GENERAL RELATIVITY

Master course for theoretical physics
1 semester, 3 hours of course and 2 hours exercises per week

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This course follows closely the corresponding chapters of the book by Norbert Straumann [16].

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Part I

Differential Geometry

Chapter 1

Differentiable Manifolds

1.1 A manifold

A manifold is a topological space which looks locally like $\mathbb{R}^n$.

Definition 1.1 (Topological manifold) *A* **topological manifold** *of dimension m is a topological space*[1] *(Hausdorff),*[2] *with countable base, which is locally homeomorphic to $\mathbb{R}^m$. This means, for each point $p \in \mathcal{M}$ there exists an open set $\mathcal{U} \subset \mathcal{M}$ with $p \in \mathcal{U}$ and a homeomorphism*[3] *$h : \mathcal{U} \to \mathcal{U}'$, for some open set $\mathcal{U}' \subset \mathbb{R}^m$. Furthermore, we require always that $\mathcal{M}$ be paracompact*[4]*. This implies:*

1. *$\mathcal{M}$ is σ-compact*[5]
2. *$\mathcal{M}$ is a countable and disjoint union of connected manifolds, each of them being a countable union compact manifolds*
3. *$\mathcal{M}$ is metrisible*

Definition 1.2 (chart, atlas) *Let $\mathcal{M}$ be a manifold of dimension m and $h : \mathcal{U} \to \mathcal{U}'$ a homeomorphism from an open set $\mathcal{U} \subset \mathcal{M}$ to an open set $\mathcal{U}' \subset \mathbb{R}^m$.*

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[1] Let X be a set $\neq \emptyset$. T, a set of subsets of X, defines a **topology** on X if it satisfies the following conditions: 1) An arbitrary union of elements of T belongs to T; 2) an intersection of finitely many elements of T belongs to T; 3) T contains X and $\emptyset$. A **topological space** is a couple (X, T) where X is a set $\neq \emptyset$ and T defines a topology on X. The elements of T are called the 'open' subsets of X.

[2] A topological space X is called Hausdorff if for any 2 points $p \neq q$ in X there exist open sets $\mathcal{U} \ni p$, $\mathcal{V} \ni q$ which are disjoint, i.e. $\mathcal{U} \cap \mathcal{V} = \emptyset$

[3] A continuous mapping which is bijective and has a continuous inverse.

[4] $\mathcal{M}$ is paracompact if every open covering of $\mathcal{M}$ admits a finer open covering which is locally finite ($\mathcal{M}$ is locally compact)

[5] A topological space $\mathcal{M}$ is σ-compact if there exist $\mathcal{M}_n \subset \mathcal{M}$ which are compact such that $\mathcal{M} = \bigcup_{n \in \mathbb{N}} \mathcal{M}_n$

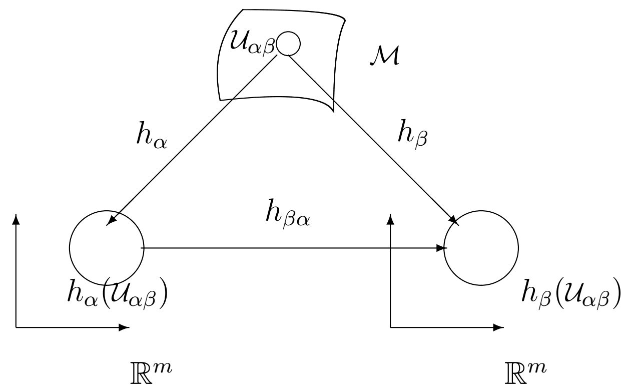


Figure 1.1: A change of charts (coordinate change).

- *The couple* $(h,\mathcal{U})$ *is called a* **chart** *of* $\mathcal{M}$ *, and* $\mathcal{U}$ *is the* **domain** *of* h*. Physicists often call a chart 'a local coordinate system'.*
- *A set of charts* $\{(h_\alpha,\mathcal{U}_\alpha) \mid \alpha \in I\}$ *is an* **atlas** *if* $\bigcup_{\alpha\in I} \mathcal{U}_\alpha = \mathcal{M}$*.*
- *In the intersection* $\mathcal{U}_{\alpha\beta} := \mathcal{U}_\alpha \cap \mathcal{U}_\beta$ *where two charts,* h_α *and* h_β *are well defined, the mapping* $h_{\beta\alpha} := h_\beta \circ h_\alpha^{-1} : h_\alpha(\mathcal{U}_{\alpha\beta}) \to h_\beta(\mathcal{U}_{\alpha\beta})$ *is a* **change of chart** *or a coordinate change. On its domain of definition* $h_{\beta\alpha}$ *is a homeomorphism and* $h_{\alpha\beta}$ *is its inverse.*

Definition 1.3 (differentiable atlas) *An atlas is called* **differentiable** *if all its changes of charts are differentiable in the ordinary sense of maps from an open set of* $\mathbb{R}^m$ *into* $\mathbb{R}^m$*.*

To simplify, in the following "differentiable" always means $\mathcal{C}^\infty$ (i.e. all partial derivatives of $h_{\beta\alpha}$ at any order exist). Evidently, for a differentiable atlas,

$$\begin{aligned} h_{\alpha\alpha} &= \text{id}, \quad h_{\gamma\beta} \circ h_{\beta\alpha} = h_{\gamma\alpha} \quad \text{on} \quad \mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma \ , \\ h_{\alpha\beta} & \quad \text{is a diffeomorphism from } \ h_\alpha(\mathcal{U}_{\alpha\beta}) \ \text{ to } \ h_\beta(\mathcal{U}_{\alpha\beta}) \ \text{ and } \ h_{\alpha\beta}^{-1} = h_{\beta\alpha} \ . \end{aligned}$$

Let $\mathcal{A}$ be a differentiable atlas of the manifold $\mathcal{M}$. Let $\mathcal{D}(\mathcal{A})$ be the atlas which contains all the charts for which the change to any chart in $\mathcal{A}$ is differentiable. Clearly, $\mathcal{A} \subset \mathcal{D}(\mathcal{A})$. Furthermore, the atlas $\mathcal{D}(\mathcal{A})$ is also differentiable because for every change $h_{\beta\gamma}$ between two charts h_β and h_γ in $\mathcal{D}(\mathcal{A})$ there exists a chart $h_\alpha \in \mathcal{A}$ such that locally $h_{\beta\gamma} = h_{\beta\alpha} \circ h_{\alpha\gamma}$.

By definition, the atlas $\mathcal{D}(\mathcal{A})$ is maximal. It is the maximal differentiable atlas that contains $\mathcal{A}$. In this way, a differentiable atlas $\mathcal{A}$ uniquely determines a maximal differentiable atlas $\mathcal{D}(\mathcal{A})$ which contains it. If $\mathcal{D}(\mathcal{A})$ and $\mathcal{D}(\mathcal{B})$ are two maximal atlases generated by $\mathcal{A}$ and $\mathcal{B}$ then $\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{B})$ if and only if the atlas $\mathcal{A} \cup \mathcal{B}$ is differentiable.

Definition 1.4 (differentiable structure) *A* **differentiable structure** *on a topological manifold is a maximal differentiable atlas.*

Definition 1.5 (differentiable manifold) *A* **differentiable manifold** *is a topological manifold with a given differentiable structure, we denote it by* $(\mathcal{M}, \mathcal{D})$.

Remark 1.1 *To define a differentiable structure on a manifold* $\mathcal{M}$ *, in general one does not give all of* $\mathcal{D}$ *but rather a minimal atlas* $\mathcal{A} \subset \mathcal{D}$ *which, as we have seen, completely determines* $\mathcal{D}$ *.*

In the following, we suppose always (without saying) that the charts and atlases considered are contained in the differentiable structure $\mathcal{D}$ of our differentiable manifold $(\mathcal{M}, \mathcal{D})$. A topological manifold $\mathcal{M}$ in general has several different differentiable structures.

Remark 1.2 *The question whether a topological manifold has inequivalent differentiable structures,* $\mathcal{D}$ *and* $\mathcal{D}'$*, that is, such that the differentiable manifolds* $(\mathcal{M}, \mathcal{D})$ *and* $(\mathcal{M}, \mathcal{D}')$ *are not diffeomorphic is in general very difficult. For example, Kervaire (Genève) and Milnor (1963) have shown that the 7-sphere admits exactly 28 inequivalent smooth (i.e.* $\mathcal{C}^\infty$*) differentiable structures. Of the 1- to 6-spheres only the 4-sphere might admit more than one differentiable structure. This remains an unsolved problem.*
Kervaire also showed that there are topological manifolds which do not admit a differentiable structure and that $\mathbb{R}^4$ *admits uncountably many while all other* $\mathbb{R}^n$*'s admit essentially just one.*

Exemples 1.1

1. $\mathcal{M} = \mathbb{R}^m$. *The chart* $(id, \mathcal{M})$*, where id denotes the identity, forms an atlas which induces the ordinary differentiable structure of* $\mathbb{R}^m$.

2. *Every open set of a differentiable manifold,* $\mathcal{U} \subset \mathcal{M}$ *has a differentiable structure induced from* $\mathcal{M}$.

3. *The sphere of dimension n,* $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ *with its differentiable structure induced by an atlas containing the stereographic projections from two different points (see exercises).*

4. *The cone (see exercises).*

Definition 1.6 (differentiable map) *A continuous map* $\varphi : \mathcal{M} \to \mathcal{N}$ *from a differentiable manifold* $\mathcal{M}$ *into a differentiable manifold* $\mathcal{N}$ *is* **differentiable** *in* $p \in \mathcal{M}$ *if for two (and hence for all!) charts* $h : \mathcal{U} \to \mathcal{U}'$, $p \in \mathcal{U}$, *and* $k : \mathcal{V} \to \mathcal{V}'$, $\varphi(p) \in \mathcal{V}$, *the map* $k \circ \varphi \circ h^{-1}$ *from* $h\left(\varphi^{-1}(\mathcal{V}) \cap \mathcal{U}\right)$ *into* $\mathcal{V}'$ *is differentiable in* $h(p)$. *Equivalently, we say that* φ *is differentiable if it is differentiable in every point* $p \in \mathcal{M}$.

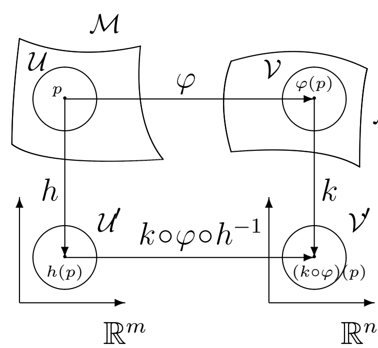


Figure 1.2: Differentiability of a map from $\mathcal{M}$ to $\mathcal{N}$

Remark 1.3

- *The identity* $id : \mathcal{M} \to \mathcal{M}$ *is differentiable.*
- *The composition of two differentiable maps is differentiable (this implies that the differentiable manifolds form a category).*
- $\mathcal{C}^\infty(\mathcal{M}, \mathcal{N})$ *is the set of differentiable maps from* $\mathcal{M}$ *to* $\mathcal{N}$ *.*
- $\mathcal{F}(\mathcal{M}) := \mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$ *is the set of differentiable functions on* $\mathcal{M}$ *.*

Definition 1.7 (diffeomorphism) *A* **diffeomorphism** *of* $\mathcal{M}$ *is a differentiable and bijective map* $\varphi : \mathcal{M} \to \mathcal{M}$ *whose inverse* φ^{-1} *is also differentiable.*

Definition 1.8 (immersion) *Let $\varphi : \mathcal{M} \to \mathcal{N}$ be a differentiable map from the differentiable manifold $\mathcal{M}$ of dimension m to the manifold $\mathcal{N}$ of dimension n, with $m \le n$. The map φ is an* **immersion** *if the charts h and k of definition 1.6 can be chosen such that $k \circ \varphi \circ h^{-1} : h(\mathcal{U}) \to k(\mathcal{V})$ is the inclusion. Here we consider $\mathbb{R}^m$ as subset of $\mathbb{R}^n$: $\mathbb{R}^m \equiv \mathbb{R}^m \times 0_{n-m} \subset \mathbb{R}^n$. In other words, in convenient coordinates φ is* **locally** *of the form $(x^1, \ldots, x^m) \mapsto (x^1, \ldots, x^m, 0, \ldots, 0)$.*

Remark 1.4

- *An immersion is* **locally***, but in general not globally injective.*
- *An injective immersion $\varphi : \mathcal{M} \to \mathcal{N}$ which is a homeomorphism of $\mathcal{M}$ to $\varphi(\mathcal{M})$ is called an* **embedding**

Definition 1.9 (sub-manifold) *Let $\mathcal{M}$ and $\mathcal{N}$ be differentiable manifolds. $\mathcal{M}$ is a* **sub-manifold** *of $\mathcal{N}$ if*

1. *$\mathcal{M} \subset \mathcal{N}$ is a topological submanifold.*
2. *The inclusion $i : \mathcal{M} \to \mathcal{N}$ is an embedding.*

As the inclusion i is also an immersion, we can choose charts $(h, \mathcal{U})$ and $(k, \mathcal{V})$ such that i is **locally** of the form

$$k \circ i \circ h^{-1} : (x^1, \ldots, x^m) \mapsto (x^1, \ldots, x^m, 0, \ldots, 0) \tag{1.1}$$

The domain $\mathcal{U}$ of the chart h is of the form $\widetilde{\mathcal{V}} \cap \mathcal{M}$ for an open set $\widetilde{\mathcal{V}} \subset \mathcal{N}$ since the topology of $\mathcal{M}$ is the one induced by $\mathcal{N}$. If k is defined on $\mathcal{V}$, the representation (1.1) of i is defined on $h\left((\mathcal{V} \cap \widetilde{\mathcal{V}}) \cap \mathcal{M}\right)$.

One can also define the **product** $\mathcal{M} \times \mathcal{N}$ of two differentiable manifolds of dimensions m and n. For two charts $h : \mathcal{U} \to \mathcal{U}' \subset \mathbb{R}^m$ and $k : \mathcal{V} \to \mathcal{V}' \subset \mathbb{R}^n$ one defines the product chart

$$(h \times k) : \begin{array}{l} \mathcal{U} \times \mathcal{V} \to \mathcal{U}' \times \mathcal{V}' \subset \mathbb{R}^m \times \mathbb{R}^n \\ (p, q) \mapsto (h(p), k(q)) \; . \end{array} \tag{1.2}$$

The set of all charts $\{h \times k \mid h \in \mathcal{A},\, k \in \mathcal{B}\}$ for the atlas $\mathcal{A}$ of $\mathcal{M}$ and $\mathcal{B}$ of $\mathcal{N}$ forms an atlas of $\mathcal{M} \times \mathcal{N}$ which defines a differentiable structure.

1.2 Vector and tensor fields

In every point $p \in \mathcal{M}$ of a differentiable manifold of dimension m one can define a tangent space, $T_p\mathcal{M}$ which is a vector space of dimension m. We will consider the tensors on this space. Choosing, in a differentiable way, a tensor of type (r, s) in each point $p \in \mathcal{M}$ we obtain a tensor field of type (r, s).

1.2.1 The tangent space

In this section we give three equivalent definitions of the tangent space in a point p. It is useful to be able to pass freely from one to the other (see exercises). First, however we have to introduce the notion of 'germs' of maps.

For two manifolds $\mathcal{M}$ and $\mathcal{N}$ we consider the maps from $\mathcal{M}$ into $\mathcal{N}$ which are differentiable in a neighborhood of a given point $p \in \mathcal{M}$. We denote this neighborhood by $\mathcal{U}_p \subset \mathcal{M}$. In other words, we define the locally differentiable maps around p,

$$\{\varphi \mid \varphi : \mathcal{U}_p \to \mathcal{N} \text{ differentiable on an open set } \mathcal{U}_p \ni p\} \ . \tag{1.3}$$

We introduce the following equivalence relation:

$$(R) \quad \varphi \sim \psi \Leftrightarrow \exists \text{ an open set } \mathcal{V} \ni p \text{ such that } \varphi|_{\mathcal{V}} = \psi|_{\mathcal{V}} \ . \tag{1.4}$$

Definition 1.10 (germ) *An equivalence class of (R) is called a* **germ** *(of a smooth map) $\mathcal{M} \to \mathcal{N}$ at the point $p \in \mathcal{M}$.*

We denote a germ which is represented by a map φ, by $\overline{\varphi} : (\mathcal{M}, p) \to \mathcal{N}$ or $\overline{\varphi} : (\mathcal{M}, p) \to (\mathcal{N}, q)$, where $q = \varphi(p)$.
Compositions of germs are defined naturally via their representatives.

Definition 1.11 (germs of functions) *A* **germ of a function** *is a germ $\overline{f} : (\mathcal{M}, p) \to (\mathbb{R}, x)$, where $x = f(p)$. The set of all germs of functions in point $p \in \mathcal{M}$ is denoted by $\mathcal{F}_{\mathcal{M}}(p)$.*

$\mathcal{F}_{\mathcal{M}}(p)$ has the structure of a real algebra (define the operations naturally with the representatives).
A germ $\overline{\varphi} : (\mathcal{M}, p) \to (\mathcal{N}, q)$ defines through composition the following algebra homomorphism $\varphi^\star$ from $\mathcal{F}_{\mathcal{N}}(q)$ to $\mathcal{F}_{\mathcal{M}}(p)$:

$$\varphi^\star : \ \mathcal{F}_{\mathcal{N}}(q) \to \mathcal{F}_{\mathcal{M}}(p) : \ \overline{f} \mapsto \overline{f \circ \varphi} \tag{1.5}$$

Obviously $\mathrm{id}^\star = \mathrm{id}$ and $\underline{(\psi \circ \varphi)}^\star = \varphi^\star \circ \psi^\star$. In particular, if φ represents an invertible germ, then $\overline{\varphi} \circ \overline{\varphi^{-1}} = \overline{\mathrm{id}}$, and

$$\varphi^\star \circ (\varphi^{-1})^\star = \mathrm{id} \quad \text{hence} \quad (\varphi^\star)^{-1} = (\varphi^{-1})^\star \tag{1.6}$$

and $\varphi^\star$ is an isomorphism.

A chart h around a point p with $h(p) = 0$ defines an invertible germ,

$$\bar{h} : (\mathcal{M}, p) \to (\mathbb{R}^m, 0) \tag{1.7}$$

and thus also an isomorphism[6]

$$h^\star : \mathcal{F}_m \to \mathcal{F}_{\mathcal{M}}(p) \qquad \mathcal{F}_m = \mathcal{F}_{\mathbb{R}^m}(0)\,. \tag{1.8}$$

We begin with the algebraic definition of tangent space. For this we need first to define the concept of a 'derivation'.

Definition 1.12 (derivation) *A* **derivation** *on $\mathcal{F}_{\mathcal{M}}(p)$ is a linear map $\mathcal{F}_{\mathcal{M}}(p) \to \mathbb{R}$ which satisfies the Leibniz rule, i.e.,*

$$X(\bar{f} \cdot \bar{g}) = \bar{g}(p)X(\bar{f}) + \bar{f}(p)X(\bar{g}), \quad \forall \bar{f}, \bar{g} \in \mathcal{F}_{\mathcal{M}}(p) \tag{1.9}$$

Definition 1.13 (tangent space (algebraic)) **The tangent space** *of $\mathcal{M}$ in p, denoted by $T_p\mathcal{M}$, is the (real) vector space of derivations on $\mathcal{F}_{\mathcal{M}}(p)$.*

Clearly, the derivations form vector space. Furthermore, from the Leibniz rule we conclude

$$X(1) = X(1 \cdot 1) = X(1) + X(1) \text{ which implies that } X(1) = 0\ . \tag{1.10}$$

where $1 : \mathcal{M} \to \mathbb{R} :\ p \mapsto 1$. Hence, by linearity, all derivations vanish on the constant functions $f(p) = c$.

A germ $\overline{\varphi} : (\mathcal{M}, p) \to (\mathcal{N}, q)$, and thus every differentiable map $\varphi : \mathcal{M} \to \mathcal{N}$, induces a linear mapping called the **tangent map** (or the differential of φ in p) denoted by $T_p\varphi$ from $T_p\mathcal{M}$ into $T_q\mathcal{N}$. It is given by

$$T_p\varphi :\ T_p\mathcal{M} \to T_q\mathcal{N} :\ X \mapsto X \circ \varphi^\star\,. \tag{1.11}$$

The differential $T_p\varphi(X)$ of a germ $\overline{f} : (\mathcal{N}, q) \to \mathbb{R}$ is given by

$$T_p\varphi(X)(\overline{f}) = X \circ \varphi^\star(\overline{f}) = X(\overline{f \circ \varphi}),\ \text{for } X \in T_p\mathcal{M}\ . \tag{1.12}$$

[6] $\mathcal{F}_m$ is the set of germs $(\mathbb{R}^m, 0) \to (\mathbb{R}, x)$.

The differential of a composition of two germs,

$$\overline{\psi} \circ \overline{\varphi} : \ (\mathcal{M}, p) \xrightarrow{\overline{\varphi}} (\mathcal{N}, q) \xrightarrow{\overline{\psi}} (\mathcal{L}, r) : \ p \mapsto \psi(\varphi(p)) \ , \tag{1.13}$$

is the product of the tangent maps $T\psi$ and $T\varphi$:

$$T_p(\psi \circ \varphi) = T_q\psi \circ T_p\varphi, \quad \text{where } q = \varphi(p) \ . \tag{1.14}$$

Proof:
Consider $X \in T_p\mathcal{M}$ and $\overline{f} \in \mathcal{F}_L(r)$.

$$T_p(\psi \circ \varphi)(X)(\overline{f}) = X\left((\psi \circ \varphi)^\star(\overline{f})\right) = X(\overline{f \circ \psi \circ \varphi}) = X\left(\varphi^\star(\overline{f \circ \psi})\right) =$$
$$(T_p\varphi)(X)(\overline{f \circ \psi}) = (T_p\varphi)(X)(\psi^\star(\overline{f})) = (T_q\psi) \circ (T_p\varphi)(X)(\overline{f}) \ .$$

□

The linearity of the differential $T_p\varphi$ (also called the tangent map of φ in the point p) follows directly from the definition.
Let now $\mathcal{N}$ be a differentiable manifold, $p \in \mathcal{N}$ and let n be the dimension of $\mathcal{N}$. We show that the dimension of $T_p\mathcal{N}$ is also n. For a chart h with $h(p) = 0$, the germ $\overline{h} : (\mathcal{N}, p) \to (\mathbb{R}^n, 0)$ induces an isomorphism $h^\star : \mathcal{F}_n \to \mathcal{F}_\mathcal{N}(p)$. Thus the map

$$T_ph : T_p\mathcal{N} \to T_0\mathbb{R}^n \tag{1.15}$$

is an isomorphism. $T_p\mathcal{N}$ and $T_0\mathbb{R}^n$ therefore have the same dimension. To caracterize $T_0\mathbb{R}^n$, we use the following lemma:

Lemma 1.1 *Let $\mathcal{U}$ be a ball around the origin in $\mathbb{R}^n$ and $f : \mathcal{U} \to \mathbb{R}$ a differentiable function. There exist differentiable functions $f_1, \ldots, f_n : \mathcal{U} \to \mathbb{R}$ such that*

$$f(x) = f(0) + \sum_{i=1}^{n} f_i(x)x^i \tag{1.16}$$

Proof:

$$f(x) - f(0) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f(tx^1, \ldots, tx^n)\,\mathrm{d}t = \sum_{i=1}^{n} x^i \underbrace{\int_0^1 \partial_i f(tx^1, \ldots, tx^n)\,\mathrm{d}t}_{f_i,\, i=1,\ldots,n},$$

where $\partial_i f$ is the partial derivative with respect to x^i.

□

Partial derivatives are special derivations of $\mathcal{F}_n$

$$\partial_i : \ \mathcal{F}_n \to \mathbb{R} : \ \overline{f} \mapsto \partial_i f(0) \ . \tag{1.17}$$

This leads us to the following lemma:

Lemma 1.2 *The $(\partial_i)_{i=1}^n$ form a basis of the vector space $T_0\mathbb{R}^n$ of derivations of $\mathcal{F}_n$.*

Proof:
a) The ∂_i are linearly independent:
For $X = \sum_{i=1}^n a^i \partial_i \equiv 0$, applying X to the germ $\overline{x}^j$ we obtain:

$$0 = X(\overline{x}^j) = \sum_{i=1}^n a^i \partial_i x^j = a^j.$$

b) The ∂_i generate $T_0\mathbb{R}^n$: For $X \in T_0\mathbb{R}^n$, we set $a^i := X(\overline{x}^i)$. We consider the derivation

$$Y = X - \sum_{i=1}^n a^i \partial_i$$

and want to show that $Y = 0$. By construction, $Y(\overline{x}^j) = 0$. For a given germ $\overline{f}$ we can write

$$\overline{f} = \overline{f}(0) + \sum_{i=1}^n \overline{f}_i \cdot \overline{x}^i.$$

From the Leibniz rule 1.9 we obtain

$$Y(\overline{f}) = \underbrace{Y(\overline{f}(0))}_{=0} + \sum_{i=1}^n \overline{f}_i(0) \underbrace{Y(\overline{x}^i)}_{=0} = 0 \ .$$

$\square$

The dimension of the tangent space $T_p\mathcal{N}$ is therefore uniquely determined and it is n, the dimension of the manifold $\mathcal{N}$.

In local coordinates $(x^1, \ldots, x^n)$ in a neighborhood of a point $p \in \mathcal{N}$, every vector $X \in T_p\mathcal{N}$ is given by its expression as linear combination of the partial derivatives ∂_i , where we interpret the ∂_i via the isomorphism $T_ph : T_p\mathcal{N} \to T_0\mathbb{R}^n$ as elements of $T_p\mathcal{N}$ ($\partial_i \equiv (T_ph)^{-1}\partial_i$) where

$$(\partial_i)_p(\overline{f}) \equiv \partial_i \left(\overline{f \circ h^{-1}}\right)(h(p)) \ . \tag{1.18}$$

Let now $\overline{\varphi} : (\mathcal{N}, p) \to (\mathcal{M}, q)$ be a germ and we choose also in a neighborhood of q local coordinates $(y_1, \ldots, y_m)$ where q is represented by 0.
With this we can interpret $\overline{\varphi}$ as a germ from $(\mathbb{R}^n, 0)$ to $(\mathbb{R}^m, 0)$, which we denote also by $\overline{\varphi}$: The tangent map $T_0\varphi$ is given as follows: Consider $\overline{f} \in \mathcal{F}_m$.
Setting $\varphi(x^1, \ldots, x^n) = (\varphi^1(x^1, \ldots, x^n), \ldots, \varphi^m(x^1, \ldots, x^n)) = (y^1, \ldots, y^m)$, we compute

$$T_0\varphi(\partial_i)(\overline{f}) = \partial_i(\overline{f} \circ \overline{\varphi}) = \sum_{j=1}^n \frac{\partial f}{\partial y^j}(0) \cdot \frac{\partial \varphi^j}{\partial x^i}(0) \tag{1.19}$$

$$
\begin{tikzcd}
(\mathcal{N}, p) \arrow[r, "\overline{\varphi}"] \arrow[d, "\text{chart } h"'] & (\mathcal{M}, q) \arrow[d, "\text{chart } k"] \\
\underset{(x^1, \ldots, x^n)}{(\mathbb{R}^n, 0)} \arrow[r, "\text{"}\overline{\varphi}\text{"}"] & \underset{(y^1, \ldots, y^m)}{(\mathbb{R}^m, 0)}
\end{tikzcd}
$$

Figure 1.3: The relation between a germ from $(\mathcal{N}, p)$ to $(\mathcal{M}, q)$ and a germ from $(\mathbb{R}^n, 0)$ to $(\mathbb{R}^m, 0)$, "$\overline{\varphi}$" $\equiv \overline{k \circ \varphi \circ h^{-1}}$

Hence $T_0\overline{\varphi}(\partial_i) = \left(\frac{\partial \varphi^j}{\partial x^i}\right)_0 \frac{\partial}{\partial y^j}$. The tangent map $T_p\varphi$ in the given coordinates is the Jacobian matrix $D\varphi := \left(\frac{\partial \varphi^j}{\partial x^i}\right)$.

For a vector $v = \sum_{i=1}^n a^i \frac{\partial}{\partial x^i}$ we obtain

$$T_0\varphi(v) = \sum_{j=1}^m b^j \frac{\partial}{\partial y^j} \quad \text{with} \quad b = (D\varphi)_0 a\,. \tag{1.20}$$

With this we have shown the following:

Proposition 1.1 *In local coordinates* $(x^1, \ldots, x^n)$ *around a point* $p \in \mathcal{N}$*, such that* p *corresponds to* $(0, \ldots, 0) = 0$ *and coordinates* $(y^1, \ldots, y^m)$ *around* $q \in \mathcal{M}$*, the derivations* $\left(\frac{\partial}{\partial x^i}\right)_{i=1}^n$ *and* $\left(\frac{\partial}{\partial y^j}\right)_{j=1}^m$ *form a basis of the vector spaces* $T_p\mathcal{N}$ *and* $T_q\mathcal{M}$*. The tangent map of a germ* $\overline{\varphi} : (\mathcal{N}, p) \to (\mathcal{M}, q)$ *in these bases is given by the Jacobian*

$$D_0\varphi : \mathbb{R}^n \to \mathbb{R}^m \;.$$

This leads us to the "physicist'" definition of a tangent vector which in short is: "A (contravariant) vector is a collection of n real numbers which transform under differential maps with the Jacobian".

We make this "definition" more precise. Let $\overline{h}$ and $\overline{k} : (\mathcal{N}, p) \to (\mathbb{R}^n, 0)$ be germs of charts. The coordinate transformation

$$\overline{g} := \overline{k} \circ \overline{h}^{-1} : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$$

is an invertible germ.
The invertible germs, i.e. the local coordinate transformations form a group $\mathcal{G}$ and for two germs of the charts $\overline{h}$ and $\overline{k}$ there exists a unique $\overline{g} \in \mathcal{G}$ such that $\overline{g} \circ \overline{h} = \overline{k}$.
To every $\overline{g} \in \mathcal{G}$ there corresponds a matrix, its Jacobian in 0, $D_0 g$.
This correspondence defines a group homomorphism

$$\mathcal{G} \to \mathcal{G}\ell(n, \mathbb{R}) \;:\; \overline{g} \mapsto D_0 g \tag{1.21}$$

where $\mathcal{G}\ell(n,\mathbb{R})$ is the group of $n\times n$ invertible matrices. The "physicist's" definition of the tangent space can now be formulated as follows:

Definition 1.14 ("physical" tangent space) *A tangent vector in a point $p \in \mathcal{N}$ is a correspondence which associates to every germ of a chart $\overline{h} : (\mathcal{N}, p) \to (\mathbb{R}^n, 0)$ in p a vector $v = (v^1, \ldots, v^n) \in \mathbb{R}^n$ such that the vector $(D_0 g)v$ corresponds to the germ $\overline{g} \circ \overline{h}$ for every $\overline{g} \in \mathcal{G}$.*

In other words: Let $\mathcal{K}_p$ be the set of germs of charts in p, $\overline{h} : (\mathcal{N}, p) \to (\mathbb{R}^n, 0)$. The tangent space of the physicist $(T_p\mathcal{N})_{\mathrm{phys}}$ is the set of maps

$$v : \mathcal{K}_p \to \mathbb{R}^n \tag{1.22}$$

which satisfy $v(\overline{g} \circ \overline{h}) \overset{(\star)}{=} (D_0 g)v(\overline{h})$ for all $\overline{g} \in \mathcal{G}$.
These maps form a vector space: for a germ $\overline{h}$, one can choose $v(\overline{h}) \in \mathbb{R}^n$ arbitrarily; for every other germ of a chart $\overline{k} = \overline{k \circ h^{-1}} \circ \overline{h}$, $v(\overline{k})$ is then determined by $(\star)$. $(T_p\mathcal{N})_{\mathrm{phys}}$ is thus isomorphic to $\mathbb{R}^n$. A choice of coordinates defines an isomorphism between $(T_p\mathcal{N})_{\mathrm{phys}}$ and $\mathbb{R}^n$.
The **canonical** isomorphism between $T_p\mathcal{N}$ and $(T_p\mathcal{N})_{\mathrm{phys}}$ is

$$\begin{aligned} T_p\mathcal{N} &\to (T_p\mathcal{N})_{\mathrm{phys}} \\ X &\mapsto v \end{aligned} \tag{1.23}$$

where v assigns to the germ of the chart $\overline{h} = (\overline{h}^1, \ldots, \overline{h}^n)$ the vector $(X(\overline{h}^1), \ldots, X(\overline{h}^n)) \in \mathbb{R}^n$. The components of this vector are exactly the coefficients of X in the basis $\left(\frac{\partial}{\partial x^i}\right)$ defined by the chart $\overline{h}$.

A physicist writes a vector in a coordinate system and takes into account how it transforms under coordinate transformations.

The most intuitive definition is the "geometrical" definition. Here one identifies tangent vectors in a point $p \in \mathcal{M}$ with velocity vectors of paths passing through p.

Definition 1.15 ("geometrical" tangent space) *Let W_p be the set of differentiable paths passing through p, i.e., $W_p = \{w : I \to \mathcal{U} \mid w(0) = p\}$, where $I \subset \mathbb{R}$ is an interval with $0 \in I$, $\mathcal{U} \subset \mathcal{M}$ is an open set containing p and w is a differentiable map. We consider two paths, w, $v \in W_p$ and define the following equivalence relation: $w \sim v$ if for every differentiable function $f \in \mathcal{F}(\mathcal{M})$*

$$\frac{\mathrm{d}}{\mathrm{d}t}(f \circ w)(0) = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ v)(0) \tag{1.24}$$

A tangent vector to $\mathcal{M}$ at the point p is an equivalence class, $[w]$. Thus we define $(T_p\mathcal{M})_{\mathrm{geo}} := W_p/\sim$.

Clearly if two functions f_1 and f_2 generate the same germ in p, i.e. $\overline{f}_1 = \overline{f}_2$, $\frac{d}{dt}(f_1 \circ w)(0) = \frac{d}{dt}(f_2 \circ w)(0)$. An equivalence class $[w]$ corresponds therefore uniquely to the derivation X_w of $\mathcal{F}_\mathcal{M}(p)$ defined by

$$X_w(\overline{f}) := \frac{\mathrm{d}}{\mathrm{d}t}(f \circ w)(0) \tag{1.25}$$

This correspondence is obviously injective:

$$(T_p\mathcal{M})_{\text{geo}} \equiv W_p/\sim \quad \to T_p\mathcal{M} : [w] \mapsto X_w \ . \tag{1.26}$$

But it is also surjective: for $X = \sum a^i \frac{\partial}{\partial x^i}$ (in a given local coordinate system) we choose $w = (ta^1, \ldots, ta^n)$ such that $X = X_w$. Hence $(T_p\mathcal{M})_{\text{geo}} \equiv W_p/\sim \ \cong T_p\mathcal{M}$.

The tangent map is also very intuitive in the geometric definition of the tangent space: a map $\varphi : \mathcal{N} \to \mathcal{M}$ with $\varphi(p) = q$ generates the mapping

$$(T_p\mathcal{N})_{\text{geo}} \to (T_q\mathcal{M})_{\text{geo}} : \ [w] \mapsto [\varphi \circ w] \ . \tag{1.27}$$

This is exactly the tangent map defined in 1.13: for $\overline{f} \in \mathcal{F}_\mathcal{M}(q)$,

$$X_{\varphi\circ w}(\overline{f}) = \frac{\mathrm{d}}{\mathrm{d}t}\left(f \circ (\varphi \circ w)\right)(0) = X_w(\overline{f \circ \varphi}) = X_w(\varphi^\star \overline{f}) = T_p\varphi(X_w)(\overline{f}) \ .$$

Hereby we have given three definitions of the tangent space of a differentiable manifold $\mathcal{M}$ in a point $p \in \mathcal{M}$, and we have shown that they are all equivalent. In the following, we shall apply them as we find convenient without distinction.

Definition 1.16 (tangent bundle) *The union* $\bigcup_{p\in\mathcal{M}} T_p\mathcal{M}$*, denoted* $T\mathcal{M}$*, is called the* **tangent bundle** *of* $\mathcal{M}$ *.*

$T\mathcal{M}$ is also a differentiable manifold (of dimension $2\dim\mathcal{M}$) with the differentiable structure induced by the one on $\mathcal{M}$: every differentiable mapping $\varphi : \mathcal{M} \to \mathcal{N}$ generates a differentiable mapping from $T\mathcal{M}$ into $T\mathcal{N}$ as follows:

$$T\varphi : T\mathcal{M} \to T\mathcal{N} : \ (p, X) \mapsto (\varphi(p), T_p\varphi X) \ . \tag{1.28}$$

Definition 1.17 (rank) *The* **rank** *of a differentiable map* $\varphi : \mathcal{N} \to \mathcal{M}$ *in a point* $p \in \mathcal{N}$ *is the rank of the tangent map at this point:*

$$\mathrm{rk}_p\varphi := Rank(T_p\varphi) \ . \tag{1.29}$$

Proposition 1.2 *An immersion is a map* $\varphi : \mathcal{M} \to \mathcal{N}$ *with* $\dim\mathcal{M} \leq \dim\mathcal{N}$ *whose rank is everywhere maximal, i.e.,* $\mathrm{rk}_p\varphi = \dim\mathcal{M}$ *for all* $p \in \mathcal{M}$*.*

Proof Choose coordinates around a given point p and apply the theorem of the rank for maps from $\mathbb{R}^m$ into $\mathbb{R}^n$. Show that the result does not depend on the chosen coordinates. □

1.2.2 Vector fields

Definition 1.18 (vector field) *A vector field is a map*

$$X: \mathcal{M} \to T\mathcal{M}: \; p \mapsto X_p \; \textit{with} \quad X_p \in T_p\mathcal{M} \; . \tag{1.30}$$

In the language of fibre bundles, a vector field is a **section** *of the fibre bundle $T\mathcal{M}$.*

Let $(x^1, \dots, x^m)$ be local coordinates on an open set $\mathcal{U} \subset \mathcal{M}$. For $p \in \mathcal{U}$, $X_p \in T_p\mathcal{M}$ can be given by

$$X_p = \xi^i(p) \cdot \frac{\partial}{\partial x^i} \; . \tag{1.31}$$

The functions ξ^i, $i = 1, \dots, m$ are the components of X in the coordinate system $(x^1, \dots, x^m)$. For an other coordinate system in $\mathcal{U}$, denoted $(\overline{x}^1, \dots, \overline{x}^m)$ we have (according the the physicist's definition)

$$X_p = \overline{\xi}^i(p)\frac{\partial}{\partial \overline{x}^i} \quad \text{where} \quad \overline{\xi}^i(p) = \frac{\partial \overline{x}^i}{\partial x^j}(p) \cdot \xi^j(p) \; . \tag{1.32}$$

We shall always denote coordinates with superscripts and hence also components of vector fields have superscripts.

Since all coordinate changes are $\mathcal{C}^\infty$ (thus also the Jacobian $\frac{\partial \overline{x}^i}{\partial x^j}$), the condition that the components ξ^i of X be $\mathcal{C}^r$ in p does not depend on the coordinate system. The following definition therefore make sense:

Definition 1.19 (vector field $\mathcal{C}^r$) *A vector field X is $\mathcal{C}^r$ in $p \in \mathcal{M}$ if its components $(\xi^i(p))$ are $\mathcal{C}^r$ functions in p. X is called $\mathcal{C}^r$ on $\mathcal{M}$ if it is $\mathcal{C}^r$ in every point $p \in \mathcal{M}$.*

In the following, we consider mainly $\mathcal{C}^\infty$ vector fields, and we denote them by $\mathcal{X}(\mathcal{M})$.

As before, $\mathcal{F}(\mathcal{M}) = \mathcal{C}^\infty(\mathcal{M})$ denotes the $\mathcal{C}^\infty$ function on $\mathcal{M}$.

For $X, Y \in \mathcal{X}(\mathcal{M})$ and $f \in \mathcal{F}(\mathcal{M})$, the maps

$$p \mapsto f(p)X(p) \text{ and } p \mapsto X(p) + Y(p) \tag{1.33}$$

define new vector fields on $\mathcal{M}$, which we denote by fX and $X+Y$. For $f, g \in \mathcal{F}(\mathcal{M})$ and $X, Y \in \mathcal{X}(\mathcal{M})$ we have

$$f(gX) = (f \cdot g)X, \quad f(X+Y) = fX + fY \quad \text{and} \quad (f+g)X = fX + gX \tag{1.34}$$

In algebraic language, $\mathcal{X}(\mathcal{M})$ is a module[7] on the algebra $\mathcal{F}(\mathcal{M})$.

Furthermore, we define the function Xf on $\mathcal{M}$ by application of the derivation X_p:

$$(Xf)(p) = X_p f, \quad \forall p \in \mathcal{M} \tag{1.35}$$

In local coordinates we have (see Eq. (1.31))

$$(Xf)(p) = \xi^i(p)\frac{\partial f}{\partial x^i}(p) \ . \tag{1.36}$$

Since ξ^i are $\mathcal{C}^\infty$ functions, also Xf is $\mathcal{C}^\infty$. Xf is the **derivation of f "in direction"** X.
The following identities hold:

$$X(f+g) = Xf + Xg \quad \text{(linearity)} \tag{1.37}$$

$$X(f \cdot g) = (Xf)\cdot g + f \cdot Xg \quad \text{(Leibniz rule (1.9))} \ . \tag{1.38}$$

The map $D_X f := Xf$ is therefore a **derivation** on $\mathcal{F}(\mathcal{M})$ in the sense of the definition 1.12.
On the other hand, the algebraic definition of tangent space tells us that every derivation of $\mathcal{F}(\mathcal{M})$ is of the form D_X for a certain $X \in \mathcal{X}(\mathcal{M})$.
For two derivations D_1 and D_2 of an algebra $\mathcal{A}$, the commutator $[D_1, D_2]$ is also a derivation. Indeed, be $a \in \mathcal{A}$. $[D_1, D_2]a := D_1(D_2(a)) - D_2(D_1(a))$. $[D_1, D_2]$ is obviously linear, as D_1 and D_2 are, and

$$\begin{aligned}
[D_1, D_2](ab) &\equiv D_1(D_2(ab)) - D_2(D_1(ab)) \\
&= D_1((D_2 a)b + a(D_2 b)) - D_2((D_1 a)b + a(D_1 b)) \\
&= (D_1(D_2 a))b + (D_2 a)(D_1 b) + (D_1 a)(D_2 b) + aD_1(D_2 b) \\
&\quad -(D_2(D_1 a))b - (D_1 a)(D_2 b) - (D_2 a)(D_1 b) - aD_2(D_1 b) \\
&= ([D_1, D_2]a)b + a([D_1, D_2]b);
\end{aligned}$$

hence the commutator obeys the Leibniz rule. Furthermore, the Jacobi identity is satisfied (exercise!), i.e.,

$$[D_1,[D_2,D_3]] + [D_3,[D_1,D_2]] + [D_2,[D_3,D_1]] = 0 \ . \tag{1.39}$$

The composition $D_1 \circ D_2$ is, however, not a derivation. According to the remarks, for two vector fields X and Y there exists a vector field $Z := [X, Y]$ defined by

$$Zf = [X,Y]f \equiv X(Y(f)) - Y(X(f)) = D_X(D_Y f)) - D_Y(D_X f)) \ . \tag{1.40}$$

[7] Let A be a ring. A set M is a module over A if M is an additive group with a map $A \times M \to M$ which assigns to each couple (x, v), with $x \in A$ and $v \in M$, an element xv of M such that the following properties are satisfied: 1) if e is the identity of A, $ev = v$, $\forall v \in M$; 2) $x(v + w) = xv + xw$, for $v, w \in M$; 3) $(x + y)v = xv + yv$, for $x, y \in A$; 4) $(xy)v = x(yv)$.

In local coordinates with $X = \xi^i \frac{\partial}{\partial x^i}$ and $Y = \eta^i \frac{\partial}{\partial x^i}$ one finds (exercise!)

$$[X, Y] = \left(\xi^i \frac{\partial \eta^j}{\partial x^i} - \eta^i \frac{\partial \xi^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} \ . \tag{1.41}$$

Applying the definition, it is easy to see that for vector fields X, Y and Z and functions f and g the following identities are satisfied:

1. $[X + Y, Z] = [X, Z] + [Y, Z]$
2. $[X, Y] = -[Y, X]$
3. $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$
4. $[aX, Y] = a[X, Y], \quad \text{where} \quad a \in \mathbb{R}$
5. $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X.$

The properties 1. to 4. tell us that $\mathcal{X}(\mathcal{M})$ together with the commutator product is a real Lie algebra.

1.2.3 Tensor fields

As we have seen, the tangent space $T_p\mathcal{M}$ forms a real vector space of dimension $m = \dim \mathcal{M}$.

Definition 1.20 (cotangent space) *The dual*[8] *the tangent in* p, $T_p^\star \mathcal{M}$, *is called* **cotangent space** *to* $\mathcal{M}$ *in the point* p. *The union* $\bigcup_{p \in \mathcal{M}} T_p^\star \mathcal{M} := T^\star \mathcal{M}$ *is* **the cotangent bundle** *of* $\mathcal{M}$.

Definition 1.21 (differential of a function) *Let* f *be a differentiable function on an open set* $\mathcal{U} \subset \mathcal{M}$ *and let* $p \in \mathcal{U}$ *be an arbitrary point in* $\mathcal{U}$. *For* $v \in T_p\mathcal{M}$ *we define*

$$(df)_p(v) := v(f) \tag{1.42}$$

The mapping $(df)_p : T_p\mathcal{M} \to \mathbb{R}$ *is obviously linear, therefore* $(df)_p \in T_p^\star \mathcal{M}$. *The map* (df) *is called the* **differential (or the gradient)** *of* f *in* p.

[8] Let V be a vector space of dimension n. Its dual $V^\star$ is the space of all linear maps from V to $\mathbb{R}$ (for $u^\star \in V^\star$, $v, w \in V$ and $\alpha, \beta \in \mathbb{R}$, $u^\star(\alpha v + \beta w) = \alpha u^\star(v) + \beta u^\star(w)$). $V^\star$ is also a vector space of dimension n. For every basis $(e_i)_{i=1}^n \subset V$ there exist the dual basis $(e^{\star\, i})_{i=1}^n \subset V^\star$ defined by $e^{\star\, j}(e_i) = \delta_i^{\ j}$, $\forall i, j$.

In a local coordinate system $(x^1, \ldots, x^m)$ and for $v = \frac{\partial}{\partial x^i}$ we have

$$(df)_p \left(\frac{\partial}{\partial x^i} \right) = \frac{\partial f}{\partial x^i} \ . \tag{1.43}$$

In particular, for the component functions $x^i : p \mapsto x^i(p)$ we have

$$(dx^i)_p \left(\frac{\partial}{\partial x^j} \right) = \delta^i_j \ . \tag{1.44}$$

Hence $(dx^i)_{i=1}^m$ is the dual basis of $\left(\frac{\partial}{\partial x^i}\right)_{i=1}^m$.
Furthermore, according to (1.43) we have

$$(df)_p = \frac{\partial f}{\partial x^i}(p) dx^i \ . \tag{1.45}$$

A co-vector, often called a '1-form', is written in coordinates as

$$\omega = \omega_i dx^i$$

we shall always write the components of co-vectors as subscript.

Definition 1.22 (tensor bundle, tensor field)

- *The space* $(T_p\mathcal{M})^r_s$ *is* **the space of tensors** *of type* (r, s) *on* $T_p\mathcal{M}$ *(*r*-times contravariant and* s*-times covariants). In other words* $(T_p\mathcal{M})^r_s$ *is the space of linear mappings*[9]

$$t_p \ : \ \underbrace{T^\star_p\mathcal{M} \otimes \cdots \otimes T^\star_p\mathcal{M}}_{r \ times} \otimes \underbrace{T_p\mathcal{M} \otimes \cdots \otimes T_p\mathcal{M}}_{s \ times} \to \mathbb{R} \ . \tag{1.46}$$

In particular $(T_p\mathcal{M})^1_0 \equiv T_p\mathcal{M}$*,* $(T_p\mathcal{M})^0_1 \equiv T^\star_p\mathcal{M}$*.*

- *The union* $\displaystyle\bigcup_{p\in\mathcal{M}} (T_p\mathcal{M})^r_s := (T\mathcal{M})^r_s$ *is the* **tensor bundle** *of type* (r, s) *on* $\mathcal{M}$.

- *A* **tensor field** *of type* (r, s) *is a map*

$$t : \mathcal{M} \to (T\mathcal{M})^r_s : p \mapsto t_p \in (T_p\mathcal{M})^r_s \ . \tag{1.47}$$

[9]Here we make use of the fact that the vector space V can be identified with $(V^\star)^\star$ via the identification of $v \in V$ with the map $v^{\star\star} : V^\star \to \mathbb{R} : \alpha \mapsto \alpha(v)$. Hence an element of V can be understood as linear map from $V^\star$ to $\mathbb{R}$, i.e. as element of $(V^\star)^\star$.

Algebraic operations of tensor fields of the same type are defined point by point. For example for t, u of type (r, s), the addition is defined by $(t+u)_p := t_p + u_p$. Equivalently for the multiplication with a function $f \in \mathcal{F}(\mathcal{M})$, we define $(ft)_p := f(p)t_p$.

In an open set $\mathcal{U} \subset \mathcal{M}$ with local coordinates $(x^1, \ldots, x^m)$ we can represent the tensors of rank (r, s) in the dual bases $\left(\frac{\partial}{\partial x^i}\right)$ and (dx^i):

$$t = t^{i_1 \cdots i_r}_{j_1 \cdots j_s} \left(\frac{\partial}{\partial x^{i_1}}\right) \otimes \cdots \otimes \left(\frac{\partial}{\partial x^{i_r}}\right) \otimes (dx^{j_1}) \otimes \cdots \otimes (dx^{j_s}) \ . \tag{1.48}$$

The functions $\left(t^{i_i \cdots i_r}_{j_1 \cdots j_s}\right)$ are called the **components** of t in the coordinates $(x^1, \ldots, x^m)$. If the coordinates $(x^1, \ldots, x^m)$ are changed to $(\overline{x}^1, \ldots, \overline{x}^m)$ the components of t transform according to

$$\overline{t}^{i_1 \cdots i_r}_{j_1 \cdots j_s} = \frac{\partial \overline{x}^{i_1}}{\partial x^{k_1}} \cdots \frac{\partial \overline{x}^{i_r}}{\partial x^{k_r}} \frac{\partial x^{l_1}}{\partial \overline{x}^{j_1}} \cdots \frac{\partial x^{l_s}}{\partial \overline{x}^{j_s}} t^{k_1 \cdots k_r}_{l_1 \cdots l_s} \ . \tag{1.49}$$

The property that the functions $t^{i_1 \cdots i_r}_{j_1 \cdots j_s}$ be $\mathcal{C}^k$ is therefore independent of the coordinate system (coordinate changes are $\mathcal{C}^\infty$). A tensor field t is of class $\mathcal{C}^k$ if the components of t are $\mathcal{C}^k$. In the following we consider mainly $\mathcal{C}^\infty$ fields. They are denoted by $\mathcal{T}^r_s(\mathcal{M})$. The spaces of vector and co-vector fields are $\mathcal{T}^1_0(\mathcal{M}) = \mathcal{X}(\mathcal{M})$, and $\mathcal{T}^0_1(\mathcal{M}) = \mathcal{X}^\star(\mathcal{M})$.
For $t \in \mathcal{T}^r_s(\mathcal{M})$, $X_1, \ldots, X_s \in \mathcal{X}(\mathcal{M})$ and $\omega^1, \ldots, \omega^r \in \mathcal{X}^\star(\mathcal{M})$ we can define $F \in \mathcal{F}(\mathcal{M})$ by

$$F(p) := t_p(\omega^1(p), \ldots, \omega^r(p), X_1(p), \ldots, X_s(p)) \ . \tag{1.50}$$

Hence, t defines a multi-linear map into $\mathcal{F}(\mathcal{M})$:

$$\begin{array}{l} \overbrace{\mathcal{X}^\star(\mathcal{M}) \otimes \cdots \otimes \mathcal{X}^\star(\mathcal{M})}^{r \text{ fois}} \otimes \overbrace{\mathcal{X}(\mathcal{M}) \otimes \cdots \otimes \mathcal{X}(\mathcal{M})}^{s \text{ fois}} \to \mathcal{F}(\mathcal{M}) : \\ (\omega^1, \ldots, \omega^r, X_1, \ldots, X_s) \mapsto t(\omega^1, \ldots, \omega^r, X_1, \ldots, X_s) \,, \\ t(\omega^1, \ldots, \omega^r, X_1, \ldots, X_s)(p) = t_p(\omega^1(p), \ldots, \omega^r(p), X_1(p), \ldots, X_s(p)) \,. \end{array} \tag{1.51}$$

Definition 1.23 (pullback) *Let $\varphi : \mathcal{M} \to \mathcal{N}$ be a differentiable map and $\omega \in \mathcal{X}^\star(\mathcal{N})$ a 1-form. The* **pullback** *$\varphi^\star\omega \in \mathcal{X}^\star(\mathcal{M})$ is defined by*

$$(\varphi^\star\omega)_p(X_p) := \omega_{\varphi(p)}(T_p\varphi X_p), \quad \textit{with} \;\; X_p \in T_p\mathcal{M} \,, \; \forall \, p \in \mathcal{M} \,. \tag{1.52}$$

Analogously for a tensor field $t \in \mathcal{T}^0_s(\mathcal{N})$ and $v_i \in T_p\mathcal{M}$, we define the **pullback** *$\varphi^\star t$ by*

$$(\varphi^\star t)_p(v_1, \ldots, v_s) := t_{\varphi(p)}(T_p\varphi v_1, \ldots, T_p\varphi v_s). \tag{1.53}$$

For a function $f \in \mathcal{F}(\mathcal{N})$ we have for $v \in T_p\mathcal{M}$

$$(\varphi^\star df)_p(v) = (df)_{\varphi(p)}(T_p\varphi v) = T_p\varphi(v)f = v(f \circ \varphi) = v(\varphi^\star f) = d(\varphi^\star f)(v)\ . \quad (1.54)$$

Hence the pullback "commutes" with d.

Definition 1.24 (pseudo-Riemannian metric) *A* **pseudo-Riemannian metric** *on a differentiable manifold is a tensor field $g \in \mathcal{T}^0_2(\mathcal{M})$ with the following properties:*

- $g(X, Y) = g(Y, X)$
- *At every point $p \in \mathcal{M}$, g_p is a non-degenerate*[10] *bilinear form*

If g_p is positive definite (i.e. all its eigenvalues are positives), it is called **Riemannian metric**. *The couple $(\mathcal{M}, g)$ is a* **(pseudo)-Riemannian manifold**. *The pair (n_+, n_-) which denotes the number of positive and negative eigenvalues of the metric is called the 'signature' of g.*

Let $(\theta^i)_{i=1}^m$, $m = \dim \mathcal{M}$ be a basis of 1-forms in an open set $\mathcal{U} \subset \mathcal{M}$. In $\mathcal{U}$ we have

$$g = g_{ij}\theta^i \otimes \theta^j\ . \quad (1.55)$$

We often also write

$$ds^2 = g_{ij}\theta^i\theta^j, \quad \text{where} \quad \theta^i\theta^j := \frac{1}{2}[\theta^i \otimes \theta^j + \theta^j \otimes \theta^i]\ . \quad (1.56)$$

Denoting the dual basis of (θ^i) by (e_i) we obtain

$$g_{ij} = g(e_i, e_j)\ . \quad (1.57)$$

Remark 1.5 *With the metric g of a pseudo-Riemannian manifold, we have an isomorphism between $T_p\mathcal{M}$ and $T_p^\star\mathcal{M}$ given by*

$$\flat : T_p\mathcal{M} \to T_p^\star\mathcal{M} : \ X_p \mapsto g(p)(X_p, \cdot) =: X_p^\flat\ . \quad (1.58)$$

Since g is not degenerate, every 1-form $\omega_p \in T_p^\star\mathcal{M}$ is of the form $\omega_p = g(p)(\omega_p^\sharp, \cdot)$, for some vector field $\omega_p^\sharp$. Hence the inverse

$$\sharp : T_p^\star\mathcal{M} \to T_p\mathcal{M} : \ \omega_p \mapsto \omega_p^\sharp \quad (1.59)$$

[10] I.e., $g_p(v_1, v_2) = 0\, \forall v_2 \in T_p$ if and only if $v_1 = 0$.

is also well defined.

In a local coordinate system with $g = g_{ij}dx^i dx^j$ and $(g^{ij}) := (g_{ij})^{-1}$, if $X = \xi^i\partial_i$ we have $X^\flat = \xi^i g_{ij}dx^j = \xi_j dx^j$, where $\xi_j \equiv \xi^i g_{ij}$ and for $\omega = \eta_i dx^i$, $\omega^\sharp = \eta_i g^{ij}\partial_j = \eta^j\partial_j$, where $\eta^j \equiv \eta_i g^{ij}$. The mapping $\flat$ is called "lowering an index", and the mapping $\sharp$ "raising an indice", therefore the signs "$\flat$" and "$\sharp$".

For an arbitrary tensor field, for example $t \in \mathcal{T}_1^1(\mathcal{M})$ we can also apply $\flat$ and $\sharp$. We define $t^\flat \in \mathcal{T}_2^0(\mathcal{M})$ by $t^\flat(X,Y) = t(X,Y^\flat)$. And analoguously for a tensor field of arbitrary type $t \in \mathcal{T}_s^r(\mathcal{M})$.

This implies that in a (pseudo-)Riemannian manifold $(\mathcal{M}, g)$ the operation of "raising" and "lowering" indices provides isomorphisms

$$\mathcal{T}^{s+r}(\mathcal{M}) \leftrightarrow \mathcal{T}_s^r(\mathcal{M}) \leftrightarrow \mathcal{T}_{s-1}^{r+1}(\mathcal{M}) \leftrightarrow \mathcal{T}_{s+1}^{r-1}(\mathcal{M}) \leftrightarrow \mathcal{T}_{s+r}(\mathcal{M}) \tag{1.60}$$

Exercice: Show that in a given coordinates system we have

$$(g)^{\sharp\sharp} = g^{ij}\partial_i \otimes \partial_j, \quad \text{and that } (g)^\sharp = \delta_i^{\ j} dx^i \otimes \partial_j \tag{1.61}$$

where (g^{ij}) is the inverse of (g_{ij}).

Chapter 2

The Lie derivative

(see Kobayashi and Nomizu [12], pp 26-34 or Matsushima [13], pp 139-145)

2.1 Integral curves, the flow of a vector field

Consider $X \in \mathcal{X}(\mathcal{M})$ (i.e. $\mathcal{C}^\infty$).

Definition 2.1 (Integral curve) *A differentiable path $\gamma : J \to \mathcal{M}$, where $J \subset \mathbb{R}$ is a open interval such that $0 \in J$, is called* **integral curve** *of X with initial point $p \in \mathcal{M}$ if (we use the geometric definition of a vector)*

$$\dot{\gamma}(t) = X_{\gamma(t)}, \quad \forall t \in J \quad \textit{and} \quad \gamma(0) = p \ .$$

Proposition 2.1 *For any point $p \in \mathcal{M}$ and any vector field $X \in \mathcal{X}(\mathcal{M})$ there exists an unique* **maximal** *integral curve with initial point p, of class $\mathcal{C}^\infty$. We call it $\Phi_p : J_p \to \mathcal{M} : t \mapsto \Phi_p(t)$. $J_p \subset \mathbb{R}$ is an interval and $p = \Phi_p(0)$.*

Proof:
In a local coordinate system this proposition follows from the same theorem for a system of ordinary differential equations on $\mathbb{R}$, and coordinate changes do not pose any real difficulty. $\square$

We set

$$\mathcal{D} := \{(t,p) \mid p \in \mathcal{M},\, t \in J_p\} \subset \mathbb{R} \times \mathcal{M} \tag{2.1}$$

and, in the same way

$$\mathcal{D}_t := \{p \in \mathcal{M} \mid (t,p) \in \mathcal{D}\} = \{p \in \mathcal{M} \mid t \in J_p\} \subset \mathcal{M} \ . \tag{2.2}$$

Definition 2.2 (complete vector field) *The vector field X is called* **complete** *if $\mathcal{D} = \mathbb{R} \times \mathcal{M}$.*

Definition 2.3 (global flow) *The mapping*

$$\Phi : \ \mathcal{D} \to \mathcal{M} \ : \ (t,p) \mapsto \Phi_p(t) \equiv \Phi(t,p)$$

is the **global flow** *of X.*

Proposition 2.2

1. *$\mathcal{D} \subset \mathbb{R} \times \mathcal{M}$ is open and $\{0\} \times \mathcal{M} \subset \mathcal{D}$ ($\Rightarrow \mathcal{D}_t$ is open in $\mathcal{M}$)*
2. *$\Phi : \mathcal{D} \to \mathcal{M}$ is $\mathcal{C}^\infty$*
3. *For all $t \in \mathbb{R}$, the map*

$$\Phi_t : \ \mathcal{D}_t \to \mathcal{M} \ : \ p \mapsto \Phi(t,p) = \Phi_p(t) \equiv \Phi_t(p)$$

is a diffeomorphism from $\mathcal{D}_t$ to $\mathcal{D}_{-t}$ and $(\Phi_t)^{-1} = \Phi_{-t}$

Proof: This and the following are a simple consequence of the corresponding proposition for flows on open sets in $\mathbb{R}^m$. Just express it in local coordinates.

Consequence 2.3 *For all $p \in \mathcal{M}$, there exists an open interval $J \subset \mathbb{R}$ and an open set $\mathcal{U} \subset \mathcal{M}$, $\mathcal{U} \ni p$, such that $J \times \mathcal{U} \subset \mathcal{D}$.*

Definition 2.4 (local flow) *The map*

$$\Psi := \Phi\Big|_{J\times\mathcal{U}} : \ J \times \mathcal{U} \to \mathcal{M} \ : \ (t,p) \mapsto \Phi(t,p)$$

is called the **local flow** *of X in p.*

Proposition 2.4 *For a local flow $\Psi : J \times \mathcal{U} \to \mathcal{M}$ of X in p the following holds:*

1. *For $q \in \mathcal{U}$, $\Psi_q : J \to \mathcal{M} : t \mapsto \Psi(t,q)$ is an integral curve of X with initial point q.*
2. *For $t \in J$, $\Psi_t : \mathcal{U} \mapsto \mathcal{M} : q \mapsto \Psi(t,q)$ is a diffeomorphism from $\mathcal{U}$ to $\Psi_t(\mathcal{U})$.*
3. *For $q \in \mathcal{U}$ with $\Psi_t(q) \in \mathcal{U}$, we have (for $(s,t) \in J$)*

$$\Psi_{s+t}(q) = \Psi_s(\Psi_t(q)), \quad \Psi_s \circ \Psi_t = \Psi_{s+t} \ .$$

As a consequence of point 3, one often calls Ψ a **one parameter group of local diffeomorphisms**.

Proposition 2.5 *Let $X \in \mathcal{X}(\mathcal{M})$ be complet and $\Phi : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ the global flow of X. For any $t \in \mathbb{R}$, then $\mathcal{D}_t = \mathcal{M}$ and we have*

1. *$\Phi_t : \mathcal{M} \to \mathcal{M}$ is a diffeomorphism*
2. *$\Phi_s \circ \Phi_t = \Phi_{s+t}$*

Proof:
This is a consequence of the propositions 2.2 and 2.4. □

Proposition 2.5 implies that for complete vector fields the map $t \mapsto \Phi_t$ is a homomorphism of the additive group $\mathbb{R}$ into diff($\mathcal{M}$), the group of diffeomorphisms of $\mathcal{M}$.
One can show that all vector fields on compact manifolds are complete.
For the explicit proofs of propositions 2.1 to 2.5, see for example Abraham & Marsden, *Foundations of classical Mechanics* [2], or Arnold, *Ordinary Differential equations* [3], or any other text book on dynamical systems.

2.2 Induced maps on tensor fields

Proposition 2.6 *Let $\varphi : \mathcal{M} \to \mathcal{N}$ be a differentiable map. The pullback (see definition 1.23) defines a linear map*

$$\varphi^\star : \bigoplus_{k=0}^{\infty} \mathcal{T}_k^0(\mathcal{N}) \to \bigoplus_{k=0}^{\infty} \mathcal{T}_k^0(\mathcal{M}) \tag{2.3}$$

$\varphi^\star$ is even an algebra homomorphism (with the tensor product).

For $p \in \mathcal{M}$, $t \in \mathcal{T}_k^0(\mathcal{N})$, $u_1, \ldots, u_k \in T_p\mathcal{M}$, we have $(\varphi^\star t)_p(u_1, \ldots, u_k) := t_{\varphi(p)}(T_p\varphi u_1, \ldots, T_p\varphi u_k)$.
Proof:
This is a simple consequence of the definition of pullback.

For $\omega \in \mathcal{T}_1^0(\mathcal{N}) = \mathcal{X}^\star(\mathcal{N})$ we write

$$(\varphi^\star\omega)_p = (T_p\varphi)^\star \omega_{\varphi(p)} \ . \tag{2.4}$$

$(T_p\varphi)^\star : T^\star_{\varphi(p)}\mathcal{N} \to T^\star_p\mathcal{M}$ this is the dual of the linear map $T_p\varphi : T_p\mathcal{M} \to T_{\varphi(p)}\mathcal{N}$. In a coordinate system it is given by the transposed of the Jacobian.

Definition 2.5 (related vector fields) *Let $\varphi : \mathcal{M} \to \mathcal{N}$ be a differentiable map. The vector fields $X \in \mathcal{X}(\mathcal{M})$ and $Y \in \mathcal{X}(\mathcal{N})$ are called* **related by** φ *if*

$$T_p\varphi X_p = Y_{\varphi(p)}, \quad \forall p \in \mathcal{M} \ .$$

Proposition 2.7 *Consider $X_1, X_2 \in \mathcal{X}(\mathcal{M})$ and $Y_1, Y_2, \in \mathcal{X}(\mathcal{N})$. If X_i, Y_i are related by φ, $i = 1, 2$, the vector field $[X_1, X_2]$ is also related by φ to the field $[Y_1, Y_2]$.*

Proof: Exercise!

Proposition 2.8 *We consider $t \in \mathcal{T}_k^0(\mathcal{N})$, $X_i \in \mathcal{X}(\mathcal{M})$, $Y_i \in \mathcal{X}(\mathcal{N})$, for $i = 1, \ldots, k$, with X_i related by φ to Y_i. Then*

$$(\varphi^\star t)_p(X_{1p}, \ldots, X_{kp}) = t_{\varphi(p)}(Y_{1\varphi(p)}, \ldots, Y_{k\varphi(p)}) \ .$$

Proof:

$$(\varphi^\star t)_p(X_1, \ldots, X_k) = t_{\varphi(p)}(T_p\varphi X_1, \ldots, T_p\varphi X_k)$$

Since the Y_i are related by φ to X_1, this gives

$$(\varphi^\star t)_p(X_{1p}, \ldots, X_{kp}) = t_{\varphi(p)}(Y_{1\varphi(p)}, \ldots, Y_{k\varphi(p)}) \ .$$

□

We consider, in particular, the case where $\varphi : \mathcal{M} \to \mathcal{N}$ is a diffeomorphism.
Let us first make the following remark from linear algebra:

Remark 2.1 *Let E and F be two real vector spaces and ${E_s}^r$, ${F_s}^r$ the vector spaces of tensors of type (r, s) on E respectively F. Consider an* **isomorphism** $A : E \to F$. *This induces an isomorphism*

$${A_s}^r : {E_s}^r \to {F_s}^r$$

defined as follows
For $y_1^\star, \ldots, y_r^\star \in F^\star$, $y_1, \ldots, y_s \in F$ and $t \in {E_s}^r$ we define:

$$({A_s}^r t)(y_1^\star, \ldots, y_r^\star, y_1, \ldots, y_s) = t(A^\star y_1^\star, \ldots, A^\star y_r^\star, A^{-1}y_1, \ldots, A^{-1}y_s) \tag{2.5}$$

It is evident that $A_0^1 \equiv A$ and $A_1^0 \equiv (A^{-1})^\star$. Furthermore, we pose $A_0^0 : \mathbb{R} \to \mathbb{R} : \lambda \mapsto \lambda$.
Here $A^\star : F^\star \to E^\star$ is defined by $A^\star y^\star(x) = y^\star(Ax)$ for all $x \in E$ and $y^\star \in F^\star$.

Definition 2.6 (push-forward and pull-back) *Let $\varphi : \mathcal{M} \to \mathcal{N}$ be a diffeomorphism. We define the two maps $\varphi^\star$ and $\varphi_\star$ (which are mutually inverse) by*

$$\varphi_\star : \mathcal{T}_s^r(\mathcal{M}) \to \mathcal{T}_s^r(\mathcal{N}) \quad \textbf{(push-forward)}$$

$$\varphi^\star : \mathcal{T}_s^r(\mathcal{N}) \to \mathcal{T}_s^r(\mathcal{M}) \quad \textbf{(pull-back)}$$

by (note that ${T_s}^r\varphi \equiv (T_p\varphi)_s^{\ r}$)

$$(\varphi_\star t)_{\varphi(p)} = (T_p\varphi)_s^{\ r} t_p \ , \quad \textit{for } t \in \mathcal{T}_s^r(\mathcal{M})$$

$$\textit{and} \quad (\varphi^\star t)_p = (T_{\varphi(p)}\varphi^{-1})_s^{\ r}(t_{\varphi(p)}) \ , \quad \textit{for } t \in \mathcal{T}_s^r(\mathcal{N}).$$

For $t \in \mathcal{T}_s^0(\mathcal{N})$, $\varphi^\star$ is simply our old pullback from definition 1.23. The push-forward on $\mathcal{T}_0^r(\mathcal{M})$ is well defined also if φ is not a diffeomorphism.
We consider also diffeomorphisms $\varphi : \mathcal{M} \to \mathcal{M}$. If for a tensor field t, $\varphi^\star t = t$, we call t **invariant under** φ.
If there is an entire one parameter group Φ_s with $\Phi_s^\star t = t$, for all $s \in \mathbb{R}$, t is called **invariant under the group of transformations** $(\Phi_s)_{s\in\mathbb{R}}$.
Exercise: Write $\varphi_\star t$ and $\varphi^\star w$ in a local coordinate system.

2.3 Lie derivative

Definition 2.7 (Lie derivative) *Let X be in $\mathcal{X}(\mathcal{M})$ and Φ_t be the flow of X. We set $\mathcal{T}(\mathcal{M}) := \bigoplus_{s,r\geq 0} \mathcal{T}_s^r(\mathcal{M})$. For $T \in \mathcal{T}(\mathcal{M})$ we define the* **Lie derivative** *of T in direction X by*

$$L_X T := \frac{\mathrm{d}}{\mathrm{d}t}\Phi_t^\star T\Big|_{t=0} = \lim_{t\to 0}\frac{1}{t}[\Phi_t^\star T - T] \ .$$

Theorem 2.1 *$L_X : \mathcal{T}(\mathcal{M}) \to \mathcal{T}(\mathcal{M})$ has the following properties:*

1. *L_X is $\mathbb{R}$-linear,*
2. *$L_X(T \otimes S) = (L_X T) \otimes S + T \otimes (L_X S)$ (Leibnitz rule),*
3. *$L_X(\mathcal{T}_s^r(\mathcal{M})) \subset \mathcal{T}_s^r(\mathcal{M})$,*
4. *L_X commutes with contractions*[1]*,*

[1]For $t \in {E_s}^r$, $s, r \geq 1$, $(e_i)_{i=1}^n$ a basis of E and $({e^\star}^i)_{i=1}^n$ its dual basis (i.e. ${e^\star}^i(e_j) = \delta_j^{\ i}$), the contraction on the indices 1 and $r+1$ is the map $C : {E_s}^r \to E_{s-1}^{r-1}$; $t \mapsto \sum_i t(e^{\star i}, \ldots, e_i, \ldots)$. Here the $\ldots$ are taken $r-1$ and $s-1$ times. This definition is independent of the basis (e_i) and is denoted by C.

5. $L_X f \stackrel{\text{geo}}{=\!=} Xf = df(X)$ *for* $f \in \mathcal{F}(\mathcal{M}) = \mathcal{T}_0^0(\mathcal{M})$,

6. $L_X Y = [X, Y]$ *for* $Y \in \mathcal{X}(\mathcal{M})$,

7. $L_{X+Y} = L_X + L_Y$, $L_{\lambda X} = \lambda L_X$, *for* $\lambda \in \mathbb{R}$,

8. $L_{[X,Y]} = [L_X, L_Y] := L_X \circ L_Y - L_Y \circ L_X$,

9. $L_X df = d(L_X f)$ *on the functions* $f \in \mathcal{F}(\mathcal{M})$.

Proof: The points 1 to 5 are immediate consequences of the definition. For 9 we use that

$$\frac{1}{t}(\Phi_t^\star df - df) = \frac{1}{t}(d\Phi_t^\star f - df) = d\left[\frac{1}{t}(\Phi_t^\star f - f)\right] .$$

Such that

$$L_X df = \lim_{t\to 0}\frac{1}{t}(\Phi_t^\star df - df) = d\left[\lim_{t\to 0}\frac{1}{t}(\Phi_t^\star f - f)\right] = d(Xf) = dL_X f .$$

To show 6 we use that $Yf = C(Y \otimes df)$:

$$\begin{aligned} X(Yf)) \stackrel{\text{def geo}}{=\!=} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Phi_t^\star(Yf) &= L_X(C(Y\otimes df)) = C(L_X Y \otimes df) + C(Y \otimes L_X df) \\ &= (L_X Y)(f) + Y(Xf) \end{aligned}$$

Hence $(L_X Y)(f) = X(Yf) - Y(Xf) = [X,Y]f = L_{[X,Y]}f$, which shows point 6.
Point 7 follows easily for vector fields because of 6 and on functions because of 5. With $L_X df = d(Xf)$ it is also verified on 1-forms of the form df. But since every 1-form $\omega \in \mathcal{X}^\star(\mathcal{M})$ is a linear combination $\lambda_i df^i$ it is true for all $\omega \in \mathcal{X}^\star(\mathcal{M})$. Because of point 2, 7 is hence valid on all $\mathcal{T}(\mathcal{M})$.
Point 8 is clear on functions and it is a consequence de the Jacobi identitiy on vector fields. Since $L_X df = dL_X f$ it is also valid on 1-forms of the form df and by the above argument therefore on all 1-forms. Again, since $t \in \mathcal{T}_s^r(\mathcal{M})$, $t = \sum X_1 \otimes \cdots \otimes X_r \otimes \omega^1 \otimes \cdots \otimes \omega^s$ using 2 we find that 8 is also valid on all $\mathcal{T}(\mathcal{M})$. □

In the proof of points 7 and 8 we have used the generic property, that two derivations D_1, D_2 on $\mathcal{T}(\mathcal{M})$ which commute with d and which coïncide on the functions and on the vector fields, agree on all of $\mathcal{T}(\mathcal{M})$.

Proposition 2.9 *We consider* $X, Y \in \mathcal{X}(\mathcal{M})$ *and we denote the flow of* X *by* Φ *and the flow of* Y *by* Ψ. *The following statements are equivalent:*

1. $[X, Y] = 0$

2. $L_X \circ L_Y = L_Y \circ L_X$

3. $\Phi_s \circ \Psi_t = \Psi_t \circ \Phi_s$ *for all* s, t *such that both sides are well defined.*

Proof: The equivalence of points 1 and 2 follows from identity 8 of theorem 2.1. Furthermore, if point 3 is satisfied, 2 follows from the definition of the Lie derivative 2.7.

To show that if $[X, Y] = 0$, point 3 is verified, we show a slightly more general statement:

Proposition 2.10 *For a diffeomorphism* ϕ *and* $Y \in \mathcal{X}(\mathcal{M})$ *with local flow* Ψ_s*, the 1-parameter group of (local) diffeomorphsims* $\phi \circ \Psi_s \circ \phi^{-1}$ *is the (local) flow of* $\phi_* Y$*.*

Proof: Obviously $\phi \circ \Psi_s \circ \phi^{-1}$ is a 1-parameter group of (local) diffeomorphsims. For $p \in \mathcal{M}$ with $q = \phi^{-1}(p)$. the vector $Y_q \in T_q\mathcal{M}$ is tangent to the curve $\gamma(s) = \Psi_s(q)$ at $q = \gamma(0)$. Hence $(\phi_* Y)_p = (T_q\phi) Y_q \in T_p\mathcal{M}$ is tangent to the curve $\tilde{\gamma}_s = \phi \circ \gamma_s$ at $q = \phi^{-1}(p)$ hence to $\phi \circ \gamma_s \circ \phi^{-1}$ at p. □

Let us now assume $[X, Y] = 0$, hence $\frac{d}{dt}\Phi_{t*}Y = 0$. Therefore $\Phi_{t*}Y$ is constant, i.e. Y is invariant under Φ_t (in its domain of definition). But according to the above proposition, this implies that $\Phi_t \circ \Psi_s \circ \Phi_t^{-1} = \Psi_s$, hence the flux Ψ_s commutes with Φ_t. □

Proposition 2.11 *Let* Φ_t *be the flow of the vector field* X *and* $T \in \mathcal{T}(\mathcal{M})$*. Hence*

$$\Phi_t^\star T = T \quad \textit{is equivalent to} \quad L_X T = 0.$$

Proof:

- " $\Rightarrow$ " is trivial (see definition 2.7)
- " $\Leftarrow$ ": for this direction we use the following lemma:

Lemma 2.1 *For* $T \in \mathcal{T}(\mathcal{M})$*, and* $X \in \mathcal{X}(\mathcal{M})$ *with flow* Φ_s *we have*

$$\Phi_s^\star(L_X T) = \frac{\mathrm{d}}{\mathrm{d}t}(\Phi_t^\star T)\Big|_{t=s} .$$

Proof: By definition

$$\Phi_s^\star L_X T = \Phi_s^\star\left(\lim_{\epsilon \to 0} \frac{1}{\epsilon}(\Phi_\epsilon^\star T - T)\right)$$

$$\overset{\Phi_s^\star \text{ continu}}{=\!=\!=} \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left(\Phi_{s+\epsilon}^\star T - \Phi_s^\star T\right) = \frac{\mathrm{d}}{\mathrm{d}t}\Phi_t^\star T\Big|_{t=s} .$$

Therefore if $L_X T = 0$,it follows that $\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t^\star T\Big|_{t=s} = 0$ for all s, and hence $\Phi_t^\star T = \text{const}$ and with $\Phi_0^\star T = T$ our statement is proven. □

Proposition 2.12 *For* $T \in \mathcal{T}_s^0(\mathcal{M})$, $X_1, \dots, X_s, Y \in \mathcal{X}(\mathcal{M})$ *we have*

$$(L_Y T)(X_1, \dots, X_s) = Y(T(X_1, \dots, X_s)) - \sum_{j=1}^{s} T(X_1, \dots, [Y, X_j], \dots, X_s)$$

Proof: We consider $T \otimes X_1 \otimes \cdots \otimes X_s \in \mathcal{T}_s^s(\mathcal{M})$:

$$L_Y(T\otimes X_1 \otimes\cdots\otimes X_s) = (L_Y T)\otimes X_1, \otimes\cdots\otimes X_s + \sum_{j=1}^{s} T\otimes X_1\otimes\cdots\otimes \underbrace{L_Y X_j}_{[Y,X_j]} \otimes\cdots\otimes X_s$$

The total contractionl over both sides then gives

$$L_Y(T(X_1, \dots, X_s)) = (L_Y T)(X_1, \dots, X_s) + \sum_{j=1}^{s} T(X_1, \dots, [Y, X_j], \dots, X_s) .$$

□

Example 2.1 *For* $\omega \in \mathcal{X}^\star(\mathcal{M})$,

$$(L_Y\omega)(X) = Y(\omega(X)) - \omega([Y, X]).$$

2.3.1 Expression of the Lie derivative in local coordinates

We choose local coordinates $(x^1, \dots, x^n)$ around a point $p \in \mathcal{M}$ with dual bases $\left(\partial_i \equiv \frac{\partial}{\partial x^i}\right)_{i=1}^n$ and $(dx^i)_{i=1}^n$ of $T_p\mathcal{M}$ and $T_p^\star\mathcal{M}$.
First we have[2] $L_X dx^i = d(L_X x^i) = d(X^j \delta_j^i) = d(X^i) = X^i{}_{,j} dx^j$ and $L_X \partial_i = [X, \partial_i] = -X^j{}_{,i}\partial_j$. For $T \in \mathcal{T}_s^r(\mathcal{M})$ we use the same notation as in proposition 2.12:

$$(L_X T)^{i_1\cdots i_r}_{j_1\cdots j_s} = (L_X T)(dx^{i_1}, \dots, dx^{i_r}, \partial_{j_1}, \dots, \partial_{j_s}) \tag{2.6}$$

But this expression is the total contraction of

$$(L_X T) \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_r} \otimes \partial_{j_1} \otimes \cdots \otimes \partial_{j_s}. \tag{2.7}$$

[2] $X^i{}_{,j}$ denotes $\partial_j X^i$

With

$$L_X(T\otimes dx^{i_1}\otimes\cdots\otimes\partial_{j_s}) = (L_XT)\otimes dx^{i_1}\otimes\cdots\otimes\partial_{j_s}+$$
$$\sum_{k=1}^{r} T\otimes dx^{i_1}\otimes\cdots\otimes\underbrace{(L_X dx^{i_k})}_{X^{i_k}_{\ ,l}dx^l}\otimes\cdots\otimes\partial_{j_s}+\sum_{k=1}^{s} T\otimes dx^{i_1}\otimes\cdots\otimes\underbrace{(L_X\partial_{j_k})}_{-X^l_{\ ,j_k}\partial_l}\otimes\cdots\otimes\partial_{j_s}\ .$$

We obtain by total contraction

$$L_X(T(dx^{i_1},\dots,\partial_{j_s})) = (L_XT)(dx^{i_1},\dots,\partial_{j_s}) + X^{i_1}_{\ ,l}T(dx^l,\dots,\partial_{j_s})+\cdots+$$
$$+X^{i_r}_{\ ,l}T(dx^{i_1},\dots,dx^l,\partial_{j_1},\dots,\partial_{j_s}) - X^l_{\ ,j_1}T(dx^{i_1},\dots,dx^{i_r},\partial_l,\dots,\partial_{j_s})-\cdots-$$
$$-X^l_{\ ,j_s}T(dx^{i_1},\dots,\partial_l)\ .$$

Or, with Eq. (2.6):

$$(L_XT)^{i_1\cdots i_r}_{j_1\cdots j_s} = X^lT^{i_1\cdots i_r}_{j_1\cdots j_s\ ,l} - X^{i_1}_{\ ,l}T^{l\cdots i_r}_{j_1\cdots j_s}-\cdots-X^{i_r}_{\ ,l}T^{i_1\cdots l}_{j_1\cdots j_s}+X^l_{\ ,j_1}T^{i_1\cdots i_r}_{l,\cdots j_s}+\cdots+X^l_{\ ,j_s}T^{i_1\cdots i_r}_{j_1\cdots l}\ . \tag{2.8}$$

In particular for $\omega\in\mathcal{X}^\star(\mathcal{M})$:

$$(L_X\omega)_j = X^l\omega_{j\ ,l} + X^l_{\ ,j}\omega_l\ . \tag{2.9}$$

For more details on the topics of chapters 1-3 of this first part, see Kobayashi-Nomizu [12], Vol I, Chap.I,par. 1-3.

Chapter 3

Affine connections and the covariant derivative

In this chapter we introduce an additional structure on differentiable manifolds which is very important in General Relativity.

3.1 Affine connections

The problem we want to address is the the parallel transport of vectors from one position $p \in \mathcal{M}$ to another position q for an arbitrary manifold $\mathcal{M}$. Let us first consider the situation in $\mathbb{R}^n$. For two points $p, q \in \mathbb{R}^n$ and two vectors $v_p \in T_p\mathbb{R}^n$ and $u_q \in T_q\mathbb{R}^n$, there is a natural way to compare the directions of v_p and u_q: we connect p and q by a straight line and we transport v_p parallel along the straight line to q. With this "parallel transport" we define a map from $T_p\mathbb{R}^n$ to $T_q\mathbb{R}^n$. In a general differentiable manifold this notion of "parallel transport" does not exist. This is an additional structure which is in general not unique, and which we define now.

Definition 3.1 (affine connection) *A (linear)* **affine connection** ∇ *in a differentiable manifold $\mathcal{M}$ is a mapping which maps a pair of vector fields $X, Y \in \mathcal{X}(\mathcal{M})$ on a vector field $\nabla_X Y \in \mathcal{X}(\mathcal{M})$ such that*

1. $(X, Y) \mapsto \nabla_X Y$ *is $\mathbb{R}$-bilinear in X and in Y;*
2. *for $f \in \mathcal{F}(\mathcal{M})$, $\nabla_{fX} Y = f\nabla_X Y$ and $\nabla_X fY = f\nabla_X Y + (Xf)Y$.*

We call $\nabla_X Y$ "the covariant derivative of X in direction Y.

Lemma 3.1 *Let ∇ be a linear connection on $\mathcal{M}$ and $\mathcal{U} \subset \mathcal{M}$ an open set. Let X, Y be two vector fields of which one vanishes on $\mathcal{U}$. Then also $\nabla_X Y$ vanishes on $\mathcal{U}$.*

Proof: Consider $Y|_{\mathcal{U}} = 0$. Let $p \in \mathcal{U}$ be fixed and let h be a function with the properties that $h(p) = 0$ and $h = 1$ on $\mathcal{M} \setminus \mathcal{U}$. Hence $hY = Y$. Therefore

$$(\nabla_X Y)_p = (\nabla_X (hY))_p = h(p)(\nabla_X Y) + X(h)(p) Y_p = 0.$$

As the choice of p has been arbitrary, $(\nabla_X Y)_p = 0$, $\forall p \in \mathcal{U}$. In the same manner one shows the statement for $X|_{\mathcal{U}} = 0$. □

This implies that an affine connection∇ on $\mathcal{M}$ induces an affine connection $\nabla|_{\mathcal{U}}$ on every open subset $\mathcal{U} \subset \mathcal{M}$. Consider $X, Y \in \mathcal{X}(\mathcal{U})$. There exist $\widetilde{X}, \widetilde{Y} \in \mathcal{X}(\mathcal{M})$ such that $\widetilde{X}|_{\mathcal{U}} = X$ and $\widetilde{Y}|_{\mathcal{U}} = Y$ (we do not show this continuation lemma). We set

$$\left(\nabla\Big|_{\mathcal{U}}\right)_X Y := (\nabla_{\widetilde{X}} \widetilde{Y})\Big|_{\mathcal{U}} \tag{3.1}$$

According to lemma 3.1 this definition does not depend on the choice of the continuations $\widetilde{Y}$ and $\widetilde{X}$.

Lemma 3.2 *We consider $X, Y \in \mathcal{X}(\mathcal{M})$. If $X_p = 0$, also $(\nabla_X Y)_p = 0$.*

Proof: Let X be given locally by $X = \xi^i \partial_i \Rightarrow \xi^i(p) = 0$. Hence $(\nabla_X Y)_p = \xi^i(p)(\nabla_{\partial_i} Y)_p = 0$. □

Definition 3.2 (Christoffel symbols) *For a given chart $(\mathcal{U}, x^1, \ldots, x^m)$, we set*

$$\nabla_{\partial_i}(\partial_j) = \Gamma^k_{ij} \partial_k, \quad \partial_i \equiv \frac{\partial}{\partial x^i} \ . \tag{3.2}$$

The m^3 symbols $\Gamma^k_{ij} \in \mathcal{F}(\mathcal{U})$ are the **Christoffel symbols** *of the connection ∇ (in the chart $(\mathcal{U}, x^1, \ldots, x^m)$).*

Proposition 3.1 *Let $(x^1, \ldots, x^m) \mapsto (\bar{x}^1, \ldots, \bar{x}^m)$ be a coordinate transformation in an open set $\mathcal{U} \subset \mathcal{M}$. Let Γ^k_{ij} be the Christoffel symbols in the coordinates $(x^i)^m_{i=1}$ and $\overline{\Gamma}^k_{ij}$ those in the coordinates $(\bar{x}^i)^m_{i=1}$. We then have*

$$\overline{\Gamma}^c_{ab} = \frac{\partial x^i}{\partial \bar{x}^a} \frac{\partial x^j}{\partial \bar{x}^b} \frac{\partial \bar{x}^c}{\partial x^k} \Gamma^k_{ij} + \frac{\partial^2 x^k}{\partial \bar{x}^a \partial \bar{x}^b} \cdot \frac{\partial \bar{x}^c}{\partial x^k} \ . \tag{3.3}$$

Proof:

$$\nabla_{\frac{\partial}{\partial \bar{x}^a}}\left(\frac{\partial}{\partial \bar{x}^b}\right) = \overline{\Gamma}^c_{ab}\cdot\frac{\partial}{\partial \bar{x}^c} = \overline{\Gamma}^c_{ab}\cdot\frac{\partial x^k}{\partial \bar{x}^c}\frac{\partial}{\partial x^k}\ .$$

On the other hand:

$$\frac{\partial}{\partial \bar{x}^a} = \frac{\partial x^i}{\partial \bar{x}^a}\frac{\partial}{\partial x^i} \Rightarrow \nabla_{\frac{\partial}{\partial \bar{x}^a}} = \frac{\partial x^i}{\partial \bar{x}^a}\cdot\nabla_{\frac{\partial}{\partial x^i}}\ .$$

Furthermore

$$\begin{aligned}\nabla_{\frac{\partial}{\partial \bar{x}^a}}\left(\frac{\partial}{\partial \bar{x}^b}\right) &= \nabla_{\frac{\partial}{\partial \bar{x}^a}}\left(\frac{\partial x^j}{\partial \bar{x}^b}\frac{\partial}{\partial x^j}\right) = \frac{\partial x^j}{\partial \bar{x}^b}\nabla_{\frac{\partial}{\partial \bar{x}^a}}\left(\frac{\partial}{\partial x^j}\right) + \frac{\partial^2 x^j}{\partial \bar{x}^a\partial \bar{x}^b}\cdot\frac{\partial}{\partial x^j} \\ &= \frac{\partial x^j}{\partial \bar{x}^b}\frac{\partial x^i}{\partial \bar{x}^a}\underbrace{\nabla_{\frac{\partial}{\partial x^i}}\left(\frac{\partial}{\partial x^j}\right)}_{\Gamma^k_{ij}\partial_k} + \frac{\partial^2 x^j}{\partial \bar{x}^a\partial \bar{x}^b}\cdot\frac{\partial}{\partial x^j}\ .\end{aligned}$$

Hence

$$\overline{\Gamma}^c_{ab}\cdot\frac{\partial x^k}{\partial \bar{x}^c} = \frac{\partial x^j}{\partial \bar{x}^b}\frac{\partial x^i}{\partial \bar{x}^a}\Gamma^k_{ij} + \frac{\partial^2 x^k}{\partial \bar{x}^a\partial \bar{x}^b}\ .$$

Multiplying by $\left(\frac{\partial \bar{x}^d}{\partial x^k}\right)$, which is the inverse of $\left(\frac{\partial x^k}{\partial \bar{x}^c}\right)$ proofs proposition 3.1. $\square$

Proposition 3.1 shows that the Γ^k_{ij} are *not* the components of a tensor! But m^3 functions which transform according to (3.3) under all coordinate transformations define a connection ∇ on $\mathcal{M}$ which satisfies (3.2).

Definition 3.3 (covariant derivative of a vector field) *The map* $\nabla : \mathcal{X}(\mathcal{M}) \to \mathcal{T}^1_1(\mathcal{M}) : X \mapsto \nabla X$, *with* $\nabla X(Y,\omega) := \omega(\nabla_Y X)$, *is the* **covariant derivative** *of* X *(or the* **absolute derivative** *of* X*).*

For $X = \xi^i\partial_i$ in a coordinate system we denote the components of ∇X by

$$\nabla X = \xi^i_{\ ;j}dx^j \otimes \partial_i. \tag{3.4}$$

We have

$$\xi^i_{\ ;j} = \nabla X(\partial_j, dx^i) = dx^i(\underbrace{\nabla_{\partial_j}(\xi^k\partial_k)}_{\xi^k_{\ ,j}\partial_k + \xi^k\nabla_{\partial_j}\partial_k}) = dx^i(\xi^k_{\ ,j}\partial_k + \xi^k\Gamma^l_{jk}\partial_l) = \xi^i_{\ ,j} + \Gamma^i_{jk}\xi^k\ .$$

Hence

$$\xi^i_{\ ;j} = \xi^i_{\ ,j} + \Gamma^i_{jk}\xi^k. \tag{3.5}$$

It is easy to see that the $\xi^i_{\ ,j}$ do not transform like a tensor, however, as follows from the definition of ∇X the functions $\xi^i_{\ ;j}$ are the components of the tensor $\nabla X \in \mathcal{T}^1_1(\mathcal{M})$.

3.2 Parallel transport along a path

Definition 3.4 (autoparallel vector field) *Let $I \subset \mathbb{R}$ be an interval and $\gamma : I \to \mathcal{M} : s \mapsto \gamma(s)$ a path.*
The vector field $X \in \mathcal{X}(\mathcal{M})$ is called **autoparallel** *along γ if $\nabla_{\dot\gamma} X = 0$ (instead of $\nabla_{\dot\gamma} X$ one sometimes writes $\frac{DX}{ds}$).*
The vector field $\nabla_{\dot\gamma} X$ is called the **covariant derivative** *of X along γ.*

In a coordinate system with $X = \xi^i \frac{\partial}{\partial x^i}$ and $\dot\gamma = \frac{\mathrm{d}x^i}{\mathrm{d}s}\frac{\partial}{\partial x^i}$ we have

$$\nabla_{\dot\gamma} X = \left(\frac{\mathrm{d}\xi^i}{\mathrm{d}s} + \Gamma^i_{jk} \xi^k \frac{\mathrm{d}x^j}{\mathrm{d}s} \right) \frac{\partial}{\partial x^i} \ . \tag{3.6}$$

This shows that $\nabla_{\dot\gamma} X$ depend only on the values of X on γ.
X is thus autoparallel along γ if

$$\frac{\mathrm{d}\xi^i}{\mathrm{d}s} + \Gamma^i_{jk} \frac{\mathrm{d}x^j}{\mathrm{d}s} \xi^k = 0 \ . \tag{3.7}$$

For a given curve γ and $X_0 \in T_{\gamma(0)}\mathcal{M}$ there exists a unique field $X(s)$ with $X(0) = X_0$ which is autoparallel along γ: the solution of (3.7) with initial condition $\xi^i(0)\frac{\partial}{\partial x^i} = X_0$.
Since the equation for $X(s)$ is linear, $X(s)$ is well defined for all $s \in I$. For two points on γ, $\gamma(s)$ and $\gamma(t)$, there exist therefore an (linear) isomorphism

$$\mathcal{T}_{t,s} : T_{\gamma(s)}\mathcal{M} \to T_{\gamma(t)}\mathcal{M} : v \mapsto \mathcal{T}_{t,s} v \tag{3.8}$$

which maps a vector $v \in T_{\gamma(s)}\mathcal{M}$ to the parallel transported vector $\mathcal{T}_{t,s} v \in T_{\gamma(t)}\mathcal{M}$.

Definition 3.5 (parallel transport) *The map $\mathcal{T}_{t,s}$ is the* **parallel transport** *along γ, from $\gamma(s)$ to $\gamma(t)$.*

The fact that the solution of equation (3.7) for given initial conditions is unique implies that

$$\mathcal{T}_{t,s} \circ \mathcal{T}_{s,r} = \mathcal{T}_{t,r} \ . \tag{3.9}$$

Obviously, $\mathcal{T}_{t,t} = \mathrm{id}$.

Proposition 3.2 *Let X be a vector field along γ. Then we have*

$$\nabla_{\dot\gamma} X(\gamma(t)) = \frac{\mathrm{d}}{\mathrm{d}s} \mathcal{T}_{t,s} X(\gamma(s)) \Big|_{s=t} \ . \tag{3.10}$$

Proof: For $v_0 \in T_{\gamma(s)}\mathcal{M}$ and $v(t) = \mathcal{T}_{t,s}v_0 \in T_{\gamma(t)}\mathcal{M}$ we have

$$\dot{v}^i + \Gamma^i_{kj}\dot{x}^k v^j = 0.$$

Since $v^i(t) = (\mathcal{T}_{t,s}v_0)^i = (\mathcal{T}_{t,s})^i{}_j v_0^j$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{T}_{t,s})^i{}_j\Big|_{t=s} = -\Gamma^i_{kj}\dot{x}^k$$

With $\mathcal{T}_{s,t} = \mathcal{T}_{t,s}^{-1}$ and $\mathcal{T}_{s,s} = \mathrm{id}$,

$$\left(\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{T}_{t,s}X(\dot{\gamma}(s))\Big|_{s=t}\right)^i = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=t}\left(\mathcal{T}_{s,t}^{-1}X(\gamma(s))\right)^i = -\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=t}(\mathcal{T}_{s,t})^i{}_j X^j(\gamma(s)) + \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=t}X^i(\gamma(s))$$
$$= \Gamma^i_{kj}X^j\dot{x}^k + X^i{}_{,k}\dot{x}^k = (\nabla_{\dot{\gamma}}X)^i \ .$$

□

3.3 Geodesics, exponential map, normal coordinates

Definition 3.6 (geodesic) *A curve γ is called a geodesic, if $\dot{\gamma}$ is autoparallel aong γ, i.e.,*

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0 \ .$$

In a coordinate system, $\gamma(s) = (x^i(s))_{i=1}^m$, this condition becomes

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0 \ . \tag{3.11}$$

Remark 3.1

- *According to the existence theorem for maximal solutions to ordinary differential equations, there exists a unique maximal geodesic for given initial values $\gamma(0)$ and $\dot{\gamma}(0)$.*
- *If $\gamma(t)$ is a geodesic, $\gamma(as)$, $a \in \mathbb{R}$, is also a geodesic with initial velocity $a\dot{\gamma}(0)$.*

For sufficiently small $a \in \mathbb{R}_+$, $\gamma(as)$ is well defined in the interval $0 \le s \le 1$. Hence, for $p \in \mathcal{M}$, there exists a neighborhood $V \subset T_p\mathcal{M}$ of zero, $0 \in T_p\mathcal{M}$ such that the geodesic $\gamma_v(1)$ with initial conditions $\gamma_v(0) = p$ and $\dot{\gamma}_v(0) = v$ is well defined for $v \in V$.

We set

$$\exp_p(v) := \gamma_v(1). \tag{3.12}$$

Since $\gamma_v(s)$ depends differentiably on the initial conditions, the map $\exp_p$ is differentiable.
From $\gamma_{sv}(t) = \gamma_v(ts)$ it follows for $t = 1$ that

$$\exp_p(sv) = \gamma_v(s), \tag{3.13}$$

hence

$$\big(T\exp_p(0)\big)\, v = \frac{d}{ds}\exp_p(sv)\Big|_{s=0} = \dot\gamma_v(0) = v,$$

such that $T\exp_p(0) = \mathrm{id}$.
With this we have shown the following:

Theorem 3.1 *The map $\exp_p$ is a diffeomorphism from a neighborhood of $0 \in T_p\mathcal{M}$ to a neighborhood of the point $p \in \mathcal{M}$.*

This theorem leads us to the definition of normal coordindates:

Definition 3.7 (normal coordindates) *Let $e_1, \ldots, e_n$ be a basis of $T_p\mathcal{M}$. For an open set $\mathcal{U} \subset \mathbb{R}^n$, neighborhood of $0 \in \mathbb{R}^n$, the chart $h : \exp_p(x^i e_i) \mapsto (x^1, \ldots, x^n)$ is well defined and maps a neighborhood of the point $p \in \mathcal{M}$ into $\mathcal{U}$.*
The coordinates $(x^1, \ldots, x^n)$ are called **normal coordinates** *or* **Gaussian coordinates** *around the point p (which has coordinates $(0, \ldots, 0)$).*

Proposition 3.3 *In a Gaussian coordinate system,*
$\Gamma^k_{ij}(0) + \Gamma^k_{ji}(0) = 0.$
Here 0 denotes the point p around which the coordinate system is defined.

Proof: For sufficiently small s, the point $\exp_0(sv)$ has normal coordinates $x^i = v^i s$, $v = v^i e_i$. But $\exp_0(sv) = \gamma_v(s)$ is the geodesic with initial velocity v, hence

$$0 = \Gamma^i_{jk}(sv^1, \cdots, sv^m) v^j v^k.$$

At $s = 0$, this gives $0 = \Gamma^i_{jk}(0) v^j v^k$ for any choice of $(v^1, \cdots v^m)$, this proofs 3.3. □

For a symmetric connexion, $\Gamma^k_{ij} = \Gamma^k_{ji}$, we then have $\Gamma^k_{ij}(0) = 0$.
Normal coordinates map geodesics through 0 onto straight lines in $T_p\mathcal{M}$.

3.4 The covariant derivative of tensor fields

We first define the parallel transport of tensor fields

Definition 3.8 (parallel transport) *We consider a curve $\gamma(s)$, and $\alpha \in T^\star_{\gamma(s)}\mathcal{M}$. We define $\mathcal{T}_{t,s}\alpha \in T^\star_{\gamma(t)}\mathcal{M}$ by*

$$\mathcal{T}_{t,s}\alpha(\mathcal{T}_{t,s}v) = \alpha(v) \tag{3.14}$$

for all $v \in T_{\gamma(s)}\mathcal{M}$. Since $\mathcal{T}_{t,s} : T_{\gamma(s)}\mathcal{M} \to T_{\gamma(t)}\mathcal{M}$ is an isomorphism, this defines $\mathcal{T}_{t,s}\alpha$ uniquely and $\mathcal{T}_{t,s} : T^\star_{\gamma(s)}\mathcal{M} \to T^\star_{\gamma(t)}\mathcal{M}$ is also isomorphism. For $w \in T_{\gamma(t)}\mathcal{M}$,

$$\left(\mathcal{T}_{t,s}\alpha\right)(w) = \alpha(\mathcal{T}^{-1}_{t,s}w). \tag{3.15}$$

For a tensor $T \in (T_{\gamma(s)}\mathcal{M})^i_j$ we define the parallel transport of T, $\mathcal{T}_{t,s}T \in (T_{\gamma(t)}\mathcal{M})^i_j$ by

$$(\mathcal{T}_{t,s}T)(v_1, \ldots, v_j, \alpha_1, \ldots, \alpha_i) = T(\mathcal{T}^{-1}_{t,s}v_1, \ldots, \mathcal{T}^{-1}_{t,s}\alpha_i) \tag{3.16}$$

for $\alpha_l \in T^\star_{\gamma(t)}\mathcal{M}$, $v_n \in T_{\gamma(t)}\mathcal{M}$, $1 \le l \le i$, $1 \le n \le j$.

Definition 3.9 *Let X be a vector field with integral curve $\gamma(t)$, $\gamma(0) = p$. For a tensor field $T \in \mathcal{T}^r_s(\mathcal{M})$, we set*

$$(\nabla_X T)_p = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{T}_t^{-1}T_{\gamma(t)}\Big|_{t=0} \quad \textit{where } \mathcal{T}_t \equiv \mathcal{T}_{t,0},\ \mathcal{T}_t^{-1} \equiv \mathcal{T}_{0,t} \tag{3.17}$$

is the parallel transport along the curve γ. $(\nabla_X T)_p \in (T_p\mathcal{M})^r_s$ is the **covariant derivative** *of T at p in direction X.*

This definition generalizes the definition 3.3 of the covariant derivative of a vector field, see also proposition 3.2.
If $X(p) = 0$ we have $(\nabla_X T)_p = 0$.
For $f \in \mathcal{F}(\mathcal{M})$, we define $\nabla_X f = Xf$.

Proposition 3.4 *∇_X defines a derivation on the algebra of tensor fields, $\mathcal{T}(\mathcal{M})$.*

Proof: The linearity follows obviously from the definition ($\mathcal{T}_t$ is a linear map). It remains to show that ∇_X satisfies the Leibniz rule 1.9: according to the definition,

$$\mathcal{T}_t(T_1 \otimes T_2) = (\mathcal{T}_t T_1) \otimes (\mathcal{T}_t T_2).$$

Hence

$$\begin{aligned}\nabla_X(T_1 \otimes T_2)_p &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(\mathcal{T}_t^{-1}T_{1\,\gamma(t)} \otimes \mathcal{T}_t^{-1}T_{2\,\gamma(t)}) \\ &= (\nabla_X T_1 \otimes T_2 + T_1 \otimes \nabla_X T_2)_p\ .\end{aligned}$$

$\square$

Proposition 3.5 ∇_X *commutes with contractions.*

Proof: We consider the special case $T = Y \otimes \omega \in \mathcal{T}_1^1(\mathcal{M})$, for $Y \in \mathcal{X}(\mathcal{M})$ and $\omega \in \mathcal{X}^\star(\mathcal{M})$. The general case is totally analog, it just is harder to write down: Let C be the contraction and $\mathcal{T}_s$ the parallel transport along an integral curve of X. We then have

$$C\mathcal{T}_s^{-1}(Y \otimes \omega)_{\gamma(s)} = C((\mathcal{T}_s^{-1}Y_{\gamma(s)}) \otimes (\mathcal{T}_s^{-1}\omega_{\gamma(s)})) = \mathcal{T}_s^{-1}\omega_{\gamma(s)}(\mathcal{T}_s^{-1}Y_{\gamma(s)})$$
$$= \omega_{\gamma(s)}(Y_{\gamma(s)})\,.$$

Hence taking the limit for $s \to 0$ in the equation

$$C\left(\frac{1}{s}\left(\mathcal{T}_s^{-1}(Y \otimes \omega)_{\gamma(s)} - (Y \otimes \omega)_p\right)\right) = \frac{1}{s}\left(\omega_{\gamma(s)}(Y_{\gamma(s)}) - \omega_p(Y_p)\right)$$

we obtain

$$C\left(\nabla_X(Y \otimes \omega)\right) = \nabla_X C(Y \otimes \omega)\,.$$

$\square$

Consequence 3.6 $\nabla_X(Y \otimes \omega) = (\nabla_X Y) \otimes \omega + Y \otimes \nabla_X \omega$ *implies after contraction*

$$\underbrace{\nabla_X(\omega(Y))}_{X(\omega(Y))} = \omega(\nabla_X Y) + (\nabla_X \omega)(Y)\,.$$

With this we obtain

$$(\nabla_X \omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y) \tag{3.18}$$

this equation determines the covariant derivative of a 1-form. Eq. (3.18) implies that $\nabla_X \omega$ is $\mathcal{F}(\mathcal{M})$-linear in X. As this is also true for the covariant derivative of a vector field, ∇_X is $\mathcal{F}(\mathcal{M})$-linear on $\mathcal{T}(\mathcal{M})$: for $T \in \mathcal{T}(\mathcal{M})$,

$$\nabla_{fX} T = f \nabla_X T, \quad \text{for } f \in \mathcal{F}(\mathcal{M}). \tag{3.19}$$

Definition 3.10 (covariant derivative of a tensor field) *We set*

$$\nabla : \mathcal{T}_r^s(\mathcal{M}) \to \mathcal{T}_{r+1}^s(\mathcal{M}) : T \mapsto \nabla T$$

where ∇T *is defined by*

$$(\nabla T)(X_1, \ldots, X_{r+1}, \alpha_1, \ldots, \alpha_s) := \nabla_{X_{r+1}} T(X_1, \ldots, X_r, \alpha_1, \ldots, \alpha_s)$$

where $X_i \in \mathcal{X}(\mathcal{M})$, $\alpha_j \in \mathcal{X}^\star(\mathcal{M})$.
∇T *is called the* **covariant derivative** *of* T.

Applying the Leibniz rule and the commutation with contractions, we obtain for $T \in \mathcal{T}^r_s(\mathcal{M})$:

$$\nabla_X \left[T(Y_1, \dots, Y_r, \alpha_1, \dots, \alpha_s)\right] = \nabla_X \left[C(T \otimes Y_1 \otimes \cdots \otimes Y_r \otimes \alpha_1 \otimes \cdots \otimes \alpha_s)\right]$$

$$= C(\nabla_X T \otimes Y_1 \cdots \otimes \alpha_s) + C(T \otimes \nabla_X Y_1 \otimes \cdots \alpha_s) + \cdots + C(T \otimes Y_1 \otimes \cdots \otimes \nabla_X \alpha_s)$$

$$= (\nabla_X T)(Y_1, \dots, \alpha_s) + T(\nabla_X Y_1, \dots, \alpha_s) + \cdots + T(Y_1, \dots, \nabla_X \alpha_s);$$

Hence

$$(\nabla_X T)(Y_1, \dots, \alpha_s) = X(T(Y_1, \dots, \alpha_s)) - T(\nabla_X Y_1, \dots, \alpha_s) - \cdots - T(Y_1, \dots, \nabla_X \alpha_s). \tag{3.20}$$

This allows us to give an explicite expression for covariant derivative in a chart $\mathcal{U} \subset \mathcal{M}$ with coordinates $(x^1, \dots, x^m)$.
Let X be given in local coordinates by $X = \xi^i \partial_i$ and Γ^i_{lm} the Christoffel symbols in the chart $(\mathcal{U}, x^1, \dots, x^m)$ such that

$$\nabla_{\partial_i} \partial_j = \Gamma^l_{ij} \partial_l \tag{3.21}$$

and $\nabla_X \partial_j = \xi^i \Gamma^l_{ij} \partial_l$. Since $dx^i(\partial_j) = \delta^i{}_j$, we have

$$(\nabla_X dx^i)(\partial_j) = 0 - \xi^k \Gamma^l_{kj} dx^i(\partial_l) = -\xi^k \Gamma^i_{kj}.$$

Thus

$$\nabla_X dx^i = -\xi^k \Gamma^i_{kj} dx^j; \quad \nabla_{\partial_k} dx^i = -\Gamma^i_{kj} dx^j \tag{3.22}$$

In coordinate representation

$$T = T^{i_1 \cdots i_r}_{j_1 \cdots j_s} dx^{j_1} \otimes \cdots \otimes dx^{j_s} \otimes \partial_{i_1} \otimes \cdots \otimes \partial_{i_r},$$

$$T^{i_1 \cdots i_r}_{j_1 \cdots j_s} = T(\partial_{j_1}, \dots, \partial_{j_r}, dx^{i_1}, \dots, dx^{i_s})$$

we set

$$T^{i_1 \cdots i_r}_{j_1 \cdots j_s\ ;k} := (\nabla_{\partial_k} T)^{i_1 \cdots i_r}_{j_1 \cdots j_s} = \nabla T(\partial_{j_1}, \dots, \partial_{j_s}, \partial_k, dx^{i_1}, \dots, dx^{i_r}) \ .$$

Eq. (3.20) together with Eqs. (3.21) and (3.22) gives

$$T^{i_1 \cdots i_r}_{j_1 \cdots j_s\ ;k} = T^{i_1 \cdots i_r}_{j_1 \cdots j_s\ ,k} + \sum_{m=1}^{r} \Gamma^{i_m}_{kl} T^{i_1 \cdots l \cdots i_r}_{j_1 \cdots j_s} - \sum_{m=1}^{s} \Gamma^l_{kj_m} T^{i_1 \cdots i_r}_{j_1 \cdots l \cdots j_s} \tag{3.23}$$

where the index l is taken at the m-ieme position.
In particular for contravariant and covariant vector fields (1-forms are sometime called 'covariant vector fields'):

$$X = \xi^i \partial_i \quad \nabla X = \xi^i{}_{;k} \partial_i \otimes dx^k \,, \qquad \xi^i{}_{;k} = \xi^i{}_{,k} + \Gamma^i_{kl} \xi^l \tag{3.24}$$

$$\text{and} \quad \omega = \eta_j dx^j \quad \nabla \omega = \eta_{j;k} dx^j \otimes dx^k \,, \quad \eta_{j;k} = \eta_{j,k} - \Gamma^l_{kj} \eta_l. \tag{3.25}$$

Furthermore,

$$\delta^i{}_{j;k} = \Gamma^i_{kl} \delta^l{}_j - \Gamma^l_{kj} \delta^i{}_l = \Gamma^i_{kj} - \Gamma^i_{kj} = 0 \ .$$

3.5 Curvature and torsion of an affine connection; Bianchi identities

Definition 3.11 (torsion, curvature) *Let* ∇ *be a connection on* $\mathcal{M}$ *. The* **torsion** *of* ∇ *is the bilinear map*

$$T : \mathcal{X}(\mathcal{M}) \times \mathcal{X}(\mathcal{M}) \to \mathcal{X}(\mathcal{M}) : (X,Y) \mapsto \nabla_X Y - \nabla_Y X - [X,Y] =: T(X,Y)$$

The **curvature** *of* ∇ *is the trilinear map*

$$R : \mathcal{X}(\mathcal{M}) \times \mathcal{X}(\mathcal{M}) \times \mathcal{X}(\mathcal{M}) \to \mathcal{X}(\mathcal{M}) : R(X,Y)Z := \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z$$

For vector fields with vanishing commutators, like e.g. the basis vector fields of a holonomic basis, $(\partial_i)_{i=1}^n$, the curvature measures the non-commutativity of the covariant derivatives in direction ∂_i denoted ∇_i.

Obviously $T(X,Y) = -T(Y,X)$ and $R(X,Y) = -R(Y,X)$ ($R(X,Y) \in \mathcal{T}_1^1(\mathcal{M})$). For $f, g \in \mathcal{F}(\mathcal{M})$,

$$T(fX, gY) = \nabla_{fX} gY - \nabla_{gY} fX - [fX, gY] = fg\nabla_X Y + fX(g)Y$$

$$-gf\nabla_Y X - gY(f)X - fg[X,Y] - fX(g)Y + gY(f)X = fgT(X,Y)$$

In the same way one finds (after a somewhat longer calculation):

$$R(fX, gY)hZ = fghR(X,Y)Z \quad \text{for } f,g,h \in \mathcal{F}(\mathcal{M}) \ .$$

The map

$$\mathcal{X}^*(\mathcal{M}) \times \mathcal{X}(\mathcal{M}) \times \mathcal{X}(\mathcal{M}) \to \mathcal{F}(\mathcal{M}) : (\omega, X, Y) \mapsto \omega(T(X,Y)) \tag{3.26}$$

is a tensor field $\in \mathcal{T}_2^1(\mathcal{M})$, the **torsion tensor**. The map

$$\mathcal{X}^*(\mathcal{M}) \times \mathcal{X}(\mathcal{M}) \times \mathcal{X}(\mathcal{M}) \times \mathcal{X}(\mathcal{M}) \to \mathcal{F}(\mathcal{M}) : (\omega, X, Y, Z) \mapsto \omega(R(X,Y)Z) \tag{3.27}$$

is a tensor field $\in \mathcal{T}_3^1(\mathcal{M})$, the **curvature tensor** . As we shall see, this curvature tensor is very important in general relativity.

In a coordinate systems the components of the torsion tensor are[1]

$$T_{ij}^k = dx^k(T(\partial_i, \partial_j)) = dx^k(\nabla_i \partial_j - \nabla_j \partial_i - \underbrace{[\partial_i, \partial_j]}_{0})$$

$$= dx^k(\underbrace{\nabla_i \partial_j}_{\Gamma_{ij}^l \partial_l} - \underbrace{\nabla_j \partial_i}_{\Gamma_{ji}^l \partial_l}) = \Gamma_{ij}^k - \Gamma_{ji}^k \ .$$

[1] $\nabla_i \equiv \nabla_{\partial_i}$

Hence

$$T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}. \tag{3.28}$$

If the torsion vanishes, $\Gamma^k_{ij} = \Gamma^k_{ji}$ in every chart. Such a connection is called **symmetric**. For a symmetric connection $\Gamma^k_{ij}(0) = 0$ in normal coordinates (see definition 3.7).
This remark is very important for the formulation of the principle of equivalence in general relativity .

For the components of the curvature tensor we find

$$R^i_{jkl} = dx^i(R(\partial_k, \partial_l)\partial_j) = dx^i(\nabla_k\nabla_l\partial_j - \nabla_l\nabla_k\partial_j)$$

$$= dx^i(\nabla_k(\Gamma^n_{lj}\partial_n) - \nabla_l(\Gamma^n_{kj}\partial_n))$$

$$= dx^i(\Gamma^n_{lj\,,k}\partial_n - \Gamma^n_{kj\,,l}\partial_n + \Gamma^n_{lj}\Gamma^m_{kn}\partial_m - \Gamma^n_{kj}\Gamma^m_{ln}\partial_m)\ .$$

Such that

$$R^i_{jkl} = \Gamma^i_{lj\,,k} - \Gamma^i_{kj\,,l} + \Gamma^n_{lj}\Gamma^i_{kn} - \Gamma^n_{kj}\Gamma^i_{ln}\ . \tag{3.29}$$

Definition 3.12 (Ricci tensor) *The Ricci tensor is the contraction of the curvature tensor.*

$$R_{jl} = R^i_{\ jil} = \Gamma^i_{lj\,,i} - \Gamma^i_{ij\,,l} + \Gamma^n_{lj}\Gamma^i_{in} - \Gamma^n_{ij}\Gamma^i_{ln}\ . \tag{3.30}$$

Remark 3.2 *For the definition of the torsion and curvature tensors we have used that a $\mathcal{F}(\mathcal{M})$-multilinear map*

$$K : \underbrace{\mathcal{X}(\mathcal{M}) \times \cdots \times \mathcal{X}(\mathcal{M})}_{p\ times} \to \mathcal{X}(\mathcal{M})$$

can be interpreted as a tensor field $\in \mathcal{T}^1_p(\mathcal{M})$ by setting

$$\widetilde{K}(\omega, X_1, \ldots, X_p) := \omega(K(X_1, \ldots, X_p)).$$

We define $\nabla_Y K$ via this identification:

$$(\nabla_Y\widetilde{K})(\omega, X_1, \ldots, X_p) =: \omega((\nabla_Y K)(X_1, \ldots, X_p)).$$

But

$$(\nabla_Y\widetilde{K})(\omega, X_1, \ldots, X_p) = Y(\widetilde{K}(\omega, X_1, \ldots, X_p)) - \widetilde{K}((\nabla_Y\omega), X_1, \ldots, X_p) -$$

$$\widetilde{K}(\omega, (\nabla_Y X_1), \ldots, X_p) - \cdots - \widetilde{K}(\omega, X_1, \ldots, (\nabla_Y X_p))$$

$$= Y(\omega(K(X_1, \ldots, X_p))) - \underbrace{(\nabla_Y\omega)(K(X_1, \ldots, X_p))}_{Y(\omega(K(X_1,\ldots,X_p))) - \omega(\nabla_Y(K(X_1,\ldots,X_p)))} - \omega(K(\nabla_Y X_1, \ldots, X_p))$$

$$- \cdots - \omega(K(X_1, \ldots, \nabla_Y X_p)) \ .$$

With this we have

$$\begin{aligned}(\nabla_Y K)(X_1, \ldots, X_p) &= \nabla_Y(K(X_1, \ldots, X_p)) - \\ &\qquad K(\nabla_Y X_1, \ldots, X_p) - \cdots - K(X_1, \ldots, \nabla_Y X_p) \ . \end{aligned} \tag{3.31}$$

Theorem 3.2 *Let T and R be the torsion and curvature of an affine connection on $\mathcal{M}$. For arbitrary vector fields X, Y, Z we have*

1.

$$\sum_{\substack{cyclic \\ in\, X,Y,Z}} R(X,Y)Z = \sum_{cyclic} [T(T(X,Y),Z) + (\nabla_X T)(Y,Z)] \ . \tag{3.32}$$

(first Bianchi identity).

2.

$$\sum_{cyclic} [(\nabla_X R)(Y,Z) + R(T(X,Y),Z)] = 0 \tag{3.33}$$

(2nd Bianchi identity).

Proof: We proof the theorem for symmetric connections, i.e., $T(X,Y) = 0$. This is the case which is relevant for general relativity. The proof for $T \neq 0$ is an exercise for interested students.

We thus suppose $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = 0$. We want to show that the left and side of (3.32) vanishes.

$$\begin{aligned}\sum_{cyclic} R(X,Y)Z &= (\nabla_X\nabla_Y - \nabla_Y\nabla_X)Z + (\nabla_Z\nabla_X - \nabla_X\nabla_Z)Y \\ &+(\nabla_Y\nabla_Z - \nabla_Z\nabla_Y)X - \nabla_{[X,Y]}Z - \nabla_{[Z,X]}Y - \nabla_{[Y,Z]}X \\ &= \nabla_X[Y,Z] - \nabla_{[Y,Z]}X + cyclique \\ &= [X,[Y,Z]] + cyclique \overset{\text{Jacobi}}{=\!=} 0 \ . \end{aligned} \tag{3.34}$$

This shows (3.32). For the 2nd and 3rd equal sign we have used the symmetry of the connection. For the second identity, still with vanishing torsion, we apply (3.31):

$$(\nabla_X R)(Y,Z) = \nabla_X(R(Y,Z)) - \underbrace{R(\nabla_X Y, Z)}_{(1)} - \underbrace{R(Y, \nabla_X Z)}_{(2)} - R(Y,Z)\nabla_X$$

The cyclic sum of (1) and (2) gives

$$R(\nabla_X Y, Z) + R(Y, \nabla_X Z) + R(\nabla_Z X, Y) + R(Z, \nabla_Y X) + R(\nabla_Y Z, X) + R(X, \nabla_Z Y) \ .$$

With the anti-symmetry of R we obtain

$$R(\nabla_X Y, Z) - R(\nabla_Y X, Z) + cyclic$$
$$\stackrel{\text{symmetry}}{=} R([X,Y], Z) + cyclic.$$

Hence

$$(\nabla_X R)(Y,Z) + cyclic = \nabla_X(R(Y,Z)) - R([X,Y],Z) - R(Y,Z)\nabla_X + cyclic$$
$$= \nabla_X(\underbrace{\nabla_Y\nabla_Z}_{--} - \underbrace{\nabla_Z\nabla_Y}_{\cdots} - \nabla_{[Y,Z]}) - \nabla_{[X,Y]}\nabla_Z + \nabla_Z\nabla_{[X,Y]} + \nabla_{[[X,Y],Z]} - (\underbrace{\nabla_Y\nabla_Z}_{--}$$
$$- \underbrace{\nabla_Z\nabla_Y}_{\cdots} - \nabla_{[Y,Z]})\nabla_X + cyclic.$$

The cyclic sum of $\nabla_{[[X,Y],Z]}$ vanishes because of the Jacob identiy. The cyclic sums of the term $\cdots$ and $--$ also vanish. It remains

$$= -\underline{\nabla_X\nabla_{[Y,Z]}} - \overline{\nabla_{[X,Y]}\nabla_Z} + \underline{\nabla_Z\nabla_{[X,Y]}} + \overline{\nabla_{[Y,Z]}\nabla_X} + cyclic.$$

The cyclic sum of the terms $\underline{\cdot}$ and $\bar{\cdot}$ vanishes also and (3.33) is proven. □

3.6 The (pseudo-)Riemannian connection

Definition 3.13 (metric connection) *Let $(\mathcal{M}, g)$ be a pseudo-Riemannian manifold. An affine connection on $\mathcal{M}$ is called* **metric** *if the parallel transport along any smooth curve $\gamma(t)$ conserves the scalar product. In other words, for $X_0, Y_0 \in T_{\gamma(0)}\mathcal{M}$ and $X(t), Y(t) \in T_{\gamma(t)}\mathcal{M}$ the prallel transported vectors along γ, with $X(0) = X_0$ and $Y(0) = Y_0$,*

$$g_{\gamma(t)}(X(t), Y(t)) = g_{\gamma(0)}(X_0, Y_0)\ . \tag{3.35}$$

Proposition 3.7 *An affine connection is metric if and only if $\nabla g = 0$.*

Proof: For γ, X, Y as in the definition 3.13, ∇ is metric if and only if

$$\frac{\mathrm{d}}{\mathrm{d}t}(g_{\gamma(t)}(\mathcal{T}_{t,0}X_0, \mathcal{T}_{t,0}Y_0)) = 0.$$

But accordion to the definition 3.9 and Eq. (3.16), this implies

$$\nabla_{\dot\gamma} g(X_0, Y_0) = 0$$

for all $X_0,\ Y_0 \in T_{\gamma(0)}\mathcal{M}$. As the curve γ, and the point $\gamma(0)$ as well as the vector $\dot\gamma(0)$ and the vectors X_0 and Y_0 are arbitrary this proves the proposition. □

Remark 3.3 *According to (3.20), $\nabla g = 0$ is equivalent to*

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \tag{3.36}$$

for arbitrary vector fields X, Y and Z. Eq. (3.36) is called **the Ricci identity**.

Theorem 3.3 *On a pseudo-Riemannian manifold there exists a unique affine connection with the following two properties:*

- *∇ is metric,*
- *the torsion $T = 0$ (∇ is symmetric).*

Proof:

1. Uniqueness: $T = 0$, thus $\nabla_X Y = \nabla_Y X + [X, Y]$. With this the Ricci identity gives

$$a) \quad X(g(Y,Z)) = g(\nabla_Y X, Z) + g([X,Y], Z) + g(Y, \nabla_X Z)\,.$$

   The cyclic permutation of X, Y and Z in a) results in

$$b) \quad Y(g(Z,X)) = g(\nabla_Z Y, X) + g([Y,Z], X) + g(Z, \nabla_Y X)$$

$$c) \quad Z(g(X,Y)) = g(\nabla_X Z, Y) + g([Z,X], Y) + g(X, \nabla_Z Y)\,.$$

   The sum $b) + c) - a)$ gives

$$\begin{aligned} 2g(\nabla_Z Y, X) &= Y(g(Z,X)) + Z(g(X,Y)) - X(g(Y,Z)) - g([Z,X],Y) \\ &\quad - g([Y,Z],X) + g([X,Y],Z). \end{aligned} \tag{3.37}$$

   The right hand side does not depend on the connection ∇. Since g is not degenerate, this proves the uniqueness.

2. Existence: For Y and Z fixed, we define the mapping

$$\omega : \mathcal{X}(\mathcal{M}) \to \mathcal{F}(\mathcal{M}) : X \mapsto \frac{1}{2}\,[\text{right hand side of (3.37)}]\,.$$

   ω is obviously $\mathbb{R}$-linear.
   A brief computation shows that also $\omega(fX) = f\omega(X)$ for any function $f \in \mathcal{F}(\mathcal{M})$.
   Since g is not degenerate, there exists a unique vector field $\nabla_Z Y$ such that

$$\omega(X) = g(\nabla_Z Y, X)\,.$$

We show that the map

$$\nabla : \mathcal{X}(\mathcal{M}) \times \mathcal{X}(\mathcal{M}) \to \mathcal{X}(\mathcal{M}) : (Z, Y) \mapsto \nabla_Z Y$$

is an affine connection:
The $\mathbb{R}$-linearity in Y and Z is evident.
$\nabla_{fZ} Y = f \nabla_Z Y$ follows from a short explicite calculation.
We verify Leibniz' rule (for $g(X, Y)$ we write $\langle X, Y \rangle$):

$$\begin{aligned} 2\langle \nabla_Z fY, X \rangle &= -X\langle fY, Z \rangle + fY\langle X, Z \rangle + Z\langle X, fY \rangle - \langle [Z, X], fY \rangle \\ &\quad -\langle [fY, Z], X \rangle + \langle [X, fY], Z \rangle \\ &= 2\langle f\nabla_Z Y, X \rangle - (Xf)\langle Y, Z \rangle + (Zf)\langle X, Y \rangle + (Zf)\langle Y, X \rangle + (Xf)\langle Y, Z \rangle \\ &= 2f\langle \nabla_Z Y, X \rangle + 2\langle (Zf)Y, X \rangle. \end{aligned}$$

Thus

$$\nabla_Z fY = f\nabla_Z Y + (Zf)Y.$$

□

Definition 3.14 (Riemannian connection) *The unique connection of theorem 3.3 is the* **Riemannian connection** *or* **Levi-Cività connection** *on* $(\mathcal{M}, g)$.

Expression in local coordinates:
For a given coordinate system we set $X = \partial_k$, $Y = \partial_j$, $Z = \partial_i$, $g(\partial_l, \partial_m) = g_{lm}$ and $(g^{lm}) = (g_{lm})^{-1}$.
Since $[\partial_i, \partial_j] = 0$, Eq. (3.37) implies

$$2\langle \nabla_i \partial_j, \partial_k \rangle = \partial_j g_{ik} + \partial_i g_{jk} - \partial_k g_{ij}.$$

With $\nabla_i \partial_j = \Gamma^l_{ij} \partial_l$ we obtain

$$\Gamma^l_{ij} = \frac{1}{2} g^{lk} \left[g_{ik\,,j} + g_{jk\,,i} - g_{ij\,,k} \right] , \quad \text{and} \tag{3.38}$$

$$\nabla_i X = [X^j_{,i} + \Gamma^j_{ik} X^k] \partial_j \,. \tag{3.39}$$

We also note the following: Let us choose Gaussian normal coordinates in a point p for a given symmetric (but not necessarily metric) connection $\bar{\nabla}$ such that at p, $\partial_k = \bar{\nabla}_k$. In this case, in p the metric and symmetric connection ∇_i is given by

$$(\nabla_i X)^k = (\bar{\nabla}_i X)^k + C^k_{ij} X^j \quad \text{with} \quad C^k_{ij} = \frac{1}{2} g^{km} (\bar{\nabla}_j g_{im} + \bar{\nabla}_i g_{jm} - \bar{\nabla}_m g_{ij}) \,. \tag{3.40}$$

But both sides of this equation are tensor fields (the covariant derivative of a tensor is again a tensor). Therefore, Eq. (3.40) is valid in all coordinate systems and this for an arbitrary point p.

From Eq. (3.40) and theorem 3.3 we obtain also the following:

Proposition 3.8 *Let $\bar{\nabla}$ be the Levi-Civita connection of the metric $\bar{g}_{ij}$ and let ∇ be the one of $g_{ij} = \bar{g}_{ij} + \delta g_{ij}$ then*

$$\Gamma^l_{ij} = \bar{\Gamma}^l_{ij} + \frac{1}{2} g^{lk} \left[\bar{\nabla}_j \delta g_{ik} + \bar{\nabla}_i \delta g_{jk} - \bar{\nabla}_k \delta g_{ij} \right]. \tag{3.41}$$

Proposition 3.9 *The curvature of the Riemannian connection satisfies the following additional symmetries:*

$$\begin{aligned} \langle R(X,Y)Z,U\rangle &= -\langle R(X,Y)U,Z\rangle \\ \langle R(X,Y)Z,U\rangle &= \langle R(Z,U)X,Y\rangle. \end{aligned} \tag{3.42}$$

Proof: Because of the $\mathcal{F}(\mathcal{M})$-linearity of R and $\langle\cdot,\cdot\rangle$, it is enough to show Eqs. (3.42) for vector fields with vanishing Lie bracket (for example for the basis fields ∂_j in a local coordinate system).
For the first equation (3.42), it is sufficient to show it for $U = Z$, i.e.,

$$\langle R(X,Y)W,W\rangle = 0.$$

This equation for $W_1 = Z + U$ and $W_2 = Z - U$ then implies our statement, since

$$2\left[\langle R(X,Y)Z,U\rangle + \langle R(X,Y)U,Z\rangle\right] = \langle R(X,Y)W_1,W_1\rangle - \langle R(X,Y)W_2,W_2\rangle.$$

For the Riemannian connection and the fields X, Y and Z,

$$\langle \nabla_X \nabla_Y Z, Z\rangle = X\langle \nabla_Y Z, Z\rangle - \langle \nabla_Y Z, \nabla_X Z\rangle \quad \text{and} \quad \langle \nabla_Y Z, Z\rangle = \frac{1}{2} Y\langle Z,Z\rangle.$$

From the definition of the curvature it then follows ($[X,Y] = 0$):

$$2\langle R(X,Y)Z,Z\rangle = XY\langle Z,Z\rangle - YX\langle Z,Z\rangle = [X,Y]\langle Z,Z\rangle = 0.$$

For the second equation (3.42) we use the first Bianchi identity, (3.32), for $T = 0$, i.e.

$$\langle R(X,Y)Z,U\rangle \overset{(3.42)\sharp 1}{=\!=} -\langle R(X,Y)U,Z\rangle \overset{(3.32)}{=\!=} \langle R(U,X)Y,Z\rangle + \langle R(Y,U)X,Z\rangle,$$

and by the definition of the curvature 3.11:

$$\langle R(X,Y)Z,U\rangle \overset{3.11}{=\!=} -\langle R(Y,X)Z,U\rangle \overset{(3.32)}{=\!=} \langle R(Z,Y)X,U\rangle + \langle R(X,Z)Y,U\rangle.$$

The sum gives

$$2\langle R(X,Y)Z,U\rangle = \langle R(X,Z)Y,U\rangle + \langle R(Z,Y)X,U\rangle + \langle R(Y,U)X,Z\rangle + \langle R(U,X)Y,Z\rangle$$

Exchanging X,Y with Z,U we arrive at

$$2\langle R(Z,U)X,Y\rangle = \langle R(Z,X)U,Y\rangle + \langle R(X,U)Z,Y\rangle + \langle R(U,Y)Z,X\rangle + \langle R(Y,Z)U,X\rangle$$

With $R(X,Y) = -R(Y,X)$ and the first eqn. of (3.42), one sees that the right hand sides of the two equations agree, and therefore $\langle R(X,Y)Z,U\rangle = \langle R(Z,U)X,Y\rangle$. □

The expression for the Riemann tensor in local coordinates follows from Eqs. (3.38) and (3.29).

We now derive the expressions for the Bianchi identities and the symmetries of the Riemann tensor in local coordinates. Metricity implies

$$\nabla_k g_{ij} \equiv g_{ij\,;k} = 0. \tag{3.43}$$

Since $g_{il}g^{lj} = \delta_i^j$ and $\delta_i^{\ j}{}_{;k} = 0$, it follows that also

$$g^{ij}{}_{;k} = 0. \tag{3.44}$$

We define lthe components of the Riemann tensor in a local coordinate system $(x^1, \cdots x^n)$ by

$$R^i_{jkl} := dx^i(R(\partial_k,\partial_l)\partial_j) \quad \text{so that} \tag{3.45}$$

$$R_{ijkl} := \langle \partial_i, R(\partial_k,\partial_l)\partial_j\rangle. \tag{3.46}$$

If $\sum\limits_{(ijk)}$ indicates the cyclic sum of the indices i, j and k, the Bianchi identities for the torsion $T \equiv 0$ can be written as:

$$\sum_{(jkl)} R^i_{jkl} = 0 \quad \text{(1st Bianchi identity)}, \tag{3.47}$$

$$\sum_{(klm)} R^i_{jkl\,;m} = 0 \quad \text{(2nd Bianchi identity)}. \tag{3.48}$$

Equations (3.42) yield

$$R_{ijkl} = -R_{jikl} \quad \text{and} \quad R_{ijkl} = R_{klij}\,. \tag{3.49}$$

Furthermore, we have antisymmetry in the first two arguments (this follows also from (3.49)):

$$R^i_{jkl} = -R^i_{jlk}. \tag{3.50}$$

Contracted Bianchi identity:
Let R_{ik} be the Ricci tensor defined in (3.12)

$$R_{ik} = R^j{}_{ijk}. \tag{3.51}$$

Definition 3.15 (the Riemann scalar) *The* **Riemann scalar** R *is defined by*

$$R = g^{ik}R_{ik} = R^k{}_k\,. \tag{3.52}$$

Proposition 3.10

$$(R_i^{\ k} - \frac{1}{2}\delta_i^{\ k} R)_{;k} = 0 \quad \textit{and} \tag{3.53}$$

$$R_{ik} = R_{ki}. \tag{3.54}$$

Proof: $R_{ik} = g^{jl} R_{lijk}$. The symmetry of the Ricci tensor follows from Eq. (3.49) and from the symmetry of g^{jl}.

$$R_{j\ ;m}^{\ m} = g^{ml} R_{jl\ ;m} = g^{ml} g^{ik} R_{ijkl\ ;m} \overset{(3.49)\sharp 2}{=\!=} g^{ml} g^{ik} R_{klij\ ;m}$$

$$\overset{(3.48)}{=\!=} -g^{ml} g^{ik} (R_{klmi\ ;j} + R_{kljm\ ;i}) = R_{\ ;j} - R^{i}_{\ j\ ;i}$$

With $R_{\ ;j} = (\delta_j^{\ m} R)_{;m}$ we obtain Eq. (3.53). □

Definition 3.16 (Einstein tensor) *The tensor* $G_{ij} := R_{ij} - \frac{1}{2} g_{ij} R$ *is called* **Einstein tensor**.

The contracted Bianchi identity (3.53) is equivalent to

$$G_{i\ ;m}^{\ m} = 0\ . \tag{3.55}$$

This identity is very important for general relativity.

The following theorem elucidates the geometrical meaning of curvature.

Theorem 3.4 *Parallel transport is independent of the path for arbitrary paths (in an open set* $\mathcal{U} \subset \mathcal{M}$*) if and only if the curvature vanishes (in* $\mathcal{U}$*).*

Heuristic consideration :
Let $\gamma : [0,1] \to \mathcal{M}$ be a closed path, $\gamma(0) = \gamma(1) = p$. Consider $v_0 \in T_p\mathcal{M}$ and $\mathcal{T}_t v_0 =: v(t)$ the parallel transport of v_0 along γ:

$$\dot{v}^i = -\Gamma^i_{jk} \dot{x}^j v^k.$$

We want to determine

$$\Delta v^i = v^i(1) - v^i(0) = \int_0^1 \dot{v}^i \mathrm{d}t = -\int_0^1 \Gamma^i_{jk}(\gamma(t)) v^k(t) \dot{x}^j(t) \mathrm{d}t\ ,$$

$$\Delta v^i = \oint_\gamma \Gamma^i_{jk}(x) v^k dx^j\ .$$

Here we make use of Stokes' theorem in the situation shown on figure 3.1:

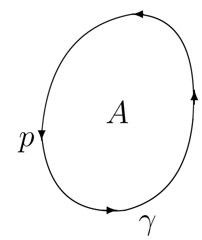


Figure 3.1: Application of Stokes' theorem

$$\oint_\gamma B_j dx^j = \int_A (B_{j\,,l} - B_{l\,,j}) dx^j dx^l \;. \tag{3.56}$$

For us $B_j = \Gamma^i_{jk}(x) v^k(x)$. Hence

$$(B_{j\,,l} - B_{l\,,j}) dx^j dx^l = \left[\Gamma^i_{jk\,,l} v^k - \Gamma^i_{lk\,,j} v^k + \Gamma^i_{jk} v^k_{,l} - \Gamma^i_{lk} v^k_{,j}\right] dx^j dx^l.$$

But

$$v^k_{,l} dx^l = \dot v^k \mathrm{d}t = -\Gamma^k_{mn} \dot x^m v^n \mathrm{d}t = -\Gamma^k_{mn} v^n dx^m.$$

Such that

$$(B_{j\,,l} - B_{l\,,j}) dx^j dx^l = \left[\Gamma^i_{jk\,,l} - \Gamma^i_{lk\,,j} - \Gamma^i_{jm}\Gamma^m_{lk} + \Gamma^i_{lm}\Gamma^m_{jk}\right] v^k dx^j dx^l \;.$$

In order for the integral (3.56) to vanish for every γ and every v_0, we must require

$$[\quad] \equiv 0 \quad \stackrel{(3.30)}{\Leftrightarrow} \quad R^i_{klj}(x) \equiv 0 \;.$$

Proof: We consider a map $H \;:\; [0,1] \times [0,1] \to \mathcal{M} : (s,t) \mapsto H(s,t)$, such that

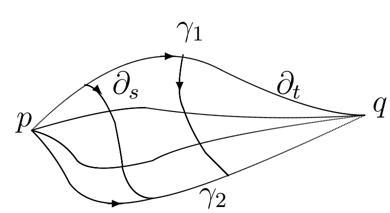


Figure 3.2: Definition of the map H

$H(0,t) = \gamma_1(t)$, $H(1,t) = \gamma_2(t)$, $H(s,0) = p$, and $H(s,1) = q$.
Let $v_0 \in T_p\mathcal{M}$ be given and let $v(s,t)$ be the vector which is parallel transported along the path $t \mapsto H(s,t)$ for arbitrary but fixed s. We set $X = H_\star(\partial_t)$ and $Y = H_\star(\partial_s)$. Then

$$\nabla_X v(s,t)\Big|_{s,t} = 0 \;\text{ and }\; \nabla_Y v(s,t)\Big|_{t=0,s} = 0. \tag{3.57}$$

If the curvature $R \equiv 0$, since $[X, Y] = H_\star([\partial_t, \partial_s]) = 0$,

$$R(X,Y)v = \nabla_X \nabla_Y v - \nabla_Y \nabla_X v = 0 \quad \text{hence} \quad \nabla_X \nabla_Y v = 0.$$

Hence $\nabla_Y v$ is parallel transported along $t \mapsto H(s,t)$. With Eq. (3.57), it then follows that $\nabla_Y v|_{t,s} = 0$ for all $t \in [0,1]$. In the limit $t \to 1$, one obtains that the value $v(s,1)$ does not depend on s:

$$\frac{\mathrm{d}}{\mathrm{d}s} v^i(s,1) + \Gamma^i_{jk}(q) Y^j(s,1) v^k(s,1) = 0.$$

But $Y(s,1) = 0$, hence $\frac{\mathrm{d}}{\mathrm{d}s} v^i(s,1) = 0$.
We now assume that $v(s,t)$ be independent of the path. It follows first that $\nabla_Y v|_{s,t} = 0$, and hence $R(X,Y)v = 0$. Since H and v_0 are arbitrary, this implies that $R \equiv 0$. □

Definition 3.17 (isometry) *Be $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$ two (pseudo-)Riemannian manifolds. A diffeomorphism $\varphi : \mathcal{M} \to \mathcal{N}$ is an* **isometry** *if*

$$\varphi^\star h = g. \tag{3.58}$$

A pseudo-Riemanniann manifold which is (locally) isometric to $(\mathbb{R}^n, \mathring{g})$,

$$\mathring{g} = \sum_{i=1}^n \varepsilon_i dx^i \otimes dx^i, \quad \varepsilon_i = \pm 1 \tag{3.59}$$

is called (locally) **flat**.

Theorem 3.5 *A pseudo-Riemannian manifold is locally flat is and only if the curvature of the Riemannian connection vanishes .*

Proof:

- " $\Rightarrow$ ": Obvious, choose a coordinate system such that the metric has the form of Eq. (3.59).
- " $\Leftarrow$ ": Consider now $R \equiv 0$. According to theorem 3.4, this implies that parallel transport is locally path-independent.

  We choose normal coordinates in p $T_p\mathcal{M}$ $(\partial_i)_{i=1}^n \subset T_p\mathcal{M}$. we can parallel transport it in a well defined way into a neighborhood. In this way we obtain a basis $(e_i)_{i=1}^n$ in an open set $\mathcal{U} \ni p$. By construction, the covariant derivatives vanish:

  $$\nabla_{e_i} e_k = 0, \quad [e_i, e_k] = \nabla_{e_i} e_k - \nabla_{e_k} e_i = 0.$$

  Therefore, (show this!) there exists a local coordinate system such that $e_i = \frac{\partial}{\partial x^i}$.
  Since $g(e_i, e_j)$ does not change under parallel transport, in this system, the metric coefficients g_{ij} are given by eq. (3.59). □

Chapter 4

Differential forms

I first develop some algebraic preparation which I assume to be more or less known from the course "compléments de mathematique 2".

4.1 Exterior algebra

Let A be a commutative, associative, unitary algebra over $\mathbb{R}$ and let E be a module on A:

- **Commutative:** $a, b \in A \Rightarrow ab = ba$
- **Associative:** $a(bc) = (ab)c$
- **Unitary:** $\exists e \in A$ tel que $ea = a$, $\forall a \in A$

We are interested mainly in the case $A = \mathbb{R}$ ou $A = \mathcal{F}(\mathcal{M})$, where $\mathcal{M}$ is a differentiable manifold and E real vector space or $E = \mathcal{X}(\mathcal{M})$.

We consider the space of p-linear forms with values in A.

Definition 4.1

1. $\Lambda_p(E) \subset T_p(E)$ is the space of totally antisymmetric p–forms on E:

$$\alpha(\cdots X \cdots Y \cdots) = -\alpha(\cdots Y \cdots X \cdots)$$

for all $\alpha \in \Lambda_p(E)$ and $X, Y \in E$.

2. *For $t \in T_p(E)$ we define the* **alternation operator** $\mathcal{A}$ *by*

$$(\mathcal{A}t)(v_1, \ldots, v_p) = \frac{1}{p!} \sum_{\sigma \in \mathcal{S}_p} (\mathrm{sgn}\sigma) t(v_{\sigma(1)}, \ldots, v_{\sigma(p)}) \tag{4.1}$$

where $v_1, \ldots, v_p \in E$ and $\mathcal{S}_p$ is the group of permutations of p elements and $\mathrm{sgn}\sigma$ *is the signature of the permutation σ.*

Proposition 4.1 *$\mathcal{A}$ is the projection from $T_p(E)$ to $\Lambda_p(E)$, i.e., $\mathcal{A}$ is a linear operator on $T_p(E)$ with $\mathcal{A}(T_p(E)) = \Lambda_p(E)$. Furthermore, $\mathcal{A} \circ \mathcal{A} = \mathcal{A}$.*

Proof: Exercice.

Definition 4.2 (exterior product) *For $\omega \in \Lambda_p(E)$, $\eta \in \Lambda_q(E)$, we define the* **exterior product**

$$\Lambda_{p+q}(E) \ni \omega \wedge \eta := \frac{(p+q)!}{p!q!} \mathcal{A}(\omega \otimes \eta) \ . \tag{4.2}$$

Proposition 4.2 *The exterior product has the following properties:*

1. $(\omega_1 + \omega_2) \wedge \eta = \omega_1 \wedge \eta + \omega_2 \wedge \eta$
2. $a(\omega \wedge \eta) = (a\omega) \wedge \eta = \omega \wedge (a\eta)$ *for $a \in A$*
3. $\omega \wedge \eta = (-1)^{pq} \eta \wedge \omega$
4. $(\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3)$

$\wedge$ is thus bilinear and associative.

Proof: Exercise.

Proposition 4.3 *Let $(\theta^i)_{i=1}^n$ be a basis of $E^\star = \Lambda_1(E)$. Then the products*

$$(\theta^{i_1} \wedge \theta^{i_2} \wedge \cdots \wedge \theta^{i_p}); \quad 1 \leq i_1 < i_2 < \cdots < i_p \leq n$$

form a basis of $\Lambda_p(E)$.
Consequently the dimension of $\Lambda_p(E)$, $p \leq n$, is

$$\dim(\Lambda_p(E)) = \binom{n}{p} = \frac{n!}{p!(n-p)!}.$$

For $p > n$, $\Lambda_p(E) = \{0\}$.

Proof: Exercise.

Definition 4.3 (Grassmann algebra) *The* **Grassmann algebra** *(or* **exterior algebra***) is the direct sum*

$$\Lambda(E) = \bigoplus_{p=0}^{n} \Lambda_p(E)$$

According to proposition 4.3, $\dim \Lambda(E) = 2^n$*.*
$\Lambda(E)$ *is a graduated algebra (associative and unitary).*

Definition 4.4 (interior product) *The* **interior product** *is the map*

$$\begin{aligned} &E \times \Lambda_p(E) \to \Lambda_{p-1}(E) \\ &(v, \omega) \mapsto i_v\omega \end{aligned}$$

where $(i_v\omega)(v_1, \ldots, v_{p-1}) := \omega(v, v_1, \ldots, v_{p-1})$*.*
For $\omega \in \Lambda_p(E)$ *we define* $i_0\omega \equiv 0$*. The interior product allows us to define the map*

$$i : E \times \Lambda(E) \to \Lambda(E) : (v, \omega) \mapsto i_v\omega.$$

Proposition 4.4

1. i_v *is A-linear*
2. $i_v(\Lambda_p(E)) \subseteq \Lambda_{p-1}(E)$
3. $i_v(\omega \wedge \eta) = (i_v\omega) \wedge \eta + (-1)^p \omega \wedge (i_v\eta)$ *for* $\omega \in \Lambda_p(E)$*.*

In other words, i_v *is an anti-derivation of degree -1 on* $\Lambda(E)$*.*

Proof: Exercice.

4.2 Differential forms and Cartan's formalism

Let $\mathcal{M}$ be a differentiable manifold of dimension m. For $p = 0, 1, \ldots, m$ and $x \in \mathcal{M}$ we consider the spaces

$$\begin{aligned} &\Lambda_p(T_x\mathcal{M}) \subset T_x(\mathcal{M})^0_p \quad \forall p \geq 1 \\ &\Lambda_0(T_x\mathcal{M}) = \mathbb{R};\ \Lambda_1(T_x\mathcal{M}) = (T_x\mathcal{M})^\star \\ &\Lambda(T_x\mathcal{M}) = \bigoplus_{p=0}^{n} \Lambda_p(T_x\mathcal{M}). \end{aligned}$$

Definition 4.5 (differential forms) *A* **differential form** *of degree p is a covariant tensor field of degree p, called ω, such that $\omega(x) \in \Lambda_p(T_x\mathcal{M})$ for all $x \in \mathcal{M}$. Often we call it simply a p-form.*
$\Lambda_p(\mathcal{M})$ is the module of p–forms on $\mathcal{F}(\mathcal{M})$.
$\Lambda(\mathcal{M}) = \bigoplus_{p=0}^{n} \Lambda_p(\mathcal{M})$ *is the exterior algebra of differential forms on $\mathcal{M}$.*

As all the elements of $\Lambda(\mathcal{M})$ are tensor fields, all our results on tensor fields are also valid for differential forms.
The algebraic operations defined in the previous section are defined point by point for the differential forms, also the exterior product. For $\omega \in \Lambda_p(\mathcal{M})$, $X_1, \ldots, X_p \in \mathcal{X}(\mathcal{M})$, the mapping

$$x \mapsto \omega(X_1(x), \ldots, X_p(x))$$

is a function on $\mathcal{M}$. The map

$$\underbrace{\mathcal{X}(\mathcal{M}) \times \cdots \times \mathcal{X}(\mathcal{M})}_{p \text{ fois}} \to \mathcal{F}(\mathcal{M}) : (X_1, \ldots, X_p) \mapsto \omega(X_1, \ldots, X_p)$$

is $\mathcal{F}(\mathcal{M})$-linear and completly anti-symmetric.
For a vector field X we define the interior product

$$(i_X\omega)_x \equiv i_{X(x)}\omega_x$$

In a local coordinate system, $(x^1, \ldots, x^n; \mathcal{U})$, $\omega \in \Lambda_p(\mathcal{M})$ can be written in the basis dx^i as

$$\begin{aligned} \omega &= \sum_{1 \le i_1 < \cdots < i_p \le n} \omega_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \\ &= \frac{1}{p!} \sum_{1 \le i_1, \cdots, i_p \le n}^{n} \omega_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \ , \end{aligned}$$

where the $\omega_{i_1 \cdots i_p}$ with arbitrary index positions are obtained from those with $i_1 < i_2 \ldots < i_p$ by anti-symmetry.

Let $\varphi : \mathcal{M} \to \mathcal{N}$ be a differentiable map. As we have seen in chapter 3 (proposition 2.6), the pull-back is linear and respects the tensor product, $\otimes$. This implies that the pull-back

$$\varphi^\star : \Lambda(\mathcal{N}) \to \Lambda(\mathcal{M})$$

respects the exterior product, $\varphi^\star(\omega \wedge \eta) = \varphi^\star\omega \wedge \varphi^\star\eta$. It is therefore an algebra homomorphism from $\Lambda(\mathcal{N})$ into $\Lambda(\mathcal{M})$. If φ is a diffeomorphism, $\varphi^\star$ is even an isomorphism with $(\varphi^\star)^{-1} = (\varphi^{-1})^\star$.

Definition 4.6 (derivation, anti-derivation) *A map $\theta : \Lambda(\mathcal{M}) \to \Lambda(\mathcal{M})$ is a* **derivation** *(respectively* **anti-derivation***) of degree $k \in \mathbb{Z}$, if*

1. *θ is $\mathbb{R}$-linear*

2. *$\theta(\omega \wedge \eta) = \theta\omega \wedge \eta + \omega \wedge \theta\eta$, for $\omega, \eta \in \Lambda(\mathcal{M})$ (anti-derivation if: $\theta(\omega \wedge \eta) = \theta\omega \wedge \eta + (-1)^p \omega \wedge \theta\eta$, $\omega \in \Lambda^p(\mathcal{M})$, $\eta \in \Lambda(\mathcal{M})$)*

3. *$\theta(\Lambda_p(\mathcal{M})) \subset \Lambda_{p+k}(\mathcal{M})$, $0 \le p \le n$.*

Proposition 4.5 *For anti-derivations θ, θ' of degree k, k', $\theta \circ \theta' + \theta' \circ \theta$ is a derivation ofe degree $k + k'$, if k and k' are both odd.*

Proof: Simple calculation.

Proposition 4.6 *The (anti-)derivations of $\Lambda(\mathcal{M})$ are* **local***, i.e. for an open set $\mathcal{U} \subset \mathcal{M}$ and $\omega \in \Lambda(\mathcal{M})$ such that $\omega|_{\mathcal{U}} = 0$ we have $\theta\omega|_{\mathcal{U}} = 0$ for every (anti-)derivation θ.*

Proof: For $x \in \mathcal{U}$ there exists a function $h \in \mathcal{F}(\mathcal{M})$ such that $h(x) = 1$ and $h|_{\mathcal{M}\setminus\mathcal{U}} = 0$. Hence $h \cdot \omega = 0$. Linearity then implies, $\theta(h\omega) = 0$, and therefore $\theta h \wedge \omega + h \cdot \theta\omega = 0$ in x which implies $(\theta\omega)_x = 0$. □

Consequently, for $\omega = \omega'$ in $\mathcal{U} \subset \mathcal{M}$ we have $\theta\omega = \theta\omega'$ in $\mathcal{U}$ for every derivation θ. We can therefore uniquely define $\theta|_{\mathcal{U}}$ on $\Lambda(\mathcal{U})$:
for $x \in \mathcal{U}$ and $\alpha \in \Lambda(\mathcal{U})$ we choose $\tilde{\alpha} \in \Lambda(\mathcal{M})$ such that $\tilde{\alpha} = \alpha$ in a neighborhood of x and we set

$$\left(\theta\Big|_{\mathcal{U}}\right)\alpha(x) = (\theta\tilde{\alpha})(x).$$

According to proposition 4.6, this definition is independent of the choice of $\tilde{\alpha}$. The existence of such an extension $\tilde{\alpha}$ is a consequence of the continuation lemma:

Lemma 4.1 (continuation lemma) *Let $\mathcal{U} \subset \mathcal{M}$ be an open set and $K \subset \mathcal{U}$ a compact set. For all $\beta \in \Lambda(\mathcal{U})$ there existe an $\alpha \in \Lambda(\mathcal{M})$ such that*

$$\beta\Big|_K = \alpha\Big|_K \quad et \quad \alpha\Big|_{\mathcal{M}\setminus\mathcal{U}} = 0.$$

Proof: There exists a function $h \in \mathcal{F}(\mathcal{M})$ with $h(x) = 1 \; \forall x \in K$ and $h(x) = 0 \, \forall x \in \mathcal{M} \setminus \mathcal{U}$. We can thus choose

$$\alpha(x) = \begin{cases} h(x)\beta(x), & x \in \mathcal{U} \\ 0, & x \in \mathcal{M} \setminus \mathcal{U} \end{cases}$$

□

We hence have the following result:

Proposition 4.7 (localisation theorem) *Let θ be an (anti-)derivation on $\Lambda(\mathcal{M})$, $\mathcal{U} \subset \mathcal{M}$ an open set. There exists a unique (anti-)derivation $\theta_{\mathcal{U}}$ on $\Lambda(\mathcal{U})$ such that*

$$(\theta\alpha)\Big|_{\mathcal{U}} = \theta_{\mathcal{U}}\Big(\alpha\Big|_{\mathcal{U}}\Big) \quad \text{for all } \alpha \in \Lambda(\mathcal{M})$$

We also need a globalisation theorem:

Proposition 4.8 (globalisation theorem) *Let $(\mathcal{U}_i)_{i\in I}$ be an open covering of $\mathcal{M}$. For $i \in I$, let θ_i be an (anti-)derivation on $\Lambda(\mathcal{U}_i)$ and θ_{ij} its restriction to $\mathcal{U}_i \cap \mathcal{U}_j$. If $\theta_{ij} = \theta_{ji}$ for every pair $(i,j) \in I \times I$ there exists a unique (anti-)derivation $\theta \in \Lambda(\mathcal{M})$ such that $\theta_i = \theta|_{\mathcal{U}_i}$.*

Proof: For $\alpha \in \Lambda(\mathcal{M})$ and $x \in \mathcal{U}_i$ we define

$$(\theta\alpha)_x = \theta_i\left(\alpha\Big|_{\mathcal{U}_i}\right)_x . \tag{4.3}$$

Since

$$\begin{aligned}\left((\theta\alpha)\Big|_{\mathcal{U}_j}\right)\Big|_{\mathcal{U}_i} = \theta_{ij}\left(\alpha\Big|_{\mathcal{U}_i\cap\mathcal{U}_j}\right) &= \theta_{ji}\left(\alpha\Big|_{\mathcal{U}_i\cap\mathcal{U}_j}\right) \\ &= \left((\theta\alpha)\Big|_{\mathcal{U}_i}\right)\Big|_{\mathcal{U}_j}\end{aligned}$$

Eq. (4.3) is independent of the choice of $\mathcal{U}_i$ as long as $x \in \mathcal{U}_i$, and hence $\theta\alpha$ is well defined. □

We shall also use the following fact:

Proposition 4.9 *Let θ be an (anti-)derivation of degree k and $\theta f = \theta df = 0$ for all $f \in \mathcal{F}(\mathcal{M})$. Then*

$$\theta \equiv 0.$$

Proof: We choose an atlas $(h_i, \mathcal{U}_i)$ of $\mathcal{M}$. We then set $\theta_i := \theta|_{\mathcal{U}_i}$. It is thus enough to show that $\theta_i \equiv 0$ for all i. But in a local coordinate system $(x^1, \ldots, x^n)$ sur $\mathcal{U}_i$ and $\alpha \in \Lambda_p(\mathcal{M})$,

$$\alpha\Big|_{\mathcal{U}_i} = \sum \alpha_{j_1\cdots j_p} dx^{j_1} \wedge \cdots \wedge dx^{j_p}.$$

And because of Leibnitz's rule (point 2 of definition 4.6)

$$(\theta\alpha)\Big|_{\mathcal{U}_i} = \theta_i\left(\alpha\Big|_{\mathcal{U}_i}\right) = 0.$$

□

Consequence 4.10 *An (anti-)derivation on $\Lambda(\mathcal{M})$ is uniquely determined by its values on the functions ($= \Lambda_0(\mathcal{M})$) and on the "gradients", $\{df \mid f \in \mathcal{F}(\mathcal{M})\} \subset \Lambda_1(\mathcal{M})\}$.*

4.3 The exterior derivative

Theorem 4.1 *There exists a unique map*

$$d : \Lambda(\mathcal{M}) \to \Lambda(\mathcal{M})$$

with the following properties:

1. *d is an anti-derivation of degree 1*
2. *$d \circ d = 0$*
3. *df is the gradient of f for all $f \in \mathcal{F}(\mathcal{M})$, i.e., $df(X) = Xf$, for $f \in \mathcal{F}(\mathcal{M})$, $X \in \mathcal{X}(\mathcal{M})$.*

Proof: The uniqueness is a consequence of proposition 4.9. Since a form $\alpha \in \Lambda_p(\mathcal{M})$ on a chart $(x^1, \ldots, x^n; \mathcal{U})$ is of the form

$$\alpha\Big|_{\mathcal{U}} = \sum_{i_1 \cdots i_p} \alpha_{i_1 \cdots i_p} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}, \quad \alpha_{i_1 \cdots i_p} \in \mathcal{F}(\mathcal{M})$$

Points 2 & 3 and the Leibniz rule determine

$$d\alpha\Big|_{\mathcal{U}} \stackrel{2, Leibn.}{=\!=} \sum_{i_1 \cdots i_p} d\alpha_{i_1 \cdots i_p} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} \in \Lambda_{p+1}(\mathcal{M}) \tag{4.4}$$

$$\stackrel{3}{=\!=} \sum_{i_1 < \cdots i_k < i_{p+1}} \sum_{k=1}^{p+1} (-1)^{k+1} \frac{\partial}{\partial x^{i_k}} \alpha_{i_1 \cdots \widehat{i_k} \cdots i_{p+1}} dx^{i_1} \wedge \cdots \wedge dx^{i_{p+1}}.$$

(The notation $i_1 \cdots \widehat{i_k} \cdots i_{p+1}$ means: "leave out the indext i_k".) The globalisation theorem 4.8 implies then the existence of d. □The components of $d\alpha$ are given by

$$(d\alpha)_{i_1 \cdots i_{p+1}} = -\sum_{k=1}^{p+1} (-1)^k \frac{\partial}{\partial x^{i_k}} \alpha_{i_1 \cdots \widehat{i_k} \cdots i_{p+1}}, \quad i_1 < i_2 < \cdots < i_{p+1} \tag{4.5}$$

Definition 4.7 (exact and closed differential forms)
A differential form $\alpha \subset \Lambda(\mathcal{M})$ is called **exact** *if there exists a form β such that $\alpha = d\beta$; α is called* **closed** *if $d\alpha = 0$.*

Since $d \circ d = 0$, every exact form is closed. Locally, the inverse is also true:

Lemma 4.2 (Poincaré Lemma) *Let $\alpha \in \Lambda(\mathcal{M})$ be closed. For all $x \in \mathcal{M}$ exists an open set $\mathcal{U} \subset \mathcal{M}$, with $x \in \mathcal{U}$ such that $\alpha|_{\mathcal{U}}$ is exact.*

Proof: See e.g. Spivak [14], "Calculus on manifolds".

Proposition 4.11 *Let $\varphi : \mathcal{M} \to \mathcal{N}$ be a differentiable map from the differentiable manifold $\mathcal{M}$ to the differentiable manifold $\mathcal{N}$. The following diagram is commutative, in other words $d \circ \varphi^\star = \varphi^\star \circ d$.*

$$\begin{tikzcd}
\Lambda(\mathcal{M}) \arrow[d, "d"'] & \Lambda(\mathcal{N}) \arrow[l, "\varphi^\star"'] \arrow[d, "d"] \\
\Lambda(\mathcal{M}) & \Lambda(\mathcal{N}) \arrow[l, "\varphi^\star"']
\end{tikzcd}$$

Proof: For functions, we have already shown this property of the pull-back, see definition 1.23, Eq. (1.54). We have thus

$$(d \circ \varphi^\star)df = (d \circ d)(\varphi^\star f) = 0 = \varphi^\star((d \circ d)f)$$

Our statement now follows with prop. 4.9. □

4.4 Relations between d, i_X and L_X

According to def. 4.6, d is an anti-derivation of degree 1, i_X, $X \in \mathcal{X}(\mathcal{M})$ is an anti-derivation of degree -1 and L_X is a derivation of degree 0 on $\Lambda(\mathcal{M})$.

Proposition 4.12 (Cartan's formula) *For $X \in \mathcal{X}(\mathcal{M})$ we have*

$$L_X = d \circ i_X + i_X \circ d \tag{4.6}$$

Proof: According to proposition 4.5, $\theta = d \circ i_X + i_X \circ d$ is a derivation of degree 0. Hence if $\theta f = L_X f$ and $\theta(df) = L_X df$ for all $f \in \mathcal{F}(\mathcal{M})$, Eq. (4.6) is shown. But for $f \in \mathcal{F}(\mathcal{M})$

$$\theta(f) = i_X df = df(X) = Xf = L_X f,$$

and

$$\theta(df) = d \circ i_X df = d(Xf).$$

On the other hand

$$\begin{aligned}(L_X df)(Y) &= L_X(df(Y)) - df(L_X Y) = L_X(Yf) - df([X,Y]) \\ &= X(Yf) - [X,Y]f = Y(Xf) = (d(Xf))\,(Y)\ .\end{aligned}$$

□With $d\circ d = 0$, Eq. (4.6) implies

$$L_X \circ d = d\circ L_X = d\circ i_X \circ d\ . \qquad (4.7)$$

Furthermore (exercise!)

$$i_{[X,Y]} = [L_X, i_Y]. \qquad (4.8)$$

Proposition 4.13 *For $\omega \in \Lambda_{p-1}(\mathcal{M})$,*

$$\begin{aligned}d\omega(X_1,\dots,X_p) &= \sum_{1\le i\le p}(-1)^{i+1}X_i\omega(X_1,\dots,\widehat{X_i},\dots,X_p) \\ &+ \sum_{1\le i<j\le p}(-1)^{i+j}\omega([X_i,X_j],X_1,\dots,\widehat{X_i},\dots,\widehat{X_j},\dots,X_p)\end{aligned} \qquad (4.9)$$

(again $\widehat{X_i}$ denote omission of X_i).

Proof: For $p = 1$, Eq. (4.9) reduces to $df(X) = Xf$.
For $\omega \in \Lambda_1(\mathcal{M})$, (4.6) gives

$$(L_X\omega)(Y) = (i_X d\omega)(Y) + d(i_X\omega)(Y) = d\omega(X,Y) + Y(\omega(X)).$$

With $(L_X\omega)(Y) = X(\omega(Y)) - \omega([X,Y])$ it follows that

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]),$$

i.e., Eq. (4.9).
By induction one can now show the step from p to $p+1$ using Eq. (4.6) and the explicit formula for $L_X\omega$. □

Proposition 4.14 *Let ∇ be a covariant derivative for a symmetric connection. For $\omega \in \Lambda_p(\mathcal{M})$ we find*

$$\mathcal{A}(\nabla\omega) = \frac{(-1)^p}{p+1}d\omega \qquad (4.10)$$

Proof: For $\omega \in \Lambda_p(\mathcal{M})$

$$\nabla\omega(X_2,\dots,X_{p+1},X_1) = (\nabla_{X_1}\omega)(X_2,\dots,X_{p+1})$$

$$= X_1(\omega(X_2, \dots, X_{p+1})) - \sum_{i=2}^{p+1} \omega(X_2, \dots, \nabla_{X_1} X_i, \dots, X_{p+1})$$

$$\mathcal{A}(\nabla\omega)(X_2, \dots, X_{p+1}, X_1) = \frac{1}{p+1} \left[\sum_{i=1}^{p+1} (-1)^{i+1} X_i \omega(X_1, \dots, \widehat{X_i}, \dots, X_{p+1}) \right.$$

$$\left. + \sum_{i<j} (-1)^{i+j} \omega(\nabla_{X_i} X_j - \nabla_{X_j} X_i, X_1, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_p) \right] .$$

But since the torsion vanishes $\nabla_{X_i} X_j - \nabla_{X_j} X_i = [X_i, X_j]$. Hence

$$\mathcal{A}(\nabla\omega)(X_1, \dots, X_{p+1}) = \frac{(-1)^p}{p+1} d\omega(X_1, \dots, X_{p+1}) .$$

□

4.5 Cartan's formalism

Definition 4.8 (Connection 1-forms) *Be ∇ an affine connection on $\mathcal{M}$ and be $(e_1, \dots, e_n)$ a basis of vector fields on an open set $\mathcal{U} \subset \mathcal{M}$. Be $(\theta_1, \dots, \theta_n)$ the dual basis of 1-forms. We define the* **connection 1-forms** *$\omega^i_{\ j} \in \Lambda_1(\mathcal{U})$ by*

$$\nabla_X e_j = \omega^i_{\ j}(X) e_i. \tag{4.11}$$

We also define the Christoffel symbols with respect to the basis $\{e_i\}$ by

$$\nabla_{e_k} e_j = \Gamma^i_{kj} e_i = \omega^i_{\ j}(e_k) e_i. \tag{4.12}$$

With this we obtain

$$\omega^i_{\ j} = \Gamma^i_{kj} \theta^k. \tag{4.13}$$

Proposition 4.15 *For a vector field $X = X^i e_i$,*

$$\nabla X = e_i \otimes (dX^i + \omega^i_{\ k} X^k). \tag{4.14}$$

For a 1-form $\alpha = \alpha_i \theta^i$,

$$\nabla \alpha = \theta^i \otimes (d\alpha_i - \omega^k_{\ i} \alpha_k). \tag{4.15}$$

Proof: Equation (4.14) follows from (4.11) and the Leibniz rule. For (4.15), we use that ∇_X commutes with contractions:

$$0 = \nabla_X(\theta^i(e_j)) = (\nabla_X \theta^i)(e_j) + \theta^i(\nabla_X e_j).$$

Therefore

$$(\nabla_X \theta^i)(e_j) = -\omega^i_{\ j}(X),$$

so that

$$\nabla_X \theta^i = -\omega^i_{\ j}(X)\theta^j. \tag{4.16}$$

With this and the Leibniz rule, equation (4.15) follows. □

Definition 4.9 (torsion and curvature 2-forms) *Since the torsion $T(X,Y)$ and the curvature $R(X,Y)Z$ are anti-symmetric in X and Y, we can define* **torsion and curvature 2-forms** Θ^i *and* $\Omega^i_{\ j}$ *by*

$$T(X,Y) = \Theta^i(X,Y)e_i \tag{4.17}$$

$$R(X,Y)e_j = \Omega^i_{\ j}(X,Y)e_i. \tag{4.18}$$

Theorem 4.2 *The torsion and curvature 2-forms satisfy the* **structure equations of Cartan***:*

$$d\theta^i + \omega^i_{\ j} \wedge \theta^j = \Theta^i \tag{4.19}$$

$$d\omega^i_{\ j} + \omega^i_{\ k} \wedge \omega^k_{\ j} = \Omega^i_{\ j} \tag{4.20}$$

Proof: For (4.19):

$$\begin{aligned}\Theta^i(X,Y)e_i &= \nabla_X Y - \nabla_Y X - [X,Y] = \nabla_X(\theta^i(Y)e_i) - \nabla_Y(\theta^i(X)e_i) - \theta^i([X,Y])e_i \\ &= \left\{X(\theta^i(Y)) - Y(\theta^i(X)) - \theta^i([X,Y])\right\} e_i + \theta^i(Y)\omega^j_{\ i}(X)e_j - \theta^i(X)\omega^j_{\ i}(Y)e_j \\ &= (d\theta^i + \omega^i_{\ l} \wedge \theta^l)(X,Y)e_i.\end{aligned}$$

And for (4.20):

$$\begin{aligned}\Omega^i_{\ j}(X,Y)e_i &= \nabla_X\nabla_Y e_j - \nabla_Y\nabla_X e_j - \nabla_{[X,Y]}e_j \\ &= \nabla_X(\omega^i_{\ j}(Y)e_i) - \nabla_Y(\omega^i_{\ j}(X)e_i) - \omega^i_{\ j}([X,Y])e_i \\ &= \left\{X(\omega^i_{\ j}(Y)) - Y(\omega^i_{\ j}(X)) - \omega^i_{\ j}([X,Y])\right\} e_i + \{\omega^i_{\ j}(Y)\omega^k_{\ i}(X) - \omega^i_{\ j}(X)\omega^k_{\ i}(Y)\}e_k \\ &= (d\omega^i_{\ j} + \omega^i_{\ l} \wedge \omega^l_{\ j})(X,Y)e_i\ .\end{aligned}$$

□

Setting $R^i_{jkl} = \theta^i(R(e_k,e_l)e_j) = \Omega^i_{\ j}(e_k,e_l)$, we obtain

$$\Omega^i_{\ j} = \frac{1}{2}R^i_{jkl}\theta^k \wedge \theta^l. \tag{4.21}$$

Equivalently

$$\Theta^i = \frac{1}{2}T^i_{kl}\theta^k \wedge \theta^l \tag{4.22}$$

where $T^i_{kl} = \theta^i(T(e_k,e_l))$.

Proposition 4.16 *A connection is metric if and only if*

$$dg_{ik} = \omega_{ik} + \omega_{ki} \tag{4.23}$$

where $\omega_{ik} := g_{il}\omega^l{}_k$; $g_{ij} = g(e_i, e_j)$.

Proof: By definition the connection ∇ is metric if $(\nabla_X g)_{ik} = X(g_{ik}) - g(\nabla_X e_i, e_k) - g(e_i, \nabla_X e_k) = 0$ for all vector fields X.Therefore, for a metric connection

$$dg_{ik}(X) \equiv X(g_{ik}) = g(\nabla_X e_i, e_k) + g(e_i, \nabla_X e_k) = g(\omega^j{}_i(X)e_j, e_k) + g(e_i, \omega^j{}_k(X)e_j)$$
$$= \omega^j{}_i(X)g_{jk} + \omega^j{}_k(X)g_{ij} = \omega_{ki}(X) + \omega_{ik}(X)\ .$$

□

For the **Riemannian connection** we therefore obtain the following equations:

$$\omega_{ij} + \omega_{ji} = dg_{ij} \tag{4.24}$$

$$d\theta^i + \omega^i{}_j \wedge \theta^j = 0 \tag{4.25}$$

$$d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j = \Omega^i{}_j = \frac{1}{2} R^i_{jkl}\theta^k \wedge \theta^l. \tag{4.26}$$

These are the Cartan structure equation for a Riemannian connection.

The formal solution of Cartan's structure equations for a Riemannian (or Levi-Civita) connection

Be $(e_i)_{i=1}^n$ and $(\theta^i)_{i=1}^n$ local bases of vector fields and 1-forms with $\theta^i(e_j) = \delta^i{}_j$, and $g_{ij} = g(e_i, e_j)$. For an orthonormal basis, $g_{ij} = \pm\delta_{ij}$.
We expand $d\theta^i$:

$$d\theta^i = -\frac{1}{2} C^i_{jl}\theta^j \wedge \theta^l \tag{4.27}$$

The choice of the basis (θ^i) determines the C^i_{jk} and the metric components g_{ij} since $g = g_{ij}\theta^i\theta^j$. We now compute the connection 1-forms, $\omega^i{}_j$ and the curvature 2-forms $\Omega^i{}_j$ from the C^i_{jl} and the metric components g_{ij}. For a holonomic basis, i.e., a basis of the form $\theta^i = dx^i$, we have $C^i_{kl} \equiv 0$.
With (4.13) and (4.25) (the first structure equation of Cartan) this yields

$$\left(-\frac{1}{2} C^i_{jl} + \Gamma^i_{jl}\right)\theta^j \wedge \theta^l = 0,$$

so that

$$\Gamma^i_{jl} - \Gamma^i_{lj} = C^i_{jl} \tag{4.28}$$

For a holonomic basis, the Γ^i_{jk} are symmetric.
We now define for an arbitrary basis

$$g_{ij,k} := e_k(g_{ij}),$$

so that $dg_{ij} = g_{ij,k}\theta^k$. Since $\omega_{ij} = g_{il}\Gamma^l_{\ kj}\theta^k$, (4.24) gives for an arbitrary basis

$$g_{il}\Gamma^l_{\ kj} + g_{jl}\Gamma^l_{\ ki} = g_{ij,k}.$$

For a orthonormal basis, the $\Gamma_{ikj} := g_{il}\Gamma^l_{\ kj}$ are therefore antisymmetric in ij. With cyclic permutation we obtain

$$g_{ki,j} = g_{kl}\Gamma^l_{\ ji} + g_{il}\Gamma^l_{\ jk}$$

$$g_{jk,i} = g_{jl}\Gamma^l_{\ ik} + g_{kl}\Gamma^l_{\ ij}$$

With eq. (4.28) this leads to

$$(g_{ij,k} + g_{kj,i} - g_{ik,j}) = g_{kl}C^l_{ij} + g_{il}C^l_{kj} + g_{jl}(\Gamma^l_{\ ki} + \Gamma^l_{\ ik})$$

Multiplication with g^{mj} gives

$$\Gamma^m_{\ ki} + \Gamma^m_{\ ik} = g^{mj}(g_{ij,k} + g_{kj,i} - g_{ik,j}) - g^{mj}g_{kl}C^l_{ij} - g^{mj}g_{il}C^l_{kj}\ .$$

With (4.28) we find

$$\Gamma^m_{\ ki} = \frac{1}{2}g^{mj}(g_{jk,i} + g_{ji,k} - g_{ik,j}) + \frac{1}{2}(C^m_{ki} - g^{mj}g_{li}C^l_{kj} - g^{mj}g_{kl}C^l_{ij}). \qquad (4.29)$$

For a holonomic basis ($\theta^i = dx^i$), only the first part of (4.29) is non-vanishing and we find again the result (3.38).
For an orthonormal basis only the second part is non-vanishing and[1]

$$\Gamma^m_{\ ki} = \frac{1}{2}(C^m_{ki} - \varepsilon_m\varepsilon_k C^k_{im} - \varepsilon_m\varepsilon_i C^i_{km})\ .$$

According to (4.13),

$$d\omega^i_{\ j} = d\Gamma^i_{\ kj}\wedge\theta^k + \Gamma^i_{\ kj}d\theta^k$$

$$d\Gamma^i_{\ kj} = e_l(\Gamma^i_{\ kj})\theta^l =: \Gamma^i_{\ kj,l}\theta^l$$

$$d\omega^i_{\ j} = \Gamma^i_{\ kj,l}\theta^l\wedge\theta^k - \frac{1}{2}\Gamma^i_{\ kj}C^k_{lm}\theta^l\wedge\theta^m$$

$$d\omega^i_{\ j} = \frac{1}{2}\left(\Gamma^i_{\ kj,l} - \Gamma^i_{\ lj,k} - \Gamma^i_{\ mj}C^m_{lk}\right)\theta^l\wedge\theta^k\ .$$

So that

$$\Omega^i_{\ j} = d\omega^i_{\ j} + \omega^i_{\ m}\wedge\omega^m_{\ j} = \left[\frac{1}{2}(\Gamma^i_{\ kj,l} - \Gamma^i_{\ lj,k} - \Gamma^i_{\ mj}C^m_{lk}) + \Gamma^i_{\ lm}\Gamma^m_{\ kj}\right]\theta^l\wedge\theta^k$$

$$= \frac{1}{2}R^i_{jlk}\theta^l\wedge\theta^k\ .$$

Hence

$$R^i_{jlk} = \Gamma^i_{\ kj,l} - \Gamma^i_{\ lj,k} - \Gamma^i_{\ mj}C^m_{lk} + \Gamma^i_{\ lm}\Gamma^m_{\ kj} - \Gamma^i_{\ km}\Gamma^m_{\ lj}\ . \qquad (4.30)$$

[1] $\varepsilon_m = g_{mm} = \pm 1$

Proposition 4.17 *The torsion and curvature forms satisfy the Bianchi identities,*

$$D\Theta^i \equiv d\Theta^i + \omega^i_l \wedge \Theta^l = \Omega^i{}_j \wedge \theta^j \ , \tag{4.31}$$

$$D\Omega^i{}_j \equiv d\Omega^i{}_j + \omega^i{}_l \wedge \Omega^l{}_j - \omega^l{}_j \wedge \Omega^i{}_l = 0 \ . \tag{4.32}$$

Proof: Of (4.31):

$$d\Theta^i + \omega^i_l \wedge \Theta^l = d(d\theta^i + \omega^i{}_j \wedge \theta^j) + \omega^i{}_j \wedge d\theta^j + \omega^i{}_l \wedge \omega^l{}_j \wedge \theta^j$$

$$= d\omega^i{}_j \wedge \theta^j + \omega^i{}_l \wedge \omega^l{}_j \wedge \theta^j = \Omega^i{}_j \wedge \theta^j \ .$$

Of (4.32):

$$d\Omega^i{}_j + \omega^i{}_l \wedge \Omega^l{}_j - \omega^l{}_j \wedge \Omega^i{}_l = d(d\omega^i{}_j + \omega^i{}_k \wedge \omega^k_j) + \omega^i{}_l \wedge (d\omega^l{}_j + \omega^l{}_k \wedge \omega^k_j) - \omega^l{}_j \wedge (d\omega^i{}_l + \omega^i{}_k \wedge \omega^k_l)$$

$$= d\omega^i{}_l \wedge \omega^l{}_j - \omega^i{}_l \wedge d\omega^l{}_j + \omega^i{}_l \wedge d\omega^l{}_j + \omega^i{}_l \wedge \omega^l{}_k \wedge \omega^k_j - \omega^l{}_j \wedge d\omega^i{}_l - \omega^l{}_j \wedge \omega^i{}_k \wedge \omega^k_l = 0 \ .$$

□

Exercise: Show that in a holonomic basis, $e_i = \partial_i$, $\theta^i = dx^i$, the Bianchi identities (4.31) et (4.32) are equivalent to (3.32) and (3.33).

The fact that with Cartan's formalism the Bianchi identities are nearly trivial shows how well this formalism is adapted to differential geometry.

Part II

General Relativity

Introduction

Among the physical theories (classical mechanics by Newton, Lagrange, Hamilton etc; electrodynamics by Faraday and Maxwell; quantum mechanics, etc.), General Relativity takes a special place, first, it is the only physical theory that has been developed by one single person, A. Einstein. Furthermore, General Relativity was not motivated by empirical facts[2], but by a contradiction between Newtonian gravity and the fundamental principles of spacetime formulated in the special theory of relativity. M. Born has made the following statement on General Relativity:
"(Die allgemeine Relativitätstheorie) erschien uns, erscheint mir auch heute als die grösste Leistung menschlichen Denkens über die Natur, die erstaunlichste Vereinigung von philosophischer Tiefe, physikalischer Intuition und mathematischer Kunst. Ich bewundere sie wie ein Kunstwerk."

Newtonian gravity ($\vec{\nabla} \wedge \vec{g} = 0$, $\vec{\nabla} \cdot \vec{g} = 4\pi G\rho$, where $\dot{\vec{v}} = \vec{g} = -\vec{\nabla}\Phi$, $\Delta\Phi = -4\pi G\rho$) is incompatible with special relativity (action at a distance): the notion of simultaneity depends on the coordinate system, and the Newtonian field $\vec{g}$ in a point p depend on the rest frame and has no physical meaning.

Einstein (like also others) has first tried to replace $\Delta\Phi$ by $\Box\Phi$ (and ρ by T^μ_μ) but in the equations of motion for test particles, that he found with this attempt, the acceleration of a particle in a vertical gravitational field depends on the kinetic energy of the particle, and therefore on its horizontal velocity.
This result was incompatible with the experience that all bodies experience the same gravitational acceleration, equality of heavy mass and inertial mass, which Einstein has immediately recognized as fundamental truth.
Einstein has worked about 10 years to find the theory of General Relativity. Once he wrote to Sommerfeld on his work concerning General Relativity:
"...Aber das eine ist sicher, dass ich mich im Leben noch nicht annähernd so geplagt habe, und dass ich grosse Hochachtung für die Mathematik eingeflösst bekommen habe, die ich bis jetzt in ihren subtileren Teilen in meiner Einfalt für puren Luxus ansah! Gegen dieses Problem ist die ursprüngliche Relativitätstheorie eine Kinderei..."

In General Relativity (GR), the structure of spacetime from special relativity is generalised. The basis of this generalisation it the **principle of equivalence**, according to which gravity can be eliminated locally in a non-rotating system in free fall. In other words, infinitesimally, in such an inertial system **locally**, special relativity (SR) is valid.
But the metric **varies** from point to point. In mathematical language. spacetime

[2]With the exception of the small perhelion advance of Mercury: after perturbative corrections due to the other planets ($\sim 530''$ per century) about $43''$ per century remain unexplained by Newtonian gravity.

is a pseudo-Riemannian manifold. The metric g of signature $(-,+,+,+)$ (Lorentz manifold) does not only determine the metrical and causal properties of spacetime but it describes also the gravitational field. It become a dynamical element and is related to the energy-momentum tensor of matter by Einstein's field equations.

GR unifies geometry and gravitation. It is a mathematical fact that for any given point x on a Lorentz manifold one can find local coordinates such that

1. $$(g_{\mu\nu}(x)) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2. $g_{\mu\nu,\lambda}(x) = 0.$

We call such coordinates a local inertial system in the point x.

In such an inertial system the gravitational field is locally (in one point) eliminated and the equations of special relativity are valid. This equivalence principle will dictate the form of the equations of motion of a particle, of Maxwell's equations, etc. in the presence of gravitational fields (see chapter 5).

Einstein's field equations relate the metric g to the masses and energies present; they form the central part of General Relativity (see chapter 6).

Einstein has found them after many years of intensive research. These are non-linear partial differential equations which relate g to the energy-momentum tensor of matter. One can show that, with some basic conditions they are (nearly) unique.

Einstein could also show that in the limit of small velocities and weak gravitational fields one recovers Newtionian gravity. Furthermore, he could derive the perihelion advance of Mercury as a relativistic correction to gravity (see chapter 7).

After all his success Einstein said once:
"Im Lichte bereits erlangter Erkenntnis erscheint das glücklich erreichte fast wie selbstverständlich, und jeder intelligente Student erfasst es ohne zu grosse Mühe. Aber das ahnungsvolle, Jahre währende Suchen im Dunkeln mit seiner gespannten Sehnsucht, seiner Abwechslung von Zuversicht und Ermattung und seinem endlichen Durchbrechen zur Wahrheit, das kennt nur, wer es selber erlebt hat."

In the beginning, GR, found in 1915, had little influence on the developments of physics. Even though, after the experimental confirmation of Einstein's formula for light deflection in 1919, Einstein became a famous public figure, GR was not very relevant for the advances of theoretical physics at that time. In the 20ties quantum electrodynamics has been developed. Later (in the 40-60) the standard

model of elementary particles has emerged, and people have shown that gravity cannot be quantized in the same perturbative manner as the interactions of the standard model (it is not renormalisible).
Since the 80 the main efforts of fundamental theoretical physics are in the direction of a quantum theory of gravity (superstrings, M-theory, the Ashtekar program, loop quantum gravity, ...).
Also in experimental and observational physics, GR has for a long time not plaid an important role. Today, this has changed drastically. Many astrophysical phenomena (X-ray emission, gamma ray bursts, quasars, pulsars, ...) are understood via interactions of neutron stars and black holes with their environnement. Also in cosmology is GR indispensable.
Nobel Prizes related to GR are:

- Penzias & Wilson (1978): discovery of the cosmic microwave background
- Chandrasekhar (1983) for the mass limit of white dwarfs.
- Hulse & Taylor (1993): indirect discovery of gravitational radiation via the study of a binary pulsar.
- Mather & Smooth (2006): for the COBE satellite experiment measuring the spectrum and fluctuations of the cosmic microwave background.
- Perlmutter, Riess and Schmidt (2011): for the discovery of the accelerated expansion of the Universe.
- (But not the one of Einstein!)

Chapter 5

The equivalence principle

5.1 Characteristic properties of gravity

Gravity is the weakest among the four interactions. To see this, let us compare the gravitational and electromagnetic force between two electrons. We find

$$\frac{Gm_e^2}{r^2} = 0.2 \cdot 10^{-42} \frac{e^2}{r^2}.$$

The gravitational equivalent of the coupling constant of electromagnetic interaction $\alpha = \frac{e^2}{\hbar c} \cong \frac{1}{137}$ is then

$$\alpha_G := \frac{Gm_e^2}{\hbar c} \cong 1.5 \cdot 10^{-45}.$$

To get a grasp of the smallness of this number, we compare the Bohr radius of a hydrogen atom with it's gravitational equivalent

$$a_B = \frac{\hbar^2}{m_e e^2} = \frac{c^{-1}\hbar}{m_e \alpha} \cong 0.5 \cdot 10^{-8}\text{cm}$$

$$(a_B)_G = \frac{\hbar^2}{m_e G m_e m_p} = \frac{c^{-1}\hbar}{m_p \alpha_G} = \frac{m_e}{m_p}\frac{\alpha}{\alpha_G} \cdot a_B \sim 10^{31}\text{cm} \cong 10^{13} \text{ light years}$$

Which is bigger than the radius of the observable universe ($R_H \sim 1.4 \cdot 10^{10}$ l.y.). This is the reason why we can safely neglect any correction from quantum gravity when we consider atomic physics. Only for large masses gravity becomes significant. Gravity is also the interaction dominating at very large distances, firstly because it has an unlimited range (like EM interaction) and secondly because it is universally attractive. Each form of energy is a source of gravitational field: matter, anti-matter, kinetic energy,... In addition, gravity acts on each form of energy.

Universality: The motion of a test body in a gravitational field is independent

of its mass and composition (equality between gravitational and inertial mass) This universality has been experimentally tested at a precision of $1 : 10^{13}$.
Equivalence principle: No local experiment can distinguish an non-rotating free falling system from a non-accelerated system with no gravity.

Remark 5.1

- *The exact mathematical formulation of this principle will follow.*
- *Equality between gravitational and inertial mass is a necessary implication of the equivalence principle, but the opposite is not true (see exercises!)*

Redshift as a consequence of the equivalence principle
It follows from the equivalence principle that all the effects of a (homogeneous) gravitational field are identical to phenomena in a constantly accelerated system with no gravity.
Let us consider two experimenters in an accelerating rocket, with constant acceleration $\vec{g}$. At the time $t = 0$, experimenter 1 sends a photon towards her friend,

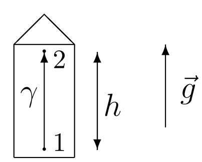


Figure 5.1: Experimenters in a rocket, with constant acceleration $\vec{g}$

experimenter 2. We suppose that in our inertial reference frame, the rocket is at rest at $t = 0$. At the time $t = \frac{h}{c}$ the photon arrives at 2 (we are neglecting corrections of order $\frac{v}{c}$). But at that moment, 2 has already acquired a velocity $v = gt = g\frac{h}{c}$. Therefore she must observe the photon with a Doppler-shift $z = \frac{\Delta\lambda}{\lambda} \cong \frac{v}{c}$, that is $z = \frac{gh}{c^2}$.
According to the equivalence principle, the same redshift is acquired in a homogeneous gravitational field $\vec{g}$. In this case we can write $z = \frac{gh}{c^2} = \frac{\Delta\Phi}{c^2}$, where Φ is the Newton potential, such that

$$z = \frac{\lambda_2 - \lambda_1}{\lambda_1} = \frac{\Phi_2 - \Phi_1}{c^2} \tag{5.1}$$

This effect has indeed been experimentally observed on Earth (Pound & Snider, 1965).

At the time when Einstein predicted this gravitational redshift (1907), experiments were not yet precise enough to observe it directly. Nonetheless, one can easily be convinced by the following energy conservation argument :
Again, let us consider the two points 1 and 2 separated by a distance h in a homogeneous gravitational field $\vec{g}$. Let us suppose that a body of mass m is in free

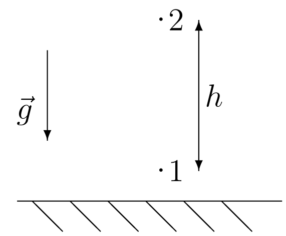


Figure 5.2: Configuration of the second experiment

fall with initial velocity 0, falling from 2 to 1. At point 1 it has a kinetic energy of mgh. Suppose that all the energy of this mass $(mc^2 + mgh)$ is transformed into a photon at 1 which is sent back to 2. If this photon would not interact with the gravitational field and so would not acquire any redshifted, it would be possible at point 2 to transform it again into a mass which has then gained an amount of energy equal to mgh. Hence, this circular process would represent a perpetual motion machine. In order to save the law of energy conservation, the photon must lose energy on its way back to point 2 :

$$E_1 = 2\pi\hbar\nu_1 = \frac{2\pi\hbar c}{\lambda_1} = E_2 + mgh = mc^2 + mgh = E_2\left(1 + \frac{gh}{c^2}\right).$$

This corresponds to the redshift

$$1 + z = \frac{\lambda_2}{\lambda_1} = \frac{E_1}{E_2} = 1 + \frac{gh}{c^2}$$

found in Eq. (5.1).

5.2 Special relativity and gravity

Here we want to briefly show that gravity, especially the redshift of photons and the deflection of light, cannot be formulated in the framework of special relativity.
Redshift: In special relativity, a clock moving along a world line $x^\mu(\lambda)$ from

$x^\mu(\lambda_1)$ to $x^\mu(\lambda_2)$ measures a time difference of

$$\Delta\tau = \int_{\lambda_1}^{\lambda_2} \sqrt{-\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda \quad \text{where} \quad (\eta_{\mu\nu}) := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.2}$$

As we will show here, this formula cannot be valid if gravitational fields are present: Suppose that there is a theory of gravity compatible with special relativity. We consider the following experiment: The gravitational field is assumed to be static

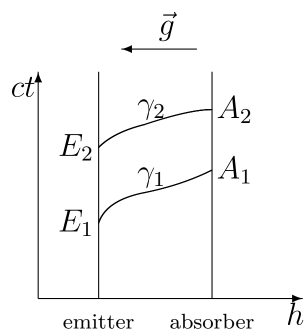


Figure 5.3: Setting of this experiment

with respect to the indicated inertial reference frame.
The emitter sends at a constant frequency during the time interval from E_1 to E_2. Because the situation is static, a photon emitted at E_1 moves parallel to the one emitted at E_2 (but not necessarily at 45^o because of the presence of the gravitational field). But if special relativity was valid for the measurement of time, the temporal separation between E_2 and E_1 would be equal to the one between A_2 and A_1, and that would be in contradiction with a redshift.
Deflection: If the relations of causality were determined by special relativity, light cones would always be straight cones and light would move along straight lines. That would contradict the deflection of light in a gravitational field, which was empirically observed (for the first time during a solar eclipse in 1919). Therefore the metric can also not be conformally flat, that is of the form $\exp(\Phi)\eta_{\mu\nu}$ with a scalar field Φ, because in this form, the light cone would remain straight.
In addition, according to the equivalence principle, one cannot empirically distinguish a freely falling system from an inertial system. But the inertial law ($\ddot{x}^\mu = 0$) is not valid in free fall. Then there is no way to operationally define what an "inertial reference frame" is. With this, we have lost the very basis of special relativity! (and also the reason to represent spacetime by an affine space which is justified by the law if inertia.)

5.3 Space-time as a Lorentzian manifold: mathematical formulation of the equivalence principle

In the previous section we argued that in presence of gravitational fields spacetime, cannot be described by Minkowski space. But we have seen that according to the equivalence principle, special relativity is nevertheless valid infinitesimally. At each point p a metric $(g_{\mu\nu}(p))$ can be specified. In general, this metric varies from one point to another, such that we cannot find any coordinate system where $g_{\mu\nu}(x) = \eta_{\mu\nu}$ for all x.
"Definition": The mathematical model for spacetime (i.e. the set of all events) is a pseudo-Riemannian manifold $\mathcal{M}$ that has a metric with lorentzian signature $(-,+,+,+)$. Such a manifold $(\mathcal{M}, g)$ is called **Lorentz manifold**.
Exercise: Show that the signature of the metric of a pseudo-Riemannian manifold cannot vary from one point to another.
The metric establishes (as in special relativity) the relations of causality: the optical signals emitted from an event $x \in \mathcal{M}$ form the future light cone , L^+. The optical signals converging towards x form the past light cone, L^-. We suppose that the distinction between the past and the future cone is possible in a continuous way, at least locally (i.e. that $(\mathcal{M}, g)$ is oriented in time).

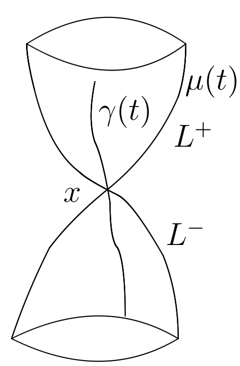


Figure 5.4: The past and future light cone. The vector $\dot\gamma(t)$ is the tangent to the world line of a massive particle, time-like, $g_{\gamma(t)}\left(\dot\gamma(t), \dot\gamma(t)\right) < 0$ and $\mu(t)$ is tangent to the light cone, and light-like, $g_{\mu(t)}\left(\dot\mu(t), \dot\mu(t)\right) = 0$.

Definition 5.1

- *A world line is called* **time-like** *if its tangent vector* $\dot\gamma$ *is at every point* *(event) inside the light cone,* $g_{\gamma(t)}\left(\dot{\gamma}(t), \dot{\gamma}(t)\right) < 0$.

- *It is called* **light-like** *if* $\dot{\gamma}$ *is everywhere tangent to the light cone,* $g_{\gamma(t)}\left(\dot{\gamma}(t), \dot{\gamma}(t)\right) = 0$.
- *It is called* **space-like** *if* $\dot{\gamma}$ *is everywhere outside the light cone,* $g_{\gamma(t)}\left(\dot{\gamma}(t), \dot{\gamma}(t)\right) > 0$.

The metric g determines the gravitational field (gravitational potential). Hence the gravitational field, the metric and causality properties of spacetime are described by the same quantity g.
In a Lorentz manifold there is in general no prefered coordinate system (apart from situations with symmetry).
Therefore the laws of physics must transform covariantly under coordinate transformations, in the sense of the following definition:

Definition 5.2 (covariant system of equations) *A system of equations is called* **covariant** *under the group of coordinate transformations,* $\mathcal{G}(\mathcal{M})$*, if each element* $\varphi \in \mathcal{G}(\mathcal{M})$ *maps the quantities in the equations to new quantities such that*

1. *This map preserves the group structure* $\mathcal{G}(\mathcal{M})$*.*
2. *The transformed quantities also satisfy the system of equations.*

Only covariant laws have an intrinsic meaning, independent of the coordinate system. By using an adapted geometrical language, it is possible to formulate them without using coordinates.
As we have shown in the first part, at each point $x_0 \in \mathcal{M}$ there exists a coordinate system (geodesic or normal system) such that

$$g_{\mu\nu}(x_0) = \eta_{\mu\nu} \quad \text{and} \quad g_{\mu\nu\,,\lambda}(x_0) = 0. \tag{5.3}$$

Such a system is called a local inertial system. In this system the laws of physics take the same form as in special relativity at the point x_0.
In the next paragraph the two following conditions will allow us to formulate the laws of physics in the presence of gravitational fields:

1. Except for the metric and its derivatives, the equations contain only quantities already present in their special-relativistic formulation.
2. The equations are covariant and they reduce to their "special-relativistic" form in a geodesic system (5.3) at the point x_0.

These two conditions represent a mathematical formulation of the **equivalence principle**.

5.4 The laws of physics in the presence of gravity

(From now on $c = 1$.)

5.4.1 Equation of motion of a test particle in a gravitational field

According to the equivalence principle and special relativity, in a locally inertial system at point $p \in \mathcal{M}$, the trajectory $x(s)$ of a free (not subject to any force) test particle obeys the following equation:

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}s^2} = 0, \tag{5.4}$$

where s is the arc length, i.e.

$$g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} = -1. \tag{5.5}$$

With $\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} (g_{\nu\alpha\ ,\beta} + g_{\nu\beta\ ,\alpha} - g_{\alpha\beta\ ,\nu})$ and $\Gamma^\mu_{\alpha\beta}(p) = 0$, equation (5.4) at the point p is also

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}s^2} + \Gamma^\mu_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} \frac{\mathrm{d}x^\beta}{\mathrm{d}s} = 0 \Leftrightarrow \nabla_{\dot{x}} \dot{x} = 0 \quad (\dot{x} = \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} \partial_\alpha) \tag{5.6}$$

But eq. (5.6) is generally covariant and thus valid in every coordinate system and at each point p of the trajectory (because we have chosen p arbitrarily).
Since $\dot{x}$ undergoes a parallel transport along the trajectory, it follows that $g(\dot{x}, \dot{x}) =$ const. $= -1$ along the trajectory.
For the trajectories of light rays $x^\mu(\lambda)$ where λ is an affine parameter, we have

$$g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 0, \tag{5.7}$$

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} = 0. \tag{5.8}$$

This equation of motion can also be derived by applying the variational principle on the action

$$S = \int g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} \mathrm{d}s\ . \tag{5.9}$$

Indeed, $\delta S = 0$ gives

$$0 = \delta \int g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} \mathrm{d}s = \int \delta x^\lambda \left[g_{\mu\nu,\lambda} \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} - 2 \frac{\mathrm{d}}{\mathrm{d}s} \left(g_{\mu\lambda} \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \right) \right] ds \tag{5.10}$$

for every variation δx^λ. It follows that

$$0 = g^{\alpha\lambda}(g_{\mu\nu},_\lambda - 2g_{\mu\lambda},_\nu)\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s} - 2\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}s^2},$$

or, with $\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\,(g_{\mu\beta},_\nu + g_{\beta\nu},_\mu - g_{\mu\nu},_\beta)$

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s} = 0\ . \tag{5.11}$$

5.4.2 "Conservation" of energy-momentum in the presence of a gravitational field

In special relativity, the energy-momentum tensor $T^{\mu\nu}$ of a closed physical system satisfies the conservation law

$$T^{\mu\nu}_{\ ,\nu} = 0.$$

In an inertial system at p ($\Gamma^\mu_{\alpha\beta}(p) = 0$) this is equivalent to

$$T^{\mu\nu}_{\ ;\nu}(p) = T^{\mu\nu}_{\ ,\nu}(p) + \Gamma^\mu_{\nu\alpha}(p)T^{\nu\alpha}(p) + \Gamma^\nu_{\nu\alpha}(p)T^{\mu\alpha}(p) = 0\,. \tag{5.12}$$

Again, since this equation is covariant it holds in every coordinate system. $T^{\mu\nu}_{\ ;\nu} = 0$ is therefore the generalization of energy-momentum conservation in the presence of gravity. In general, this law does not lead to conserved quantities. This is not so surprising, since we expect energy and momentum the be exchanged with the gravitational field. However, apart from special cases, there is no meaningful general definition of energy and momentum of the gravitational field.

Example 5.1 *For a perfect fluid with energy density ρ and pressure p where u^μ is the energy flux,*

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu} \quad (g_{\mu\nu}u^\mu u^\nu = -1)\,. \tag{5.13}$$

Show that for $p = 0$, $(\rho u^\mu)_{;\mu} = 0$ and u satisfies the geodesic equation. In this case the fluid is called "dust".

5.4.3 Electrodynamics in the presence of gravitational fields

As usual, the electromagnetic field-strength tensor (the Faraday tensor) is given by

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix} \tag{5.14}$$

where $F^{ij} = -\varepsilon^{ijk}B_k\,,\ \ F^{i0} = E^i$. Maxwell's equations in special relativity are

$$F^{\mu\nu}_{\ \ ,\nu} = -4\pi j^\mu \qquad (\delta F = -4\pi j)\,, \tag{5.15}$$

$$F_{\mu\nu\,,\lambda} + F_{\lambda\mu\,,\nu} + F_{\nu\lambda\,,\mu} = 0 \qquad (dF = 0)\,, \tag{5.16}$$

$$(j^\mu) = (\rho, \vec{J}) \text{ is the electric four-current (and } F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu \quad j = j_\mu dx^\mu)\,. \tag{5.17}$$

In the presence of gravitational fields we define $F_{\mu\nu}$ and j^μ in such a way that they transform like tensor fields and they reduce to (5.14) and (5.17) in a locally inertial system at the point p.
When gravitational fields are present, (5.15) and (5.16) become

$$F^{\mu\nu}_{\ \ ;\nu} = -4\pi j^\mu \tag{5.18}$$

$$F_{\mu\nu\,;\lambda} + F_{\lambda\mu\,;\nu} + F_{\nu\lambda\,;\mu} = 0 \tag{5.19}$$

(5.19) is identical to (5.16)!
The condition of integrability (5.16) or (5.19) allows the representation by an electromagnetic potential :

$$F_{\mu\nu} = A_{\mu\,,\nu} - A_{\nu\,,\mu} = A_{\mu\,;\nu} - A_{\nu\,;\mu} \qquad (\ F = dA\ )\ .$$

With the electromagnetic potential A_μ, the equation (5.18) becomes

$$A^{\mu\,;\nu}{}_{;\nu} - A^{\nu\,;\mu}{}_{;\nu} = -4\pi j^\mu. \tag{5.20}$$

The electromagnetic energy-momentum tensor is

$$T^{\mu\nu} = -\frac{1}{4\pi}\left[F^\mu_\lambda F^{\lambda\nu} + \frac{1}{4}g^{\mu\nu}F^{\sigma\lambda}F_{\sigma\lambda}\right]. \tag{5.21}$$

When gravitational fields are present, the Lorentz equation becomes

$$m\left(\frac{\mathrm{d}^2x^\mu}{\mathrm{d}s^2} + \Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\frac{\mathrm{d}x^\beta}{\mathrm{d}s}\right) = -eF^\mu{}_\nu\frac{\mathrm{d}x^\nu}{\mathrm{d}s}. \tag{5.22}$$

Ambiguities:
In the absence of gravitational fields (5.20) is

$$A^{\mu\,,\nu}{}_{,\nu} - A^{\nu\,,\mu}{}_{,\nu} = -4\pi j^\mu \tag{5.23}$$

Since partial derivatives commute, this is equivalent to

$$A^{\mu\ \,,\nu}{}_{,\nu} - A^\nu{}_{,\nu}{}^{,\mu} = -4\pi j^\mu\ . \tag{5.24}$$

But in a gravitational field, the second expression gives

$$A^{\mu\,;\nu}{}_{;\nu} - A^{\nu\,;\mu}{}_{;\nu} + R^{\mu}_{\nu} A^{\nu} = -4\pi j^{\mu} \tag{5.25}$$

where R^{μ}_{ν} is the Ricci tensor. For this we made use of the definition of the Riemann tensor which for a coordinate basis (∂_μ) gives

$$\begin{aligned} (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) Z &= \left(Z^{\beta}{}_{;\nu\mu} - Z^{\beta}{}_{;\mu\nu} \right) \partial_\beta = R^{\beta}{}_{\alpha\mu\nu} Z^{\alpha} \partial_\beta \quad \text{which implies} \\ A^{\nu\,;\mu}{}_{;\nu} - A^{\nu}{}_{;\nu}{}^{;\mu} &= R^{\nu}{}_{\alpha\nu}{}^{\mu} A^{\alpha} = R_{\alpha}{}^{\mu} A^{\alpha} \,. \end{aligned}$$

The equivalence principle does not say which of the equations (5.20) and (5.25) is the right one when gravitational fields are present.
The transition from special to general relativity contains such possibilities of "non-minimal couplings" once we encounter higher derivatives. This ambiguity is comparable to the one about the order of operators in the transition from classical to quantum mechanics.
In practice, such problems are rare but there is no general prescription to solve them (in the case of (5.20) and (5.25) one must look at the original Maxwell equations, (5.18), and so decide for (5.20) and not for (5.25)).

5.4.4 The Newtonian limit

For slowly moving particles, $(u^\mu) = (\dot{x}^\mu) = \gamma(1, \vec{v})$, $\gamma \simeq \frac{1}{\sqrt{1-v^2}}$, $v^2 \ll 1$ in a weak gravitational field

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad (\eta_{\mu\nu}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad |h_{\mu\nu}| \ll 1$$

we can neglect $|\frac{\mathrm{d}x^i}{\mathrm{d}t}| \sim |v^i|$ which is small compared to $\frac{\mathrm{d}x^0}{\mathrm{d}t} \sim 1$.
The equation (5.6) then gives us ($\gamma \sim 1$)

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} \cong \frac{\mathrm{d}^2 x^i}{\mathrm{d}s^2} = -\Gamma^i_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} \frac{\mathrm{d}x^\beta}{\mathrm{d}s} \cong -\Gamma^i_{00} \cong +\frac{1}{2} h_{00\,,i} - h_{0i\,,0} \,. \tag{5.26}$$

If we also suppose that the gravitational field varies slowly in time, we can neglect the quantity $h_{0i\,,0}$, and we arrive at

$$\frac{\mathrm{d}^2 \vec{x}}{\mathrm{d}t^2} = \frac{1}{2} \vec{\nabla} h_{00}. \tag{5.27}$$

For

$$h_{00} = -2\Phi, \quad g_{00} = -(1 + 2\frac{\Phi}{c^2}) \tag{5.28}$$

we find Newton's equation
$$\frac{\mathrm{d}^2\vec{x}}{\mathrm{d}t^2} = -\vec{\nabla}\Phi.$$
Eq. (5.28) can only by determined up to a constant. That constant can be specified by the boundary conditions,

$$h_{00}(\vec{x}) \stackrel{|\vec{x}|\to\infty}{\longrightarrow} 0 \quad \text{and} \quad \Phi(\vec{x}) \stackrel{|\vec{x}|\to\infty}{\longrightarrow} 0$$

(in g_{00} we have reinserted c just for once).

Exemples 5.2 $\frac{\Phi}{c^2}$ *is approximately:*

- 10^{-9} *on Earth,*
- $2 \cdot 10^{-6}$ *on the surface of the Sun,*
- 10^{-4} *on a white dwarf,*
- 10^{-1} *on a neutron star,*
- 10^{-39} *"on a proton".*

5.4.5 Redshift in a gravitational field

We define (an intrinsic definition will come later) a **static** gravitational field as a spacetime with a coordinate system (t, x^i) such that the metric takes the form

$$\mathrm{d}s^2 = g_{00}(\vec{x})\mathrm{d}t^2 + g_{ij}(\vec{x})dx^i dx^j.$$

There exists a "foliation" of spacetime $\mathcal{M} = \mathbb{R} \times \mathcal{S} \ni (t, \vec{x})$ such that $\mathcal{S}$ is a Riemannian manifold with the metric

$$g_{ij}(\vec{x})dx^i dx^j \quad \text{and} \quad g(\partial_t, X) = 0, \forall X \in \mathcal{X}(\mathcal{S}) \ .$$

The proper time τ of an observer on the trajectory $(t, \vec{x}(t))$ satisfies

$$-\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2 = g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t} \tag{5.29}$$

(according to the equivalence principle).
For a watch at rest in the system (t, x^i), i.e. $\frac{\mathrm{d}x^i}{\mathrm{d}t} = 0$, this gives

$$\mathrm{d}\tau = \sqrt{-g_{00}}\,\mathrm{d}t$$

At the point $\vec{x}_2$ we periodically emit light flashes with temporal separation $\Delta\tau = \sqrt{-g_{00}}\,\Delta t_2$ (the above differential identity is also valid for finite time intervals since g_{00} is time independent).
As the metric does not depend on the time t, the time difference between the arrival of two flashes at the point $\vec{x}_1$, $\Delta t_1'$ is equal to the difference of the time of emission

$$\Delta t_1' = \Delta t_2 = \frac{\Delta\tau}{\sqrt{-g_{00}(\vec{x}_2)}}.$$

On the other hand, the same physical process at the point $\vec{x}_1$ has the same "proper period" $\Delta\tau$ but then the amount of time Δt_1 that elapses is given by

$$\Delta t_1 = \frac{\Delta\tau}{\sqrt{-g_{00}(\vec{x}_1)}}.$$

If we compare the frequency ν_2 of a signal arriving from point 2 with the frequency ν_1 of the same signal produced at point 1 we get

$$\nu_2 \propto \frac{1}{\Delta t_1'}, \quad \nu_1 \propto \frac{1}{\Delta t_1}, \quad \frac{\nu_2}{\nu_1} = \sqrt{\frac{g_{00}(\vec{x}_2)}{g_{00}(\vec{x}_1)}}. \tag{5.30}$$

For weak gravitational fields $g_{00} \cong -(1+2\Phi)$, this gives the shift

$$z := \frac{\nu_1}{\nu_2} - 1 \cong \Phi(\vec{x}_1) - \Phi(\vec{x}_2). \tag{5.31}$$

The light coming from the Sun arrives on Earth with a shift $z \sim 2 \cdot 10^{-6}$. This effect has been measured convincingly in the gravitational field of the sun rather late. The measurement is difficult since velocities on the surface of the sun also induce a Doppler redshift [11].

5.4.6 Fermat's principle for static gravitational fields

We consider again a metric of the form

$$\mathrm{d}s^2 = g_{00}(\vec{x})\mathrm{d}t^2 + g_{ik}(\vec{x})dx^i dx^k.$$

Let $x^\mu(\lambda)$ be a light ray with $x_1 = x(0)$ and $x_2 = x(1)$. The geodesic equation for $x^\mu(\lambda)$ leads to (5.11),

$$\delta \int_0^1 g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\mathrm{d}\lambda = 0\ . \tag{5.32}$$

First we consider only the variation of $t(\lambda)$:

$$0 = \delta \int_0^1 g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\mathrm{d}\lambda = \int_0^1 2g_{00}\frac{\mathrm{d}t}{\mathrm{d}\lambda}\delta\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)\mathrm{d}\lambda \tag{5.33}$$

$$= -\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(2g_{00}\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)\delta t \mathrm{d}\lambda \ .$$

Because δt can arbitrarily be chosen, it follows that

$$g_{00}\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \text{const.}$$

Let us choose the parameter λ such that

$$g_{00}\frac{\mathrm{d}t}{\mathrm{d}\lambda} = 1 \ . \tag{5.34}$$

For any (light-like) trajectory we have

$$-g_{00}\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 = g_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}$$

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \frac{\sqrt{g_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}}}{\sqrt{-g_{00}}} \ .$$

With eq. (5.34), eq. (5.33) implies

$$0 = \delta\int\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)\mathrm{d}\lambda = \delta\int\frac{\mathrm{d}\sigma}{\sqrt{-g_{00}}} \quad \text{where} \quad \mathrm{d}\sigma^2 = g_{ij}dx^i dx^j \ .$$

In other words, $1/\sqrt{-g_{00}}$ plays the role of a refractive index. In a static spacetime, the **spatial** trajectory of a light ray is a geodesic of the metric $\frac{g_{ij}}{-g_{00}}$ on $\mathcal{S}$!

5.4.7 Static and stationary gravitational fields

Naively, a gravitational field is called stationary if there exists a coordinate system in which the metric does not depend on the time $t = x^0$.
Here we want to translate this intuitive notion into a geometrical description. Setting $K = \partial_t$,

$$\partial_t g_{\mu\nu} = 0 \qquad \Leftrightarrow \qquad K^\lambda g_{\mu\nu,\lambda} + g_{\lambda\nu}K^\lambda_{\ ,\mu} + g_{\mu\lambda}K^\lambda_{\ ,\nu} \equiv L_K g = 0$$

This leads us to the important definition of a Killing vector:

Definition 5.3 (Killing field) *A vector field K that satisfies*

$$L_K g = 0 \tag{5.35}$$

is called a **Killing field** *for the metric g.*

Each Killing field generates an isometry group of one parameter:

$$\text{The flow of } K \text{ denoted by } \Phi_{s,t} \text{ satisfies } \Phi^\star_{s,t} g = g. \tag{5.36}$$

Definition 5.4 (stationary metric) *The metric of a Lorentz manifold* $(\mathcal{M}, g)$ *is called* **stationary** *if it admits a* **time-like** *Killing field.*

From this definition, it follows that there exists a local coordinate system in which g is time-independent. In order to construct it, we consider in a neighborhood of a point p_0 a three dimensional hypersurface $\mathcal{S}$ that is not tangent to K, i.e. $K(p) \notin T_p\mathcal{S}$ for every p in a neighborhood of p_0.
Let $x^1(p)$, $x^2(p)$ and $x^3(p)$ be some coordinates on $\mathcal{S}$ and $\Phi_t = \Phi_{t,0}$ the flow of K. At a point $q = \Phi_t(p)$ we choose the coordinates $(t, x^1(p), x^2(p), x^3(p))$. With this construction $K = \partial_t = \partial_{x^0}$ and

$$L_K g = 0 \quad \text{ is equivalent to } \quad g_{\mu\nu\,,0} = 0 \ .$$

It may happen that a Killing field is time-like in a certain domain, space-like in another one and light-like at the boundary (see black holes). In this sense the definition of stationarity given here is a local one.

Definition 5.5 (static metric) *A stationary metric is called* **static** *if the 1-form* $K^\flat$ *satisfies*

$$K^\flat \wedge dK^\flat = 0 \ . \tag{5.37}$$

We would like to show that in this case the surfaces $\mathcal{S}$ can be chosen such that K is normal to $\mathcal{S}$ and thus $g_{i0} = 0$.
This is indeed a consequence of Frobenius' theorem: let us introduce $K(p)^\perp =: \{X_p \in T_p\mathcal{M} \mid g_p\,(X_p, K(p)) = 0\}$. If eq. (5.37) is satisfied, the distribution $\{K(p)^\perp \mid p \in \mathcal{M}\}$ is locally integrable. That is, locally

$$K^\flat = \langle K, K\rangle df \tag{5.38}$$

for some function f. The surfaces $\{f = \text{const}\}$ are then normal to K. We now prove the existence of the function f in (5.38). In a coordinate system which is normal at the point p, eq. (5.35) leads to $K_{\mu\,,\nu} + K_{\nu\,,\mu} = 0$. For an arbitrary coordinate system, this implies

$$\text{the } \textbf{Killing equation}, \quad K_{\mu\,;\nu} + K_{\nu\,;\mu} = 0\,. \tag{5.39}$$

$$K^\flat \wedge dK^\flat = (K_\mu K_{\nu\,,\lambda} + K_\nu K_{\lambda\,,\mu} + K_\lambda K_{\mu\,,\nu}) dx^\mu \wedge dx^\nu \wedge dx^\lambda = 0 \ .$$

Because $K_{\nu},_{\lambda}\, dx^\nu \wedge dx^\lambda = K_{\nu};_{\lambda}\, dx^\nu \wedge dx^\lambda$, we can replace in this expression every ordinary derivatives by covariant derivatives. This gives

$$\mathcal{A}\,(K_\mu K_{\nu\,;\lambda} + K_\nu K_{\lambda\,;\mu} + K_\lambda K_{\mu\,;\nu}) = 0 \ .$$

($\mathcal{A}$ denotes antisymmetrization in all indices.) Using the Killing equation, $K_{\lambda;\nu}$ is antisymmetric, this is equivalent to

$$-K_\mu K_{\lambda\,;\nu} + K_\nu K_{\lambda\,;\mu} + \frac{1}{2}K_\lambda(K_{\mu\,;\nu} - K_{\nu\,;\mu})\,.$$

It is easy to check that the above expression is antisymmetric in all its indices. By multiplying it by K^λ we get

$$-K_\mu \underbrace{K^\lambda K_{\lambda\,;\nu}}_{\frac{1}{2}\langle K,K\rangle_{;\nu}} + K_\nu \underbrace{K^\lambda K_{\lambda\,;\mu}}_{\frac{1}{2}\langle K,K\rangle_{;\mu}} + K^\lambda K_\lambda \cdot \underbrace{K_{\mu\,;\nu}}_{\frac{1}{2}(K_{\mu\,;\nu} - K_{\nu\,;\mu})} = 0$$

or equivalantly

$$-K_\mu\langle K,K\rangle_{;\nu} + K_\nu\langle K,K\rangle_{;\mu} + \langle K,K\rangle(K_{\mu\,;\nu} - K_{\nu\,;\mu}) = 0\ .$$

This can be written as

$$\left(\frac{K_\mu}{\langle K,K\rangle}\right)_{;\nu} - \left(\frac{K_\nu}{\langle K,K\rangle}\right)_{;\mu} = 0 \quad \text{such that} \quad d\left(\frac{K^\flat}{\langle K,K\rangle}\right) = 0\ .$$

This implies

$$K^\flat = \langle K,K\rangle df =: \langle K,K\rangle dt$$

□

Conclusions: (for $df \equiv dt$):

- The flow Φ_s of K maps the hypersurfaces $t = \text{const}$ in an isometric way.
- An observer at rest propagates along integral curves of K.
- If there exists a time-like Killing field satisfying eq. (5.37), there exists a prefered time t with $dt = \frac{K^\flat}{\langle K,K\rangle}$.
- For $\mathcal{S} = \{t = \text{const}\}$, the Lagrangian coordinates introduced for the stationary case lead, in the static situation, to a metric of the form

  $$\mathrm{d}s^2 = g_{00}(\vec{x})\mathrm{d}t^2 + g_{ij}(\vec{x})dx^i dx^j, \tag{5.40}$$

  that coincides with our naive definition of a static metric.

5.5 Local frames and Fermi transport

Here we discuss the questions of reference frames and inertial forces.
Let us consider an observer in a space ship that moves along a time-like world line in a gravitational field, not necessarily along a geodesic. This observer defines a coordinate system fixed to the space ship in which all his apparatus is at rest. We ask the following questions:

1. What are the equations of motion of a free falling test particle in this coordinate system?

2. How should we orient the space ship to avoid any centrifugal and Coriolis forces?

3. How does a free top spin?

4. In a stationary gravitational field, can we find a preferred rest frame? In which case does the top precess in this frame and what is its equation of motion?

5.5.1 Equation of motion of a spin in a gravitational field

The spin can be the expectation value of a particle's spin operator or a classical angular momentum vector (a gyroscope).
The spin is firstly defined in the rest frame of the particle (or the gyroscope). In this frame, it is given by a three-vector, $\vec{S}$. In the absence of external forces, it does not vary in the rest frame

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{S}(t) = 0 \quad \text{(in the rest frame).} \tag{5.41}$$

We define a four-vector S that reduces to $(0, \vec{S})$ in the rest frame of a particle moving with the four-velocity u. We have then

$$\langle u, S\rangle = 0. \tag{5.42}$$

In the absence of gravitational fields and in the rest frame where $u = (1, \vec{0})$

$$(\nabla_u S)_R = \frac{\mathrm{d}}{\mathrm{d}t}S = \left(\frac{\mathrm{d}}{\mathrm{d}t}S^0, \frac{\mathrm{d}}{\mathrm{d}t}\vec{S}\right) = \left(\frac{\mathrm{d}}{\mathrm{d}t}S^0, \vec{0}\right) \ .$$

But

$$0 = \nabla_u\langle u, S\rangle = \langle\underbrace{\nabla_u u}_{a}, S\rangle + \langle u, \nabla_u S\rangle$$

where a is the acceleration. We then have

$$\langle u, \nabla_u S\rangle = -\langle a, S\rangle \tag{5.43}$$

But in the rest frame,

$$\langle\nabla_u S, u\rangle_R = -\frac{\mathrm{d}S^0}{\mathrm{d}t} = -\langle a, S\rangle \ .$$

This gives
$$(\nabla_u S)_R = (\langle a, S\rangle, \vec{0}) = (\langle a, S\rangle u)_R \ . \tag{5.44}$$
Equation (5.44) is covariant. Hence it is valid in any reference frame, and according to the equivalence principle, it is also valid when gravitational fields are present. The covariant equation that we wanted to find is
$$\nabla_u S = \langle a, S\rangle u, \quad a := \nabla_u u. \tag{5.45}$$
It is obvious that (5.45) is compatible with (5.42).

Application: The Thomas precession (in special relativity):
We consider a particle (top) that moves with four-velocity u in the **absence** of gravitational fields. Then we have
$$u = \gamma(1, \vec{\beta}) \quad , \gamma = \frac{1}{\sqrt{1-\beta^2}} \quad , \beta^2 < 1 \tag{5.46}$$
In the rest frame $S_R = (0, \vec{S}(t))$.
In the reference frame of the laboratory, S_L is obtained by the boost $\Lambda(-\vec{\beta}(t))$:
$$S \equiv S_L = (\gamma\vec{\beta}\cdot\vec{S}, \vec{S} + \vec{\beta}\left(\frac{\gamma^2}{\gamma+1}\right)(\vec{\beta}\cdot\vec{S})) \ . \tag{5.47}$$
With
$$a \equiv \dot{u} = (\dot{\gamma}, \dot{\gamma}\vec{\beta} + \gamma\dot{\vec{\beta}}) = \frac{\dot{\gamma}}{\gamma}u + \gamma(0, \dot{\vec{\beta}}) \tag{5.48}$$
we get, using $\langle S, u\rangle = 0$:
$$\langle S, a\rangle = \langle S, \dot{u}\rangle = \gamma\langle(0, \dot{\vec{\beta}}), S\rangle = \gamma(\dot{\vec{\beta}}\cdot\vec{S} + \frac{\gamma^2}{\gamma+1}(\dot{\vec{\beta}}\cdot\vec{\beta})(\vec{\beta}\cdot\vec{S}))$$
Eq. (5.45) gives for the 0 component
$$(\gamma\vec{\beta}\cdot\vec{S})^{\cdot} = \gamma^2\left(\dot{\vec{\beta}}\cdot\vec{S} + \frac{\gamma^2}{\gamma+1}(\dot{\vec{\beta}}\cdot\vec{\beta})(\vec{\beta}\cdot\vec{S})\right) \tag{5.49}$$
and for the spatial components
$$(\vec{S} + (\vec{\beta}\cdot\vec{S})\frac{\gamma^2}{\gamma+1}\vec{\beta})^{\cdot} = \vec{\beta}\gamma^2\left(\dot{\vec{\beta}}\cdot\vec{S} + \frac{\gamma^2}{\gamma+1}(\dot{\vec{\beta}}\cdot\vec{\beta})(\vec{\beta}\cdot\vec{S})\right) \ . \tag{5.50}$$
Combining (5.49) with (5.50) gives (exercise!):
$$\dot{\vec{S}} = \vec{S}\wedge\vec{\omega}_T, \quad \text{with} \tag{5.51}$$
$$\vec{\omega}_T = \frac{\gamma-1}{\beta^2}\vec{\beta}\wedge\dot{\vec{\beta}}. \tag{5.52}$$
$\vec{\omega}_T$ is the **Thomas precession frequency**.

5.5.2 Fermi transport

Definition 5.6 (Fermi derivative) *Let $\gamma(s)$ be a time-like path with tangent vector $u = \dot{\gamma}, \quad \langle u, u\rangle = -1$.*
Let $X \in \mathcal{X}(\mathcal{M})$. The **Fermi derivative** *of X along γ is defined by*

$$\mathbb{F}_u X := \nabla_u X - \langle X, a\rangle u + \langle X, u\rangle a, \tag{5.53}$$

with $a := \nabla_u u$.

As $\langle S, u\rangle = 0$, the equation (5.45) is equivalent to

$$\mathbb{F}_u S = 0. \tag{5.54}$$

It can easily be seen that the Fermi derivative has the following properties:

1. If γ is a geodesic, $\mathbb{F}_u = \nabla_u$.
2. $\mathbb{F}_u u = 0$
3. If $\langle u, X\rangle$ =constant along γ, then $\langle \mathbb{F}_u X, u\rangle = 0$.
4. If $\mathbb{F}_u X = \mathbb{F}_u Y = 0$ along γ, $\langle X, Y\rangle$ is constant along γ.
5. If $\mathbb{F}_u X = 0$, $\langle X, u\rangle$ is constant along γ. (This is not the case when X is parallel transported unless γ is a geodesic for which the two notions coincide)
6. If $\langle X, u\rangle = 0$ along γ,
$$\mathbb{F}_u X = (\nabla_u X)_\perp \tag{5.55}$$
where $\perp$ means orthogonal projection on u,
$$v_\perp := v + \langle u, v\rangle u.$$

Definition 5.7 (Fermi-transported fields) *A vector field $X \in \mathcal{X}(\mathcal{M})$ is called Fermi-transported along γ if*

$$\mathbb{F}_{\dot{\gamma}} X = 0.$$

This is a linear (ordinary) differential equation of X, and it has thus a global solution. Similarly to parallel transport, there exits a group of isomorphisms

$$\mathcal{T}^F_{t,s} : T_{\gamma(s)}\mathcal{M} \to T_{\gamma(t)}\mathcal{M} : X(\gamma(s)) \mapsto \mathcal{T}^F_{t,s} X(\gamma(s))$$

with

$$\mathbb{F}_{\dot{\gamma}} X(\gamma(t)) = \frac{\mathrm{d}}{\mathrm{d}s} \mathcal{T}^F_{t,s} X(\gamma(s))\Big|_{s=t} . \tag{5.56}$$

Much like with parallel transport, we can extend this definition to tensor fields of arbitrary rank, such that

$$\mathcal{T}^F_{t,s} : (T_{\gamma(s)}\mathcal{M})^r_q \to (T_{\gamma(t)}\mathcal{M})^r_q$$

is a linear isomorphism and

$$\mathbb{F}_{\dot\gamma} T_{\gamma(t)} = \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{T}^F_{t,s} T_{\gamma(s)}\Big|_{s=t} \quad \text{for } T \in \mathcal{T}^r_q\mathcal{M}. \tag{5.57}$$

Thus defined, the Fermi derivative of tensor fields has the following properties:

1. For $T \in \mathcal{T}^r_q\mathcal{M}$, $\mathbb{F}_u T \in \mathcal{T}^r_q\mathcal{M}$
2. $\mathbb{F}_u(S\otimes T) = \mathbb{F}_u S \otimes T + S \otimes \mathbb{F}_u T$
3. $\mathbb{F}_u$ commutes with contractions
4. $\mathbb{F}_u f = \frac{\mathrm{d}f}{\mathrm{d}s}$ along γ, for $f \in \mathcal{F}(\mathcal{M})$.

We consider the world line $\gamma(\tau)$ of an accelerated observer with four-velocity $u = \dot\gamma$. The quantity τ is proper time such that we have $u^2 = -1$.
At each point $\gamma(\tau)$ along γ we define 3 orthonormal vectors $e_i(\tau) \in T_{\gamma(\tau)}\mathcal{M}\mathcal{M}$ which are normal to u such that $\langle e_i u\rangle = 0$ and $\langle e_i e_j\rangle = \delta_{ij}$, The vectors $(e_i)_{i=1}^3$ define an arbitrary orthonormal reference frame along γ, normal to $u =: e_0$ (the reference frame of the space ship).
We then have $\langle e_\mu, e_\nu\rangle = \eta_{\mu\nu}$. Clearly,

$$\langle a, u\rangle = \frac{1}{2}\nabla_u\langle u, u\rangle = 0, \quad a = \nabla_u u \ .$$

We define

$$\omega_{ij} := \langle \nabla_u e_i, e_j\rangle = -\omega_{ji} \ . \tag{5.58}$$

With $e^\mu := \eta^{\mu\nu} e_\nu$, we obtain

$$\begin{aligned} \nabla_u e_i &= \langle \nabla_u e_i, e^\mu\rangle e_\mu = -\langle \nabla_u e_i, u\rangle u + \langle \nabla_u e_i, e_j\rangle e_j \\ &= \langle e_i, a\rangle u + \omega_{ij} e_j \ . \end{aligned} \tag{5.59}$$

Defining

$$\omega_{\alpha\beta} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \omega_{ij} \end{array}\right) , \tag{5.60}$$

this leads us to

$$\nabla_u e_\alpha = -\langle e_\alpha, u\rangle a + \langle e_\alpha, a\rangle u + \omega_{\alpha\beta} e^\beta \tag{5.61}$$

or, using (5.53)

$$\mathbb{F}_u e_\alpha = \omega_{\alpha\beta} e^\beta \ . \tag{5.62}$$

The quantity $\omega_{\alpha\beta}$ therefore defines the deviation from Fermi transport. For a free gyroscope, we have $\mathbb{F}_u S = 0$. Because $(S, u) = 0$, $S = S^i e_i$, and we obtain

$$0 = \frac{\mathrm{d}S^i}{\mathrm{d}\tau} e_i + S^i \mathbb{F}_u e_i = \frac{\mathrm{d}S^i}{\mathrm{d}\tau} e_i + S^i \omega_{ij} e^j,$$

which gives

$$\frac{\mathrm{d}S^j}{\mathrm{d}\tau} = \omega_{ji} S^i \ . \tag{5.63}$$

In other words, a free gyroscope precesses relative to $\{e_i\}$ with the angular velocity $\vec{\Omega}$ given by

$$\begin{aligned} \omega_{ji} &= \varepsilon_{jik} \Omega^k \\ \frac{\mathrm{d}\vec{S}}{\mathrm{d}\tau} &= \vec{S} \wedge \vec{\Omega}. \end{aligned} \tag{5.64}$$

If the reference frame (e_i) is Fermi-transported, $\vec{\Omega} = 0$, and our gyroscope does not precess.

5.5.3 Staticity and stationarity

We consider a gravitational field with a time like Killing vector K and an observer **at rest** with respect to K, i.e. an observer who moves along a world line $\gamma(\tau)$ tangent to K. The four-velocity u is then given by

$$u = \frac{1}{\sqrt{-\langle K, K \rangle}} K. \tag{5.65}$$

We consider again an orthonormal reference frame, $(e_i)_{i=1}^3$, along γ. We demand that

$$L_K e_i = 0 \quad i = 1, 2, 3 \ . \tag{5.66}$$

From the Killing equation, it follows that the orthogonality of the vectors e_i is conserved along γ:

$$0 = (L_K g)(X, Y) = K\left(g(X, Y)\right) - g(L_K X, Y) - g(X, L_K Y)$$

using (5.66) we get $\partial_\tau \left(g_{\gamma(\tau)}(e_i(\gamma(\tau)), e_j(\gamma(\tau))) \right) = 0$.
In the same way, the orthogonality with K and therefore with u is preserved.
The e_i are axes at rest. They define a "Copernican system" of the fixed stars.
We compute the change in the components of the spin $S = S^i e_i$ in this reference frame. With (5.65) and (5.58) we have

$$\omega_{ij} = \frac{1}{\sqrt{-\langle K, K \rangle}} \langle e_j, \nabla_K e_i \rangle$$

But

$$0 = L_K e_i = [K, e_i] = \nabla_K e_i - \nabla_{e_i} K \ ,$$

such that

$$\omega_{ij} = \frac{1}{\sqrt{-\langle K, K\rangle}}\langle e_j, \nabla_{e_i} K\rangle = \frac{1}{\sqrt{-\langle K, K\rangle}}\nabla K^\flat(e_j, e_i) \ .$$

In order to render the antisymmetry of ω_{ij} explicit we write

$$\omega_{ij} = \frac{1}{2}k^{-1}[\nabla K^\flat(e_j, e_i) - \nabla K^\flat(e_i, e_j)] \quad \text{with} \quad k = \sqrt{-\langle K, K\rangle} \ . \tag{5.67}$$

In general, we saw that (geometry, eq. (4.10)) for a 1-form α

$$(\nabla\alpha)(X, Y) - (\nabla\alpha)(Y, X) = -d\alpha(X, Y).$$

Hence ω_{ij} can be written as

$$\omega_{ij} = \frac{1}{2}k^{-1} dK^\flat(e_i, e_j) \ . \tag{5.68}$$

We consider the dual basis of (K, e_i), $(K^\flat, \theta^i)$. The 2-form $dK^\flat$ has, in general, the form

$$dK^\flat = \kappa_{ij}\theta^i \wedge \theta^j + \alpha \wedge K^\flat \tag{5.69}$$

for antisymmetric functions κ_{ij} and a 1-form α. The condition of staticity, $K^\flat \wedge dK^\flat = 0$ then is equivalent to $\kappa_{ij} \equiv 0$. But $dK^\flat(e_i, e_j) = 2\kappa_{ij}$ and so

$$\omega_{ij} = k^{-1}\kappa_{ij}. \tag{5.70}$$

This implies that $\omega_{ij} = 0$ if and only if $K^\flat \wedge dK^\flat = 0$.
According to definition 5.5 therefore, a gyroscope does not precess relative to a Copernician system if and only if the gravitational field is static.
In other words, a reference system at rest with respect to a **non-static** stationary gravitational field is rotating. With $\omega_{ij} = \varepsilon_{ijk}\Omega^k$ it follows that

$$\varepsilon^{ijk}\omega_{ij} = 2\Omega^k, \text{ and}$$

$$\Omega^k = \frac{1}{2k}\varepsilon^{ijk}\kappa_{ij}. \tag{5.71}$$

Explicitly in coordinates, for $K = \partial_t$:

$$\mathrm{d}s^2 = g_{00}(\vec{x})\mathrm{d}t^2 + 2g_{0i}(\vec{x})dx^i\mathrm{d}t + g_{ij}(\vec{x})dx^i dx^j \ ,$$

$$K^\flat = g_{00}\mathrm{d}t + g_{0i}dx^i \ .$$

Anti-symmetrisation of $\nabla K^\flat$ yields

$$dK^\flat = g_{00\,,k}dx^k \wedge \mathrm{d}t + \frac{1}{2}(g_{0i\,,j} - g_{0j\,,i})dx^j \wedge dx^i$$

$$= \frac{g_{00\,,k}}{g_{00}} dx^k \wedge (g_{00}\mathrm{d}t + g_{0i}dx^i) + \frac{1}{2}\left(g_{0i\,,j} - g_{0j\,,i} - \frac{g_{00\,,j}g_{0i}}{g_{00}} + \frac{g_{00\,,i}g_{0j}}{g_{00}}\right) dx^j \wedge dx^i$$
$$= \alpha \wedge K^\flat + \kappa_{ji} dx^j \wedge dx^i$$

with

$$\kappa_{ij} = \frac{g_{00}}{2}\left(\left(\frac{g_{0i}}{g_{00}}\right)_{,j} - \left(\frac{g_{0j}}{g_{00}}\right)_{,i}\right).$$

Generally, these κ_{ij} do not correspond to the κ_{ij} defined in eq. (5.69) because the ∂_i do not form an orthonormal basis. But if we consider the weak gravitational field limit, $g_{00} \cong -1$. $g_{i0} \ll 1$, and $g_{ii} \cong 1$, $g_{ij} \ll 1$ for $i \neq j$, the basis (∂_i) is almost orthonormal. In this case, we get, to the first order,

$$\vec{\Omega} \cong \frac{1}{2}\varepsilon_{ijk} g_{0i\,,j}\partial_k \tag{5.72}$$

where

$$\vec{\Omega} \cong \frac{1}{2}\vec{\nabla} \wedge \vec{g}, \quad \text{with} \quad \vec{g} := (g_{01}, g_{02}, g_{03}).$$

This is the **Lense-Thirring** effect, often called the 'frame dragging' or 'gravitomagnetic effect'. The components g_{0i} of the gravitational field of a rotating star are proportional to the angular momentum $\vec{J}$ of the star. To the frame dragging one has to add the geodesic precession, which is of the order of $\vec{v} \wedge \vec{\nabla}\phi$. We can compare these two effects for the terrestrial gravitational field :

$$\frac{\text{Lense-Thirring}}{\text{geodesic}} \simeq \frac{GJ_\oplus}{3(M_\oplus G)^{3/2} R_\oplus^{1/2}} = 6 \times 10^{-3}\ .$$

The 'Gravity-Probe-B' experiment performed by the NASA has measured the Lense-Thirring effect. The final results which perfectly confirm GR are finally available (see e.g. http://einstein.stanford.edu/highlights/ and Ref. [9]). In 2011, after 6 years of data analysis, the geodesic precession has finally been confirmed with a precision of about 0.3%, and the Lense-Thirring (frame dragging) effect is measured with a significance of more than 5 standard deviation:

$$\begin{aligned} \omega_{\text{geo}} &= -6601.8 \pm 18.3 \text{ milli-arc-seconds per year,} \\ \omega_{\text{fd}} &= -37.2 \pm 7.2 \text{ milli-arc-seconds per year.} \end{aligned}$$

The theoretical predictions are 6.606 arc-seconds per year for the geodesic precession of and 39 milli-arc-seconds per year for 'frame dragging' (Lense-Thirring effect).

Summary

- A gyroscopic compass has constant components relative to a reference frame $(e_i)_{i=1}^3$ if and only if the (e_i) are Fermi transported along the world line of the observer (who carries the compass).

- In a frame at rest relative to a stationary gravitational field (Copernician reference frame, fixed stars), a gyroscope rotates with an angular velocity $\vec{\Omega}$ defined in eqs. (5.68), (5.70) and (5.71).
- The gyroscope does not rotate in a rest frame relative to a stationary gravitational field, $\vec{\Omega} = 0$, if and only if the gravitational field is static.

5.5.4 Local reference frames

We now wish to determine the equations of motion of a free falling test particle, seen by an observer on an arbitrary world line $\rho(\tau)$. We want to work out the effects of general relativity. We therefore determine the local reference frame of the observer and the metric in a neighborhood of the world line.

As before, let $\rho(\tau)$ be the world line of an (accelerated) observer, $u = \dot{\rho}$ her 4-velocity, $u^2 = -1$ and $a = \nabla_u u$. Let $(e_i)_{i=1}^3$ an arbitrary orthonormal reference frame, $\langle e_i, e_j\rangle = \delta_{ij}$, normal to $u = e_0$, $\langle u, e_i\rangle = 0$.
At each point $\rho(\tau)$ we consider a geodesic $\alpha(s)$ normal to u, where s is the arc length, i.e. $\dot{\alpha}^2 = 1$ ($\dot{\alpha} = \frac{\mathrm{d}\alpha}{\mathrm{d}s}$) and $\langle\dot{\alpha}(0), \dot{\rho}(\tau)\rangle = 0$ ($\alpha(0) = \rho(\tau)$).
By $\alpha(s, n, \tau)$ we designate the geodesic passing through $\rho(\tau)$ in direction n, $\dot{\alpha}(0) = n$, $\alpha(0) = \rho(\tau)$ with arc length parameter s

$$n = \left(\frac{\partial}{\partial s}\right)_{\alpha(0,n,\tau)} \quad , \quad n^2 = 1 \ .$$

Every point $p \in \mathcal{M}$ in the neighborhood of $\rho(\tau)$ is on one and only one of these geodesics α (see figure 5.5).
We fix the coordinates of $p = \alpha(s, n, \tau)$, $n = n^i e_i$

$$(x^0(p), x^1(p), x^2(p), x^3(p)) = (\tau, sn^1, sn^2, sn^3) \tag{5.73}$$

That is,

$$x^0(\alpha(s, n, \tau)) = \tau \tag{5.74}$$
$$x^i(\alpha(s, n, \tau)) = sn^i = s\langle n, e_i\rangle \ . \tag{5.75}$$

5.5.5 The Christoffel symbols and developement of the metric along a world line

Along $\rho(\tau)$, $s = 0$, we have by construction

$$\frac{\partial}{\partial x^i} = e_i, \quad \frac{\partial}{\partial x^0} = u \quad \text{(along } \rho). \tag{5.76}$$

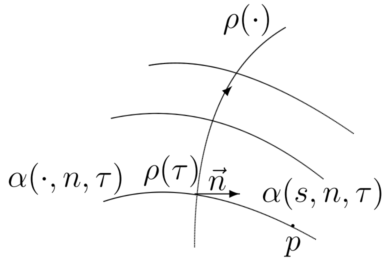


Figure 5.5: A world line $\rho(\tau)$ with transversal geodesics $\alpha(s, n, \tau)$.

This yields

$$g_{\mu\nu} = \langle \partial_\mu, \partial_\nu \rangle = \eta_{\mu\nu} \quad \text{(along } \rho\text{)}. \tag{5.77}$$

Along ρ we then have

$$\nabla_u e_\alpha = \nabla_{e_0} e_\alpha = \Gamma^\beta_{0\alpha} e_\beta$$

and

$$\langle e_\beta, \nabla_u e_\alpha \rangle = \eta_{\beta\gamma} \Gamma^\gamma_{0\alpha} \ .$$

Hence

$$\left.\begin{array}{l} \Gamma^0_{00} = -\langle u, \nabla_u u \rangle = 0 \\ \Gamma^j_{00} = \langle e_j, \nabla_u u \rangle = \langle e_j, a \rangle = a^j \\ \Gamma^0_{0j} = -\langle u, \nabla_u e_j \rangle = \langle \nabla_u u, e_j \rangle = a^j \end{array} \right\} (\text{ along } \gamma). \tag{5.78}$$

Moreover,

$$\omega_{ij} \overset{(5.58)}{=\!=} \langle \nabla_u e_i, e_j \rangle =: \varepsilon_{ijk} \Omega^k = \Gamma^j_{0i} \ . \tag{5.79}$$

Along a geodesic $\alpha(\tau, s, n)$, $\tau = \text{const}$, $n = \text{const}$ we have

$$0 = \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}s^2} + \Gamma^\mu_{\nu\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} \frac{\mathrm{d}x^\lambda}{\mathrm{d}s}$$

Since $(x^\mu) = (\tau, sn^i)$, such that $d^2x^\mu/ds^2 = 0$, this gives

$$\Gamma^\mu_{ij} n^i n^j = 0 \qquad \text{(along } \rho\text{)}. \tag{5.80}$$

Because n^i is arbitrary this implies

$$\Gamma^\mu_{ij} = 0 \qquad \text{(along } \rho\text{)}. \tag{5.81}$$

With the help of these Christoffel symbols we are able to find the partial derivatives of the metric:

$$0 = g_{\mu\nu\,;\lambda} = g_{\mu\nu\,,\lambda} - \Gamma^\alpha_{\mu\lambda} g_{\nu\alpha} - \Gamma^\alpha_{\nu\lambda} g_{\mu\alpha} \ .$$

Using equations (5.77) to (5.79) and (5.80) this gives along ρ

$$g_{\mu\nu\,,0} = \Gamma^\alpha_{\mu 0} \eta_{\alpha\nu} + \Gamma^\alpha_{\nu 0} \eta_{\mu\alpha} \quad \Rightarrow \quad \left\{ \begin{array}{l} g_{00\,,0} = 0 \\ g_{0i\,,0} = -a^i + a^i = 0 \\ g_{ij\,,0} = \varepsilon_{ijk} \Omega^k + \varepsilon_{jik} \Omega^k = 0 \end{array} \right.$$

Along ρ

$$\begin{aligned}
g_{\mu\nu\,,0} &= 0 \\
g_{ij\,,l} &= \Gamma^\mu_{il}\eta_{\mu j} + \Gamma^\mu_{jl}\eta_{\mu i} = 0 \\
g_{00\,,j} &= 2\Gamma^\mu_{0j}\eta_{\mu 0} = -2a^j \\
g_{0j\,,k} &= \Gamma^\mu_{0k}\eta_{\mu j} + \Gamma^\mu_{jk}\eta_{0\mu} = \varepsilon_{kji}\Omega^i
\end{aligned} \tag{5.82}$$

In the neighborhood of $\rho(t)$, the metric is then given by

$$\mathrm{d}s^2 = -(1+2\vec{a}\cdot\vec{x})\mathrm{d}t^2 + 2\varepsilon_{kji}\Omega^i x^k \mathrm{d}t dx^j + \delta_{ij}dx^i dx^j + \mathcal{O}(|\vec{x}|^2)dx^\mu dx^\nu \ . \tag{5.83}$$

The acceleration contributes as $\delta g_{00} = -2\vec{a}\cdot\vec{x}$.
If $\Omega \neq 0$, the axes e_i of the observer are rotating, we have the non-diagonal components

$$g_{0j} = \varepsilon_{jik}\Omega^i x^k = (\vec{\Omega}\wedge\vec{x})^j$$

If $a = \nabla_u u = 0$ and $\vec{\Omega} = 0$ we obtain an inertial reference frame along $\rho(t)$ with $g_{\mu\nu} = \eta_{\mu\nu}$, $\Gamma^\mu_{\alpha\beta} = 0$.
In other words, an observer finds himself in an inertial frame of reference if he is in free fall, $\nabla_u u = 0$, and he Fermi-transports his coordinate axes, (e_i), $(\Leftrightarrow \omega_{ij} = 0)$.
As $a = 0$, for a free falling observer, Fermi transport is equivalent to the parallel transport.

5.5.6 The equations of motion of a test particle, inertial forces

As before, $\rho(t)$ is the world line of our observer (astronaut in a space ship).
She observes a freely falling particle:

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} = 0 \tag{5.84}$$

She measures the velocity of the particle using her time coordinate, t. We define $\gamma = \frac{\mathrm{d}t}{\mathrm{d}\lambda}$.
With $\frac{\mathrm{d}x^0}{\mathrm{d}\lambda} = \gamma$, $\frac{\mathrm{d}x^l}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^l}{\mathrm{d}t}\gamma =: v^l\gamma$, (5.84) becomes

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}t^2} + \frac{1}{\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}t}\frac{\mathrm{d}x^\mu}{\mathrm{d}t} + \Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x^\beta}{\mathrm{d}t} = 0 \ .$$

For $\mu = 0$ this gives, using (5.81)

$$-\frac{1}{\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}t} = 2\Gamma^0_{i0}v^i + \Gamma^0_{00} \ . \tag{5.85}$$

For $\mu = j$, we find, using (5.85)

$$\frac{\mathrm{d}v^j}{\mathrm{d}t} - v^j\left(\Gamma^0_{00} + 2\Gamma^0_{i0}v^i\right) + \Gamma^j_{00} + 2\Gamma^j_{0i}v^i + \Gamma^j_{ki}v^iv^k = 0\ . \tag{5.86}$$

We evaluate (5.86) in the neighborhood of the world line $\rho(t)$ and we consider slowly moving particles, $\mid v \mid\ll 1$. We expand (5.86) to the first order in $\vec{x}$ and $\vec{v}$:

$$\begin{aligned}\frac{\mathrm{d}v^j}{\mathrm{d}t} &= v^j\Gamma^0_{00}\Big|_{\vec{x}=0} - \Gamma^j_{00}\Big|_{\vec{x}=0} - \left(\Gamma^j_{00\,,k}\Big|_{\vec{x}=0}\right)\cdot x^k - 2\Gamma^j_{0i}\Big|_{\vec{x}=0}v^i \\ &= -a^j - 2\varepsilon_{ijk}\Omega^k v^i - \left(\Gamma^j_{00\,,k}\Big|_{\vec{x}=0}\right)x^k\end{aligned} \tag{5.87}$$

$\Gamma^i_{00\,,k}$ can be related to the Riemann tensor:

$$R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha{}_{\beta\delta\,,\gamma} - \Gamma^\alpha{}_{\beta\gamma\,,\delta} + \Gamma^\alpha_{\gamma\mu}\Gamma^\mu_{\beta\delta} - \Gamma^\alpha_{\delta\mu}\Gamma^\mu_{\beta\gamma}\ .$$

Then

$$\Gamma^j_{00\,,k} = R^j{}_{0k0} + \underbrace{\Gamma^j_{0k\,,0}}_{\varepsilon_{kjl}\dot{\Omega}^l} - \underbrace{\Gamma^j_{k\mu}\Gamma^\mu_{00}}_{0} + \underbrace{\Gamma^j_{0\mu}\Gamma^\mu_{0k}}_{a^ja^k+\varepsilon_{ijl}\Omega^l\varepsilon_{kim}\Omega^m}$$

at $\vec{x} = 0$.
Substituting everything in (5.87), we finally get ($\dot{} = \frac{d}{dt}$)

$$\dot{\vec{v}} = -\vec{a}(1 + \vec{a}\cdot\vec{x}) - \underbrace{2\vec{\Omega}\wedge\vec{v}}_{(1)} - \dot{\vec{\Omega}}\wedge\vec{x} - \underbrace{\vec{\Omega}\wedge(\vec{\Omega}\wedge\vec{x})}_{(2)} + \vec{f} \tag{5.88}$$

where

$$f^j = R^j{}_{00k}x^k. \tag{5.89}$$

The first term is the inertial acceleration with the relativistic correction $\vec{a}\cdot\vec{x}$ whose origin is in the correction of the metric (5.83).
The terms with $\vec{\Omega}$ are identical to those in classical mechanics, (1) is the “Coriolis force” and (2) is the “centrifugal force”.
The force $R^j{}_{0k0}x^k$ is the consequence of the inhomogeneity of the gravitational field. It is the “tidal force”.
If the axes (e_i) are Fermi-transported, the inertial forces that involve $\vec{\Omega}$ vanish, $\vec{\Omega} = 0$. If the observer moves along a geodesic, $a = \nabla_u u = 0$, only the tidal force remains. It is not possible to get rid of it by a coordinate transformation. Therefore it represents the “true”, coordinate independent, gravitational force.

Chapter 6

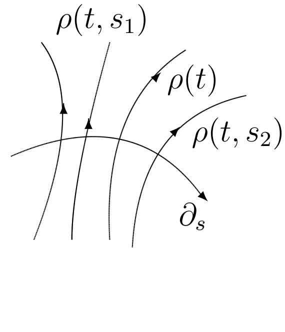
Einstein's field equations

Until here we have studied the laws of physics in the presence of an external gravitational field using the equivalence principle.

In addition, the gravitational field (potential) $(g_{\mu\nu})$ is determined by the amount of energy and momentum of the universe (together with boundary conditions). To find the equations that govern the relationship between $g_{\mu\nu}$ and energy-momentum tensor $T_{\mu\nu}$ we use the equivalence principle, the Newtonian limit and the covariant version of the "conservation" of energy and momentum.

6.1 Heuristic derivation of the Einstein field equation

We first reformulate the result we have found in the previous paragraph for an observer in free fall and an orthonormal basis $(e_i)_{i=1}^3$ that is parallel transported: let $\rho(t)$ be a geodesic and $\rho(t,s)$ a congruence of the neighboring geodesics, $\rho(t) = \rho(t,0)$. For each value s, the path $t \mapsto \rho(t,s)$ is a geodesic. Let



$$u = \dot{\rho} = \rho_\star \partial_t \ , u^2 \equiv \langle u, u \rangle = -1 \ , \quad \text{and} \quad w = \rho_\star \partial_s \tag{6.1}$$

Let $n = w + \langle w, u\rangle u$ be the distance vector normal to u:

$$n = n^i e_i .$$

We wish to establish an equation for the evolution of this distance vector n.

Because of eq. (6.1) $[u, w] = 0$, which implies that $\nabla_u w = \nabla_w u$. As $\nabla_u u = 0$ this leads to

$$\nabla_u^2 w = \nabla_u \nabla_w u = (\nabla_u \nabla_w - \nabla_w \nabla_u)u = R(u, w)u \ . \tag{6.2}$$

Similarly with n we have

$$L_u n = [u, n] = [u, \langle w, u\rangle u] = u(\langle w, u\rangle)u \ .$$

But

$$u\langle w, u\rangle = \langle \nabla_u w, u\rangle = \langle \nabla_w u, u\rangle = \frac{1}{2}\nabla_w \langle u, u\rangle = 0,$$

that is

$$L_u n = 0 = \nabla_u n - \nabla_n u \tag{6.3}$$

By analogy with (6.2) we get

$$\nabla_u^2 n = R(u, n)u \ .$$

In our basis $n = n^i e_i$, and because the e_i are parallel transported along u,

$$\nabla_u n = u(n^i)e_i = \frac{\mathrm{d}n^i}{\mathrm{d}t} e_i \quad \text{and} \quad \nabla_u^2 n = \frac{\mathrm{d}^2 n^i}{\mathrm{d}t^2} e_i \ .$$

With $e_0 \equiv u$ we then obtain

$$\frac{\mathrm{d}^2 n^i}{\mathrm{d}t^2} e_i = n^j R(e_0, e_j)e_0 = n^j R^i{}_{00j} e_i \ , \qquad \text{or}$$

$$\frac{\mathrm{d}^2 n^i}{\mathrm{d}t^2} = n^j R^i{}_{00j} \ . \tag{6.4}$$

This equation describes the relative acceleration of neighbouring test particles in free fall. It is called the 'geodesic deviation equation'.

We compare this with the Newtonian theory. There we have

$$\ddot{x}^i = -(\partial_i \Phi)_{\vec{x}}$$

$$\ddot{x}^i + \ddot{n}^i = -(\partial_i \Phi)_{\vec{x}+\vec{n}} \ .$$

By substracting one from the other, we get

$$\ddot{n}^i = -(\partial_i \Phi)_{\vec{x}+\vec{n}} + (\partial_i \Phi)_{\vec{x}} \cong -(\partial_i \partial_j \Phi)_{\vec{x}} n^j . \tag{6.5}$$

Comparing this with equation (6.4) suggests the following analogy

$$R^i{}_{00j} \Longleftrightarrow -(\partial_i\partial_j\Phi) \ .$$

By taking the trace on both sides we find the correspondence

$$R_{00} \Longleftrightarrow \Delta\Phi.$$

In the Newtonian theory, the gravitational potential is determined by the Poisson equation:

$$\Delta\Phi = 4\pi G\rho \cong 4\pi G T_{00} \ .$$

Leading to

$$R_{00} = 4\pi G T_{00} \ . \tag{6.6}$$

This motivates the covariant equation

$$R_{\mu\nu} = 4\pi G T_{\mu\nu}. \tag{6.7}$$

Actually, this equation cannot be correct, because $R^{\mu\nu}{}_{;\nu} \neq 0$ and $T^{\mu\nu}{}_{;\nu} = 0$ in generality. Therefore, (6.7) needs the following modification:

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T), \quad \text{where} \quad T = T^\mu_\mu \ . \tag{6.8}$$

In the Newtonian limit, $T_{00} \cong \rho$, $T \cong T^0_0 \cong -\rho$, $g_{00} \cong -1$, so that, for $\mu\nu = 00$ eq. (6.8) is again reduced to (6.6). Eq. (6.8) is equivalent to

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}. \tag{6.9}$$

The equations (6.9) or (6.8) are the Einstein field equations.
We have defined the Einstein tensor, $G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ so that we obtain

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \ . \tag{6.10}$$

We have already shown that the contracted Bianchi identities are equivalent to $G^{\mu\nu}{}_{;\nu} = 0$.

Uniqueness: One can show the following fact:
let $\mathscr{D}_{\mu\nu}[g]$ be a symmetric tensor which is built only with $g_{\mu\nu}$ and with its first and second derivatives and that satisfies $\mathscr{D}_{\mu\nu}[g]^{;\nu} = 0$. We then have

$$\mathscr{D}_{\mu\nu}[g] = aG_{\mu\nu} + bg_{\mu\nu}, \qquad \text{where } a, \text{ and } b \text{ are constants}$$

Therefore, the Einstein field equations necessarily take the form

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \quad \kappa = 8\pi G \,, \qquad \Lambda = b/a \,. \tag{6.11}$$

The value of κ is determined by the Newtonian limit

$$\Delta\Phi = 4\pi G\rho + \Lambda.$$

The constant Λ plays the role of a **homogeneous** mass density, $\rho_{\text{eff}} = \frac{\Lambda}{4\pi G}$.
This quantity plays an important role in current cosmology. But because $\left|\frac{\Lambda}{4\pi G}\right| \lesssim 10^{-29}\frac{\text{g}}{\text{cm}^3}$, we can neglect it when we are mainly interested in the gravitational effects over stellar distances with $\rho \gtrsim \rho_\odot \cong 1.4\frac{\text{g}}{\text{cm}^3}$.
Equations (6.10) and (6.11) are, even in vacuum, ($T_{\mu\nu} = 0$), non-linear partial differential equations. I believe that even today, many of their consequences are still not fully understood. In some cases, the non-linearity can be interpreted as the coupling of the gravitational field with its own energy-momentum density.
A complete analysis of the Einstein field equations is difficult (I will only mention some of the main results). Exact solutions are only known for symmetric situations. Fortunately they are the most relevant in astrophysics and cosmology:

1. The Schwarzschild metric (see next chapter): a static, spherically symmetric solution; it describes spherical stars and black holes.
2. The Kerr metric: a stationary solution with rotational symmetry; it describes rotating stars and black holes
3. The Robertson and Walker metric: a homogeneous and isotropic solution, describes Friedmann-Lemaître universes that are expanding or contracting.
4. (Anti-)de Sitter spacetime. $T_{\mu\nu} = 0$ and $\Lambda > 0$ (for de Sitter) or $\Lambda < 0$ (for Anti-de Sitter).

There are many more exact solutions, see [15].

6.2 The Cauchy problem

The problem is the following: let $\mathcal{S}$ be a 3-dimensional Riemannian manifold with metric γ_{ij}. In addition, let the functions $\gamma_{0\mu}$, $\gamma_{\mu\nu,0}$ be given ($ij = 1,2,3$; $\mu,\nu = 0,1,2,3$). Is there a Lorentz manifold $(\mathcal{M}, g)$ and an embedding $\sigma : \mathcal{S} \to \mathcal{M}$ such that

$$\sigma_\star\gamma_{\mu\nu} = g_{\mu\nu} \quad \text{and} \quad \sigma_\star\gamma_{\mu\nu,0} = g_{\mu\nu,0} \;\; \text{on} \;\; \sigma(\mathcal{S})$$

and

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{6.12}$$

on $\mathcal{M}$?
In other words, does a solution to the problem (6.12) exist, with inital conditions

$(\mathcal{S}, \gamma, \gamma_{,0})$?
We first note that the Bianchi identity gives

$$G^{\mu 0}{}_{,0} = -G^{\mu i}{}_{,i} - \Gamma^{\mu}_{\nu\lambda} G^{\lambda\nu} - \Gamma^{\nu}_{\nu\lambda} G^{\mu\lambda}. \tag{6.13}$$

Because the right hand side of these equations contains at most second derivatives with respect to time ($t = x^0$), the components $G^{\mu 0}$ contain at most first derivatives of $g_{\mu\nu}$ with respect to time. The equations

$$G_{\mu 0} = 8\pi G T_{\mu 0} \tag{6.14}$$

are therefore **constraints** on $g_{\mu\nu}$ and $g_{\mu\nu\,,0}$, constraints that have to be satisfied on $\sigma(\mathcal{S})$ for a solution to exist.
We are then left with 6 evolution equations,

$$G_{ij} = 8\pi G T_{ij} \tag{6.15}$$

that determine the evolution of the ten components of $g_{\mu\nu}$ in time.The solution is therefore not uniquely determined. One must impose four gauge conditions, which precisely corresponds to a choice of coordinates!
For a "compatible" gauge choice and initial conditions $(g_{\mu\nu}, g_{\mu\nu\,,0})$ that satisfy the constraints (6.14), with a "reasonable" energy-momentum tensor one can always obtain a local solution of (6.15). Equation (6.15) is a second order system of differential equations that is **hyperbolic** for the six $g_{\mu\nu}$ that are not fixed by the gauge choice.
The existence of a local solution is not so difficult to prove, see e.g. [7]. But any global results are rather deep and limited.
An example of a gauge choice is the **harmonic gauge** defined by the four conditions

$$\left(\sqrt{-g}\, g^{\mu\nu}\right)_{,\nu} = 0 \; . \tag{6.16}$$

By taking the time derivative of these conditions, we obtain

$$\left(\sqrt{-g}\, g^{\mu 0}\right)_{,00} = -\left(\sqrt{-g}\, g^{\mu i}\right)_{,0i} \; . \tag{6.17}$$

With this and the six equations $G_{ij} = 8\pi G T_{ij}$, that are differential equations of second order in time for the variables g_{ij}, we then have 10 second order equations that determine the 10 functions $g_{\mu\nu}$ for given initial conditions.

6.3 Lagrangian formulation

One can also obtain Einstein's equations by a variational principle from the so called Einstein-Hilbert action,

$$S = \int d^4x \sqrt{-g} \left(\frac{R}{16\pi G} + L_m \right) \; . \tag{6.18}$$

Here R is the Riemann curvature scalar and L_m is the Lagrangian density of matter. By definition, its variation with respect to the metric gives the energy momentum tensor,

$$2\frac{\delta\,(\sqrt{-g}L_m)}{\delta g_{\mu\nu}} =: \sqrt{-g}T^{\mu\nu}\,. \tag{6.19}$$

Exercise: Show this for the Maxwell Lagrangian,

$$L_m = L_{\mathrm{elm}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = -\frac{1}{4}F_{\mu\nu}F_{\alpha\beta}g^{\mu\alpha}g^{\beta\nu}\,.$$

(in Heavyside units) where as

$$T^{\mu\nu}_{\mathrm{elm}} = g^{\mu\alpha}F_{\alpha\beta}F^{\beta\nu} + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\,.$$

We now show that if we may disregard boundary terms,

$$\frac{\delta\,(\sqrt{-g}R)}{\delta g_{\mu\nu}} = -\sqrt{-g}G^{\mu\nu}\,. \tag{6.20}$$

To see this we proceed in three steps:

$$\delta\left(\sqrt{-g}R\right) = (\delta\sqrt{-g})R_{\alpha\beta}g^{\alpha\beta} + \sqrt{-g}(\delta R_{\alpha\beta})g^{\alpha\beta} + \sqrt{-g}R_{\alpha\beta}\delta g^{\alpha\beta}\,.$$

1. We first determine $\delta\,(\sqrt{-g})$.
   To simplify the notation we denote the matrix $g_{\mu\nu}$ here by g, and its inverse by $g^{-1} = (g^{\mu\nu})$. We consider a path of metrics through $\bar{g}$ and parameterize it as $g(\lambda) = \exp(\lambda C)\bar{g}$, where C is a 4×4 matrix. In the vicinity of $\bar{g}$ every matrix is of this form, just set $g(\lambda)\bar{g}^{-1} = \exp(\lambda C)$. We then have $\det g = \exp(\lambda \mathrm{tr} C)\det\bar{g}$, such that

   $$\frac{d}{d\lambda}\det g = \mathrm{tr}C\exp(\lambda\mathrm{tr}C)\det\bar{g} = \mathrm{tr}\left(\frac{dg}{d\lambda}g^{-1}\right)\det g = g^{\mu\nu}\frac{dg_{\mu\nu}}{d\lambda}\det g\,. \tag{6.21}$$

   Since every small variation δg is of this form, we obtain

   $$\delta\sqrt{-\det g} = \frac{1}{2}\sqrt{-\det g}g^{\mu\nu}\delta g_{\mu\nu}\,. \tag{6.22}$$

2. $R_{\alpha\beta}\delta g^{\alpha\beta}$.
   We use $g^{\alpha\beta}g_{\alpha\mu} = \delta^\beta_\mu$ such that $0 = \delta g^{\alpha\beta}g_{\alpha\mu} + g^{\alpha\beta}\delta g_{\alpha\mu}$ which yields

   $$\delta g^{\alpha\beta} = -g^{\alpha\nu}g^{\beta\mu}\delta g_{\nu\mu}\,. \tag{6.23}$$

   These two terms put together already give $-\sqrt{-g}G^{\mu\nu}\delta g_{\mu\nu}$ it remains to show that $g^{\mu\nu}\delta R_{\mu\nu}$ is a surface term.

3. $g^{\mu\nu}\delta R_{\mu\nu}$.
To show this is a surface term, we remember (3.41) which implies that the Christoffel symbols of the metric connections of the metrics $\bar{g}_{\mu\nu}$ and $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$ are related by

$$\Gamma^{\mu}_{\alpha\beta} = \bar{\Gamma}^{\mu}_{\alpha\beta} + \frac{1}{2}g^{\mu\nu}\left[\bar{\nabla}_{\beta}\delta g_{\alpha\nu} + \bar{\nabla}_{\alpha}\delta g_{\beta\nu} - \bar{\nabla}_{\nu}\delta g_{\alpha\beta}\right] = \bar{\Gamma}^{\mu}_{\alpha\beta} + C^{\mu}_{\alpha\beta}. \qquad (6.24)$$

To linear order in $\delta g_{\mu\nu}$ we then obtain the following relation for the Riemann tensors,

$$R^{\nu}{}_{\mu\alpha\beta} = \bar{R}^{\nu}_{\mu\alpha\beta} - \bar{\nabla}_{\alpha}C^{\nu}_{\beta\mu} + \bar{\nabla}_{\beta}C^{\nu}_{\alpha\mu}\,. \qquad (6.25)$$

Contraction gives

$$R_{\mu\beta} = \bar{R}_{\mu\beta} + \delta R_{\mu\beta} = \bar{R}_{\mu\beta} - \bar{\nabla}_{\alpha}C^{\alpha}_{\beta\mu} + \bar{\nabla}_{\beta}C^{\alpha}_{\alpha\mu}\,. \qquad (6.26)$$

With this we obtain

$$\bar{g}^{\mu\beta}\delta R_{\mu\beta} = \bar{\nabla}_{\beta}\left[\bar{g}^{\mu\beta}C^{\alpha}_{\alpha\mu} - \bar{g}^{\mu\alpha}C^{\beta}_{\alpha\mu}\right] = \bar{\nabla}_{\beta}v^{\beta}\,. \qquad (6.27)$$

If we can neglect surface terms[1], this term does not contribute to the field equation. This proves our assertion. □

This Lagrangian formulation of Einstein's field equations is especially important for the problem of quantizing gravity, but also for the development of modified theories of gravity. Furthermore, the definition (6.19) automatically gives the symmetrical energy momentum tensor of matter which appears as source on the rhs. of Einstein's equations. Remember that the canonical energy momentum tensor need not be symmetric.

[1] A covariant divergence of a vector field (but not of a 2-tensor!), $v^{\beta}_{;\beta}$ is a surface term since $\int_M d^nx\sqrt{-g}v^{\beta}_{;\beta} = \int_M d^nx\left(\sqrt{-g}v^{\beta}\right)_{,\beta} = \int_{\partial M} d\sigma v^{\beta}n_{\beta}$, where n is the normal to ∂M. To verify the first equal sign one has to show that $\sqrt{-g}\Gamma^{\mu}_{\mu\nu} = \partial_{\nu}\sqrt{-g}$. We leave this as an exercise.

Chapter 7

The Schwarzschild solution and the classical tests of GR

7.1 Derivation

We search for a static spherically symmetric solution of Einstein's equations in the vacuum ($T_{\mu\nu} = 0$). This solution describes the exterior of a static star with vanishing (small) angular momentum, like e.g. the sun.
We choose the manifold

$$\mathcal{M} = \underbrace{\mathbb{R}}_{t} \times \underbrace{\mathbb{R}_+}_{r} \times \underbrace{\mathbb{S}^2}_{(\vartheta,\varphi)}$$

and make the following ansatz for the metric:

$$g = -e^{2a(r)}\mathrm{d}t^2 + e^{2b(r)}\mathrm{d}r^2 + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2). \tag{7.1}$$

The variable r is chosen such that a sphere of radius r has the surface $4\pi r^2$.
The functions $a(r)$, $b(r)$ should obey the boundary condition $a(r), b(r) \xrightarrow{r\to\infty} 0$ such that the spacetime is asymptotically flat. With

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}\left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}\right) ,$$

we find

$$\Gamma^{\vartheta}_{\varphi\varphi} = -\sin\vartheta\cos\vartheta \qquad \Gamma^{\varphi}_{\vartheta\varphi} = \Gamma^{\varphi}_{\varphi\vartheta} = \cot\vartheta \tag{7.2}$$

$$\Gamma^{r}_{\varphi\varphi} = -e^{-2b} r\sin^2\vartheta \qquad \Gamma^{\varphi}_{r\varphi} = \Gamma^{\varphi}_{\varphi r} = \frac{1}{r} \tag{7.3}$$

$$\Gamma^{r}_{\vartheta\vartheta} = -e^{-2b} r \qquad \Gamma^{\vartheta}_{r\vartheta} = \Gamma^{\vartheta}_{\vartheta r} = \frac{1}{r} \tag{7.4}$$

$$\Gamma^{r}_{tt} = e^{2(a-b)} a' \qquad \Gamma^{t}_{rt} = \Gamma^{t}_{tr} = a' \tag{7.5}$$

$$\Gamma^{r}_{rr} = b' . \tag{7.6}$$

All other Christoffel symbols vanish. Inserting this in (see eq. (3.27)

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\mu \Gamma^\alpha_{\alpha\nu} + \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta} - \Gamma^\alpha_{\beta\mu}\Gamma^\beta_{\alpha\nu}\,,$$

we obtain

$$\begin{aligned}
R_{tt} &= \left(e^{2(a-b)}a'\right)' + e^{2(a-b)}a'\left(b' + a' + \frac{2}{r}\right) - 2e^{2(a-b)}a'^2 \\
&= e^{2(a-b)}\left[a'' + a'^2 + \frac{2a'}{r} - a'b'\right] \\
R_{rr} &= -a'' - a'^2 + \frac{2b'}{r} + a'b' \\
R_{\vartheta\vartheta} &= 1 - e^{-2b} + (b' - a')re^{-2b} \\
R_{\varphi\varphi} &= \sin^2\vartheta\left[1 - e^{-2b} + (b' - a')re^{-2b}\right]\,.
\end{aligned}$$

The vacuum Einstein equations imply $R_{\mu\nu} = 0$. Using $e^{2(b-a)}R_{tt} + R_{rr} \equiv 0$ we find $a' + b' = 0$, with the boundary condition this implies $a = -b$. Inserting this in $R_{\vartheta\vartheta}$ we obtain $1 = e^{-2b} - 2b're^{-2b} = (re^{-2b})'$ which we integrate to

$$r - 2m = re^{-2b}\,, \qquad e^{-2b} = 1 - \frac{2m}{r} = e^{2a}\,. \tag{7.7}$$

Here m is an integration constant. With (7.1) and (7.7) we now have the solution

$$g = -\left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\left(1 - \frac{2m}{r}\right)} + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2). \tag{7.8}$$

Eq. (7.8) is the famous Schwarzschild solution (K. Schwarzschild, 1916).

In order to interpret the integration constant m, we remember that in the weak field limit, $r \gg m$,

$$-g_{00} \cong 1 + \frac{2\Phi}{c^2} = 1 - 2\frac{GM}{rc^2},$$

therefore

$$m = \frac{GM}{c^2}. \tag{7.9}$$

One can show (Birkhoff's theorem) that the Schwarzschild metric is the **only** spherically symmetric solution of the vacuum Einstein equations. Hence a spherically symmetric vacuum solution is necessarily static.

The metric (7.8) is often also written in the following form:

We set

$$\rho = \frac{1}{2}(r - m + (r^2 - 2mr)^{1/2}).$$

$$\text{Where} \quad r = \rho\left(1 + \frac{m}{2\rho}\right)^2, \quad 1 - \frac{2m}{r} = \left(\frac{1 - \frac{m}{2\rho}}{1 + \frac{m}{2\rho}}\right)^2\,.$$

With (7.8) this yields

$$
\begin{aligned}
g &= -\left(\frac{1-\frac{m}{2\rho}}{1+\frac{m}{2\rho}}\right)^2 \mathrm{d}t^2 + \left(1+\frac{m}{2\rho}\right)^4 (\mathrm{d}\rho^2 + \rho^2 \mathrm{d}\vartheta^2 + \rho^2 \sin^2\vartheta \mathrm{d}\varphi^2) \\
&= -h^2(|\vec{x}|)\mathrm{d}t^2 + f^2(|\vec{x}|)\mathrm{d}\vec{x}^2 \qquad (7.10)
\end{aligned}
$$

or

$$
\begin{aligned}
h^2(\vec{x}) &= \left(\frac{1-\frac{m}{2|x|}}{1+\frac{m}{2|x|}}\right)^2 = 1 - 2\alpha\frac{m}{|x|} + 2\beta\left(\frac{m}{|x|}\right)^2 + \mathcal{O}\left(\frac{m}{|x|}\right)^3 \\
f^2(\vec{x}) &= \left(1+\frac{m}{2|x|}\right)^4 = 1 + 2\gamma\frac{m}{|x|} + 3\gamma'\left(\frac{m}{|x|}\right)^2 + \mathcal{O}\left(\frac{m}{|x|}\right)^3 .
\end{aligned}
\qquad (7.11)
$$

where $\alpha = \beta = \gamma = \gamma' = 1$.
Tests of General Relativity are often in terms of limits on these "post-Newtonian parameters".
The value of $\alpha = 1$ is simply a consequence of the Newtonian limit. At present, the other parameters are experimentally constrained by $|\gamma - 1| < 2\cdot 10^{-5}$, $|\gamma' - 1| < 2 \cdot 10^{-3}$ and $|\beta - 1| < 3\cdot 10^{-3}$ (see e.g. [18]).

The solution (7.8) seems to have a singularity at the Schwarzschild radius, $r = R_S$, given by

$$
R_S = 2m = \frac{2GM}{c^2}\,, \qquad R_S(M_\odot) = \frac{2GM_\odot}{c^2} = 2.9\mathrm{km}. \qquad (7.12)
$$

We shall later see that $r = R_S$ is simply a coordinate singularity. An indication of this fact is that the components of the Riemann tensor rest finite at $r = R_S$.
Nevertheless, the sphere $r = R_S$ has an important physical meaning: no information can penetrate from $r < R_S$ to $r > R_S$. The Schwarzschild sphere, $r = R_S$ forms a **horizon**: all information which passes through $r = R_S$ is lost from the outside (loss of information? unitarity?).

7.2 Test particles in the Schwarzschild metric

The equation of motion of a test particle is the Euler-Lagrange equation for the Lagrange function

$$
\mathcal{L} = \frac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu .
$$

In our Schwarzschild geometry this results in

$$
2\mathcal{L} = -\left(1 - \frac{2m}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{1-\frac{2m}{r}} + r^2(\dot{\vartheta}^2 + \sin^2\vartheta\dot{\varphi}^2)\ . \qquad (7.13)
$$

Along the path $x^\mu(\tau)$ we have (for a massive particle)

$$
-2\mathcal{L} = 1. \qquad (7.14)
$$

The equation for ϑ, $\left(\frac{\partial\mathcal{L}}{\partial\dot\vartheta}\right)^{\cdot} - \frac{\partial\mathcal{L}}{\partial\vartheta} = 0$, yields

$$(r^2\dot\vartheta)^{\cdot} = r^2\sin\vartheta\cos\vartheta\dot\varphi^2\ . \tag{7.15}$$

By choosing the initial condition (which corresponds to a coordinate choice (ϑ,φ)) $\vartheta = \frac{\pi}{2},\ \dot\vartheta = 0$, we obtain

$$\ddot\vartheta = 0 \quad \text{and so} \quad \dot\vartheta \equiv 0\ ,\ \text{therefore} \quad \vartheta \equiv \frac{\pi}{2} \tag{7.16}$$

solves the ϑ-equation. With $\vartheta = \frac{\pi}{2}$ we have

$$2\mathcal{L} = -\left(1-\frac{2m}{r}\right)\dot t^2 + \frac{\dot r^2}{1-\frac{2m}{r}} + r^2\dot\varphi^2. \tag{7.17}$$

Here t and φ are cyclic variables; they determine the following first integrals:

$$\frac{\partial\mathcal{L}}{\partial\dot\varphi} = r^2\dot\varphi = \text{const} = L \tag{7.18}$$

(Kepler's area constant) and

$$-\frac{\partial\mathcal{L}}{\partial\dot t} = \left(1-\frac{2m}{r}\right)\dot t = \text{const} = E\ . \tag{7.19}$$

With (7.14) and (7.17) this yields

$$\frac{E^2}{\left(1-\frac{2m}{r}\right)} - \frac{\dot r^2}{\left(1-\frac{2m}{r}\right)} - \frac{L^2}{r^2} = 1 \tag{7.20}$$

$$\dot r^2 + V(r) = E^2, \tag{7.21}$$

With

$$V(r) = \left(1-\frac{2m}{r}\right)\left(1+\frac{L^2}{r^2}\right). \tag{7.22}$$

We wish to determine the curve $r(\varphi)$. We use the notation $' \equiv \frac{\mathrm{d}}{\mathrm{d}\varphi}$. So that $r' = \frac{\dot r}{\dot\varphi}$, $\dot r = r'\dot\varphi = \frac{r'L}{r^2}$. With this, equation (7.21) leads to

$$\frac{r'^2L^2}{r^4} = E^2 - V(r).$$

We set $u = \frac{1}{r}$. Then $r' = -\frac{u'}{u^2}$ and $u' = -r'/r^2$. Inserting this above yields

$$\begin{aligned} L^2u'^2 &= E^2 - (1-2mu)(1+L^2u^2)\ , \\ u'^2 + u^2 &= \tfrac{E^2-1}{L^2} + \tfrac{2mu}{L^2} + 2mu^3\ . \end{aligned} \tag{7.23}$$

Differentiating eq. (7.23) with respect to φ we obtain

$$2u'u'' + 2uu' = \frac{2m}{L^2}u' + 6mu^2u'.$$

Either

$$u' = 0 \quad \text{(circular motion)} \tag{7.24}$$

or

$$u'' + u = \frac{m}{L^2} + 3mu^2. \tag{7.25}$$

Let us compare (7.25) with the equivalent equation of motion for Newtonian gravity, $\Phi = -\frac{m}{r}$,

$$\mathcal{L} = \frac{1}{2}\left[\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + r^2\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2\right] - \Phi\ .$$

Here φ is cyclic and $\frac{\mathrm{d}\varphi}{\mathrm{d}t}r^2 = L = \text{const}$. The equation for r yields

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = r\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 - \frac{d\Phi}{dr}\ . \tag{7.26}$$

With $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\varphi}\cdot\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{r'}{r^2}L = -Lu'$ and $\frac{\mathrm{d}^2r}{\mathrm{d}t^2} = -Lu''\frac{\mathrm{d}\varphi}{\mathrm{d}t} = -L^2u''u^2$, the eq. (7.26) leads to

$$u'' + u = \frac{1}{L^2u^2}\frac{d\Phi}{dr} = \frac{m}{L^2}. \tag{7.27}$$

The relativistic correction to this is the term $3mu^2$. This "perturbation" is very small,

$$\begin{aligned}\frac{3mu^2}{m/L^2} &= 3u^2L^2 = \frac{3}{r^2}(r^2\dot\varphi)^2 \cong 3\frac{(r\dot\varphi)^2}{c^2}\\ &= 3\left(\frac{v_\perp}{c}\right)^2 \cong 7.7\cdot 10^{-8} \text{ for the planet Mercury.}\end{aligned}$$

We can interpret (7.25) as the equation of a Newtonian motion in the potential

$$\Phi = -\frac{m}{r} - \frac{mL^2}{r^3}. \tag{7.28}$$

7.3 The perihelion advance of a planet

From classical mechanics we know that the only central potentials for which all the orbits are **closed** are the Newton potential, $\Phi_N = -\frac{m}{r}$ and the harmonic oscillator, $\Phi_H = \kappa^2 r^2$ (Bertrand's theorem). We expect that

$$r(\varphi + 2\pi + \Delta\varphi) = r(\varphi)$$

with $\Delta\varphi \neq 0$. The value $\Delta\varphi$ is the perihelion advance. We calculate $\Delta\varphi$ to the first order in the perturbation $\frac{mL^2}{r^3}$.

The solution of (7.27) (Kepler's ellipse) is

$$u = \frac{1}{p}(1 + e\cos\varphi) \tag{7.29}$$

with $p = \frac{L^2}{m} = a(1-e^2)$, where a is the semi major axis of the ellipse and e is its excentricity. Introducing this solution for the perturbation in eq. (7.25), we obtain

$$u'' + u = \frac{m}{L^2} + \frac{3m^3}{L^4}(1 + 2e\cos\varphi + e^2\cos^2\varphi)\ . \tag{7.30}$$

Particular solutions of the equations

$$u'' + u = \begin{cases} \frac{3m^3}{L^4} \\ \frac{6m^3}{L^4} e\cos\varphi \\ \frac{3m^3}{L^4} e^2\cos^2\varphi \end{cases}$$

are

$$u_1 = \begin{cases} \frac{3m^3}{L^4} \\ \frac{3m^3}{L^4} e\varphi\sin\varphi \\ \frac{3m^3}{L^4} e^2\left(\frac{1}{2} - \frac{1}{6}\cos 2\varphi\right)\ . \end{cases}$$

Adding them together we obtain

$$u_2(\varphi) = \frac{m}{L^2}\left[1 + e\cos\varphi + \frac{3m^2}{L^2}\left(1 + \frac{e^2}{2}\right) - \frac{m^2}{2L^2}e^2\cos 2\varphi + \frac{3m^2}{L^2}e\varphi\sin\varphi\right]\ .$$

Let us consider a first perihelion passage at $\varphi = 0$. The next one will be close to 2π but not exactly at $\varphi = 2\pi$. The perihelion is defined by $r' = u' = 0$. Inserting the solution u_2 we obtain

$$u_2' = -\frac{me}{L^2}\left[\sin\varphi - \frac{m^2}{L^2}\left(e\sin 2\varphi + 3\sin\varphi + 3\varphi\cos\varphi\right)\right]\ .$$

Setting $\varphi = 2\pi + \Delta\varphi$ for the second perihelion passage we obtain to lowest order in the perturbation

$$\sin\Delta\varphi - \frac{3m^2}{L^2}2\pi = 0\,.$$

Since $\Delta\varphi$ is small we obtain

$$\Delta\varphi \cong \frac{6\pi m^2}{L^2} \cong \frac{6\pi m}{a(1-e^2)} \quad \text{for each period.} \tag{7.31}$$

For Mercury a is the smallest, so that in the solar system this effect is largest for Mercury's orbit. Inserting numbers we obtain

$$(\Delta\varphi)_{\text{rel}} \cong 42,92'' \quad \text{per century.} \tag{7.32}$$

The perihelion advance of Mercury due to Newtonian perturbations caused by the other planets (mainly Jupiter) is about

$$\begin{aligned}(\Delta\varphi)_{\text{pert. newt.}} &\simeq 531'' \quad \text{per century,}\\ (\Delta\varphi)_{\text{measured}} &\simeq 574'' \quad \text{per century.}\end{aligned}$$

The less than 10% discrepancy between the Newtonian theory of gravity and the observations was the only **observable** indication of general relativity **before** 1915.

There is also a perihelion advance of Mercury's orbit that is caused by the solar **quadrupole**. But according to recent measurements of the Sun's quadrupole, this effect causes a perihelion advance of less than 1″ per century.

"... dass die Gleichungen die Perihelbewegungen Merkurs richtig liefern! Ich war einige Tage fassungslos vor freudiger Erregung."
(A. Einstein to P.Ehrenfest, January 1916).

7.4 The deflection of light

For light rays we have $\mathcal{L}=\frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$. Instead of (7.23) we obtain

$$u'^2 + u^2 = \frac{E^2}{L^2} + 2mu^3$$

and, after taking the derivative with respect to φ,

$$u'' + u = 3mu^2. \tag{7.33}$$

The right hand side of (7.33) is very small

$$\frac{3mu^2}{u} = \frac{3R_S}{2R} \lesssim \frac{R_S}{R_\odot} \cong 10^{-6}$$

By substituting it with 0 we obtain

$$u = \frac{1}{b}\sin\varphi \;, \tag{7.34}$$

the straight line with impact parameter b indicated in the following figure.

Inserting (7.34) in the perturbation $3mu^2$ in (7.33) we find

$$u'' + u = \frac{3m}{b^2}(1-\cos^2\varphi) = \frac{3m}{2b^2}\,(1-\cos 2\varphi)$$

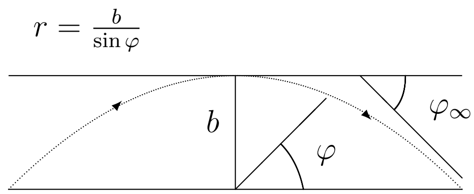


Figure 7.1: Deflection of light

with particular solution

$$u_1 = \frac{3m}{2b^2}\left(1 + \frac{1}{3}\cos 2\varphi\right) \ .$$

To first order in the perturbation, we obtain

$$u = \frac{1}{b}\sin\varphi + \frac{3m}{2b^2}\left(1 + \frac{1}{3}\cos 2\varphi\right) \ .$$

In the limit $r \to \infty$, the angle φ becomes very small, so we can use $\sin\varphi \sim \varphi$ and $\cos 2\varphi \sim 1$. In that case, for $r \to \infty$, $u \to 0$ this yields

$$0 = \frac{1}{b}\varphi_\infty + \frac{2m}{b^2}, \quad \varphi_\infty = -\frac{2m}{b}. \tag{7.35}$$

$$\delta = 2|\varphi_\infty| = \frac{4m}{b} = \frac{2R_S}{b} \ . \tag{7.36}$$

For the Sun, this gives

$$\delta_\odot = \frac{1,75''}{b/R_\odot} \ . \tag{7.37}$$

This result can also be obtained by using the **linearized** theory, contrary to the perihelion advance, that is sensitive to the non-linear behavior of the theory. In order to observationally verify the deflection of light, one must compare the apparent positions of nearby stars during a solar eclipse with their usual position on the nocturnal sky. This was successfully performed for the first time in 1919 by A. Eddington and F. Dyson.

7.5 The time delay of a radar echo (Shapiro time delay)

Another relativistic effect that has been tested very precisely is the delay of a radar signal emitted from Earth towards a satellite which it is reflected back to

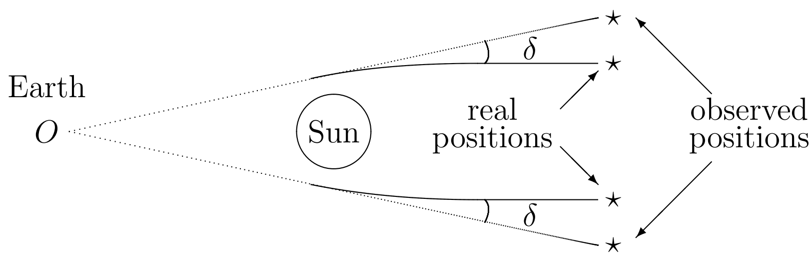


Figure 7.2: Apparent change in the angular distance of two stars in the neighborhood of the Sun.

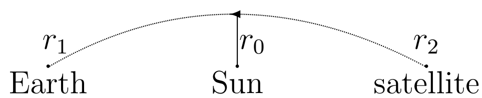


Figure 7.3: Delay of a radar signal when it passes close to the Sun.

the Earth, in the case when the signal passes close to the Sun (see figure 7.3). Let us calculate the delay of the signal on its round trip. We first determine the time coordinate t_{12} that elapses during the propagation of a photon from r_1 to r_2. For an observer far from the mass m, $r/m \gg 1$, coordinate time is approximately his proper time. $\mathcal{L}= 0$ yields (for $\vartheta = \frac{\pi}{2}$, $\dot{\vartheta} = 0$):

$$\dot{r}^2 = E^2 - \left(1 - \frac{2m}{r}\right)\frac{L^2}{r^2} \; . \tag{7.38}$$

$$\text{where} \quad \dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t} \cdot \frac{E}{1 - \frac{2m}{r}} \; .$$

In eq. (7.38) this gives

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 \left(1 - \frac{2m}{r}\right)^{-3} = \left(1 - \frac{2m}{r}\right)^{-1} - \frac{L^2}{E^2}\frac{1}{r^2}. \tag{7.39}$$

At r_0, we have a radial minimum, $\frac{\mathrm{d}r}{\mathrm{d}t} = 0$. Such that

$$\left(\frac{L}{E}\right)^2 = \frac{r_0^2}{1 - \frac{2m}{r_0}},$$

that leads us to

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 \left(1 - \frac{2m}{r}\right)^{-3} + \left(\frac{r_0}{r}\right)^2 \frac{1}{1 - \frac{2m}{r_0}} - \frac{1}{1 - \frac{2m}{r}} = 0. \tag{7.40}$$

We multiply it with $\left(1-\frac{2m}{r}\right)^3$ and isolate dt. Integrating the result gives us the time elapsed for a photon travelling from r_0 to r (or the other way round),

$$t(r,r_0)=\int_{r_0}^{r}\frac{\mathrm{d}r}{1-\frac{2m}{r}}\frac{1}{\left(1-\frac{1-\frac{2m}{r}}{1-\frac{2m}{r_0}}\left(\frac{r_0}{r}\right)^2\right)^{1/2}}\ . \tag{7.41}$$

We expand the integrated function, using the small quantity $\frac{2m}{r}$ ($\frac{2m}{r}\ll 1$, $\frac{2m}{r_0}\ll 1$):

$$\begin{aligned}\frac{1}{1-\frac{2m}{r}}\left(1-\frac{1-\frac{2m}{r}}{1-\frac{2m}{r_0}}\left(\frac{r_0}{r}\right)^2\right)^{-1/2} &\cong \left(1+\frac{2m}{r}\right)\left(1-\left(1-\frac{2m}{r}+\frac{2m}{r_0}\right)\left(\frac{r_0}{r}\right)^2\right)^{-1/2}\\ &=\left(1+\frac{2m}{r}\right)\left(1-\left(\frac{r_0}{r}\right)^2\right)^{-1/2}\left(1-\frac{2m(1/r_0-1/r)r_0^2}{r^2(1-(r_0/r)^2)}\right)^{-1/2}\\ &\cong\left(1-\left(\frac{r_0}{r}\right)^2\right)^{-1/2}\left(1+\frac{2m}{r}+\frac{mr_0}{r(r+r_0)}\right)\ .\end{aligned}$$

We insert this approximation in (7.41), and obtain

$$t(r,r_0)\cong\int_{r_0}^{r}\mathrm{d}r\left(1-\left(\frac{r_0}{r}\right)^2\right)^{-1/2}\left[1+\frac{2m}{r}+\frac{mr_0}{r(r+r_0)}\right]\ .$$

This integral has the solution

$$t(r,r_0)\cong\sqrt{r^2-r_0^2}+2m\ln\left(\frac{r+\sqrt{r^2-r_0^2}}{r_0}\right)+m\left(\frac{r-r_0}{r+r_0}\right)^{1/2}\ .$$

For the trip from r_1 to r_2 and back we obtain the delay of the time coordinate

$$\begin{aligned}\Delta t\ &=2\left[t(r_1,r_0)+t(r_2,r_0)-\sqrt{r_1^2-r_0^2}-\sqrt{r_2^2-r_0^2}\right]\\ &=4m\ln\left(\frac{\left(r_1+\sqrt{r_1^2-r_0^2}\right)\left(r_2+\sqrt{r_2^2-r_0^2}\right)}{r_0^2}\right)\\ &\quad+2m\left(\sqrt{\frac{r_1-r_0}{r_1+r_0}}+\sqrt{\frac{r_2-r_0}{r_2+r_0}}\right)\ .\end{aligned}\tag{7.42}$$

For large distances, r_1, $r_2\gg r_0$, we have

$$\Delta t\cong 4m\left[\ln\left(\frac{4r_1r_2}{r_0^2}\right)+1\right]\ . \tag{7.43}$$

For the Earth-Mars distance we get

$$\Delta t_{\max} \cong 72\,[\mathrm{km}] \cong 240\,[\mu\mathrm{sec}]. \tag{7.44}$$

This delay was first measured with the missions of Viking and Mariner 6,7,9 for Mars and Venus, with a precision of 3%.
The three classical tests of general relativity allowed the determination of the β and γ parameters at a precision of $|\gamma - 1| \lesssim 2\cdot 10^{-3}$ (Shapiro time delay, ∼1980). $|\beta - 1| \lesssim 6\cdot 10^{-4}$ (Lunar Laser Ranging) $|\beta - 1| \lesssim 3\cdot 10^{-3}$ (perihelion advance of Mercury, 1990).
A substantial improvement was made possible with the help of the satellite 'Cassini' which was in conjunction with the Earth and the Sun in July 2002 during its journey to Saturn [5]. These measurements give the limit $|\gamma - 1| < 2.3\times 10^{-5}$.

7.6 The precession of a geodesic gyroscope

Let us consider a gyroscopic compass moving along a circular geodesic around a spherically symmetric star (a Schwarzschild solution). In this case S is parallel transported and we have

$$\langle S, u\rangle = 0, \quad \nabla_u S = 0, \quad \nabla_u u = 0. \tag{7.45}$$

We consider a geodesic with $\vartheta = \frac{\pi}{2}$ = constant and r = constant. Such that $u^r = u^\vartheta = 0$ and $u = u^t\partial_t + u^\varphi\partial_\phi$. Orthogonality of S and u implies

$$0 = e^{2a}u^tS^t - r^2u^\varphi S^\varphi\,. \tag{7.46}$$

Normalization of u gives

$$1 = e^{2a}(u^t)^2 - r^2(u^\varphi)^2 \tag{7.47}$$

Using the Christoffel symbols computed in eqs. (7.2) to (7.6), the geodesic equation for u gives

$$\begin{aligned} u^\mu\partial_\mu u^t \equiv \dot u^t &= 0\,, \quad \dot u^\varphi = 0 \\ \Gamma^r_{\alpha\beta}u^\alpha u^\beta &= -e^{2a}r(u^\varphi)^2 + a'e^{4a}(u^t)^2 = 0\,. \end{aligned} \tag{7.48}$$

Combining (7.47) and (7.48) we can determine u:

$$e^{2a}(u^t)^2 = \frac{1}{1-ra'}\,, \qquad r^2(u^\varphi)^2 = \frac{ra'}{1-ra'}\,. \tag{7.49}$$

Again using the Christoffel symbols computed in eqs. (7.2) to (7.6), the geodesic transport of S leads to

$$\begin{aligned} \dot S^t &= -\Gamma^t{}_{\alpha\beta}u^\alpha S^\beta = -a'u^tS^r \\ \dot S^r &= -\Gamma^r{}_{\alpha\beta}u^\alpha S^\beta = -\left[a'e^{4a}u^tS^t - re^{2a}u^\varphi S^\varphi\right] = re^{2a}u^\varphi S^\varphi[1-ra'] \\ \dot S^\vartheta &= -\Gamma^\vartheta{}_{\alpha\beta}u^\alpha S^\beta = 0 \\ \dot S^\varphi &= -\Gamma^\varphi{}_{\alpha\beta}u^\alpha S^\beta = -\tfrac{1}{r}u^\varphi S^r\ . \end{aligned} \tag{7.50}$$

We now choose the following orthonormal basis, normal to $u = e_0$:

$$e_1 = e^a \partial_r, \quad e_2 = r^{-1}\partial_\vartheta, \quad e_3 = re^{-a}u^\varphi \partial_t + r^{-1}e^a u^t \partial_\varphi \ .$$

In this orthonormal basis, $\langle e_\mu, e_\nu \rangle = \eta_{\mu\nu}$, we have $S = S^i e_i$ with $S^i = \langle S, e_i \rangle$ we obtain

$$S^3 = e^a r(u^t S^\varphi - u^\varphi S^t), \quad S^1 = e^{-a}S^r, \quad S^2 = rS^\vartheta \ .$$

Equations (7.50) yield for the components S^i, using (7.49) and (7.46)

$$\begin{aligned}
\dot{S}^1 &= re^a u^\varphi S^\varphi [1 - a'r] = \frac{a'}{r}e^{2a}\frac{u^t}{u^\varphi}S^3 \\
\dot{S}^2 &= 0 \\
\dot{S}^3 &= e^a \left[1 - a'r\right] u^t u^\varphi S^r = -\frac{a'}{r}e^{2a}\frac{u^t}{u^\varphi}S^1 \\
\frac{\mathrm{d}S^i}{\mathrm{d}t} &= \frac{\dot{S}^i}{\dot{t}} = \frac{\dot{S}^i}{u^t} \quad (u^t = \dot{t})
\end{aligned}$$

Such that

$$\frac{\mathrm{d}S^1}{\mathrm{d}t} = \Omega S^3 \,, \qquad \frac{\mathrm{d}S^3}{\mathrm{d}t} = -\Omega S^1 \qquad \Omega = \frac{a'}{r}e^{2a}\frac{1}{u^\varphi}. \tag{7.51}$$

Using $u^\varphi = \dot{\varphi}$, the angular frequency of the orbit is

$$\omega := \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\dot{\varphi}}{\dot{t}} = \frac{u^\varphi}{u^t}.$$

$$\omega^2 = \left(\frac{u^\varphi}{u^t}\right)^2 \overset{(7.48)}{=\!=} \frac{a'}{r}e^{2a} = \frac{1}{2r}\left(e^{2a}\right)'.$$

With $e^{2a} = 1 - \frac{2m}{r}$ we obtain

$$\omega^2 = \frac{m}{r^3} \tag{7.52}$$

which is Kepler's 3rd law.

For the rotation frequency Ω which describes the rotation given by eq. (7.51) we have

$$\begin{aligned}
\Omega^2 &= \frac{a'^2}{r^2(u^\varphi)^2}e^{4a} \overset{(7.49)}{=\!=} \frac{a'}{r}\left(1 - a'r\right)e^{4a} \\
&= \omega^2 e^{2a}(1 - a'r) = \omega^2 e^{2a}\left[1 - \frac{m}{r}\frac{1}{1 - \frac{2m}{r}}\right] \\
&= \omega^2 e^{2a}\left(\frac{1 - \frac{3m}{r}}{1 - \frac{2m}{r}}\right) = \omega^2\left(1 - \frac{3m}{r}\right) \ ,
\end{aligned}$$

$$\Omega^2 = e^2\omega^2 \quad \text{with} \quad e^2 = 1 - \frac{3m}{r}. \tag{7.53}$$

We can also write (7.51) in a vectorial notation

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{S} = \vec{\Omega} \wedge \vec{S} \quad \text{with} \quad \vec{\Omega} = (0, e\omega, 0) \ . \tag{7.54}$$

In the Newtonian limit we have $\Omega = \omega$, which means that in the Newtonian limit the components of $\vec{S}$ do not change in a cartesian system and the precession with ω is caused by a variation in the basis e_1, e_2, e_3 along the orbit.
In a gravitational field, $m \neq 0$, this variation in the basis is no more entirely compensated and the spin precesses only with frequency

$$e\omega < \omega.$$

After one period, $\vec{S}$ has turned by an angle

$$2\pi(1-e) =: \omega_G \frac{2\pi}{\omega}.$$

The **geodesic precession frequency** is

$$\begin{aligned} \omega_G &= \omega(1-e) = \sqrt{\frac{m}{r^3}}\left[1 - \sqrt{1 - \frac{3m}{r}}\right] \\ &\cong \left(\frac{GM}{r^3}\right)^{1/2} \frac{3}{2}\frac{GM}{r} = \frac{3}{2}\frac{(GM)^{3/2}}{r^{5/2}} \ . \end{aligned} \tag{7.55}$$

For the mass of the Earth this gives (for the precession of a satellite at a distance r from the center of the Earth)

$$\omega_G \cong 8.4 \left(\frac{R_\oplus}{r}\right)^{5/2} \quad \text{arc-seconds per year} \tag{7.56}$$

The geodesic precession (the spin-orbit coupling) is added to the effect caused by the rotation of Earth (Lense Thirring effect, spin-spin coupling) and is approximately 10^3 times stronger than the latter. Recently, the spin-orbit coupling has measured with a precision of 0.27% for the orbit of the "Gravity Probe B" satellite around the Earth. For the polar orbit of gravity probe B at an altitude of 642km, we obtain $\omega_G = 6.6061$ arc-sec per year. The measurement gave 6.6018 ± 0.018 arc-sec per year. To discern the spin-spin coupling, the data analysis had to be improved in order to reduce the error of about a 10 factor as discussed in Sec. 5.5.3.

The Earth-Moon system can be seen as a natural gyroscope that undergoes this precession on its orbit around the Sun (in the solar gravitaional field). The mean distance of the Earth-Moon system from the Sun is about $a = 1.5 \times 10^8$km. The gravitational potential is $\phi = GM_\odot/a \simeq 9.87\times10^{-9}$. This gives a contribution of $(3/2)(GM_\odot/a)^{3/2}/a \simeq 1.92\times10^{-2}$arc-sec/yr when the Moon is at its perigee. This contribution is measured in the 'lunar laser ranging' experiments with a precision of 0.1%, [4], so somewhat better than the spin-orbit coupling measurement of the "Gravity Probe B". The spin-orbit coupling is also observed in the binary pulsar PSR1916+13 (see next chapter) where it is approximately 1.1^o/year.

7.7 The Kruskal extension

As we have already mentioned, the Schwarzschild metric

$$g = -\left(1-\frac{2m}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-\frac{2m}{r}} + r^2\left(\mathrm{d}\vartheta^2+\sin^2\vartheta\mathrm{d}\varphi^2\right)$$

seems to possess a singularity at $r=2m$. However, all components of the Riemann tensor remain finite at $r=2m$. They typically are of the order

$$\begin{aligned}R^1{}_{212} = R^1{}_{313} = e^{-2b}\frac{b'}{r} &= -\frac{1}{2r}(\underbrace{e^{-2b}}_{1-\frac{2m}{r}})' \\ &= -\frac{m}{r^3}.\end{aligned}$$

Thus, the relative acceleration of free falling particles remains finite at $r=2m$.

Before searching for the coordinates in which we can extend the Schwarzschild metric beyond $r=2m$, let us discuss what happens when a particle comes from far away towards $r\to 2m$. We consider a radial geodesic, $L=r^2\dot\varphi=0$. With (7.21) and (7.22) we have

$$\dot r^2 = \frac{2m}{r} + E^2 - 1\ . \tag{7.57}$$

Let us consider a particle initially at rest at a distance $r=R>2m$ from the center. At $r=R$, we have $\dot r = 0$: $E^2-1 = -\frac{2m}{R}$. This yields

$$\mathrm{d}\tau = \frac{-\mathrm{d}r}{\left(\frac{2m}{r}-\frac{2m}{R}\right)^{1/2}}. \tag{7.58}$$

We have chosen the minus sign because we are considering a particle that is falling towards $r\to0$, so r must decrease (it is 'ingoing'). Eq. (7.58) is the equation of a cycloid. It can be sloved in the following parametric representation:

$$\begin{aligned} r &= \tfrac{R}{2}(1+\cos\eta)\\ \tau &= \tfrac{R}{2}\left(\tfrac{R}{2m}\right)^{1/2}(\eta+\sin\eta)\ .\end{aligned} \tag{7.59}$$

For $r = R$ we have $(\eta = 0)$ $\tau = 0$. Nothing special happens at $r = 2m$. The center $r = 0\,(\eta = \pi)$ is reached at the proper time value

$$\tau = \frac{\pi}{2} R \left(\frac{R}{2m}\right)^{1/2} \cong \left(\frac{R}{R_S}\right)^{3/2} \left(\frac{R_S}{R_{S\odot}}\right) \times 10^{-5}\text{sec}.$$

To compare, we consider r as a function of the time coordinate t (the proper time of an observer situated at $r \to \infty$).

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t}\dot{t} = \frac{\mathrm{d}r}{\mathrm{d}t}\frac{E}{1 - \frac{2m}{r}} \ . \tag{7.60}$$

It is useful to introduce the variable

$$r^\star = r + 2m \ln\left(\frac{r}{2m} - 1\right) \tag{7.61}$$

$$\frac{\mathrm{d}r^\star}{\mathrm{d}t} = \frac{1}{1 - \frac{2m}{r}}\frac{\mathrm{d}r}{\mathrm{d}t}, \quad \text{so} \quad \dot{r} = E\frac{\mathrm{d}r^\star}{\mathrm{d}t}. \tag{7.62}$$

From (7.62) it follows that

$$E^2\left(\frac{\mathrm{d}r^\star}{\mathrm{d}t}\right)^2 = E^2 - 1 + \frac{2m}{r} = \frac{2m}{r} - \frac{2m}{R}. \tag{7.63}$$

When $r \to 2m$, $r^\star \to -\infty$ and the right hand side of (7.63) approaches E^2.
For $r \cong 2m$ we therefore have

$$\frac{\mathrm{d}r^\star}{\mathrm{d}t} \cong -1, \quad \text{i.e.} \quad r^\star \cong -t + \text{const.}$$

$$2m + 2m \ln\left(\frac{r}{2m} - 1\right) \cong -t + \text{const.}$$

For $r \to 2m$, hence $r(t)$ behaves like

$$r \cong 2m + \text{const} \cdot e^{-\frac{t}{2m}}. \tag{7.64}$$

The Schwarzschild radius $r = 2m$ is reached at the time coordinate $t = \infty$. This also follows from

$$\dot{t} = \frac{E}{1 - \frac{2m}{r}} \quad \text{where} \quad \frac{\mathrm{d}t}{\mathrm{d}\tau} \quad \text{diverges for } r \to 2m.$$

Therefore, for a far away observer, the particle never disappears $(t \to \infty)$ behind the horizon $r = 2m$.
Now let us consider light-like radial geodesics. For these, $\mathrm{d}s^2 = 0$ which implies that

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \pm\left(1 - \frac{2m}{r}\right) \tag{7.65}$$

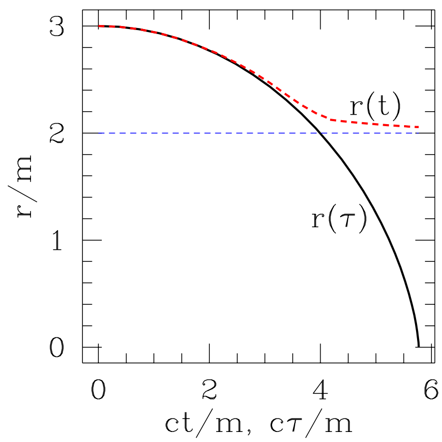


Figure 7.4: $r(t)$ and $r(\tau)$ for a radial geodesic.

For $\mathbf{r > 2m}$ the + sign is for rays with increasing radius, $r \to \infty$ for $t \to \infty$ ("outgoing" rays), and the minus − sign is for "ingoing" rays.
For $\mathbf{r \to 2m}$ the light cone becomes more and more narrow. (see figure 7.5)
Physically, it is important to note that **the redshift** of a photon emitted at $r = R > 2m$ is at r

$$z + 1 \overset{(5.30)}{=\!=} \left(\frac{g_{00}(R)}{g_{00}(r)}\right)^{1/2} = \left(\frac{1 - \frac{2m}{R}}{1 - \frac{2m}{r}}\right)^{1/2} \overset{r \to 2m}{\longrightarrow} \infty. \tag{7.66}$$

This diverges for $r \to 2m$. Hence, even though a massive particle (or a photon) never reaches the horizon from the point of view of a far away observer, its light becomes more and more redshifted such that it soon becomes invisible.

Let us introduce the so-called Eddington-Finkelstein coordinate,

$$w = t + r^* = t + r + 2m \ln\left(\frac{r}{2m} - 1\right) \ ,$$

such that

$$dw = dt + dr + \frac{1}{\frac{r}{2m} - 1} dr = dt + \frac{dr}{1 - \frac{2m}{r}} \ .$$

Replacing t with w, the metric becomes

$$g = -\left(1 - \frac{2m}{r}\right) \mathrm{d}w^2 + 2\mathrm{d}w\mathrm{d}r + r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2\right) \ .$$

Light-like radial geodesics are then given by

$$-\left(1 - \frac{2m}{r}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^2 + 2\frac{\mathrm{d}w}{\mathrm{d}r} = 0 \ .$$

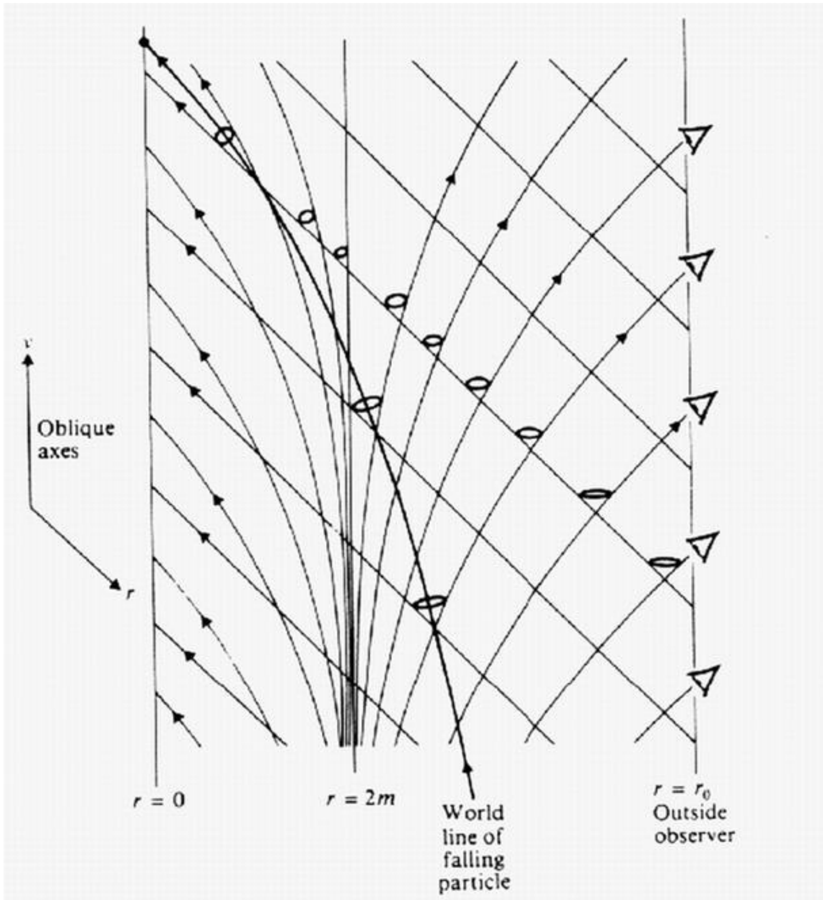


Figure 7.5: "Ingoing" and "outgoing" geodesics in the Eddington Finkelstein coordinate (vertical, called v) and r (oblique). From J. Foster, J.D Nightingale, *A short course in General Relativity*, Springer, 1995

With solutions $\frac{\mathrm{d}w}{\mathrm{d}r} = 0$, describing geodesics with decreasing r (ingoing), and $\frac{\mathrm{d}w}{\mathrm{d}r} = \frac{2}{1-\frac{2m}{r}} > 0$ describing 'outgoing' geodesics.

The fact that there are events happening at finite proper time τ which is later than $t = \infty$ shows us that t is not a good coordinate for $r < 2m$. For $r > 2m$, t is the temporal coordinate relative to which the metric is static : $K = \partial_t$ is the Killing field that satisfies $K^\flat \wedge dK^\flat = 0$. This uniquely determines t (up to a constant). For $r < 2m$, the r coordinate becomes timelike.

We wish to extend the metric beyond $r = 2m$. For this, we follow Kruskal's approach (1956) by trying the ansatz $u(r,t)$, $v(r,t)$ such as

$$g = f^2(u,v)(-\mathrm{d}v^2 + \mathrm{d}u^2) + r^2 \left(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2\right) \ . \quad (7.67)$$

For such a coordinate system, we have $\left(\frac{\mathrm{d}u}{\mathrm{d}v}\right)^2 = 1$ for radial light rays, like in Minkowski spacetime. The radial part of the metric, $\vartheta = \text{const}$, $\varphi = \text{const}$, is conformal to the 2 dimensional Minkowski metric and thus has the same light cones. In this kind of coordinate system we no longer encounter the problem of the light cones which become more and more narrow.
The relation

$$g_{\alpha\beta} = \frac{dx'^{\mu}}{dx^{\alpha}}\frac{dx'^{\nu}}{dx^{\beta}}g'_{\mu\nu}$$

leads to the following differential equations for the transformation $(t,r) \mapsto (v,u)$:

$$\left(1 - \frac{2m}{r}\right) = f^2\left[\left(\frac{\partial v}{\partial t}\right)^2 - \left(\frac{\partial u}{\partial t}\right)^2\right] , \tag{7.68}$$

$$-\frac{1}{1-\frac{2m}{r}} = f^2\left[\left(\frac{\partial v}{\partial r}\right)^2 - \left(\frac{\partial u}{\partial r}\right)^2\right] , \tag{7.69}$$

$$0 = \frac{\partial u}{\partial t}\frac{\partial u}{\partial r} - \frac{\partial v}{\partial t}\frac{\partial v}{\partial r}. \tag{7.70}$$

Motivated by the staticity of the Schwarzschild solution, we try to find a function f that does not depend on t. We first set

$$F(r^\star) = \frac{1-\frac{2m}{r}}{f^2(r)} \tag{7.71}$$

where $r^\star = r + 2m\ln\left(\frac{r}{2m} - 1\right)$ as before in eq. (7.60). With (7.68) and (7.69) this yields

$$\left(\frac{\partial v}{\partial t}\right)^2 - \left(\frac{\partial u}{\partial t}\right)^2 = F(r^\star) , \tag{7.72}$$

$$\left(\frac{\partial v}{\partial r^\star}\right)^2 - \left(\frac{\partial u}{\partial r^\star}\right)^2 = -F(r^\star) , \tag{7.73}$$

$$\frac{\partial u}{\partial t}\frac{\partial u}{\partial r^\star} = \frac{\partial v}{\partial t}\frac{\partial v}{\partial r^\star} . \tag{7.74}$$

We used the fact that

$$\frac{dr^\star}{dr} = \frac{1}{1-\frac{2m}{r}} .$$

The sums (7.72)+(7.73)∓2·(7.74) yield

$$\left(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial r^\star}\right)^2 = \left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial r^\star}\right)^2 , \tag{7.75}$$

$$\left(\frac{\partial v}{\partial t} - \frac{\partial v}{\partial r^\star}\right)^2 = \left(\frac{\partial u}{\partial t} - \frac{\partial u}{\partial r^\star}\right)^2 . \tag{7.76}$$

The signs of $\frac{\partial u}{\partial t}$, $\frac{\partial v}{\partial t}$, etc. are not determined by (7.68) to (7.70).
When taking the square roots of (7.75) and (7.76) we choose the positive sign in (7.75) and and the negative sign in (7.76).
If we chose the same sign in (7.75) as in (7.76), the determinant $\left|\frac{\partial(u,v)}{\partial(t,r^\star)}\right|$ would vanish.
With this we find ($\sqrt{(7.75)}+\sqrt{(7.76)}$; $\sqrt{(7.75)}-\sqrt{(7.76)}$)

$$\frac{\partial v}{\partial t}=\frac{\partial u}{\partial r^\star} \quad ; \quad \frac{\partial v}{\partial r^\star}=\frac{\partial u}{\partial t} \; . \tag{7.77}$$

This gives

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial r^{\star 2}}=\frac{\partial^2 v}{\partial t^2}-\frac{\partial^2 v}{\partial r^{\star 2}}=0 \; . \tag{7.78}$$

With the general solutions

$$\begin{aligned} v &= h(r^\star+t)+g(r^\star-t) \; , \\ u &= h(r^\star+t)-g(r^\star-t) \; . \end{aligned} \tag{7.79}$$

The equations (7.72) and (7.73) demand that

$$-4h'(r^\star+t)g'(r^\star-t)=F(r^\star) \; . \tag{7.80}$$

Taking the derivative with respect to $r^\star$ yields

$$\frac{h''(r^\star+t)}{h'(r^\star+t)}+\frac{g''(r^\star-t)}{g'(r^\star-t)}=\frac{F'(r^\star)}{F(r^\star)} \; ,$$

and taking the derivative with respect to t leads to

$$\frac{h''(r^\star+t)}{h'(r^\star+t)}-\frac{g''(r^\star-t)}{g'(r^\star-t)}=0 \; .$$

The sum of these two equations implies

$$[\ln F(r^\star)]'=2\,[\ln h'(r^\star+t)]' \; . \tag{7.81}$$

Here we consider $r^\star$ and $y=r^\star+t$ as independent variables. The two sides of (7.81) therefore have to be constant. It follows that (with the choice of 2η for $[\ln(F(r^\star)]'$ and the choice of the integration constant in such a way that the (7.80) constraint is satisfied),

$$F(r^\star)=\eta^2 e^{2\eta r^\star} \; , \quad h(r^\star+t)=\frac{1}{2}e^{\eta(r^\star+t)} \; , \quad g(r^\star-t)=-\frac{1}{2}e^{\eta(r^\star-t)} .$$

For $r>2m$ the sign of h and g is determined by the condition $F>0$. We finally obtain

$$\begin{aligned} u &= h-g=\frac{1}{2}e^{\eta(r^\star+t)}+\frac{1}{2}e^{\eta(r^\star-t)} \\ &= e^{\eta r^\star}\cosh\eta t=\left(\frac{r}{2m}-1\right)^{2m\eta} e^{\eta r}\cosh\eta t \; . \end{aligned}$$

And similarly,

$$v = \left(\frac{r}{2m} - 1\right)^{2m\eta} e^{\eta r} \sinh \eta t.$$

For f we find

$$f^2 = \frac{2m}{\eta^2 r} \left(\frac{r}{2m} - 1\right)^{1-4m\eta} e^{-2\eta r}.$$

We want to choose the integration constant η such that $f^2 \neq 0$ at $r = 2m$. This requires $\eta = \frac{1}{4m}$. This choice defines the **Kruskal transformation** $(r, t) \mapsto (u, v)$

$$u = \left(\frac{r}{2m} - 1\right)^{1/2} \exp(r/4m) \cosh\left(\frac{t}{4m}\right) \tag{7.82}$$

$$v = \left(\frac{r}{2m} - 1\right)^{1/2} \exp(r/4m) \sinh\left(\frac{t}{4m}\right) \tag{7.83}$$

$$f^2 = \frac{32m^3}{r} \exp(-r/2m). \tag{7.84}$$

For $r > 2m$, This is simply a coordinate transformation. The domain $(r > 2m, t)$ is mapped on the region $u > |v|$ of the plane (u, v).

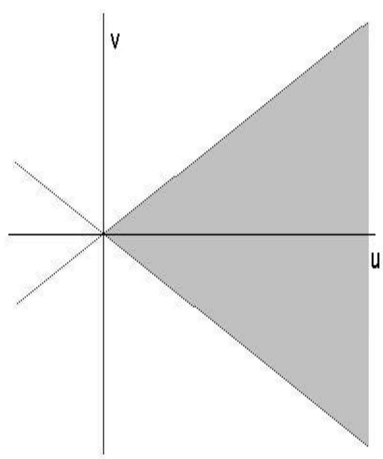


Figure 7.6: The domain $(r > 2m, t)$ is the domain $u > |v|$.

We have

$$u^2 - v^2 = \left(\frac{r}{2m} - 1\right) e^{\frac{r}{2m}} \quad \text{and} \quad \frac{v}{u} = \tanh\left(\frac{t}{4m}\right).$$

The curves r =constant are then hyperbolas, $u^2 - v^2 = \text{const.}$ For $r \to 2m$, they approach the lines $u = |v|$. The lines $t = \text{const}$ are radial lines. The diagonal $u = v$ corresponds to $t = \infty$ and $u = -v$ to $t = -\infty$. The diagonals $t = \pm\infty$ also correspond to $r = 2m$.

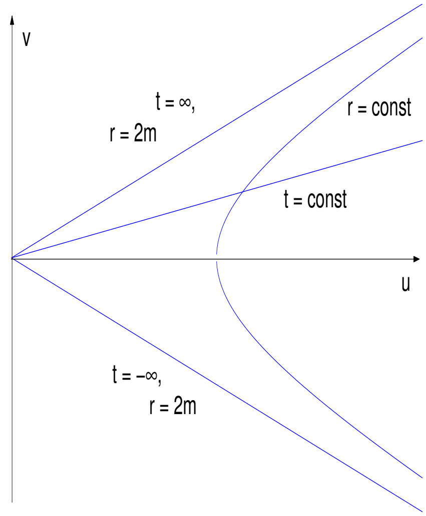


Figure 7.7: Lines with r =const. and t =const. in the plane (u, v).

The metric

$$g = f^2(u,v)\left(-\mathrm{d}v^2 + \mathrm{d}u^2\right) + r^2(u,v)\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2\right) \tag{7.85}$$

is well defined not only for $u > |v|$, but everywhere where $f(u,v) = f(r(u,v))$ is well defined and $r(u,v) > 0$. The function $r(u^2 - v^2)$ given by

$$u^2 - v^2 = \left(\frac{r}{2m} - 1\right) e^{\frac{r}{2m}} \ , \tag{7.86}$$

is monotonous (see graphic 7.8). The domain $r > 0$ is given by

$$u^2 - v^2 > -1, \quad v^2 - u^2 < 1 \ . \tag{7.87}$$

The new Kruskal manifold (u, v) contains the Schwarzschild manifold (r, t) for $r > 2m$. On the Kruskal manifold we also have $G_{\mu\nu} = 0$. Therefore it is also a solution of Einstein's field equations in vacuum.
When we derived the Kruskal transformation, we arbitrarily chose the sign of h, $h > 0$. The choice $h < 0$ would also have been possible. It corresponds to $(u,v) \mapsto (-u, v)$. This transformation maps the domains I and III on figure 7.9 isometrically.
Consider the Schwarzschild solution for $r < 2m$. It satisfies the vacuum equations, $G_{\mu\nu} = 0$. For $r < 2m$, r is a time coordinate. The domain $r < 2m$ of the Schwarzschild solution is isometric to the domain II in the Kruskal manifold. To see this, one can repeat the derivation that led us to (7.82) and to (7.83) with $r < 2m$. Now $F(r^\star)$ is negative and the relative sign of h and g must be positive. We obtain

$$u = \left(1 - \frac{r}{2m}\right)^{1/2} \exp\left(\frac{r}{4m}\right) \sinh\left(\frac{t}{4m}\right) \ , \tag{7.88}$$

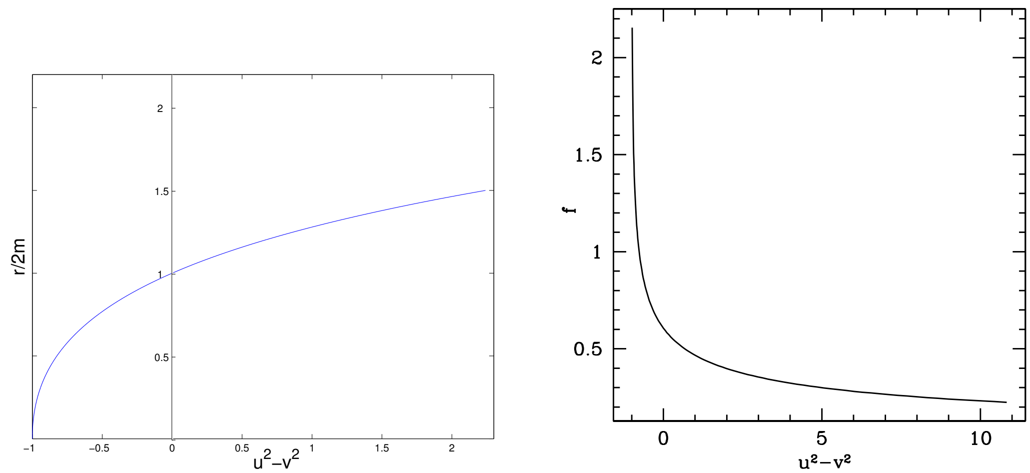


Figure 7.8: The functions $r(u^2 - v^2)$ and $f(r(u^2 - v^2))$.

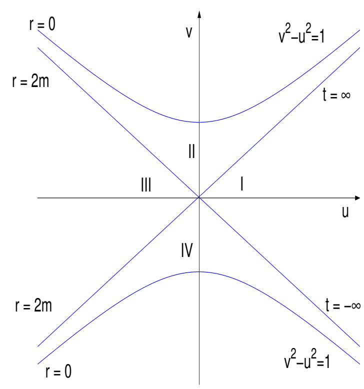


Figure 7.9: The 4 domains in the Kruskal manifold separated by straight lines at 45°.

$$v = \left(1 - \frac{r}{2m}\right)^{1/2} \exp\left(\frac{r}{4m}\right) \cosh\left(\frac{t}{4m}\right) \ , \tag{7.89}$$

and f is (7.84) like before.
We now have $v^2 - u^2 = \left(1 - \frac{r}{2m}\right) e^{\frac{r}{2m}} > 0$ and $\frac{u}{v} = \tanh\left(\frac{t}{4m}\right)$, which corresponds to the domain II. On $v^2 - u^2 = 1$, $r = 0$, the geometry has a true singularity. For example, the Riemann scalar R diverges. The Kruskal manifold is maximal. Every geodesic encounters the singularity $v^2 - u^2 = 1$, or its affine parameter goes to infinity.

The causal relations of the Kruskal metric are simple, because light cones are at 45^o as in Minkowski space:

- The observers I and III can receive signals from IV. (I and III are themselves causally disconnected)

- Part I and III are equivalent to the Schwarzschild manifold at $r > 2m$. Part III is obtained from I by the transformation $u \to -u$ which corresponds to a change in the sign of the square roots of eqs. (7.75) and (7.76)

- No signal can stay in IV.

- No signal can leave II.

- Each time-like or light-like geodesic in II meets the singularity $u^2 - v^2 = -1$ within finite proper time.

- Each time-like or light-like geodesic in IV has left the singularity a finite proper time in the past. The singularity $v^2 - u^2 = 1$ is separated from the asymptotically flat regions, I and III, by the horizon $r = 2m$. [1]

- A signal that comes from IV towards I (or III) has left the singularity before $t = -\infty$. The domains IV of the Kruskal manifold corresponds to $t < -\infty$ and is (perhaps) not observable. In general an astrophysical black hole is represented by the domains I and II of the Kruskal manifold (the part IV is sometimes called "white hole").

- The singularity at $v^2 - u^2 = 1$, $r = 0$ is space-like. In I and III the Killing field $K = \partial_t$ is time-like, while in II and IV it is space-like and at the horizon, $r = 2m$, it is light-like:

$$(K, K) = -\left(1 - \frac{2m}{r}\right) = \begin{cases} < 0 & \text{if } r > 2m \\ = 0 & \text{if } r = 2m \\ > 0 & \text{if } r < 2m \ . \end{cases} \tag{7.90}$$

- In II (IV) no observer is at rest, that is with $r =$ constant. Every observer either approaches or recedes from the singularity.

Generalised Birkhoff's theorem: Any solution of the vacuum Einstein field equations that has a spherical symmetry is a part of the Kruskal manifold.
Proof: See, for example, Straumann [16].
Much more difficult to prove but also true is that every static vacuum solution is a part of the Kruskal manifold (i.e. static vacuum solutions are necessarily spherically symmetric, this is Israel's theorem, see also [16] for a proof).

[1] A horizon is a light-like closed surface with the property that every light cone with vertex on the surface is partially tangent to the horizon surface, and the non-tangent part only opens into the interior of the horizon.

7.8 The spherical collapse to a black hole

As soon as a star's radius is less than $2m$ there exists no static solution. Because the world lines of the surface of the star are inside the light cone, its surface will inevitably precipitate towards $r = 0$ (quantum effects at $r \sim 0$?).
A signal passing to the inside of $r = 2m$ cannot "come back out" (see figure 7.9 and remember that light cones are at 45^o). All the light rays also fall into the singularity. The horizon $r = 2m$ is the limit of the domain that is causally connected to infinity, $r \to \infty$. It acts like a semipermeable membrane: energy and information can enter but nothing can come out.

The existence of horizons in the universe is a remarkable consequence of general relativity without any Newtonian analog. The singularity of the Schwarzschild solution is behind a horizon and thus invisible. There is a conjecture that this is so for every stable solution possessing a singularity:

'The cosmic censorship conjecture': Every singularity of a **stable** solution of Einstein's field equations is separated from an asymptotic observer by a horizon.

There does not yet exist a counter-example (what we would call a naked singularity) that is stable if the energy-momentum tensor satisfies some 'reasonable' conditions.

7.8.1 The redshift of an asymptotic observer

We consider an emitter that radially approaches the Schwarzschild horizon with four-velocity v and emits signals with frequency ω_e.
If k is the wave vector of the emitted photons, we have (special relativity):

$$\begin{aligned} \omega_e &= -(k, v) \\ \omega_0 &= -(k, u)\,, \end{aligned} \qquad \text{the redshift is} \qquad z + 1 = \frac{\omega_e}{\omega_0} = \frac{(k, v)}{(k, u)} \tag{7.91}$$

We set $a = t - r^\star$, $r^\star = r + 2m \ln\left(\frac{r}{2m} - 1\right)$ (a is the "retarded" time). With this

$$g = -\left(1 - \frac{2m}{r}\right)(\mathrm{d}a)^2 - 2\mathrm{d}a\mathrm{d}r + r^2 \mathrm{d}\Omega^2. \tag{7.92}$$

For a radial ray of light we have

$$k = k^a \partial_a + k^r \partial_r = k^t \partial_t + \left[1 - \frac{1}{1 - \frac{2m}{r}}\right] k^r \partial_r \ .$$

With (7.92) the lagrangian of a radial geodesic is

$$-\mathcal{L} = \left[\frac{1}{2}\left(1 - \frac{2m}{r}\right) k^a + k^r\right] k^a = 0 \ . \tag{7.93}$$

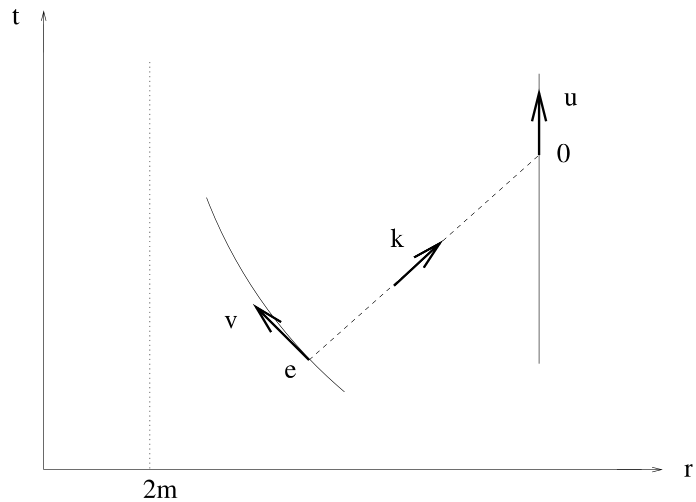


Figure 7.10: An emitter 'e' with four-velocity v approaches the horizon, $r = 2m$, of a black hole . It emits photons with four-momentum k towards an observer '0', who is at rest (four-vector u).

$\mathcal{L} = 0$ implies $k_a = 0$ or $\frac{1}{2}\left(1 - \frac{2m}{r}\right) k^a + k^r = 0$. For an *outgoing* radial light ray we have

$$k^a = k^t - k^r \left(\frac{1}{1 - \frac{2m}{r}}\right) = 0 \ .$$

As $\epsilon = (1 - \frac{2m}{r})k^t$ is a constant of motion, it follows that k^r =constant, such that $k = \text{const} \cdot \partial_r$ and

$$1 + z = \frac{(v, \partial_r)}{(u, \partial_r)}. \tag{7.94}$$

For $v = v_a\partial_a + v_r\partial_r$ we have $(v, \partial_r) = g_{ar}v_a = -v_a$. For a far away observer at rest we have $(u, \partial_r) = g_{ar}u_a = g_{ar}u_t \cong -1$. So that

$$1 + z = v_a = v_t - \frac{1}{1 - \frac{2m}{r}} v_r \ . \tag{7.95}$$

With $E = v_t\left(1 - \frac{2m}{r}\right)$, $2\mathcal{L} = 1 = (E^2 - v_r^2)\left(1 - \frac{2m}{r}\right)^{-1}$ we obtain

$$1 + z = \frac{E - v_r}{1 - \frac{2m}{r}} = \left(1 - \frac{2m}{r}\right)^{-1}\left[E + \left(E^2 - 1 + \frac{2m}{r}\right)^{1/2}\right] \overset{r \to 2m}{\longrightarrow} \infty \ . \tag{7.96}$$

The proper time of the observer is $\mathrm{d}\tau_o \cong \mathrm{d}a$.
The coordinate of the emitter $r_e(\tau_o)$ behaves like

$$\frac{\mathrm{d}r_e}{\mathrm{d}\tau_o} = \frac{\mathrm{d}r_e}{\mathrm{d}a} = \left(\frac{\dot{r}}{\dot{a}}\right)\frac{v_r}{v_a} = \frac{v_r}{1+z} = -\left(1 - \frac{2m}{r_e}\right)\frac{(E^2 - 1 + \frac{2m}{r_e})^{1/2}}{E + (E^2 - 1 + \frac{2m}{r_e})^{1/2}} \ . \tag{7.97}$$

Very close to $r_e \sim 2m$ we obtain

$$-\frac{\mathrm{d}r_e}{\mathrm{d}\tau_o} \cong \frac{1}{2}\left(1-\frac{2m}{r_e}\right) \quad \text{or} \quad -\frac{\mathrm{d}}{\mathrm{d}\tau_o}(r_e-2m) \cong \frac{1}{4m}(r_e-2m) \tag{7.98}$$

with solution

$$r_e - 2m \cong e^{-\frac{\tau_o}{4m}} \ . \tag{7.99}$$

This implies

$$1+z \cong \frac{4mE}{r_e-2m} \propto e^{\frac{\tau_o}{4m}}. \tag{7.100}$$

Eq. (7.99) shows that the transmitter reaches $r_e = 2m$ only for $\tau_o \to \infty$, where τ_o is the proper time of the observer. But according to (7.100) the redshift diverges for $r \to 2m$.
The caracteristic time is $4m \cong 10^{-5}[\mathrm{sec}]\left(\frac{M}{M_\odot}\right)$. Therefore the transmitter "disappears" quickly when it approaches the Schwarzschild radius, because of the redshift that shifts the signal's frequency out of the telescope's sensitivity range.

7.8.2 Falling into a black hole

The typical components of the Riemann tensor of the Schwarzschild solution are

$$R_{1212} = \frac{m}{r^3}\,, \quad R_{0101} = \frac{2m}{r^3} \ . \tag{7.101}$$

The relative acceleration of freely falling particles is $\sim 2\frac{m}{r^3}$ (in the radial direction). Let's consider a small cube with dimensions $\ell \ll m$ and mass μ. The difference of the gravitational force et the two ends of the cube is

$$F \cong \frac{2m}{r^3}\mu\ell.$$

Yielding a tension $T = \frac{F}{\ell^2} \cong \frac{2m}{r^3}\frac{\mu}{\ell}$. The question is at which radius r this tension will tear apart the object. For $\mu \sim 60\,[\mathrm{kg}]$, $\ell \sim 170\,[\mathrm{cm}]$ we find

$$T(r) \sim 10^{13}\left(\frac{R_S R_S^2(M_\odot)}{r^3}\right)[\mathrm{dyne\ cm}^{-2}] \ . \tag{7.102}$$

(compare to 1 [atm] $\sim 10^6$ [dyne cm^{-2}]),

$$T(R_S) = 10^{13}\left(\frac{R_S(M_\odot)}{R_S}\right)^2 [\mathrm{dyne\ cm}^{-2}]$$

For large masses, $R_S \gg R_S(M_\odot)$, the tension can be bearable at $r = R_S$. But then, the approach $r \to 0$, $T \to \infty$ is inescapable ...

7.8.3 Observational evidence

It has been proven that there is no stable configuration of matter when $M \gtrsim 2.5M_\odot$. Any star that is unable to get rid of its excess mass will eventually form a black hole. Because a black hole is not directly visible (no photon can escape from the inside of the horizon), we can only observe it using its gravitational effects. Very often a black hole is surrounded by a disc of 'dust', the accretion disc that emits X-rays, or it is in a binary system. In these cases, we have means to estimate its mass and/or its 'size' (the size of its horizon, R_S):

- We have found several binary systems where one of the bodies has a mass superior than 5 $M_\odot$, 7 $M_\odot$ and even 10 $M_\odot$, and is not visible in the optical spectrum, and so is probably a black hole. Despite this, the abundance of stellar black holes in our galaxy is very difficult to estimate.

- The coalescing black hole binary from which the first gravitational waves have been detected by the LIGO experiment [10] consisted of two original black holes of masses $M \sim 30M_\odot$ forming a final black hole with mass $M \sim 60M_\odot$.

- It is also very likely that massive black holes, $M \sim 10^{6-7}\, M_\odot$, are found at the center of most galaxies. At the center of the Milky Way there is most probably a massive black hole with a mass $M \simeq 3.6 \times 10^6 M_\odot$. This mass has been measured by observing the very small stellar orbits near the galactic center.

- The activity of quasars and the nuclei of active galaxies is understood as an accretion around a supermassive black hole, $M \gtrsim 10^8\, M_\odot$, at the center of those galaxies.

7.9 The Carter-Penrose diagram for the Kruksal spacetime

In this section we will construct the 'conformal compactification' of Kruksal spacetime. Such constructions enable a discussion of what happens at infinity using the local tools of differential geometry. This is most important for example when studying gravitational radiation.

7.9.1 Conformal compactification of Minkowski spacetime

As a simple example, we construct the 'conformal compactification' of Minkowski spacetime. In polar coordinates, the Minkowski metric is given by

$$ds^2 = -dt^2 + dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2) \ .$$

By introducing the coordinates $u = t - r$ and $v = t + r$ we can write

$$ds^2 = -dudv + \frac{1}{4}(v-u)^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2) \ .$$

The hypersurfaces $\{u =\text{constant}\ \}$ and $\{v =\text{constant}\ \}$ are zero-surfaces, that is

$$g\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = g\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) = 0 \ .$$

These are the surfaces normal to light rays propagating in radial direction towards $r \to \infty$ for $u =$constant or towards $r \to 0$ for $v =$constant. The coordinates (u, v) of the region $u \le v$ of $\mathbb{R}^2$ represent all the domain (t, r) with $r \ge 0$. We map them to the bounded domain $(U, V) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with $U \le V$ given by

$$u = \tan U \ , \quad v = \tan V \ . \tag{7.103}$$

With

$$dudv = \frac{dUdV}{\cos^2 U \cos^2 V}$$

and

$$(u-v)^2 = \frac{\sin^2(U-V)}{\cos^2 U \cos^2 V} \ ,$$

the metric becomes

$$\begin{aligned} ds^2 &= \frac{1}{4\cos^2 U \cos^2 V}\left(-4dUdV + \sin^2(U-V)(d\vartheta^2 + \sin^2\vartheta d\varphi^2)\right) \\ &= \frac{1}{4\cos^2 U \cos^2 V} d\tilde{s}^2 \ , \quad \text{with} \\ d\tilde{s}^2 &= -4dUdV + \sin^2(U-V)(d\vartheta^2 + \sin^2\vartheta d\varphi^2) \ . \end{aligned}$$

The metrics ds^2 and $d\tilde{s}^2$ are related by the 'conformal factor' Ω^2,

$$ds^2 = \Omega^2 d\tilde{s}^2 \ , \quad \Omega^2 = 4\cos^2 U \cos^2 V \ .$$

Unlike ds^2, the metric $d\tilde{s}^2$ is also defined at the boundary, $U = \pm\pi/2$ or $V = \pm\pi/2$. It is a regular metric on $\widetilde{\mathcal{M}} = [-\pi/2, \pi/2] \times [-\pi/2, \pi/2] \times \mathbb{S}^2$. Indeed $(\widetilde{\mathcal{M}}, d\tilde{s}^2)$ is a part of the Einstein universe given by $\mathbb{R} \times \mathbb{S}^3$ with its metric induced by the one of $\mathbb{R}^5$. By setting $\tau = U + V$ and $\chi = V - U$ we obtain, $\chi \in [0, \pi]$

$$d\tilde{s}^2 = -d\tau^2 + d\chi^2 + \sin^2\chi(d\vartheta^2 + \sin^2\vartheta d\varphi^2) ,$$

which is the metric of $\mathbb{R} \times \mathbb{S}^3$. The part $\chi > 0$ of the shaded area of the cylinder $\mathbb{R} \times \mathbb{S}^3$ on figure 7.11 corresponds to $\widetilde{\mathcal{M}}$. The image of Minkowski space is the inside of $\widetilde{\mathcal{M}}$ ($\widetilde{\mathcal{M}}$ without the boundary) . The border is called the 'conformal infinity'. It consists of the following parts: the two zero-hypersurfaces given by $\{2V = \tau + \chi = \pi, |2U| = |\tau - \chi| < \pi\} \equiv \mathcal{J}^+$ corresponding to '**future null infinity**' and $\{2U = \tau - \chi = -\pi, |2V| = |\tau + \chi| < \pi\} \equiv \mathcal{J}^-$ corresponding to '**past null infinity**'. The two points $i^\pm = (U, V) = \pm(\pi/2, \pi/2)$ indicated on the graph corresponding to time-like future infinity (i^+) and past infinity (i^-) and $i^0 = (U, V) = (-\pi/2, \pi/2)$ represents the space-like infinity.

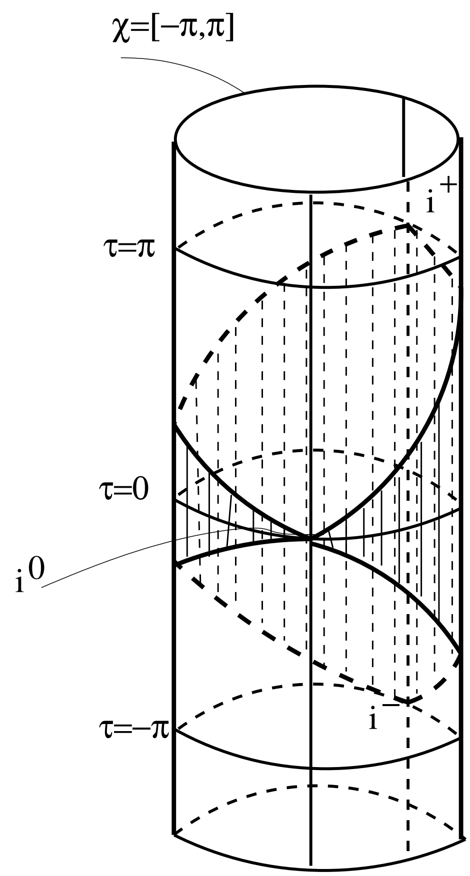


Figure 7.11: The Einstein univers with the manifold $\widetilde{\mathcal{M}}$ (shaded area) which is conformally isometric to a Minkowski space. The boundaries of $\widetilde{\mathcal{M}}$ correspond to infinity in Minkowski space.

Any time-like geodesic that is directed towards the future ends at i^+ and any time-like geodesic that is directed towards the past ends at i^-. Any space-like geodesic ends at i^0. The conformal diagram with coordinates (τ, χ) is presented in figure 7.12. This figure is the Penrose-Carter diagram of Minkowski space.

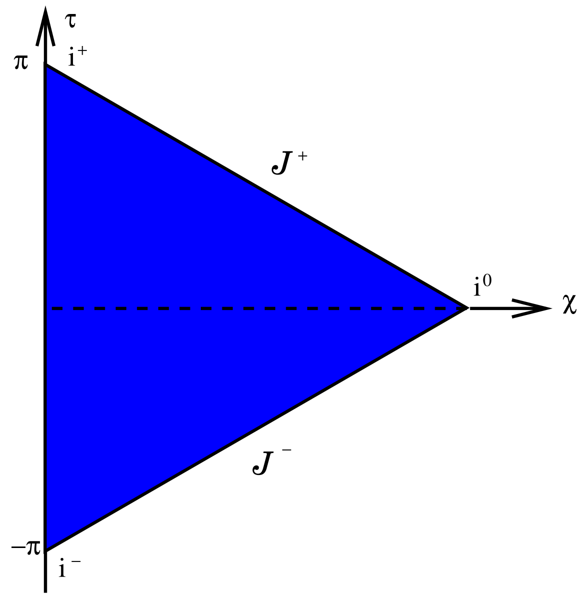


Figure 7.12: The conformal diagram of Minkowski space. The inside corresponds to points at finite (t, r) whereas the boundary represents points at infinity, as described in the text. The angular coordinates (ϑ, φ) are suppressed.

7.9.2 The Carter-Penrose diagram for the Schwarzschild-Kruskal spacetime

We proceed in the same way to construct the Carter-Penrose diagram of Schwarzschild-Kruskal spacetime. We transform the Kruskal coordinates, (u, v), given in (7.82) and (7.83) to

$$\tilde{u} = v - u \quad \text{and} \quad \tilde{v} = v + u \ .$$

As in the case of the Minkowski spacetime, we perform the transformation (7.103),

$$\tilde{u} = \tan U \quad \text{and} \quad \tilde{v} = \tan V \ ,$$

with $(U, V) \in \,]-\pi/2, \pi/2[\, \times \,]-\pi/2, \pi/2[$.

$$\left(1 - \frac{r}{2m}\right) \exp\left(\frac{r}{2m}\right) = v^2 - u^2 = \tilde{u}\tilde{v} \quad \begin{cases} > 0 & \text{for} \quad r < 2m \\ < 0 & \text{for} \quad r > 2m\,. \end{cases}$$

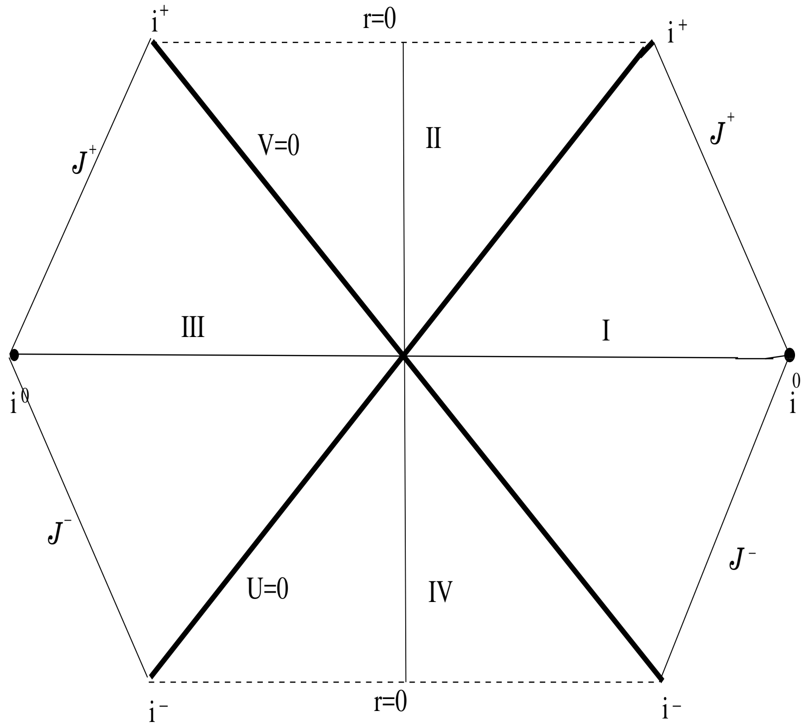


Figure 7.13: The conformal Carter-Penrose diagram of Kruskal spacetime in the plane (τ, χ). The inside corresponds to points at finite values of (u, v) whereas the border represents points at infinity, as described in the text. The four regions I, II, III and IV are also indicated. The horizon $r = 2m$ corresponds to the diagonals $V = 0$ and $U = 0$. As in the case of Minkowski, the light-like infinities, $\mathcal{J}^{\pm}$ correspond to U and $V = \pm\pi/2$ respectively, i^0 is space-like infinity and $i^{\pm}$ represent the time-like future and past infinity respectively.

As $\tilde{u}\tilde{v} = \tan U \tan V$, the sign of UV is positive inside the horizon, $r < 2m$ and negative outside, $r > 2m$. At the singularity, $r = 0$, we have $\tilde{u}\tilde{v} = \tan U \tan V = 1$.

As for Minkowski, we set $\tau = U + V$ and $\chi = V - U$. The entire Kruskal spacetime in the plane (τ, χ) is shown on the diagram 7.13. The horizon is represented by the lines $V = 0$ and $U = 0$. Inside the horizon (regions II and IV) U and V have the same sign. Outside the horizon they have opposite signs. The singularity is given by the lines $\tau = U + V = \pm\pi/2$. The boundaries (light like infinity are at $U = \pm\pi/2$ and $V = \pm\pi/2$. The intersection points of light like infinities correspond to spacelike infinity. Timelike infinity is at the intersection of light like infinity with the singularity $r = 0$.

In this Carter-Penrose diagram, the hyperboloids

$$1 = v^2 - u^2 \equiv \left(1 - \frac{r}{2m}\right) e^{r/2m}$$

which correspond to the singularity $r = 0$ are transformed in $\tan U \cdot \tan V = \tilde{u} \cdot \tilde{v} = 1$, which gives $\cos(U + V) = (1 - \tan U \cdot \tan V) \cos U \cdot \cos V = 0$, such that $\tau = U + V = \pm\pi/2$. Hence the two dashed lines represent the singularity. As for the Minkowski case, for a certain conformal factor Ω^2, the metric $\tilde{g} = \Omega^2 g$ can be extended differentiably to $\mathcal{J}^{\pm}$ (but at i^0 only continuously).

Exercise: Show that in the region $r > 2m$ we have

$$\chi = \begin{cases} \operatorname{arctg}\left(\frac{2\left(\frac{r}{2m}-1\right)^{1/2} e^{r/4m}\cosh(t/4m)}{1-\left(\frac{r}{2m}-1\right)e^{r/2m}}\right) & \text{if } \left(\frac{r}{2m}-1\right)e^{r/2m} < 1 \\ \pi - \operatorname{arctg}\left(\frac{2\left(\frac{r}{2m}-1\right)^{1/2} e^{r/4m}\cosh(t/4m)}{\left(\frac{r}{2m}-1\right)e^{r/2m}-1}\right) & \text{if } \left(\frac{r}{2m}-1\right)e^{r/2m} > 1 \\ \pi/2 & \text{if } \left(\frac{r}{2m}-1\right)e^{r/2m} = 1\ , \end{cases} \tag{7.104}$$

$$\tau = \operatorname{arctg}\left(\frac{2\left(\frac{r}{2m}-1\right)^{1/2} e^{r/4m}\sinh(t/4m)}{1+\left(\frac{r}{2m}-1\right)e^{r/2m}}\right) . \tag{7.105}$$

Some of the curves $t =$ constant and $r =$ constant in region I ($r > 2m$) are shown in fig. 7.14. Draw an analogous figure for region II, $0 < r < 2m$. For this, first derive the formulas corresponding to (7.104) and (7.105) in this region.

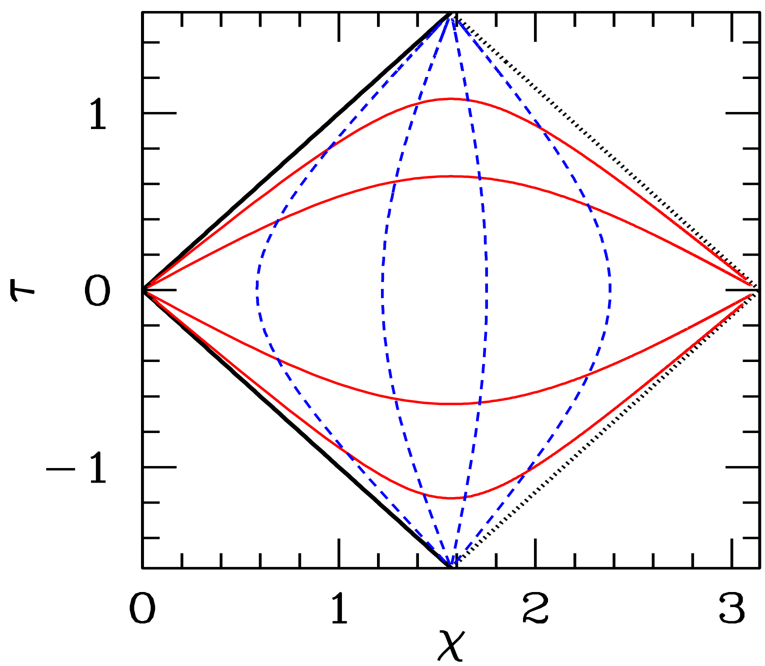


Figure 7.14: Region I of the Carter-Penrose diagram in the plane (τ, χ) is represented. Lines t =constant (red, solid), and r =constant (blue dashed) are indicated. All t =constant lines start in $(\tau, \chi) = (0, 0)$ corresponding to $r = 2m$ and end in i^0, i.e. $r = \infty$ From top to bottom the times $t = 4m\ln(4),\ 4m\ln(2),\ -4m\ln(2),\ -4m\ln(5)$ are chosen. The lines r =constant start in i^- and end in i^+. From left to right the values of r are such that $(r/2m - 1)\exp(r/2m) = 0.1,\ 0.7,\ 1.1,\ 2.5$.

Chapter 8

Weak gravitational fields, gravitational waves, gravitational lenses

We consider Lorentzian manifolds $(\mathcal{M}, g)$ with weak gravitational fields. In this case there exist coordinates where

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad \text{with} \quad |h_{\mu\nu}| \sim \varepsilon \ll 1 \; . \tag{8.1}$$

For instance, in the solar system $|h_{\mu\nu}| \cong \frac{2|\Phi|}{c^2} \lesssim \frac{2GM_\odot}{R_\odot c^2} \cong 4 \times 10^{-6}$.

8.1 Linearized gravitation

We expand the Ricci curvature and we only keep the terms that are of the first order in $h_{\mu\nu}$, $\mathcal{O}(h_{\mu\nu} = \mathcal{O}(\varepsilon)$.
According to (3.30),

$$R_{\mu\nu} = \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\alpha\mu,\nu} + \mathcal{O}(\varepsilon^2) \tag{8.2}$$

$$\begin{aligned} \Gamma^\alpha_{\mu\nu} &= \frac{1}{2}\eta^{\alpha\beta}\left(h_{\beta\mu,\nu} + h_{\beta\nu,\mu} - h_{\mu\nu,\beta}\right) + \mathcal{O}(\varepsilon^2) \\ &= \frac{1}{2}\left(h^\alpha_{\ \mu,\nu} + h^\alpha_{\ \nu,\mu} - h_{\mu\nu}^{\ \ ,\alpha}\right) \; . \end{aligned} \tag{8.3}$$

Here we "raise" and "lower" the indices of $h_{\mu\nu}$ with the Minkowski metric $\eta_{\mu\nu}$.
This gives

$$R_{\mu\nu} = \frac{1}{2}\left[h^\alpha_{\ \mu,\nu\alpha} + h^\alpha_{\ \nu,\mu\alpha} - \Box h_{\mu\nu} - h^\alpha_{\ \alpha,\mu\nu}\right] \tag{8.4}$$

and

$$R = h^{\alpha\beta}_{\ \ \ ,\alpha\beta} - \Box h \quad \text{where} \quad h := h^\alpha_{\ \alpha} \; . \tag{8.5}$$

The linearized Einstein tensor is

$$2G_{\mu\nu} = -\Box h_{\mu\nu} - h_{,\mu\nu} + h_{\mu}{}^{\alpha}{}_{,\alpha\nu} + h_{\nu}{}^{\alpha}{}_{,\alpha\mu} + \eta_{\mu\nu}\left(\Box h - h^{\alpha\beta}{}_{,\alpha\beta}\right). \tag{8.6}$$

The Bianchi identity to 1st order, $G^{\mu\nu}{}_{,\nu} = 0$, are identically satisfied by (8.6). Inserting the Einstein tensor (8.6), Einstein's field equations give

$$-\Box h_{\mu\nu} - h_{,\mu\nu} + h_{\mu}{}^{\alpha}{}_{,\alpha\nu} + h_{\nu}{}^{\alpha}{}_{,\alpha\mu} + \eta_{\mu\nu}\left(\Box h - h^{\alpha\beta}{}_{,\alpha\beta}\right) = 16\pi G T_{\mu\nu}\ . \tag{8.7}$$

In the linearized theory $T^{\mu\nu}_{\ \ ,\nu} = 0$, that is $T_{\mu\nu}$ creates a gravitational field but this field does not affect $T_{\mu\nu}$ to first order.
We work also with

$$\gamma_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h, \tag{8.8}$$

$$h_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\gamma, \quad \text{where } \gamma = \gamma^{\alpha}{}_{\alpha}. \tag{8.9}$$

Using $\gamma_{\mu\nu}$ Einstein's field equations become

$$-\Box\gamma_{\mu\nu} - \eta_{\mu\nu}\gamma_{\alpha\beta}{}^{,\alpha\beta} + \gamma_{\mu\alpha}{}^{,\alpha}{}_{\nu} + \gamma_{\nu\alpha}{}^{,\alpha}{}_{\mu} = 16\pi G T_{\mu\nu}\ . \tag{8.10}$$

Let ξ^μ be a vector field. A simple but somewhat lengthy calculation shows that the Einstein tensor (8.6) is invariant under the transformation

$$h_{\mu\nu} \mapsto h_{\mu\nu} + \varepsilon\xi_{\mu\,,\nu} + \varepsilon\xi_{\nu\,,\mu}. \tag{8.11}$$

Eq. (8.11) can be interpreted as an **infinitesimal** coordinate transformation:
let us set $x'^{\mu} = x^\mu + \varepsilon\xi^\mu(x)$. This transformation induces changes of the order ε in every quantity. In the curvature (and the energy-momentum tensor), which is already of first order in ε, we can neglect this change. But for the change in the metric we obtain

$$\begin{aligned} g_{\mu\nu}(x) &= \frac{\partial x'^{\alpha}}{\partial x^\mu}\frac{\partial x'^{\beta}}{\partial x^\nu} g'_{\alpha\beta}(x') \\ &= \left(\delta^\alpha{}_\mu + \varepsilon\xi^\alpha{}_{,\mu}\right)\left(\delta^\beta{}_\nu + \varepsilon\xi^\beta{}_{,\nu}\right) g'_{\alpha\beta} + \mathcal{O}(\varepsilon^2) \\ &= g'_{\mu\nu}(x') + \varepsilon g'_{\nu\alpha}(x')\xi^\alpha{}_{,\mu} + \varepsilon g'_{\mu\beta}\xi^\beta{}_{,\nu} + \mathcal{O}(\varepsilon^2)\,. \end{aligned}$$

To first order,

$$h_{\mu\nu}(x) = h'_{\mu\nu}(x) + \varepsilon\xi_{\mu\,,\nu} + \varepsilon\xi_{\nu\,,\mu}. \tag{8.12}$$

On the other hand, an infinitesimal diffeomorphism is the infinitesimal flow Φ_ε of some vector field ξ. The change in the metric under Φ_ε is the pull-back,

$$g' = \Phi_\varepsilon^\star g = g + \varepsilon L_\xi g + \mathcal{O}(\varepsilon^2)\,.$$

For $g = \eta + h$, $\mathcal{O}(h) \sim \mathcal{O}(\varepsilon)$, this yields

$$h \mapsto h + \varepsilon L_\xi \eta + \mathcal{O}(\varepsilon^2) \tag{8.13}$$

but $(L_\xi \eta)_{\mu\nu} = \xi_{\mu,\nu} + \xi_{\nu,\mu}$. Under linearized coordinate transformations, also called 'gauge transformations' in this context, the curvature and energy-momentum tensors are invariant whereas the metric transforms according to Eq. (8.12). We can use gauge invariance in order to simplify the Einstein field equations (8.10).

Lemma 8.1 *We can always find a gauge such that*

$$\gamma^{\mu\nu}{}_{,\nu} = 0\,. \tag{8.14}$$

This gauge is called Hilbert gauge.

Proof: Under a gauge transformation $h_{\mu\nu} \mapsto h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu}$ (we absorb the factor ε in ξ_μ), $\gamma_{\mu\nu}$ transforms according to

$$\gamma_{\mu\nu} \mapsto \gamma_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu} - \eta_{\mu\nu}\xi^{\alpha}{}_{,\alpha} = \gamma'_{\mu\nu} \tag{8.15}$$

and so

$$\gamma^{\mu\nu}{}_{,\nu} \mapsto \gamma^{\mu\nu}{}_{,\nu} + \Box\xi^{\mu} = \gamma'^{\mu\nu}{}_{,\nu}\ . \tag{8.16}$$

For

$$\Box\xi^{\mu} = -\gamma^{\mu\nu}{}_{,\nu} \tag{8.17}$$

this yields $\gamma'^{\mu\nu}{}_{,\nu} = 0$.
But (8.17) always has a solution (retarded Green's function). □

In the class of the Hilbert gauges (8.14) Einstein's field equations simply become

$$\Box\gamma_{\mu\nu} = -16\pi G T_{\mu\nu}. \tag{8.18}$$

The general solution of (8.18) that satisfies the condition (8.14) is

$$\gamma_{\mu\nu} = -16\pi G D_R \star T_{\mu\nu} + \text{homogeneous solution} \tag{8.19}$$

where D_R is retarded Green's function of the d'Alembert operator (see Compl. Math. II)

$$D_R(\vec{x}) = \frac{-1}{4\pi|\vec{x}|}\delta(t - |\vec{x}|)\theta(t).$$

(where $(D_R \star T^{\mu\nu})_{,\nu} = D_R \star T^{\mu\nu}{}_{,\nu} = 0$.)
We then obtain

$$\gamma_{\mu\nu}(\vec{x},t) = 4G\int \frac{T_{\mu\nu}(t - |\vec{x}-\vec{x}'|, \vec{x}')}{|\vec{x}-\vec{x}'|}\,\mathrm{d}^3x' + \quad \text{hom. solution.} \tag{8.20}$$

We can interpret this field as the field generated by the source $T_{\mu\nu}$. As in electrodynamics, the reaction of the metric to the source is **causal**; i.e., $\gamma_{\mu\nu}(\vec{x},t)$ only depends on the values of the source on the background light cone of $(t, \vec{x})$.

The homogeneous solution represents a wave coming from infinity (e.g. a plane wave).

8.2 Quasi-newtonian sources

Let us consider an energy-momentum tensors with $|T_{00}| \gg |T_{0j}|$, $|T_{ij}|$ and with small velocities such that we can neglect retardation effects inside the source, i.e. $T_{00}(t - |\vec{x} - \vec{x}'|) \cong T_{00}(t)$ in the source, $|v| \ll 1$.
We then have

$$h_{00} + \frac{1}{2}h = \gamma_{00} = -4\Phi, \quad \gamma_{0j} = \gamma_{ij} = h_{ij} - \frac{1}{2}\delta_{ij}h = 0, \tag{8.21}$$

where Φ is the Newton potential,

$$\Phi(t, \vec{x}) = -G \int \frac{\rho(t, \vec{x}')}{|\vec{x} - \vec{x}'|} \mathrm{d}^3x' \ = G \int \frac{T_{00}(t, \vec{x}')}{|\vec{x} - \vec{x}'|} \mathrm{d}^3x' \ . \tag{8.22}$$

with $\gamma = +4\Phi$ and (8.9), it follows that

$$g_{00} = -(1 + 2\Phi), \quad g_{0i} = 0, \quad g_{ij} = (1 - 2\Phi)\delta_{ij} \ ,$$

that is,

$$g = -(1 + 2\Phi)\mathrm{d}t^2 + (1 - 2\Phi)\mathrm{d}\vec{x}^2 \ . \tag{8.23}$$

Far away from the source (or in a spherically symmetric system) we can restrict our consideration to the monopole contribution,

$$g = -\left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 + \left(1 + \frac{2m}{r}\right)\mathrm{d}\vec{x}^2, \tag{8.24}$$

With $m = \frac{GM}{c^2}$, which is the linearization of the Schwarzschild metric.

The approximations in (8.23) are

- The terms of order Φ^2 and higher are neglected.
- The terms γ_{0i} are of the order $v\Phi$ where $v \sim \left|\frac{T_{0i}}{T_{00}}\right|$ is the typical velocity of the source.
- We neglect the $\gamma_{ij} \sim \Phi \times \left|\frac{T_{ij}}{T_{00}}\right| \sim \Phi v^2$.

In the Newtonian approximation, $v^2 \sim \Phi$ and the most important corrections are then of the order $\Phi^{3/2}$. In the solar system, $\Phi \sim 10^{-6}$, every corrections are $\lesssim 10^{-9}$.

8.3 Free gravitational waves in the linearized theory

8.3.1 The TT gauge

We first consider the linearized equations in the vacuum:

$$\Box\gamma_{\mu\nu} = 0. \tag{8.25}$$

Lemma 8.2 *If $T_{\mu\nu} \equiv 0$, there exists a gauge with $\gamma = 0$ in the class of the Hilbert gauges.*

Proof: The gauge transformations

$$\gamma_{\mu\nu} \mapsto \gamma_{\mu\nu} + \xi_{\mu\,,\nu} + \xi_{\nu\,,\mu} - \eta_{\mu\nu}\xi^{\alpha}{}_{,\alpha} = \gamma'_{\mu\nu}$$

in the Hilbert class satisfy

$$\gamma^{\mu\nu}{}_{,\nu} = \gamma'^{\mu\nu}{}_{,\nu} = 0 \quad \Leftrightarrow \quad \Box\xi^{\mu} = 0 \ .$$

We have $\gamma' = \gamma - 2\xi^{\alpha}{}_{,\alpha}$. We search for a field ξ^{μ} with $\Box\xi^{\mu} = 0$ and $\xi^{\alpha}{}_{,\alpha} = \frac{1}{2}\gamma$.
Construction: Let η^{μ} be a solution of $\eta^{\mu}{}_{,\mu} = \frac{1}{2}\gamma$. Let $\zeta^{\mu} = \Box\eta^{\mu}$. Because

$$\zeta^{\mu}{}_{,\mu} = \Box\eta^{\mu}{}_{,\mu} = \frac{1}{2}\Box\gamma \stackrel{(8.25)}{=} 0 \ ,$$

there exists an antisymmetric tensor $f^{\mu\nu}$ with

$$\zeta^{\mu} = f^{\mu\nu}{}_{,\nu}.$$

Let $\sigma^{\mu\nu} = -\sigma^{\nu\mu}$ be a solution of $\Box\sigma^{\mu\nu} = f^{\mu\nu}$. We set

$$\xi^{\mu} = \eta^{\mu} - \sigma^{\mu\nu}{}_{,\nu}.$$

This satisfies $\xi^{\mu}{}_{,\mu} = \eta^{\mu}{}_{,\mu} = \frac{1}{2}\gamma$ and

$$\Box\xi^{\mu} = \Box\eta^{\mu} - \Box\sigma^{\mu\nu}{}_{,\nu} = \zeta^{\mu} - f^{\mu\nu}{}_{,\nu} = 0. \qquad \Box$$

In this class of gauges we have $\gamma_{\mu\nu} = h_{\mu\nu}$.
The general solution of (8.25) is a superposition of plane waves

$$h_{\mu\nu} = \mathrm{Re}\left(\varepsilon_{\mu\nu}e^{-ik\cdot x}\right) \quad \text{with} \quad k^2 = 0 \quad \text{where} \quad k = (\omega, \vec{k}) \quad \text{et} \quad k^2 = \omega^2 - \vec{k}^2. \tag{8.26}$$

The gauge conditions require

$$\gamma^{\mu\nu}{}_{,\nu} = h^{\mu\nu}{}_{,\nu} = 0, \quad h = \gamma = 0.$$

So that

$$\begin{aligned} k_\mu \varepsilon^\mu{}_\nu &= 0 \\ \varepsilon^\mu{}_\mu &= 0 \end{aligned} \tag{8.27}$$

The matrix $\varepsilon_{\mu\nu}$ is called the polarisation tensor. The five conditions (8.27) leave free 5 components of the symmetric matrix $\varepsilon_{\mu\nu}$. We now show that only two degrees of freedom are physically relevant, the other three can be eliminated by additional gauge transformations inside the class of gauges (8.27).
A gauge transformation that is accepted in this class of gauges must satisfy

$$\Box \xi^\mu = 0 \quad \text{and} \quad \xi^\mu{}_{,\mu} = 0. \tag{8.28}$$

We set

$$\xi^\mu = \mathrm{Re}\left(i\varepsilon^\mu e^{-ik\cdot x}\right) \tag{8.29}$$

(8.28) implies that $k^2 = 0$ and $k_\mu \varepsilon^\mu = 0$.
Under a gauge transformation $\varepsilon_{\mu\nu}$ transforms according to

$$\varepsilon_{\mu\nu} \mapsto \varepsilon_{\mu\nu} + k_\mu \varepsilon_\nu + k_\nu \varepsilon_\mu. \tag{8.30}$$

Let us consider a wave propagating in the z direction:

$$(k^\mu) = (k, 0, 0, k).$$

In this case (8.27) yields

$$\varepsilon_{0\nu} = \varepsilon_{3\nu}, \quad \varepsilon_{00} = \varepsilon_{30} = \varepsilon_{33}, \quad \varepsilon_{01} = \varepsilon_{31}, \quad \varepsilon_{02} = \varepsilon_{32}$$

$$\text{and } -\varepsilon_{00} + \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = 0 \Rightarrow \varepsilon_{11} + \varepsilon_{22} = 0$$

Every components are then determined by

$$\varepsilon_{00},\ \varepsilon_{11},\ \varepsilon_{01},\ \varepsilon_{02},\ \varepsilon_{12}.$$

Under a gauge transformation of the form (8.30) these components change into

$$\varepsilon_{00} \mapsto \varepsilon_{00} + 2k\varepsilon_0$$

$$\varepsilon_{11} \mapsto \varepsilon_{11}, \quad \varepsilon_{12} \mapsto \varepsilon_{12}$$

$$\varepsilon_{01} \mapsto \varepsilon_{01} + k\varepsilon_1, \quad \varepsilon_{02} \mapsto \varepsilon_{02} + k\varepsilon_2$$

$k_\mu \varepsilon^\mu = 0$ implies that $\varepsilon_0 = \varepsilon_3$.
By choosing $\varepsilon_0 = -\frac{\varepsilon_{00}}{2k}$, $\varepsilon_1 = -\frac{\varepsilon_{01}}{k}$ et $\varepsilon_2 = -\frac{\varepsilon_{02}}{k}$ we can cancel $\varepsilon_{0\mu}$ and the only remaining components are $\varepsilon_{12} = \varepsilon_{21}$ and $\varepsilon_{11} = -\varepsilon_{22}$.

This gauge is the TT gauge (transverse, traceless). In this gauge

$$h_{\mu 0} = 0, \quad h^i{}_i = 0 \quad \text{and} \quad h_{ij\,,j} = 0. \tag{8.31}$$

These gauge conditions can by satisfied for any gravitational wave (the conditions are linear).

To determine the effect of a gravitational wave on the distance between two free falling particles, we must calculate the linearized Riemann tensor. Equation (3.29) gives

$$R^{\mu}{}_{\sigma\nu\rho} = \Gamma^{\mu}_{\rho\sigma\,,\nu} - \Gamma^{\mu}_{\nu\sigma\,,\rho} + \text{terms of } (\Gamma^{\alpha}_{\beta\gamma})^2 \ .$$

So that

$$R_{\mu\sigma\nu\rho} = \frac{1}{2}\left(h_{\nu\sigma\,,\mu\rho} + h_{\rho\mu\,,\sigma\nu} - h_{\mu\nu\,,\sigma\rho} - h_{\rho\sigma\,,\mu\nu}\right) \ . \tag{8.32}$$

In particular for the components

$$R_{i0j0} = R_{0i0j} = R_{0j0i} = -R_{i00j} = -R_{0ij0}$$

we get, in the TT gauge,

$$\begin{aligned} R_{i0j0} &= \frac{1}{2}\left(h_{0j\,,0i} + h_{0i\,,0j} - h_{ij\,,00} - h_{00\,,ij}\right) \\ &= -\frac{1}{2}h_{ij\,,00} \ . \end{aligned} \tag{8.33}$$

As R_{i0j0} is a gauge invariant, (8.33) implies that it is impossible to reduce $h_{\mu\nu}$ further than the two degrees of freedom of the TT gauge.

8.3.2 Geodesic deviation generated by a linearized gravitational wave

The separation vector $\vec{n}$ between two neighboring geodesics satisfies (see equation (6.4))

$$\frac{\mathrm{d}^2\vec{n}}{\mathrm{d}\tau^2} = K\vec{n}, \quad \text{avec} \quad K_{ij} = R^{i}{}_{00j}. \tag{8.34}$$

To first order this gives, in the TT coordinate system

$$\frac{\mathrm{d}^2 n^i}{\mathrm{d}t^2} = \frac{1}{2}\ddot{h}^i_j n^j. \tag{8.35}$$

where $\dot{} = \partial_t$. We consider two timelike geodesics with vanishing velocity in the absence of a gravitational wave. In the absence of a gravitational wave therefore $\vec{n}$ is constant.To first order in h_{ij}, we obtain

$$n^i(\tau) \cong n^i(0) + \frac{1}{2}h_{ij}(\tau)n^j(0). \tag{8.36}$$

In particular, let us consider a plane wave in z direction.

$$\begin{aligned} h_{xx} &= -h_{yy} = A(t-z) \\ h_{xy} &= h_{yx} = B(t-z) \end{aligned} \tag{8.37}$$

are the non vanishing components in TT gauge.
If the separation vector $\vec{n}$ is parallel to the wave's direction of propagation, $n(0) = (0, 0, a)$, it is not affected, $h_{ij}n^j(0) = 0$. Only the transverse separations oscillate. The transverse part of n, $n_\perp = (n_x, n_y)$ satisfies

$$\ddot{n}_\perp = K_\perp n_\perp$$

with

$$K_\perp = \frac{1}{2}\begin{pmatrix} \ddot{h}_{xx} & \ddot{h}_{xy} \\ \ddot{h}_{xy} & -\ddot{h}_{xx} \end{pmatrix} .$$

We then have

$$n_\perp \cong n_\perp(0) + \frac{1}{2}\begin{pmatrix} h_{xx} & h_{xy} \\ h_{xy} & -h_{xx} \end{pmatrix} n_\perp(0) . \tag{8.38}$$

Let R be the rotation of the plane (x, y) that diagonalizes the symmetric matrix $K_\perp$,

$$\frac{1}{2}\begin{pmatrix} h_{xx} & h_{xy} \\ h_{xy} & -h_{xx} \end{pmatrix} = R\begin{pmatrix} \Omega & 0 \\ 0 & -\Omega \end{pmatrix} R^T ,$$

and let (η, ξ) the components of $n_\perp$ in the rotated system,

$$n_\perp =: R\begin{pmatrix} \eta \\ \xi \end{pmatrix}, \quad n_\perp(0) = R\begin{pmatrix} \eta_0 \\ \xi_0 \end{pmatrix} .$$

In a small time interval, the evolution of $n_\perp$ is then given by

$$\begin{aligned} \eta &\cong \eta_0 + \Omega(t)\eta_0 \\ \xi &\cong \xi_0 - \Omega(t)\xi_0 . \end{aligned}$$

Hence Eq. (8.38) corresponds to a shear. To first order, a circle is transformed into an ellipse with the same area. Fig. 8.1 shows the distortion of a circle of test particles. The distortion is generated by a periodic wave with period ω :

$$\begin{aligned} h_{xx} &= \mathrm{Re}\left(A_0 e^{-i\omega(t-z)}\right) \\ h_{xy} &= \mathrm{Re}\left(B_0 e^{-i\omega(t-z)}\right) . \end{aligned}$$

8.3.3 The energy radiated by a gravitational wave

Within the framework of special relativity, the energy-momentum tensor is conserved,

$$T^{\mu\nu}_{\mathrm{tot}\ ,\mu} = 0. \tag{8.39}$$

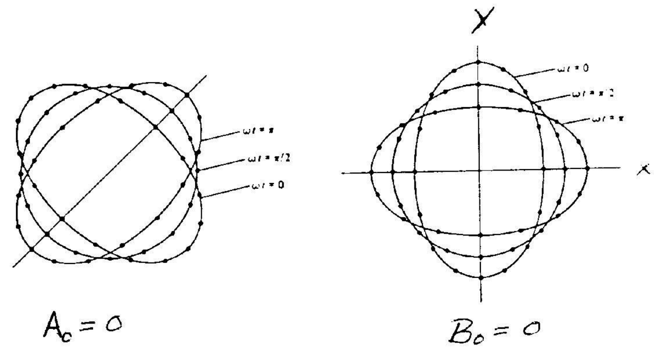


Figure 8.1: The distortion caused by a purely diagonal gravitational wave (on the right) and a purely "off diagonal" one (on the left).

Here $T^{\mu\nu}_{\rm tot}$ can, for instance, represent the sum of the energy-momentum of particles and electromagnetic waves. By integrating the 0 component of (8.39) on a fixed 3-volume V with surface Σ we find

$$0 = \int_V T^0_{0\ ,0} \mathrm{d}^3x + \int_V T^i_{0\ ,i}\, \mathrm{d}^3x$$

After using the Gauss theorem, this yields

$$-\frac{\mathrm{d}E(V)}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_V T^0_0 \mathrm{d}^3x = \int_\Sigma T^i_0 n^i\, \mathrm{d}\sigma\ , \tag{8.40}$$

where $\vec{n}$ is the normal to the surface Σ. The second term is the energy flow through the surface Σ (towards the outside).

In general relativity, we also want to find symmetric quantities

$$\tau^{\mu\nu} = g\left(T^{\mu\nu} + t^{\mu\nu}\right) \tag{8.41}$$

that are conserved, that is, with $\tau^{\mu\nu}_{\ \ ,\nu} = 0$. Here, $g = |\det(g_{\mu\nu})|^{1/2}$, and $T^{\mu\nu}$ is the energy-momentum tensor of matter. The quantities $t^{\mu\nu} = t^{\nu\mu}$ are the specific contributions of the gravitational field. Because the equation $\tau^{\mu\nu}_{\ \ ,\nu} = 0$ is not covariant, $t^{\mu\nu}$ cannot be a tensor. The quantity $t^{\mu\nu}$ is called the energy-momentum pseudo-tensor of the gravitational field. There are several versions (Landau-Lifschitz pseudo-tensor, Einstein pseudo-tensor) that lead to the same conserved quantities in the linearized theory. Here we only consider the linearized theory. As before, we set

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\ . \tag{8.42}$$

We separate the linearized part of the Einstein tensor from the rest,

$$G_{\mu\nu} = G^{(1)}_{\mu\nu} + G^{(2)}_{\mu\nu}\ , \tag{8.43}$$

where $G^{(1)}$ contains the terms linear in $h_{\mu\nu}$ and $G^{(2)}$ contains the rest. With $g = \sqrt{-\det(g_{\mu\nu})} = 1 + \delta g \simeq 1 + \frac{1}{2}h$ the field equations yield

$$G^{(1)}_{\mu\nu} = 8\pi G g T_{\mu\nu} - g G^{(2)}_{\mu\nu} - \delta g G^{(1)}_{\mu\nu} + \mathcal{O}(h^3) \ . \tag{8.44}$$

As $G^{(1)}_{\mu\nu}$ is conserved, $G^{(1)\,\mu\nu}{}_{,\nu} = 0$, we obtain up to terms of order h^3 which we shall neglect in what follows,

$$\left(g T_{\mu\nu} - \frac{1}{8\pi G}\left(G^{(2)}_{\mu\nu} + \delta g G^{(1)}_{\mu\nu} \right) \right)^{,\nu} = 0 \ . \tag{8.45}$$

We define

$$\tau_{\mu\nu} \equiv g T_{\mu\nu} - \frac{1}{8\pi G}\left(g G^{(2)}_{\mu\nu} + \delta g G^{(1)}_{\mu\nu} \right) = g \left(T_{\mu\nu} + t_{\mu\nu} \right) \ . \tag{8.46}$$

The quantity $\tau_{\mu\nu}$ is symmetric and possesses the required properties to be considered as the energy-momentum pseudo-tensor, $\tau^{\mu\nu}{}_{,\nu} = 0$.
Using Hilbert gauge we find after a little calculation (to the order of h^2),

$$\begin{aligned} g t^{\mu\nu} = \frac{1}{32\pi G} \Bigg[& 2\gamma^{\mu}{}_{\alpha\,,\beta}\gamma^{\nu\alpha\,,\beta} - 2\gamma^{\mu}{}_{\alpha\,,\beta}\gamma^{\alpha\beta\,,\nu} - 2\gamma^{\nu}{}_{\alpha\,,\beta}\gamma^{\alpha\beta\,,\mu} + \gamma_{\alpha\beta}{}^{,\mu}\gamma^{\alpha\beta\,,\nu} - \frac{1}{2}\gamma^{,\mu}\gamma^{,\nu} \\ & - \frac{1}{2}\eta^{\mu\nu}\left(\gamma_{\alpha\beta\,,\sigma}\gamma^{\alpha\beta\,,\sigma} - \frac{1}{2}\gamma^{,\alpha}\gamma_{,\alpha} - 2\gamma_{\alpha\beta\,,\sigma}\gamma^{\beta\sigma}{}_{,\alpha} \right) \Bigg] \end{aligned} \tag{8.47}$$

Taking the divergence of this expression we obtain

$$\left(g t^{\mu\nu} \right)_{,\nu} = \frac{1}{32\pi G}\left(h_{,\nu}\Box\gamma^{\mu\nu} - 2h^{\mu}{}_{\alpha\,,\nu}\Box\gamma^{\alpha\nu} + h_{\alpha\nu}{}^{,\mu}\Box\gamma^{\alpha\nu} \right) . \tag{8.48}$$

Similarly, by using $g_{,\beta} = 2g\Gamma^{\alpha}_{\alpha\beta}$ and the Einstein field equations, we find (still to the second order in h)

$$0 = \nabla_{\beta}T^{\alpha\beta} = T^{\alpha\beta}{}_{,\beta} + \Gamma^{\alpha}_{\sigma\beta}T^{\sigma\beta} + \Gamma^{\beta}_{\sigma\beta}T^{\alpha\sigma}$$

$$0 = g\nabla_{\beta}T^{\alpha\beta} = (gT^{\alpha\beta})_{,\beta} + \frac{1}{8\pi G}\left(\Gamma^{\alpha}_{\sigma\beta}G^{(1)\,\sigma\beta} - \Gamma^{\beta}_{\sigma\beta}G^{(1)\,\alpha\sigma} \right)$$

$$0 = (gT^{\alpha\beta})_{,\beta} + \frac{1}{32\pi G}\left(h_{,\beta}\Box\gamma^{\alpha\beta} - 2h^{\alpha}{}_{\sigma\,,\beta}\Box\gamma^{\sigma\beta} + h_{\sigma\beta}{}^{,\alpha}\Box\gamma^{\sigma\beta} \right)$$

which proves that $\tau^{\alpha\beta}{}_{,\beta}$ is zero, given the already established identity (8.48).

For a plane wave in x^1 direction, in TT gauge, equation (8.47) yields

$$h_{\mu\nu} = h_{\mu\nu}(t - x^1), \quad \gamma^{0\mu} = \gamma^{1\mu} = 0, \quad \gamma = 0 \quad \gamma^{\mu\nu}{}_{,\nu} = 0$$

$$t^{01} = \frac{1}{16\pi G}\left[\left(\dot{h}_{23} \right)^2 + \frac{1}{4}\left(\dot{h}_{22} - \dot{h}_{33} \right)^2 \right] = t^{00} = t^{11} \ . \tag{8.49}$$

All other components of $t^{\mu\nu}$ vanish.

8.4 Emission of gravitational waves

We consider the solution

$$\gamma_{\mu\nu}(\vec{x},t) = 4G\int \frac{T_{\mu\nu}(t-|\vec{x}-\vec{x}'|,\vec{x}')}{|\vec{x}-\vec{x}'|}\,\mathrm{d}^3x' \tag{8.50}$$

at large distance from the source, $|\vec{x}| \gg |\vec{x}'|$, we can neglect $|\vec{x}'|$ in the retarded time and in $\frac{1}{|\vec{x}-\vec{x}'|}$, i.e. we consider the 'wave zone'. This yields, with $|\vec{x}| \equiv r$,

$$\gamma_{\mu\nu}(\vec{x},t) \cong \frac{4G}{r}\int T_{\mu\nu}(t-r,\vec{x}')\,\mathrm{d}^3x'. \tag{8.51}$$

To transform (8.51) we use

$$0 = \int x^k \partial_\nu T_\mu{}^\nu \mathrm{d}^3x = \partial_t\left(\int x^k T_\mu{}^0 \mathrm{d}^3x\right) + \int x^k \partial_l T_\mu{}^l \mathrm{d}^3x$$

Integrating by parts the second term yields

$$\partial_t\left(\int x^k T_\mu{}^0 \mathrm{d}^3x\right) = \int T_\mu^k \mathrm{d}^3x\ . \tag{8.52}$$

In addition, Gauss' theorem gives $0 = \int \partial_j\left(T^{j0}x^kx^l\right)\mathrm{d}^3x$, so

$$\int \partial_\mu\left(T^{\mu0}x^kx^l\right)\mathrm{d}^3x = \partial_t\int T^{00}x^kx^l\mathrm{d}^3x\ . \tag{8.53}$$

But

$$\int \partial_\mu\left(T^{\mu0}x^kx^l\right)\mathrm{d}^3x \overset{\partial_\mu T^{\mu0}=0}{=\!=\!=} \int T^{\mu0}\partial_\mu\left(x^kx^l\right)\mathrm{d}^3x = \int\left(T^{k0}x^l + T^{l0}x^k\right)\mathrm{d}^3x. \tag{8.54}$$

With (8.53) and (8.52) this yields

$$\frac{1}{2}\partial_t^2\int T^{00}x^kx^l\mathrm{d}^3x = \frac{1}{2}\partial_t\int\left(T^{k0}x^l + T^{l0}x^k\right)\mathrm{d}^3x = \int T^{kl}\mathrm{d}^3x\ . \tag{8.55}$$

If we insert this equation in (8.51) we obtain

$$\gamma^{kl}(\vec{x},t) = \frac{2G}{r}\left[\partial_t^2\int \rho(\vec{x}',t')x'^k x'^l \mathrm{d}^3x'\right]_{t'=t-r}, \tag{8.56}$$

where $\rho = T^{00}$.

The solution (8.56) is not in TT gauge. To obtain the TT part of the metric we have to act with the following projection tensor on γ_{ij}.

$$\Lambda_{lk}{}^{ij} = P_l^iP_k^j - \frac{1}{2}P_{lk}P^{ij}\,, \qquad P^{ij} = \left(\delta_{ij} - n^in^j)\right)\,, \qquad n^i = x^i/r, \tag{8.57}$$

$$\gamma_{kl}^{(TT)}(\vec{x},t) = h_{kl}^{(TT)}(\vec{x},t) = \Lambda_{kl}{}^{ij}(\vec{x})\gamma_{ij}(\vec{x},t)\,.$$

Defining the quadrupole tensor of a (non-relativistic) matter distribution as

$$Q_{kl} = \int \left(3x'^k x'^l - r'^2\delta^{kl}\right)\rho(\vec{x}',t)\mathrm{d}^3x'\ , \tag{8.58}$$

the metric in TT gauge can be expressed as

$$h_{lk}^{(TT)}(\vec{x},t) = \frac{2G}{3r}\Lambda_{lk}{}^{ij}\partial_t^2 Q_{ij}(t-r) = \frac{2G}{3r}\partial_t^2 Q_{lk}^{(TT)}(t-r)\ . \tag{8.59}$$

We now use this in order to calculate the energy emitted by a gravitational wave. To the lowest (quadratic) order in $h_{\mu\nu}$ we have found the result (8.49). Inserting the above expression, we find

$$t^{01} = \frac{G}{36\pi}\frac{1}{r^2}\left[\dddot{Q}_{23}^2 + \frac{1}{4}(\dddot{Q}_{22} - \dddot{Q}_{33})^2\right]. \tag{8.60}$$

Here we have used that for a wave in x^1-direction $Q_{ij}^{(TT)} = Q_{ij}$ for $i \neq 1$ and $j \neq 1$. Setting $\vec{x}/r = \vec{n} = (1,0,0)$ we find

$$t^{0j}n^j = -\frac{G}{72\pi}\frac{1}{r^2}\Lambda^{klij}\ \dddot{Q}_{kl}\dddot{Q}_{ij} = -\frac{G}{36\pi}\frac{1}{r^2}\left[\frac{1}{2}\ \dddot{Q}_{kl}\dddot{Q}_{kl} - \dddot{Q}_{kl}\dddot{Q}_{km}\ n^l n^m + \frac{1}{4}\left(\dddot{Q}_{kl}\ n^l n^k\right)^2\right]. \tag{8.61}$$

This formula is valid in a arbitrary direction $\vec{n}$. The emitted power per unit solid angle in direction $\vec{n}$ is $r^2 t^{0j}n^j = \frac{\mathrm{d}P}{\mathrm{d}\Omega}(\vec{n})$. Such that

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{G}{36\pi}\left[\frac{1}{2}\ \dddot{Q}_{kl}\dddot{Q}_{kl} - \dddot{Q}_{km}\dddot{Q}_{kl}\ n^m n^l + \frac{1}{4}(\dddot{Q}_{kl}\ n^l n^k)^2\right]$$

The loss of energy is obtained by integrating this expression over all the directions. Using

$$\frac{1}{4\pi}\int n^k n^l \mathrm{d}\Omega = \frac{1}{3}\delta^{kl}$$

and

$$\frac{1}{4\pi}\int n^k n^l n^j n^i \mathrm{d}\Omega = \frac{1}{15}\left(\delta^{kl}\delta^{ij} + \delta^{ki}\delta^{lj} + \delta^{kj}\delta^{li}\right)$$

we find the famous quadrupole formula of Albert Einstein:

$$\boxed{-\frac{\mathrm{d}E}{\mathrm{d}t} = P = \frac{G}{45c^5}\ \dddot{Q}_{lk}\dddot{Q}_{lk}}\ . \tag{8.62}$$

The experimental verification of (8.62) on the pulsar 1913+16 made by Hulse and Taylor deserved a Nobel price in 1993. In the meantime, the quadrupole formula has been confirmed on several other binary pulsars. Until 2016, this indirect 'proof' represented the only verification of the existence of gravitational waves. In January 2016, the LIGO Collaboration announced the direct discovery of gravitational waves emitted from a binary black hole system. We will discuss both discoveries in more detail in the next paragraph.

Exercice: Setting $n^i = x^i/r$ we define the following moments of the matter distribution:

$$\begin{aligned}
M(t) &= \textstyle\int T^{00}(t,\vec{y})d^3y & M_j(t) &= \int T^{00}(t,\vec{y})y_j d^3y \\
P^i(t) &= \textstyle\int T^{0i}(t,\vec{y})d^3y & P^i{}_j(t) &= \int T^{0i}(t,\vec{y})y_j d^3y \\
S^{ij}(t) &= \textstyle\int T^{ij}(t,\vec{y})d^3y & M_{ij}(t) &= \int T^{00}(t,\vec{y})y_i y_j d^3y\,.
\end{aligned}$$

Using the same methods as in the previous paragraph show that

$$\begin{aligned}
\dot{M} &= 0 & \dot{M}^i &= P^i & \dot{M}^{ij} &= P^{ij} + P^{ji} \\
\dot{P}^i &= 0 & \dot{P}^{ij} &= S^{ij} & \ddot{M}^{ij} &= 2S^{ij}
\end{aligned}$$

Using these identities show that in the wave zone

$$\begin{aligned}
\gamma_{00}(t,\vec{x}) &= \frac{4G}{r}\left[M(t-r) + P^j(t-r)n_j + \frac{1}{4}S^{ij}(t-r)n_i n_j + \cdots\right], \\
\gamma_{0j}(t,\vec{x}) &= \frac{-4G}{r}\left[P_j(t-r) + S_{ij}(t-r)n^i + \cdots\right], \\
\gamma_{ij}(t,\vec{x}) &= \frac{4G}{r}\left[S_{ij}(t-r) + \cdots\right],
\end{aligned}$$

where for each component the neglected terms are of order (v/c) smaller than the once taken into account.

Verify that the solution satisfies the Hilbert gauge condition.

Perform a gauge transformation with the vector field

$$\xi^0 = -\frac{G}{r}[P^i_i + P^{ij}n_i n_j]\,, \quad \xi^k = -\frac{G}{r}\left[4M^k + 4P^{ki}n_i - P^i_i n^k - P^{ij}n_i n_j n^k\right].$$

Show that this transforms the metric components explicitly into TT gauge, up to time independent contributions like $h^{(TT)}_{00} = 4GM/r = -2\Phi$, where Φ is the (time independent) Newtonian potential (and up to terms $\mathcal{O}(1/r^2)$ which we can neglect in the wave-zone).

8.5 Application: gravitational radiation of a binary star system

We consider a binary star system, where M_1 and M_2 are the masses of the stars, and the energy is $E < 0$. We set $m_1 = GM_1$, $m_2 = GM_2$. We approximate the orbits of the stars by the Newtonian solution.
The parameters a (semi major axis) and e (excentricity) of the Newtonian orbit are related to the first integrals E (energy) and L (angular momentum) by

$$a = -\frac{GM_1M_2}{2E}\,, \quad E = \mu(\dot{r}^2 + r^2\dot{\vartheta}^2) - \frac{G\mu(M_1+M_2)}{r} < 0\,, \tag{8.63}$$

$\mu = M_1M_2/(M_1+M_2)$ is the reduced mass of the system.

$$e^2 = 1 + \frac{2EL^2(M_1+M_2)}{G^2M_1^3M_2^3}\,, \quad L = \mu r^2\dot{\vartheta}. \tag{8.64}$$

These results can be found in any text book on classical mechanics. Furthermore, the period of the system is given by Kepler's 3rd law. Setting $m_1 = GM_1$ and $m_2 = GM_2$, the period of the system is given by

$$T = \frac{2\pi a^{3/2}}{\sqrt{m_1+m_2}}. \tag{8.65}$$

We wish to calculate the change in the period, $\dot{T}$ caused by the emission of gravitational waves. The binary system orbits in a plane that we choose to be (x, y).

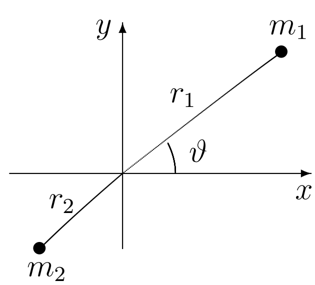


Figure 8.2: The gravitational wave emission problem in a binary star system

Then, the quadrupole tensor of our system has only x and y components: if I_{ij} is the inertia tensor,

$$I_{ij} = \int \rho(\vec{x})x_ix_j\,\mathrm{d}^3x,$$

we have

$$Q_{ij} = 3(I_{ij} - \frac{1}{3}I), \quad I = \sum_{i=1}^{3} I_{ii}.$$

The quadrupole formula (8.62) then yields

$$-\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{G}{5}\left(\dddot{I}_{kl}\dddot{I}_{kl} - \frac{1}{3}\dddot{I}^{2}\right) = \frac{G}{5}\left(\dddot{I}^{2}_{xx} + \dddot{I}^{2}_{yy} + 2\,\dddot{I}^{2}_{xy} - \frac{1}{3}\dddot{I}^{2}\right). \tag{8.66}$$

The semi-major axis a changes because of this energy loss, and this leads to a change in the period T. To determine this, one must compute the derivatives $\dddot{I}_{ij}$. We adopt the coordinates (x, y) as indicated on Fig. 8.2. The distance r between the two masses is

$$r = \frac{a(1-e^2)}{1+e\cos\vartheta}\ . \tag{8.67}$$

The positions r_1 and r_2 are

$$r_1 = \frac{m_2}{m_1+m_2}r, \quad r_2 = \frac{m_1}{m_1+m_2}r.$$

The components of the inertia tensor are

$$I_{ij} = \int \rho(\vec{x})x_i x_j\,\mathrm{d}^3x.$$

For our mass distribution, $\rho(\vec{x}) = M_1\delta(\vec{x}-\vec{x}_1) + M_2\delta(\vec{x}-\vec{x}_2)$, so that we obtain

$$\begin{aligned}
GI_{xx} &= m_1x_1^2 + m_2x_2^2 = \frac{m_1m_2}{m_1+m_2}r^2\cos^2\vartheta,\\
GI_{yy} &= \frac{m_1m_2}{m_1+m_2}r^2\sin^2\vartheta,\\
GI_{xy} &= \frac{m_1m_2}{m_1+m_2}r^2\cos\vartheta\sin\vartheta,\\
GI &= GI_{xx} + GI_{yy} = \frac{m_1m_2}{m_1+m_2}r^2\ .
\end{aligned}$$

With $L = \frac{m_1m_2}{m_1+m_2}r^2\dot{\vartheta}$ and (8.64) we find

$$\dot{\vartheta} = \frac{\sqrt{(m_1+m_2)a(1-e^2)}}{r^2}\ . \tag{8.68}$$

Using (8.67), this leads to

$$\dot{r} = e\sin\vartheta\left(\frac{m_1+m_2}{a(1-e^2)}\right)^{1/2}\ . \tag{8.69}$$

With (8.67), (8.68) and (8.69) we can determine the derivatives of the inertia tensor:

$$G\dot{I}_{xx} = -\frac{2m_1m_2}{\sqrt{(m_1+m_2)a(1-e^2)}}r\cos\vartheta\sin\vartheta$$

$$
\begin{aligned}
G\ddot{I}_{xx} &= -\frac{2m_1m_2}{a(1-e^2)}(\cos 2\vartheta + e\cos^3\vartheta) \\
G\dddot{I}_{xx} &= \frac{2m_1m_2}{a(1-e^2)}(2\sin 2\vartheta + 3e\cos^2\vartheta\sin\vartheta)\dot{\vartheta} \\
G\dot{I}_{yy} &= \frac{2m_1m_2}{\sqrt{(m_1+m_2)a(1-e^2)}}r(\sin\vartheta\cos\vartheta + e\sin\vartheta) \\
G\ddot{I}_{yy} &= \frac{2m_1m_2}{a(1-e^2)}(\cos 2\vartheta + e\cos\vartheta + e\cos^3\vartheta + e^2) \\
G\dddot{I}_{yy} &= -\frac{2m_1m_2}{a(1-e^2)}(2\sin 2\vartheta + e\sin\vartheta + 3e\cos^2\vartheta\sin\vartheta)\dot{\vartheta} \\
G\dot{I}_{xy} &= \frac{m_1m_2}{\sqrt{(m_1+m_2)a(1-e^2)}}r(\cos^2\vartheta - \sin^2\vartheta + e\cos\vartheta) \\
G\ddot{I}_{xy} &= -\frac{2m_1m_2}{a(1-e^2)}(\sin 2\vartheta + e\sin\vartheta + e\sin\vartheta\cos^2\vartheta) \\
\dddot{I}_{xy} &= -\frac{2m_1m_2}{a(1-e^2)G}(2\cos 2\vartheta - e\cos\vartheta + 3e\cos^3\vartheta)\dot{\vartheta} \\
G\dddot{I} &= -\frac{2m_1m_2}{a(1-e^2)}e\sin\vartheta\dot{\vartheta}\ .
\end{aligned}
$$

In (8.66) this finally gives

$$
\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{8m_1^2m_2^2}{15a^2G(1-e^2)^2}\left[12(1+e\cos\vartheta)^2 + e^2\sin^2\vartheta\right]\dot{\vartheta}^2.
$$

For the averaged loss of energy over a period, we obtain

$$
\begin{aligned}
\left\langle\frac{\mathrm{d}E}{\mathrm{d}t}\right\rangle &= \frac{1}{T}\int_0^T \frac{\mathrm{d}E}{\mathrm{d}t}\,\mathrm{d}t = \frac{1}{T}\int_0^{2\pi}\frac{\mathrm{d}E}{\mathrm{d}t}\frac{1}{\dot{\vartheta}}\,\mathrm{d}\vartheta \\
&= -\frac{32}{5G}\frac{m_1^2m_2^2(m_1+m_2)}{a^5(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)
\end{aligned}
$$

With (8.63) this leads to the following change of the semi-major axis

$$
\left\langle\frac{\mathrm{d}a}{\mathrm{d}t}\right\rangle = \frac{2a^2G}{m_1m_2}\left\langle\frac{\mathrm{d}E}{\mathrm{d}t}\right\rangle = -\frac{64}{5}\frac{m_1m_2(m_1+m_2)}{a^3}f(e)
$$

where

$$
f(e) = \frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{7/2}}.
$$

Using Kepler's 3rd law, we then find

$$
\frac{\dot{T}}{T} = \frac{3}{2}\frac{\dot{a}}{a} = -\frac{96}{5}\frac{m_1m_2}{a^4}(m_1+m_2)f(e).
$$

Substituting a by T we find with (8.65)

$$\frac{\dot{T}}{T} = -\frac{96}{5}\frac{m_1 m_2}{(\frac{T}{2\pi})^{8/3}(m_1+m_2)^{1/3}} f(e) \ . \tag{8.70}$$

This formula has been confirmed in several systems. The most remarkable is the PSR 1913+16 binary pulsar system which is a system of two white dwarfs of masses $M_1 \sim M_2 \sim 1.4M_\odot$. This system is under observation for more than 40 years and the formula (8.70) is confirmed at a precision of half a percent. The measured value is $\dot{T}_{\rm exp} = (-2.408 \pm 0.01) \times 10^{-12}$, whereas the theory predicts $\dot{T}_{\rm theo} = (-2.40243 \pm 0.00005) \times 10^{-12}$. In the experimental value, the effects caused by the relative acceleration of the binary pulsar with respect to our solar system is taken into account. This changes the result by about 1%. The uncertainty of the theoretical value is a consequence of the uncertainty of the masses, the ellipticity and the semi-major axis of the system. For more details, see C.M. Will [18]
For this indirect observation of the existence of gravitational waves, the radio-astronomer Joe Taylor and his student R. Hulse have received the Nobel price in 1993.

In January 2016 the LIGO collaboration has announced the first direct detection of a gravitational wave with the Advanced LIGO interferometer. The LIGO experiment consists of two interferometers, one in Hanford, Washington State (US) and one in Livingston, Luisiana (US), both with 4km arm-length optical cavities. The observation was made on September 14, 2015 while the Advanced LIGO interferometers where still being tested. The detected signal (GW150914) is in excellent agreement with two inspiralling black holes with masses $36^{+5}_{-4}M_\odot$ and $(29 \pm 4)M_\odot$. The final black hole has a mass of $(62 \pm 4)M_\odot$ with $(3 \pm 0.3)M_\odot$ radiated in gravitational waves, see Fig. 8.4. The distance to the event is about 400^{+160}_{-180}Mpc corresponding to a cosmological redshift of $z \simeq 0.09$. In the mean time (until the end of 2015) there have been one other detection (GW151226, $14M_\odot$ and $8M_\odot$ at more than 5σ, GW) and a candidate event (LVT151012, $23M_\odot$ and $13M_\odot$ less than 3σ).

The waveforms during the inspiral phases are calculated using a post-Newtonian approximation (up to second order) combined with numerical relativity. This allows to determine the parameters of the two original black holes. Approximately 250'000 template waveforms are used to cover the parameter space of different mass ratios and spins. The 'ring down' phase after coalescence is modelled with black hole perturbation theory. This allows to determine the parameters of the final black hole.

Many more detections not only with the LIGO experiment but also with the European VIRGO detector which consists of a 3km interferometer in Pisa, and others are expected in the near future. Gravitational wave astronomy has just begun.

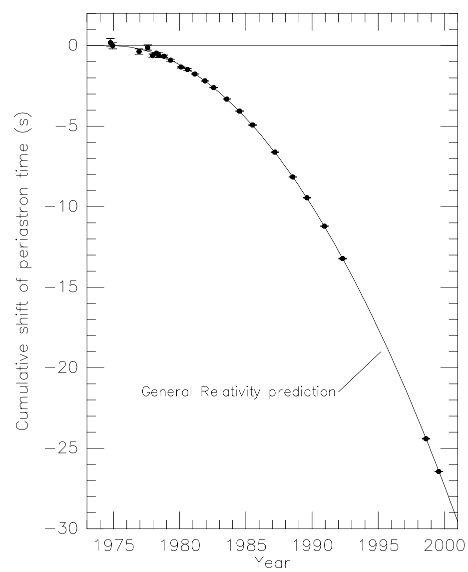


Figure 8.3: The decay of the orbital period of the PRS1913+16 binary pulsar corresponding to about 3.2mm per period is shown as the displacement of the phase of the orbit at the time of periastron passage, compared to a system whose period does not decrease. This decrease is explained entirely by the emission of gravitational waves, as shown by the comparison of the experimental results (points) with the theoretical curve from the quadrupole formula derived in the text.

A simple quantity, the so called 'chirp mass' can, be obtained from (8.70). Replacing T by $\omega = 2\pi/T$ we find

$$\frac{\dot{\omega}}{\omega} = \frac{96}{5}\frac{m_1 m_2}{\omega^{-8/3}(m_1+m_2)^{1/3}} f(e) \ .$$

With $M_i = m_i/G$ this yields for $e = 0$, so that $f(e) = 1$

$$\mathcal{M} := \frac{(M_1 M_2)^{3/5}}{(M_1+M_2)^{1/5}} = \frac{c^3}{G}\left[\frac{5}{96}\frac{\dot{\omega}}{\omega^{11/3}}\right]^{3/5} \ .$$

Noting that the frequency of the gravitational mass is given by $f = \pi\omega$ (the frequency of quadrupole radiation is twice the orbital frequency) , we can write this as

$$\mathcal{M} = \frac{c^3}{G}\left[\frac{5}{96}\frac{\dot{f}}{\pi^{8/3} f^{11/3}}\right]^{3/5} \ .$$

$\mathcal{M}$ is the so called chirp mass which can be measured directly from f and $\dot{f}$ in the perturbative regime. One assumes that back reaction from gravitational radiation

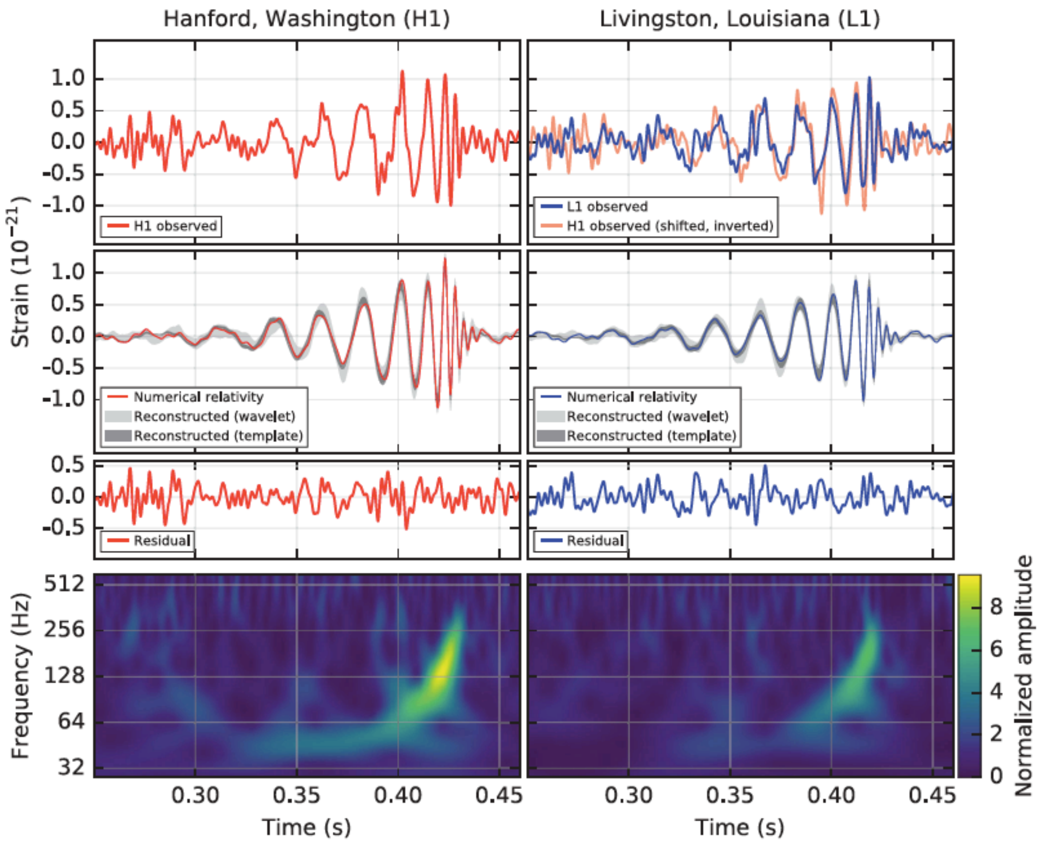


Figure 8.4: The first gravitational wave signal detected by the LIGO collaboration (GW150914) . Figure from Ref. [10].

leads to the decay of the ellipticity of the orbit so that $e = 0$ can be assumed at late stages.

For the event GW150914 a chirp mass $\mathcal{M} \simeq 30M_\odot$ was measured. Setting $M_2 = rM_1$ with $0 < r$ we obtain

$$\frac{\mathcal{M}}{M_1} = \frac{r^{3/5}}{(1+r)^{1/5}}$$

or

$$M_1 + M_2 = (1+r)M_1 = \frac{(1+r)^{6/5}}{r^{3/5}}\mathcal{M}\,.$$

The function $(1+r)^{6/5}/r^{3/5}$ for $0 < r$ has a minimum at $r = 1$ where it is $2^{6/5} \simeq 2.297$. Hence

$$M_1 + M_2 > 2.3\mathcal{M} = 70M_\odot$$

is a simple consequence which can be obtained in the well understood perturbative regime. To go beyond this requires higher order perturbation theory, both for the metric and motion of the binary system as well as in matching to the wave zone.

This calculation has been performed up to second post Newtonian approximation for spinning black holes with arbitrary mass ratios. This goes far beyond the level of this course, see [6] and references therein.

8.6 Gravitational lensing

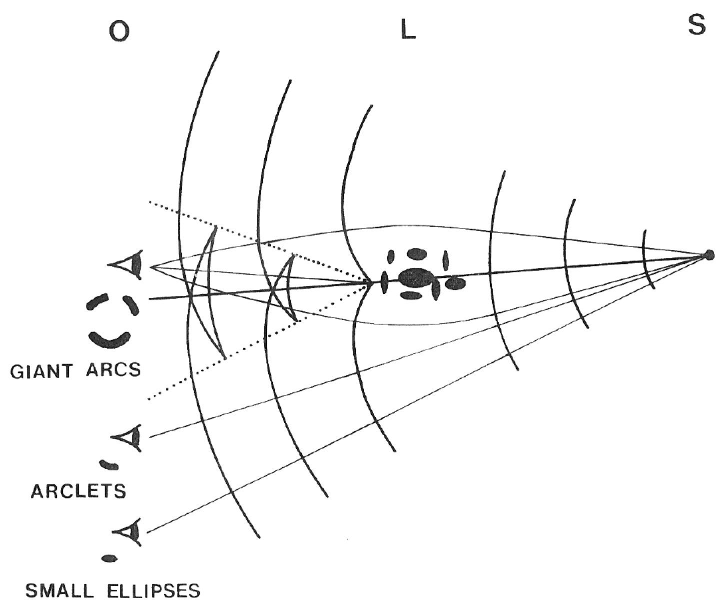


Figure 8.5: The phase-constant surfaces and the orthogonal light rays are indicated. The lensing effects in the inside region of the dashed cone are strong and create multiple images, in the center region they are intermediate and lead to the distortion of the image (medium lensing) whereas in the further region they are weak and only induce a small distortion of the image (weak lensing).

In section 5.4.6 we saw that in a statical gravitational field, the photon paths, $\mathbf{x}(\lambda)$, follow geodesics in the geometry $g^F_{ij} = g_{ij}/(-g_{00})$. In a quasi-Newtonian situation, in a weak potential, $\phi \ll 1$, the metric is

$$g = -(1+2\phi)dt^2 + (1-2\phi)d\mathbf{x}^2$$

so Fermat's principle is reduced to

$$S \equiv \int \sqrt{\left(\frac{1-2\phi}{1+2\phi}\right)\dot{\mathbf{x}}^2 d\lambda} = \int n(\mathbf{x}(\lambda))|\dot{\mathbf{x}}(\lambda))|d\lambda = \int L(\mathbf{x},\dot{\mathbf{x}})d\lambda \ , \qquad \delta S = 0 \ , \tag{8.71}$$

where $n(\mathbf{x}) \simeq 1 - 2\phi(\mathbf{x})$. In other words, the gravitational potential plays the role of a refractive index and the light propagation in a static gravitational field is identical to the one in an inhomogeneous medium with refractive index $n(\mathbf{x})$. If the metric is static but not Newtonian, such that g^F_{ij} is not diagonal, we have a situation analogous to an inhomogeneous and anisotropic medium.

As in the optical limit, with

$$F_{\mu\nu} = \mathrm{Re}(F_{\mu\nu}e^{i\psi})\,,$$

Maxwell's equations imply that a photon's four-momentum is given by the gradient of the phase ψ,

$$\partial_\mu\psi = \dot{x}_\mu \quad \text{and} \qquad g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi = 0\ .$$

For $d\lambda = ds =$ the arc length, such that $\dot{\mathbf{x}}^2 = 1$, the Euler-Lagrange equation from (8.71) yields

$$\frac{d}{ds}\left(n\frac{d\mathbf{x}}{ds}\right) = \boldsymbol{\nabla} n\ . \tag{8.72}$$

This is the optical propagation equation of light rays in an inhomogeneous medium. The different possibilities depending on the distance between the line of sight and the lens are shown in fig. 8.5. In a region where a ray can cross a constant phase surface ('wavefront') more than once, a source produces more than one image. In the regions outside the dashed cone, the image is only distorted.

Already Einstein realized the possibility of gravitational lensing, but he did not believe that we would ever observe it. Fritz Zwicky, in 1937 was the first to predict the observation of gravitational lenses. The first candidate gravitational lens was discovered in 1979 and since then, several hundreds of systems have been found, where multiple images with almost perfect arcs or Einstein rings can be seen (see e.g. http//www.cfa.harvard.edu/glensdata/).

The vector $\frac{d\mathbf{x}}{ds} \equiv \mathbf{e}$ in eq. (8.72) is a unitary vector and (8.72) is equivalent to

$$\frac{d}{ds}\mathbf{e} = -2\boldsymbol{\nabla}\phi + 2\mathbf{e}(\mathbf{e}\cdot\boldsymbol{\nabla})\phi = -2\boldsymbol{\nabla}_\perp\phi\ . \tag{8.73}$$

Here $\boldsymbol{\nabla}_\perp$ denotes the gradient in the plane normal to $\mathbf{e}$. For the deflection angle defined by $\hat{\boldsymbol{\alpha}} = \mathbf{e}_{\rm in} - \mathbf{e}_{\rm fin}$ this yields

$$\hat{\boldsymbol{\alpha}} = 2\int_{s_{\rm in}}^{s_{\rm fin}} \boldsymbol{\nabla}_\perp\phi\, ds\ . \tag{8.74}$$

Note that $\hat{\boldsymbol{\alpha}}$ is a vector in the plane normal to $\mathbf{e}$ and the integral is to be taken along the unperturbed path, $\mathbf{x}(s) = \mathbf{e}(s - s_{\rm in})$ (this is the 'Born approximation' correct to first order in $\hat{\boldsymbol{\alpha}}$). By taking the divergence of (8.74) we find

$$\boldsymbol{\nabla}_\perp\cdot\hat{\boldsymbol{\alpha}} = 2\int_{s_{\rm in}}^{s_{\rm fin}}\Delta_\perp\phi\, ds = 2\int_{s_{\rm in}}^{s_{\rm fin}}\Delta\phi\, ds = \frac{8\pi G}{c^2}\int_{s_{\rm in}}^{s_{\rm fin}}\rho\, ds = \frac{8\pi G}{c^2}\Sigma\,, \tag{8.75}$$

where

$$\Sigma = \int_{s_{\rm in}}^{s_{\rm fin}} \rho$$

is the density per unit surface projected onto a surface normal to $\mathbf{e}$. In Eq. (8.75) we use the Newtonian approximation for the gravitational potential. After the second equal sign, we have replaced $\Delta_\perp$ by Δ because the additional terms are of the form $d^2\phi/ds^2$ can be integrated and we neglect the boundary terms $\frac{d\phi}{ds}(s_{\rm in/fin})$. Defining the lens potential by

$$\hat{\psi} \equiv \int_{s_{\rm in}}^{s_{\rm fin}} \phi\, ds\,, \qquad \text{we have} \;\; \hat{\boldsymbol{\alpha}} = 2\boldsymbol{\nabla}_\perp \hat{\psi} \;\; \text{and} \;\; \Delta_\perp \hat{\psi} = 4\pi G \Sigma\ . \tag{8.76}$$

The Green function of the two dimensional Laplacian is[1]

$$\mathcal{G}(\boldsymbol{\xi}) = \frac{1}{2\pi}\ln(|\boldsymbol{\xi}|)\ , \tag{8.77}$$

such that

$$\hat{\psi}(\boldsymbol{\xi}) = 2G\int_{\mathbb{R}^2} \ln|\boldsymbol{\xi} - \boldsymbol{\xi}'|\Sigma(\boldsymbol{\xi}')d\boldsymbol{\xi}'\ . \tag{8.78}$$

The deflection angle is then

$$\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = 4G\int_{\mathbb{R}^2} \frac{\boldsymbol{\xi} - \boldsymbol{\xi}'}{|\boldsymbol{\xi} - \boldsymbol{\xi}'|^2}\Sigma(\boldsymbol{\xi}')d\boldsymbol{\xi}'\ . \tag{8.79}$$

For a point mass M at the origin, $\Sigma(\boldsymbol{\xi}') = M\delta^2(\boldsymbol{\xi}')$, this yields

$$\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = 4GM\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}\ . \tag{8.80}$$

For a more general mass distribution but with cylindrical symmetry around the $\mathbf{e}$ axis which corresponds to $\boldsymbol{\xi} = 0$ we obtain (exercise!)

$$\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = 4G\frac{M(|\boldsymbol{\xi}|)\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}\ . \tag{8.81}$$

8.6.1 The lens map

In the situation depicted in fig. 8.6, Euclidean geometry yields in the limit of small deflection angle $\hat{\boldsymbol{\alpha}}$

$$\boldsymbol{\eta} = \frac{D_s}{D_d}\boldsymbol{\xi} - D_{ds}\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi})\ . \tag{8.82}$$

[1] see complément de math. II

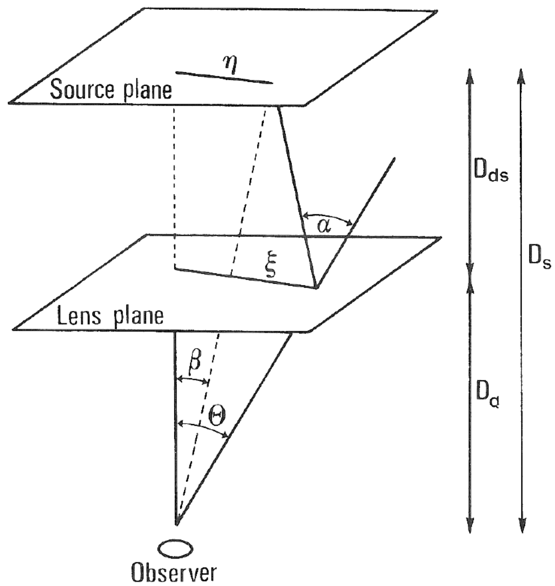


Figure 8.6: The notation used in the text

Here $\boldsymbol{\eta}$ is the position of the source and $\boldsymbol{\xi}$ is the position of the image. This defines the map $\boldsymbol{\xi} \to \boldsymbol{\eta}$. According to fig. 8.6, we also have

$$\boldsymbol{\xi} = D_d \boldsymbol{\theta} \ , \ \text{and} \ \ \boldsymbol{\eta} = D_s \boldsymbol{\beta} \ , \tag{8.83}$$

and we can write (8.82) in the form

$$\boldsymbol{\beta} = \boldsymbol{\theta} - \frac{D_{ds}}{D_s} \hat{\boldsymbol{\alpha}} \ . \tag{8.84}$$

Eq. (8.84) is the lens equation. It is useful to write this equation in dimensionless form. To do so we introduce a length scale ξ_0 in the plane of the lens. We set $\eta_0 = (D_s/D_d)\xi_0$, $\mathbf{x} = \boldsymbol{\xi}/\xi_0$ and $\mathbf{y} = \boldsymbol{\eta}/\eta_0$. In addition, we set

$$\kappa(\mathbf{x}) = \frac{\Sigma(\xi_0 \mathbf{x})}{\Sigma_{\rm crit}} \ , \quad \boldsymbol{\alpha}(\mathbf{x}) = \frac{D_d D_{ds}}{\xi_0 D_s} \hat{\boldsymbol{\alpha}}(\xi_0 \mathbf{x}) \ , \ \ \text{and} \ \ \psi(\mathbf{x}) = \frac{D_d D_{ds}}{\xi_0^2 D_s} \hat{\psi}(\xi_0 \mathbf{x}) \ , \tag{8.85}$$

with

$$\Sigma_{\rm crit} = \frac{c^2}{4\pi G} \frac{D_s}{D_d D_{ds}} = 0.35 \text{g} \cdot \text{cm}^{-2} \left(\frac{1\text{Gpc}}{D_d D_{ds}/D_s} \right) \ . \tag{8.86}$$

The length 1Gpc (giga parsec) $= 10^9$pc $= 3.26 \times 10^9$ light-years is a cosmological distance. The radius of the observable universe is approximately $3H_0^{-1} \simeq 13$Gpc. Considering a galaxy with mass density $\rho_{\rm gal} \sim 10^{11} M_\odot/(10\text{kpc})^3$ we obtain

$$\Sigma_{\rm gal} \simeq \rho_{\rm gal} \times 10\text{kpc} \simeq 0.2 \frac{g}{\text{cm}^2}$$

which is of the same order of magnitude, hence $\kappa_{\rm gal} \sim 1$ close to the centre of a galaxy.

In the rescaled variables, the lens map becomes

$$\mathbf{y} = \mathbf{x} - \boldsymbol{\alpha}(\mathbf{x}) \quad \text{with} \tag{8.87}$$

$$\boldsymbol{\alpha}(\mathbf{x}) = \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^2}\kappa(\mathbf{x}')d^2x' = \boldsymbol{\nabla}\psi(\mathbf{x}) \ , \tag{8.88}$$

where $\boldsymbol{\nabla} \equiv \boldsymbol{\nabla}_\perp$ is 2-dimensional gradient. The potential ψ satisfies the 2-dimensional Poisson equation,

$$\Delta\psi = 2\kappa \ . \tag{8.89}$$

The lens map, $\boldsymbol{\varphi} : x \mapsto y$ is a gradient map,

$$\mathbf{y} = \boldsymbol{\varphi}(\mathbf{x}) = \boldsymbol{\nabla}\left(\frac{1}{2}\mathbf{x}^2 - \psi(\mathbf{x})\right) \ , \tag{8.90}$$

with

$$\psi(\mathbf{x}) = \frac{1}{\pi}\int_{\mathbb{R}^2} \ln|\mathbf{x}-\mathbf{x}'|\kappa(\mathbf{x}')d^2x' \ . \tag{8.91}$$

The differential $(D\boldsymbol{\varphi})_{ij} = \delta_{ij} - \partial_i\partial_j\psi$ is symmetric. The standard parameterization is

$$D\boldsymbol{\varphi} = \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix} \ . \tag{8.92}$$

Here $\gamma_1 = \frac{1}{2}(\partial_1^2 - \partial_2^2)\psi$ is $\gamma_2 = \partial_1\partial_2\psi = \partial_2\partial_1\psi$. The variable $\gamma = \gamma_1 + i\gamma_2$ is the complex shear and $2(1-\kappa) = 2 - \Delta\psi$ is the trace. One can show [8] that, if $\kappa > 1$ for some values of $\mathbf{x}$, there are always multiple images for certain positions $\mathbf{y}$ of the source. (This condition is sufficient but not necessary). It actually follows straight from the consideration of the eigen-values of $D\varphi$ which are given by

$$\lambda_{1,2} = 1 - \kappa \pm \sqrt{|\gamma|^2} \, . \tag{8.93}$$

Far away from the lens, both κ and $|\gamma|$ are much smaller than one, hence $\lambda_{1,2} > 0$. If $\kappa = 1$ then $\lambda_1 > 0$ and $\lambda_2 < 0$ hence at some point λ_2 and thus the determinant of $D\boldsymbol{\varphi}$ must change sign. In this occasion the number of images changes by (at least) 2 since an image of negative parity must be present when $\det D\boldsymbol{\varphi} < 0$.

The amplification μ of an image is [8]

$$\mu = \frac{1}{|\det D\boldsymbol{\varphi}|} = \frac{1}{(1-\kappa)^2 - |\gamma|^2} \ . \tag{8.94}$$

The **critical lines** are given by

$$\det D\boldsymbol{\varphi}(\mathbf{x}) = 0 \ ,$$

and the images (under φ) of the critical lines are the **caustics**. These are the positions of the sources where the amplification diverges. Of course this is a formal divergence because for an extended source the amplification remains finite and for a point source close to a caustic, the light ray approximation is no more valid. Nevertheless, a source that lies in the neighborhood of a caustic is strongly amplificated.

8.6.2 The Schwarzschild lens

We finish this chapter with the simple example of a point mass, $\Sigma(\boldsymbol{\xi}) = M\delta^2(\boldsymbol{\xi})$. Even if the Schwarzschild lens is generated by a singular distribution and so contradicts some general theorems (odd image theorem, see [8]), it is mostly useful for the study of 'microlensing'.

We will see that a good choice for the length ξ_0 is the Einstein radius R_E defined by

$$R_E = \sqrt{\frac{4GM}{c^2}\frac{D_d D_{ds}}{D_s}} = 610 R_\odot \left(\frac{M}{M_\odot}\frac{D_d}{D_s}\frac{D_{ds}}{\mathrm{kpc}}\right)^{1/2} . \tag{8.95}$$

Because the situation is symmetric under rotations, the deflection angle is of the form $\boldsymbol{\alpha}(\mathbf{x}) = \alpha(x)\mathbf{x}/x$ and $\mathbf{y} = y \cdot \mathbf{x}/x$, $x \equiv \pm|\mathbf{x}|$. Furthermore $\psi = \ln(x)$ so that $\alpha = x^{-1}$, and the lens map becomes

$$y = x - \frac{1}{x} . \tag{8.96}$$

If the source is on the symmetry axis, $y = 0$, the solutions are $x = \pm 1$ and the image forms a circle of radius 1, $|\boldsymbol{\xi}| = R_E$. This is an Einstein ring with aperture angle

$$\theta_E = \frac{R_E}{D_d} = \sqrt{\frac{4GM}{c^2}\frac{D_{ds}}{D_d D_s}} .$$

For any other position $y \neq 0$, cylindrical symmetry is broken and there are two images situated at

$$x_\pm = \frac{1}{2}\left(y \pm \sqrt{y^2+4}\right) \tag{8.97}$$

The differential of the lens map is

$$(D\boldsymbol{\varphi})_{ij} = \left(1 - \frac{1}{x^2}\right)\delta_{ij} + 2\frac{x_i x_j}{x^4} ,$$

with

$$\mu^{-1} = |\det D\varphi| = \left|1 - \frac{1}{x^4}\right| . \tag{8.98}$$

For the two images this gives an amplification of

$$\mu_{\pm} = \frac{1}{4}\left|\frac{y}{\sqrt{y^2+4}} + \frac{\sqrt{y^2+4}}{y} \pm 2\right| . \tag{8.99}$$

If y is large, $x_- \sim 0$ and $\mu_- \ll 1$ while $x_+ \sim y$ and $\mu_+ \sim 1$. The image x_- is close to the axis and very weak, whereas x_+ is very little modified by the lens. On the contrary, if y is small, the source is close to the axis, and both images can be strongly magnified. If they are unresolvable, the significant quantity is the total amplification,

$$\mu_p = \mu_+ + \mu_- = \frac{y^2+2}{y\sqrt{y^2+4}} .$$

For quasars that undergo a lensing effect by a foreground galaxy, this so called 'micro-lensing' effect is caused by the stars inside the galaxy which pass very close to the line of sight from us to the quasar. It is variable in time. This effect has also been observed for stars in the Large Magellanic Cloud and close to the galactic centre. It most probably is caused mainly by light stars, either in our galaxy or in the Large Magellanic Cloud, that passed through the line of sight between us and the source star ($y = 0$). Also planets can be discovered in this way, see Fig. 8.7.

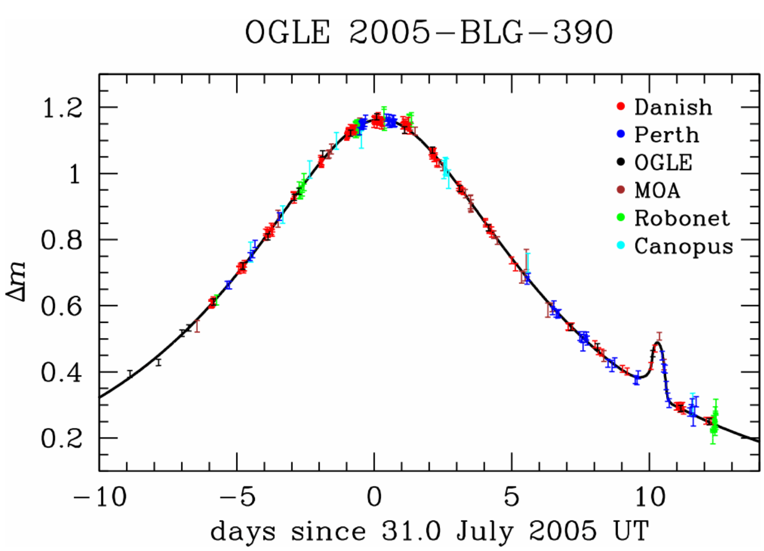


Figure 8.7: A microlensing event involving a planet of about 5 earth masses in a Keplerian orbit around the lens. Observed in 2005 by the OGLE collaboration.

8.6.3 The odd number of images theorem

Let us now assume that the mass distribution κ is regular and finite so that $\boldsymbol{\nabla}\psi$ decays at infinity and the vector field

$$\mathbf{X}(\mathbf{x}) = \mathbf{x} - \boldsymbol{\nabla}\psi(\mathbf{x}) - \mathbf{y} = \boldsymbol{\nabla}\Phi\,, \quad \Phi(\mathbf{x}) = \frac{1}{2}(\mathbf{x}-\mathbf{y})^2 - \psi(\mathbf{x})$$

is regular with the asymptotic behavior $\mathbf{X}(\mathbf{x}) \rightarrow \mathbf{x}$ for $|\mathbf{x}| \rightarrow \infty$. Images of the source at $\mathbf{y}$ are the zeros of $\mathbf{X}$. Let us consider such a zero, $\mathbf{x}_0$ and draw a small smooth path around it which does not pass trough any other zero (see Fig. 8.8).

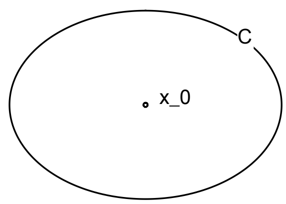


Figure 8.8: A smooth closed path around $\mathbf{x}_0$

Without loss of generality we set $\mathbf{x}_0 = 0$ and choose the polar angle as the parameter of our curve, $C(\phi) = (\epsilon_1 \sin\phi, \epsilon_2 \cos\phi)$. On the curve we parameterize the vector field by

$$\mathbf{X}(\phi) = |\mathbf{X}|(\cos\vartheta, \sin\vartheta)\,.$$

The integral

$$N(\mathbf{x}_0) = \frac{1}{2\pi}\int_C d\vartheta = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\vartheta(\phi)}{d\phi} d\phi$$

must be an integer. It is called the index (or winding number) of the vector field $\mathbf{X}$ at the critical point $\mathbf{x}_0$.

We can orient our coordinate system such that $D\boldsymbol{\varphi} \equiv D\mathbf{X}$ is diagonal and to lowest order and Φ can be approximated by

$$\Phi = \Phi_0 + \frac{1}{2}\Phi_{,11}(0)x_1^2 + \frac{1}{2}\Phi_{,22}(0)x_2^2\,.$$

close to $\mathbf{x}_0 = 0$. We assume $\det D\boldsymbol{\varphi}(0) = \Phi_{,11}(0)\Phi_{,22}(0) \neq 0$ so that 0 is a simple zero. To lowest order therefore (we assume our curve to lie very close around $\mathbf{x}_0$)

$$\mathbf{X}(\phi) = (\Phi_{,11}x_1, \Phi_{,22}x_2) = (\Phi_{,11}\epsilon_1\cos\phi, \Phi_{,22}\epsilon_2\sin\phi)\,.$$

For $\Phi_{,11}$ and $\Phi_{,22} > 0$, i.e. when $\mathbf{x}_0$ is a minimum of Φ, we may choose $\epsilon_1 = \Phi_{,22}\epsilon$ and $\epsilon_2 = \Phi_{,11}\epsilon$ for some small but positive ϵ so that

$$\tan\vartheta = \frac{\mathbf{X}_2}{\mathbf{X}_1} = \frac{\epsilon\Phi_{,11}\Phi_{,22}\sin\phi}{\epsilon\Phi_{,11}\Phi_{,22}\cos\phi} = \tan\phi\,.$$

Hence $\vartheta = \phi$ and $N(\mathbf{x}_0) = 1$. The same is possible up to a sign in both, the choice of ϵ_1 and the choice of ϵ_2 hence the ratio remains unchanged, if both, $\Phi_{,11}$ and $\Phi_{,22} < 0$, i.e. when $\mathbf{x}_0$ is a maximum of Φ. Hence both, (simple) minima and maxima of Φ lead to an index $N(\mathbf{x}_0) = 1$.

The situation is different for saddle points where one of the eigenvalues, say $\Phi_{,11} > 0$ and the other is negative. Then we have to choose $\epsilon_1 = -\Phi_{,22}\epsilon$ and $\epsilon_2 = \Phi_{,11}\epsilon$ which yields

$$\tan\vartheta = -\tan\phi\,,$$

and therefore $N(\mathbf{x}_0) = -1$. See Fig 8.9 for the configuration of $\mathbf{X}$ around different types of zeros.

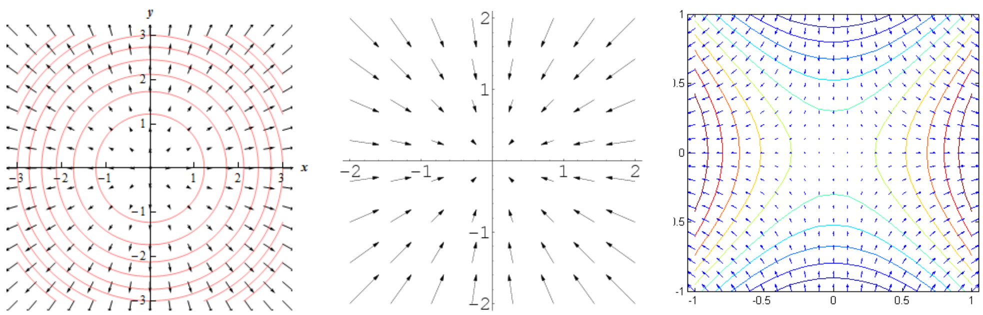


Figure 8.9: The vector field $\mathbf{X}$ around a maximum, a minimum and a saddle of Φ.

As the index is an integer it cannot change under continuous deformations of the curve C as long as the curve C does not pass through a zero of $\mathbf{X}$. On zeros ϑ is not well defined and we cannot compute our index. Let us assume that we have encircled each of the finite number of zeros $\mathbf{x}_i$ of the vector field $\mathbf{X}$, with such a curve C_i which does not cross any other zero. Without crossing a zero we can now continuously combine these curves to one large curve C encircling all the images (zeros of $\mathbf{X}$). We can let $|\mathbf{x}|$ become very large on the large curve C such that $\mathbf{X}$ is well approximated by $\mathbf{X} \sim \mathbf{x}$ on it. Hence the index of the large curve is equal to 1. Since it is the same as the sum of all the indices, as C is obtained from the sum of all C_i's by a continuous deformation which does not cross any zero, it has the same index as the sum of all C_i. This leads to

$$1 = n_+ + n_- - n_s$$

where n_+ denotes the number of minima of Φ, n_- denotes the number of maxima and n_s is the number of saddle points. For the total number of images this yields

$$n_{\text{tot}} = n_+ + n_- + n_s = 1 + 2n_s\,.$$

Hence this number is odd. This and much more in gravitational lensing can be found in Ref. [8], an excellent book even though the observational part is very much outdated.

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