_\zeta^{(UV)} - P_\zeta^{(2)} = \frac{1}{2M_{\rm pl}^2}\left(\frac{H_v}{2\pi}\right)^2\frac{1}{\epsilon_v}\left[1 + v^2 + f(v)\epsilon_v + g(v)\eta_v - \frac{v^3}{(v^2+3\eta_v)^{1/2}} - \left(1+\frac{5}{2}\epsilon_v\right)\frac{v^3}{(v^2+3\eta_v)^{3/2}} + \right.$$
$$\left. - \frac{9\eta_v v^3}{4(v^2+3\eta_v)^{5/2}} + \frac{45\eta_v^2 v^3}{8(v^2+3\eta_v)^{7/2}}\right] , \tag{32}$$

$$P_t = P_t^{(UV)} - P_t^{(2)} = \frac{8}{M_{\rm pl}^2}\left(\frac{H_v}{2\pi}\right)^2\left[f_t(v)\,\epsilon_v + \frac{\epsilon_v}{2}\right] . \tag{33}$$

Let us first analyze the expression for P_ζ. For $v = 1$, we can expand with respect to the slow-roll parameters and find, at leading order,

$$P_\zeta = \frac{1}{2M_{\rm pl}^2}\left(\frac{H}{2\pi}\right)^2\left(\frac{3\alpha\epsilon + 3\beta\eta}{\epsilon}\right) , \tag{34}$$

where $\alpha \simeq 0.903$, $\beta \simeq 0.449$, and all the parameters are calculated at the time when $k = aH$. We roughly agree with [7] as long as $v = 1$. Note that in Eq. (9) of [7] there is just α instead of 3α. This comes from the fact that we use the Mukhanov variable, while in [7] they use the scalar inflaton perturbation in a space-time without metric fluctuations.

When $v < 1$, our result is substantially different. In fact, in this case we simply cannot expand with respect to the slow-roll parameter. The reason is that $3\eta_v$ is no longer much smaller than v^2. For example, in the case $V = m^2\phi^2/2$, as shown in Appendix B, we have

$3\epsilon_v \equiv 3\eta_v \simeq 1/(2N)$. So, if $N = 50$ and $v = 1/5$, then $v^2 = 1/25$ and $3\eta_v = 3/100$.

For P_t we obtain a result similar to what was found in [7] also in the case $v < 1$ [17]. Most importantly, the resulting tensor-to-scalar ratio strongly depends on v for $v < 1$, as is shown in Fig. 1.

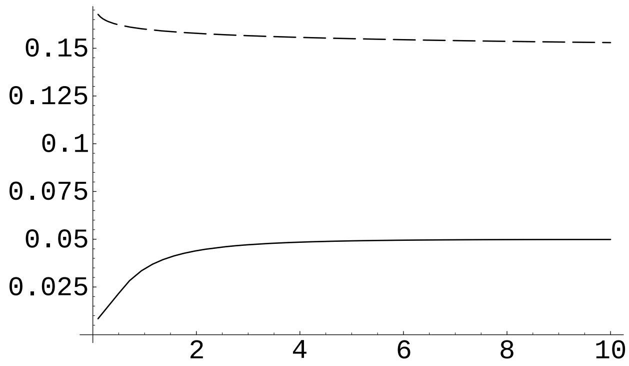

FIG. 1: Tensor-to-scalar ratio r for $N = 50$ and $V = \frac{m^2\phi^2}{2}$, for the case with the adiabatic subtraction (solid line) and without adiabatic subtraction (dashed line), in the range $1/10 < v < 10$.

Let us now consider the far IR, $v \ll 1$. In this case, as already pointed out, the expansion of the universe is not adiabatic. The expansion parameter v^{-1} is not small and such an adiabatic expansion does not seem physically meaningful. Because of this, there is no physical reason to subtract this adiabatic expansion to the standard result in the IR. One might argue, however, that this expansion scheme still produces a finite result and therefore provides a way to renormalize the IR modes. In fact, we find this argument not very convincing, as there might be many different schemes to renormalize the IR modes (for example, see the recent paper [15]). In contrast, in the UV, where space-time curvature becomes negligible, the physical spectrum is independent of the regularization scheme, and this reflects the fact that the UV singularity structure of the two-point function is always of the Hadamard form. Furthermore, since inflation has not started in the infinite past, there is a natural infrared cutoff, namely the horizon scale at the beginning of inflation. Also, the long wave modes do not contribute significantly to the Green function at coincident points, in realistic inflationary models, as noted, for example, in [16]. We therefore conclude that one should not subtract the adiabatic contribution in the IR in realistic inflationary models.

In addition to these considerations, we also find that the ratio $P_\zeta^{(2)}/P_\zeta^{(IR)}$ becomes negligible, as shown in Fig. 2. This is also easily verified by comparing Eq. (23) with Eq. (30). Therefore, the adiabatic subtraction does not modify the scalar spectrum in the infrared. However, for the tensor case, the adiabatic spectrum $P_t^{(2)}$ remains of considerable size, also when the mode has entered the IR. However, we argue that subtracting it from the bare spectrum P_t^{IR} is unphysical, and that P_t^{IR} *is* the physically relevant spectrum for $v \ll 1$. As said before, the main reason is that the adiabatic expansion is not meaningful in this regime where the oscillations are far from adiabatic: they are much slower than the expansion of the Universe.

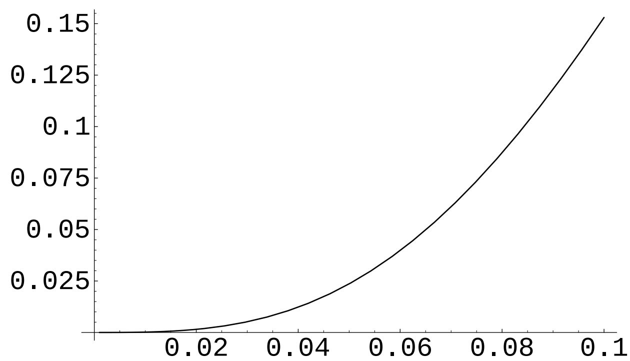

FIG. 2: Ratio $P_\zeta^{(2)}/P_\zeta^{(IR)}$, for $N = 50$ and $V = \frac{m^2\phi^2}{2}$, in the range $10^{-3} < v < 10^{-1}$.

IV. CONCLUSIONS

In this paper we have first determined the renormalized perturbation spectra in a slow-roll inflationary model in UV domain. In agreement with Ref. [7], we have found that adiabatic subtraction can lead to a substantial reduction of power in the UV. Namely, the larger k/a in comparison to H, the closer the weight of the adiabatic counterterm becomes to the unrenormalized spectra. This is reasonable, since we do not expect that the expansion of the Universe is "energetic enough" to excite physical modes in the UV.

On the contrary, in the IR, the adiabatic expansion is no longer valid, and there is no convincing physical argument to subtract this term to the standard power spectrum. As there is a natural IR cutoff to inflation, we propose that for cosmologically relevant scales, which have been amplified by inflation but which are in the far IR at the end of inflation, no adiabatic subtraction should be performed.

Even though the adiabatic calculation does not apply in the IR, it is interesting to note that for scalar perturbations the adiabatic counterterm becomes much smaller than the unrenormalized spectrum in the IR. On the contrary, for tensor modes, the adiabatic counterterm is still large with respect to the unrenormalized spectrum, in large range of v in the IR. However, we advocate the idea that only the unrenormalized spectrum is the physical one, and no subtraction should be performed.

At the end of inflation, all cosmologically relevant scales are in the far IR, hence the adiabatic subtraction, which is a possible prescription for the UV does not affect the associated spectra. This is the main conclusion of this work.

The adiabatic subtraction does, however provide a clean means to derive the shape of the physical spectrum in the UV, where it actually tends to zero: at any given time, fluctuations with $v > 1$ are significantly suppressed by the adiabatic counterterm. In this sense the adiabatic subtraction provides a UV cutoff of the spectrum which is roughly given by the scale k_{UV} which reaches $v = 1$ at the end of inflation, $k_{UV} = a_f H_f$.

The reason why it is usually sensible to compute P_ζ and P_t at the Hubble exit, $v \simeq 1$ instead of evaluating them at the end of inflation, is that we have simple and sound formulae for them, which are valid "inside the Hubble scale", while the growing modes of the perturbations are nearly constant "outside the Hubble scale". Therefore, in general we do not need to calculate their evolution until the end of inflation. This is different for the adiabatic counterterm $P^{(2)}$, which is strongly time-dependent, and which decreases with v in the IR. Therefore, it seems reasonable to perform the adiabatic subtraction at the end of inflation, or at least far in the IR, where, however, it becomes irrelevant for all scales of cosmological interest.

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Appendix A: Definitions

In Section II B, we present the renormalized power spectrum of curvature and tensor perturbations, expanded with respect to the slow-roll parameters. The three functions f, g, and f_t appearing in these expressions are defined by

$$f(v) = -2 - 3\sqrt{2\pi}\, v^{5/2} \left\{ \cos(v) \left[\frac{\partial}{\partial\nu} J_\nu(v)\right]_{\nu=3/2} + \sin(v) \left[\frac{\partial}{\partial\nu} J_\nu(v)\right]_{\nu=3/2} \right\} + +3\sqrt{2\pi}\, v^{3/2} \left\{ \sin(v) \left[\frac{\partial}{\partial\nu} J_\nu(v)\right]_{\nu=3/2} - \cos(v) \left[\frac{\partial}{\partial\nu} J_\nu(v)\right]_{\nu=3/2} \right\} , \tag{A1}$$

$$g(v) = \sqrt{2\pi}\, v^{5/2} \left\{ \cos(v) \left[\frac{\partial}{\partial\nu} J_\nu(v)\right]_{\nu=3/2} + \sin(v) \left[\frac{\partial}{\partial\nu} J_\nu(v)\right]_{\nu=3/2} \right\} + -\sqrt{2\pi}\, v^{3/2} \left\{ \sin(v) \left[\frac{\partial}{\partial\nu} J_\nu(v)\right]_{\nu=3/2} - \cos(v) \left[\frac{\partial}{\partial\nu} J_\nu(v)\right]_{\nu=3/2} \right\} , \tag{A2}$$

$$f_t(v) = -2 - \sqrt{2\pi}\, v^{5/2} \left\{ \cos(v) \left[\frac{\partial}{\partial\nu} J_\nu(v)\right]_{\nu=3/2} + \sin(v) \left[\frac{\partial}{\partial\nu} J_\nu(v)\right]_{\nu=3/2} \right\} + +\sqrt{2\pi}\, v^{3/2} \left\{ \sin(v) \left[\frac{\partial}{\partial\nu} J_\nu(v)\right]_{\nu=3/2} - \cos(v) \left[\frac{\partial}{\partial\nu} J_\nu(v)\right]_{\nu=3/2} \right\} . \tag{A3}$$

These functions are plotted in the range $1/10 < v < 10$ in Fig. 3.

Appendix B: Spectral indices

We consider the case $V = m^2\phi^2/2$. During slow-roll, we can neglect the terms $\dot\phi^2$ in Eq. (3) and $\ddot\phi$ in Eq. (2). It follows that $H \simeq H_i + \dot H t$, where H_i is the initial value of the Hubble factor, and $\dot H \simeq -m^2/3$. Thus, the scale factor satisfies the equalities

$$\ln\frac{a(t)}{a_i} = \left(H_i t - \frac{m^2}{6}t^2\right) = \frac{3}{2m^2}\left[H_i^2 - H^2(t)\right] , \tag{B1}$$

where a_i is its initial value. If we assume that inflation finishes approximately when $H(t) \simeq 0$, it follows that

$$N \equiv \ln\frac{a_f}{a_N} = \frac{3}{2}\frac{H_N^2}{m^2} , \quad a_N = e^{-N} a_f , \tag{B2}$$

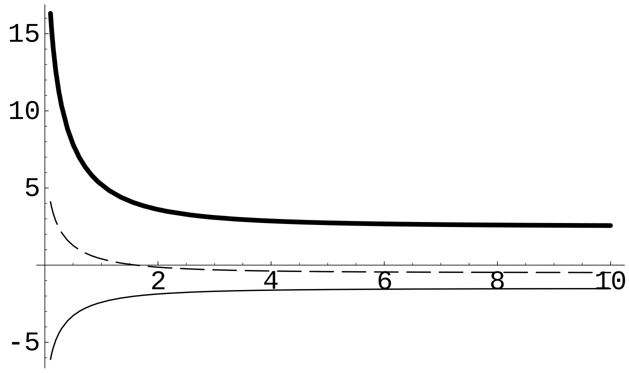

FIG. 3: Plots of the behaviour of f (thick line), g (solid line) and f_t (dashed line) in function of $v = k/(aH)$ in the range from 1/10 to 10.

where H_N and a_N are the values of the Hubble and scale factors N e-folds before the end of inflation. Thus, we can write the momentum k_N, associated to the mode that exit the Hubble scale at N e-folds before the end of the inflation, as

$$k_N \equiv a_N H_N = m \left(\frac{2}{3}N\right)^{1/2} a_i \exp\left(\frac{3H_i^2}{2m^2} - N\right) \ . \quad \text{(B3)}$$

Let s be v or c, according to whether we are dealing with the UV or IR respectively. Let t_s the time when

$$k_N = s a(t_s) H(t_s) \, , \quad \text{(B4)}$$

with the help of Eqs. (B1) and (B3), we find a quadratic equation in t_s, which gives

$$t_s = \frac{3H_i}{m^2} - \frac{\sqrt{6}}{m}\sqrt{N + \ln s} \ , \quad \text{(B5)}$$

where we have replaced the slowly varying function $H(t_s)$ with the constant value H_N in order to obtain an analytical solution of Eq. (B4). Then, as $H = H_i - m^2 t/3$, we find

$$H(t_s) = m\sqrt{\frac{2}{3}(N + \ln s)} \ , \quad \text{(B6)}$$

from which follows that

$$\epsilon_s = \eta_s = 2\left(\frac{M_{\rm pl}}{\phi_s}\right)^2 = \frac{m^2}{3H_s^2} = \frac{1}{2(N + \ln s)} \ . \quad \text{(B7)}$$

A possible way to evaluate n_s is to express, at leading order, the derivative with respect to $\ln k$ as

$$\frac{d}{d\ln k} \simeq -\frac{d}{dN} \ , \quad \text{(B8)}$$

which yields, in both regimes,

$$n_s^{(UV)} = 1 - 4\epsilon_v \ , \quad n_s^{(IR)} = 1 - 4\epsilon_c \ . \quad \text{(B9)}$$

In the UV, we obtain the standard result, while in the IR we have a slightly different expression. In fact, even if the relation between r and n_s is the same in the two regimes, namely $r = 4 - 4n_s$, in the IR we have $n_s = 1 - 2/(N + \ln c)$, while in the UV we have $n_s = 1 - 2/(N + \ln v)$. This difference is, however, quite small since $N \gg 1$ for scales which exit during slow-roll.

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