ELECTRODYNAMICS

A course for the 2nd year of the Bachelor in Physics

by

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Preamble

"Höchste Aufgabe der Physiker ist das Aufsuchen jener allgemeinsten, elementaren Gesetze, aus denen durch reine Deduktion das Weltbild zu gewinnen ist. Zu diesen elementaren Gesetzen führt kein logischer Weg, sondern nur die auf Einfühlung in die Erfahrung sich stützende Intuition..."

(*A. Einstein, excerpt from his address at the Physical Society, Berlin, for Max Planck's 60th birthday*)

In English (approximately):

"The supreme task of the physicist is to arrive at those universal elementary laws from which the cosmos can be built up by pure deduction. There is no logical path to these laws; only intuition, resting on sympathetic understanding of experience, can reach them"

I would like you to realise from this quotation that there are no genuine *derivations* of Maxwell's laws: even if driven by some of them, these relations are not totally implied by any experiment whatsoever. In this course we will in fact postulate Maxwell's equations as the point of departure, and we will show thereafter that these laws describe phenomena very well.

As Einstein puts it, finding such sort of laws is the creative act of the highest rank of theoretical physics. This has happened no more than five or six times in the history of physics:

- Gravitation (Newton)

- Thermodynamics (Clausius, Carnot)
- Electrodynamics (Maxwell, Faraday)
- Statistical mechanics (Boltzmann, Gibbs, Einstein)
- General relativity (Einstein)
- Quantum mechanics (Einstein, Schrödinger, Planck, Pauli, Heisenberg, Bohr, Dirac,...)

1.1 Microscopic and Macroscopic Maxwell equations and the Lorentz force

1.1.1 Macroscopic equations

The Lorentz force determines the action of the electromagnetic field on a point charge q with velocity $\mathbf{v}$,

$$\mathbf{K} = q\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \wedge \mathbf{B}\right) . \tag{1.1}$$

Here $\mathbf{E}$ and $\mathbf{B}$ are the electric and the magnetic field and c is the speed of light (as we shall see later). If we define the charge density (charge per unit volume), $\rho = dq/dV$, as well as the current density, $\mathbf{J} = d(q\mathbf{v})/dV$, and use the symbol $\mathbf{k}$ to denote the force per unit volume, we then obtain:

$$\mathbf{k} = \rho\mathbf{E} + \frac{1}{c}\mathbf{J} \wedge \mathbf{B}. \tag{1.2}$$

The fields $\mathbf{E}$ and $\mathbf{B}$ themselves are originated by the charge distribution. The relation between the fields and the charges and currents is determined by Maxwell's equations, the homogeneous version of which are the expressions

$$\mathrm{div}\mathbf{B} \equiv \nabla \cdot \mathbf{B} \equiv \sum_i \partial_i B^i \equiv \frac{\partial B^1}{\partial x^1} + \frac{\partial B^2}{\partial x^2} + \frac{\partial B^3}{\partial x^3} = 0, \tag{1.3}$$

which tells us that *there are no magnetic charges*;

$$\mathrm{rot}\mathbf{E} + \frac{1}{c}\dot{\mathbf{B}} \equiv \nabla \wedge \mathbf{E} + \frac{1}{c}\dot{\mathbf{B}} = \sum_{\ell m} \varepsilon_{i\ell m}\partial_\ell E_m + \frac{1}{c}\dot{B}_i = 0, , \tag{1.4}$$

that expresses the *law of induction*, with $\cdot \equiv \frac{\partial}{\partial t}$ and

$$\varepsilon_{i\ell m} = \begin{cases} \text{sign of the permutation } i\ell m \to 123, & \text{if } i\ell m \text{ are all different} \\ 0 \text{ otherwise.} \end{cases}$$

The two other equations are, on the one hand, *Coulomb's law*,

$$\mathrm{div}\mathbf{D} \equiv \nabla \cdot \mathbf{D} = 4\pi\rho \tag{1.5}$$

and, on the other, *Ampère's law* with the *displacement current*,

$$\mathrm{rot}\mathbf{H} \equiv \nabla \wedge \mathbf{H} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\dot{\mathbf{D}}. \tag{1.6}$$

Here $\mathbf{D}$ and $\mathbf{H}$ are the electric and magnetic fields in a ponderable medium, and ρ and $\mathbf{J}$ are the macroscopic charge and current densities.

Using Gauss and Stokes' theorems, Maxwell equations can be rewritten as integral equations:

- Gauss' theorem:
$$\int_V (\nabla \cdot \mathbf{W}) dv = \int_{\partial V} (\mathbf{W} \cdot \mathbf{e})\, ds \tag{1.7}$$
Here V is an arbitrary (finite) volume of boundary ∂V. $\mathbf{W}$ is a vector field, also arbitrary, and $\mathbf{e}$ is the outward pointing unit normal vector to the surface element ds.

- Stokes' theorem:
$$\int_S (\nabla \wedge \mathbf{W}) \cdot \mathbf{e}\, ds = \oint_{\partial S} \mathbf{W} \cdot \mathbf{n}\, dl \tag{1.8}$$
Here S is some (bounded) surface and ∂S is its boundary. $\mathbf{W}$ and $\mathbf{e}$ are as before and $\mathbf{n}$ is the unit vector parallel to ∂S.

These mathematical theorems allow then to rewrite (1.5) as an integral equation,

$$\int_{\partial V} (\mathbf{D} \cdot \mathbf{e})\, ds = 4\pi Q \tag{1.9}$$

where Q is the total charge in the interior of the volume V. The flow of $\mathbf{D}$ through the boundary of a volume V determines the total charge in the interior. Similarly, Eq. (1.3) gives

$$\int_{\partial V} (\mathbf{B} \cdot \mathbf{e}) ds = 0\ . \tag{1.10}$$

There are no magnetic charges.

With (1.8) and the second Maxwell equation, (1.4), we find

$$\oint_{\partial S} \mathbf{E} \cdot \mathbf{n} dl = -\frac{d}{dt} \int_S (\mathbf{B} \cdot \mathbf{e}) ds$$

which is nothing but Faraday's law, also known as law of induction: a magnetic field that changes along a surface delimited by a metallic wire induces a current in the wire.

The fourth Maxwell equation gives

$$\oint_{\partial S} \mathbf{H} \cdot \mathbf{n} dl = \frac{4\pi}{c} \int_S (\mathbf{J} \cdot \mathbf{e}) ds + \frac{1}{c} \frac{d}{dt} \int_S (\mathbf{D} \cdot \mathbf{e}) ds \tag{1.11}$$

The first two terms of this equation are just Ampère's law. The last term, usually negligible ($\frac{1}{c}$ factor), is the famous 'displacement current'.

When, by interpreting the experimental results—particularly those of Faraday—Maxwell arrived for the first time at these relations, they were all stationary laws, with the only exception of the one of induction. The displacement current (the term $\frac{1}{c}\dot{\mathbf{D}}$) was yet to appear. Maxwell was well aware of the fact that, without that term, the equations were not in agreement with the conservation of electric charge. To get the final form (1.6) he therefore invented, based on complicated and dubious arguments, the reaction of an *ether* (carrier of the electromagnetic field) on the currents, which he named *displacement current*.

For us, however, the term *displacement current* is nothing more than a name, and Ampère's law with the displacement current is justified by its agreement with experiments (which, by the way, were not available at Maxwell's era; in order to measure the displacement current, very rapidly changing electric fields are needed).

In vacuum, $\mathbf{E} = \mathbf{D}$ and $\mathbf{B} = \mathbf{H}$. For ponderable media one has, in addition, some phenomenological relations that link $\mathbf{E}$ to $\mathbf{D}$ and $\mathbf{B}$ to $\mathbf{H}$. In their simplest form, these are linear and isotropic relations with constants ε and μ:

$$\mathbf{D} = \varepsilon \mathbf{E} \quad \text{and} \quad \mathbf{H} = \frac{1}{\mu} \mathbf{B} , \tag{1.12}$$

where ε and μ are known as the dielectric constant and the magnetic permeability.

In a good conductor one also has

$$\mathbf{J} = \sigma \mathbf{E} , \qquad \sigma \text{ is the conductivity.} \tag{1.13}$$

1.1.2 Derivation of the macroscopic equations

To remind you the relation between the microscopic and macroscopic fields, we proceed to derive the equation $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$ (for further details see Jackson, §6.7).

The microscopic laws, the authentically fundamental ones, are

$$\begin{aligned} \nabla\cdot\mathbf{b} &= 0 & \nabla\wedge\mathbf{e}+\tfrac{1}{c}\dot{\mathbf{b}} &= 0 & &\text{(homogeneous eqs.)} \quad (1.14)\\ \nabla\cdot\mathbf{e} &= 4\pi\eta & \nabla\wedge\mathbf{b}-\tfrac{1}{c}\dot{\mathbf{e}} &= \frac{4\pi}{c}\mathbf{j} & &\text{(inhomogeneous eqs.).} \quad (1.15)\end{aligned}$$

Here $(\mathbf{e}, \mathbf{b})$ are the fields and $(\eta, \mathbf{j})$ are the microscopic charge and current densities, which vary over length scales down to atomic distances ($\sim 10^{-8}$cm) (or less) and time scales as short as 10^{-17}s (orbital motion of an electron). The fields $\mathbf{E}$ and $\mathbf{B}$ are averaged

$$\mathbf{E} = \langle\mathbf{e}\rangle, \quad \mathbf{B} = \langle\mathbf{b}\rangle, \tag{1.16}$$

where we define

$$\langle a\rangle(\mathbf{x}) := \int f(\mathbf{x}-\mathbf{y})a(\mathbf{y})d^3y \tag{1.17}$$

with $\int f(\mathbf{y})d^3y = 1$. Here f is a function that averages over a region of size $R \sim 10^{-4}$cm. For example

$$\begin{aligned} f(\mathbf{x}) &= \frac{1}{(\pi R^2)^{\frac{3}{2}}}e^{-|\mathbf{x}|^2/R^2} & &\text{(Gaussian)}\\ &\text{or}\\ f(\mathbf{x}) &= \begin{cases} \frac{3}{4\pi R^3}, & |\mathbf{x}| < R\\ 0, & |\mathbf{x}| \geqslant R\end{cases} & &\text{(top hat).}\end{aligned} \tag{1.18}$$

For the derivatives of the macroscopic field, one finds the following

$$\begin{aligned} \partial_i E_j &= \int \frac{\partial}{\partial x^i} f(\mathbf{x}-\mathbf{y})e_j(\mathbf{y})d^3y = -\int (\frac{\partial}{\partial y^i} f(\mathbf{x}-\mathbf{y}))e_j(\mathbf{y})d^3y\\ &= \int f(\mathbf{x}-\mathbf{y})\frac{\partial}{\partial y^i}e_j(\mathbf{y})d^3y = \langle\partial_i e_j\rangle.\end{aligned}$$

Where the second line is reached after integrating by parts. The derivatives and the average commute. The macroscopic fields, $\mathbf{E}$ and $\mathbf{B}$, satisfy therefore the homogeneous equations (1.14).

In order to determine the inhomogeneous macroscopic equations, it is necessary to take the average of both the charge and current densities. The total charge density can be split up into a density of free charges η_{free}, and a density of the charges of the nuclei and the electrons bound to the molecules, η_{bound}. We thus have, $\langle\eta\rangle = \langle\eta_{\text{free}}\rangle + \langle\eta_{\text{bound}}\rangle$, where

$$\eta_{\text{bound}}(\mathbf{x}) = \sum_n \sum_{\ell_n} q_{\ell_n}\delta(\mathbf{x} - \mathbf{x}_{\ell_n}) \tag{1.19}$$

is the summation over all the molecules (n) and over all the charges (ℓ_n) in a given molecule. Let $\mathbf{x}_n$ be the center of charge of the molecule n. We then have $\mathbf{x}_{\ell_n} = \mathbf{x}_n + \Delta\mathbf{x}_{\ell_n}$, with $|\Delta\mathbf{x}_{\ell_n}| \ll |\mathbf{x}_n|$. For the average $\langle\eta_{\text{bound}}\rangle$ we obtain

$$\begin{aligned}
\langle\eta_{\text{bound}}\rangle(\mathbf{x}) &= \sum_n \sum_{\ell_n} q_{\ell_n} \int d^3y f(\mathbf{x} - \mathbf{y})\delta(\mathbf{y} - \mathbf{x}_n - \Delta\mathbf{x}_{\ell_n}) \\
&= \sum_n \sum_{\ell_n} q_{\ell_n} f(\mathbf{x} - \mathbf{x}_n - \Delta\mathbf{x}_{\ell_n}).
\end{aligned}$$

Let us now Taylor expand this expression in terms of the small quantity $\Delta\mathbf{x}_{\ell_n}$:

$$\langle\eta_{\text{bound}}\rangle(\mathbf{x}) = \sum_n \sum_{\ell_n} q_{\ell_n} f(\mathbf{x} - \mathbf{x}_n) - q_{\ell_n}\Delta\mathbf{x}_{\ell_n} \cdot \nabla f(\mathbf{x} - \mathbf{x}_n) + \cdots. \tag{1.20}$$

With $q_n := \sum_{\ell_n} q_{\ell_n}$ and $\mathbf{p}_n := \sum_{\ell_n} q_{\ell_n}\Delta\mathbf{x}_{\ell_n}$ the charge and dipole moment of the n-th molecule, we find:

$$\langle\eta_{\text{bound}}\rangle(\mathbf{x}) = \sum_n \left(q_n f(\mathbf{x} - \mathbf{x}_n) - \mathbf{p}_n \cdot \nabla f(\mathbf{x} - \mathbf{x}_n) + \ldots\right).$$

The first term is the average of a point charge q_n located at $\mathbf{x}_n$, and the second term is the average of a point dipole $\mathbf{p}_n$ at $\mathbf{x}_n$:

$$\begin{aligned}
\langle\eta_{\text{bound}}\rangle(\mathbf{x}) &= \sum_n \langle\eta_n\rangle \\
\langle\eta_n\rangle &= \langle q_n\delta(\mathbf{x} - \mathbf{x}_n)\rangle - \nabla \cdot \langle\mathbf{p}_n\delta(\mathbf{x} - \mathbf{x}_n)\rangle + \ \ldots
\end{aligned}$$

Defining the macroscopic charge density as

$$\rho(\mathbf{x}) := \langle\eta_{\text{free}}\rangle + \sum_n \langle q_n\delta(\mathbf{x} - \mathbf{x}_n)\rangle$$

and the polarisation as

$$\mathbf{P}(\mathbf{x}) := \sum_n \langle\mathbf{p}_n\delta(\mathbf{x} - \mathbf{x}_n)\rangle$$

it is found that

$$\langle\eta\rangle = \rho - \nabla \cdot \mathbf{P}$$

If we insert here the microscopic Coulomb equation (the first of equations (1.15)), we obtain:

$$\nabla \cdot \mathbf{E} = 4\pi(\rho - \nabla \cdot \mathbf{P}). \tag{1.21}$$

Making

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} \tag{1.22}$$

this gives the macroscopic Coulomb equation (1.5). To the lowest order, $\mathbf{P}$ is a linear function of $\mathbf{E}$, and $\mathbf{P}$ is proportional to $\mathbf{E}$ if the medium is isotropic. Hence

$$\mathbf{P} = \chi_e \mathbf{E}, \quad \mathbf{D} = (1 + 4\pi\chi_e)\mathbf{E} = \varepsilon\mathbf{E}. \tag{1.23}$$

χ_e is the *electric susceptibility* and ε is the *dielectric constant* of the medium. The quadripolar term that we have neglected by writing ... is indeed almost always very small.

An analogous procedure leads to the following expression for the average of the current density

$$\langle \mathbf{j} \rangle = \mathbf{J} + c\nabla \wedge \mathbf{M} + \dot{\mathbf{P}} + ...$$

(the rather lengthy derivation of this result can be found in Jackson, §6.7).

Here $\mathbf{M}$ is the magnetisation, which is usually proportional to $\mathbf{B}$, $\mathbf{M} = \chi_m \mathbf{B}$. Writing

$$\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M} = (1 - 4\pi\chi_m)\mathbf{B} = \frac{1}{\mu}\mathbf{B}, \tag{1.24}$$

we arrive at the macroscopic Ampère's law, (1.6).

If $\mu > 1$, $\chi_m > 0$ and $\mathbf{M}$ is parallel to $\mathbf{B}$, we speak of *paramagnetism*. If, instead, $\mu < 1$, $\chi_m < 0$ and $\mathbf{M}$ is antiparallel to $\mathbf{B}$, we speak of *diamagnetism*. Typically, the *magnetic susceptibility*, χ_m, is very small, so much as to have $|\mu - 1| \sim 10^{-5}$. μ is known as the *magnetic permeability*. In the case of *ferromagnetism* the relation between $\mathbf{B}$ and $\mathbf{H}$ is more complicated because it is time-dependent and non-linear (*hysteresis phenomena*). (Ferromagnetic materials are paramagnetic, but they present in addition some spontaneous magnetisation, which means that the magnetisation $\mathbf{M}$ does not vanish when the field $\mathbf{B}$ is off).

1.1.3 Some simple solutions

Electrostatics

We consider the case $\mathbf{D} = \mathbf{E}$ and $\dot{\mathbf{E}} = \dot{\mathbf{B}} = 0$. Faraday equation (1.4) implies then $\nabla \wedge \mathbf{E} = 0$, and so there exists a function ϕ (the electrostatic potential) with

$\mathbf{E} = -\nabla\phi$, such that

$$\Delta\phi = -4\pi\rho\,, \qquad \text{where} \quad \Delta = \sum_{i=1}^{3} \partial_i^2$$

(Poisson equation). The solution of this equation (which decreases for $r \to \infty$ if the source is contained in a finite volume) is given by

$$\phi(\mathbf{x}) = \int d^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \qquad \text{and} \qquad \mathbf{E} = \int d^3x' \frac{\rho(\mathbf{x}')(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\,. \tag{1.25}$$

Magnetostatics

Once again, we consider $\mathbf{D} = \mathbf{E}$ and $\mathbf{H} = \mathbf{B}$ and $\dot{\mathbf{E}} = \dot{\mathbf{B}} = 0$. From eq. (1.3) we conclude that $\mathbf{B}$ can be written in the form $\mathbf{B} = \nabla \wedge \mathbf{A}$. Here $\mathbf{A}$ is the magnetic vector potential. Additionally, we can choose $\mathbf{A}$ such that $\nabla \cdot \mathbf{A} = 0$. This choice is referred to as the Coulomb gauge.

Exercise: *Show that for any vector field* $\mathbf{V}$

$$\nabla \wedge (\nabla \wedge \mathbf{V}) = \nabla(\nabla \cdot \mathbf{V}) - \Delta\mathbf{V}$$

For $\dot{\mathbf{E}} = 0$ we then obtain, from Ampère's law, that

$$-\nabla \wedge (\nabla \wedge A) = -\nabla(\nabla \cdot \mathbf{A}) + \Delta\mathbf{A} = \Delta\mathbf{A} = -\frac{4\pi}{c}\mathbf{J}$$

The solution of this equation (which also decreases for $r \to \infty$ if the source is contained in a finite volume) is given by

$$\mathbf{A}(\mathbf{x}) = \frac{1}{c}\int d^3x' \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,. \tag{1.26}$$

Equations (1.25) and (1.26) determine the electro- and magneto-static potentials and, consequently, the fields $\mathbf{E}$ and $\mathbf{B}$ for a given static distribution of charges and currents.

Maxwell equations in vacuum

In vacuum, $\rho = 0$, $\mathbf{J} = 0$, and Maxwell equations reduce to

$$\frac{1}{c}\dot{\mathbf{E}} \;=\; \nabla \wedge \mathbf{B} \tag{1.27}$$

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 &(1.28)\\ \frac{1}{c}\dot{\mathbf{B}} &= -\nabla \wedge \mathbf{E} &(1.29)\\ \nabla \cdot \mathbf{B} &= 0. &(1.30)\end{aligned}$$

If we differentiate (1.27) with respect to time and introduce (1.29) for $\dot{\mathbf{B}}$, we find

$$\frac{1}{c^2}\ddot{\mathbf{E}} = \frac{1}{c}\nabla \wedge \dot{\mathbf{B}} = -\nabla \wedge (\nabla \wedge \mathbf{E}) = \Delta\mathbf{E} - \nabla(\underbrace{\nabla \cdot \mathbf{E}}_{=0}).$$

Therefore

$$\frac{1}{c^2}\ddot{\mathbf{E}} - \Delta\mathbf{E} = 0, \tag{1.31}$$

Taking the derivative with respect to time of (1.29) and replacing $\dot{\mathbf{E}}$ by (1.27), we obtain the same equation (1.31) for $\mathbf{B}$. Note that the sign difference between (1.27) and (1.29) is responsible for the appearance of the minus sign in (1.31). We finally arrive at the wave equations

$$\left.\begin{aligned} (\Delta - \frac{1}{c^2}\partial_t^2)\mathbf{E} &= 0\\ (\Delta - \frac{1}{c^2}\partial_t^2)\mathbf{B} &= 0 \end{aligned}\right\}. \tag{1.32}$$

As we will see in more detail in the chapters to come, the Maxwell equations in vacuum describe waves that travel at the speed c (see (1.32)).

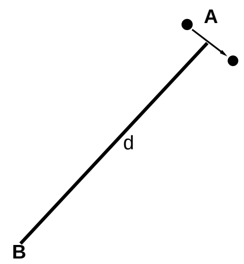
Exercise: *Show that the* ansatz

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_0 f(ct \pm \mathbf{n}\cdot\mathbf{x})\\ \mathbf{B} &= \mathbf{B}_0 g(ct \pm \mathbf{n}\cdot\mathbf{x}) \end{aligned}$$

with $\mathbf{n}^2 = 1$, *and* $\mathbf{E}_0$, $\mathbf{B}_0$ *and* $\mathbf{n}$ *constant vectors, solves equations (1.32). Show that eqs. (1.27) to (1.30) are all also satisfied if and only if* $f = g$, $\mathbf{n}\cdot\mathbf{E}_0 = 0$ *and* $\mathbf{B}_0 = \pm\mathbf{n} \wedge \mathbf{E}_0$.

A first consequence is that the speed c at which the fields $\mathbf{E}$ and $\mathbf{B}$ propagate is finite and hence there is no *action at a distance* in electrodynamics. If a charge located at A moves, a time $\delta t = \frac{d}{c}$ will elapse before this modification is received by an antenna at B at a distance d. Maxwell predicted (after finding equations (1.32)) that light is an electromagnetic phenomenon, something that has been definitely verified. Hertz was the first to detect electromagnetic waves experimentally.

As we will see later on, the electromagnetic field carries energy as well as linear and angular momenta in vacuum, the transport of which takes place at the speed



of light, c, a constant of nature. If Maxwell's equations are valid in all systems of reference, then the speed c is the same in all frames. It is precisely this remark that led Einstein to the theory of special relativity (his original 1905 article was entitled "*Zur Elektrodynamik bewegter Körper*").

Before entering fully into our subject, I would like to discuss once and for all the issue of units.

1.2 Units

(See Jackson, Appendix)

You have probably noticed that Maxwell's equations (1.3–1.6), as well as the relations for the fields **H** and **D**, (1.23) and (1.24), are not exactly equal to those you had already encountered in the first year. The reason for this is that I have adopted the Gaussian system of units, whereas in your previous year you had expressed these equations in MKSA units. For fundamental problems and specially later for quantum electrodynamics, the Gaussian (or Heaviside-Lorentz) units prove to be better adapted. For engineering problems of a more practical nature, however, MKSA units turn out to be very well suited. It is therefore convenient to know both systems (and perhaps even others).

In principle, there is absolute freedom when choosing the units, but it is very important to be aware of and benefit from the choice of units that is better adapted to the problem under study. In physics, it makes no sense to state that the outcome of a measurement of a length is 2. It is imperative to specify if one is speaking of 2 mm, 2 inches, 2 km, 2 light years, etc.

In elementary particle physics, it is customary to set $c = 1$ and $\hbar = 1$. There then remains a basic unit that is in general chosen to be the energy (e.g. 1 electron-Volt, 1eV). From $c = 1$ it is easy to deduce that time has the same dimension as length and that mass has the same dimension as energy and momentum. From $i\hbar = [x, p] = i$ (see quantum mechanics) it follows that energy, momentum and mass have all the dimension of inverse length and inverse time.

In electrostatics, the Coulomb force between two charges q and q' in rest at a distance r is given by

$$F = k_1 \frac{qq'}{r^2}. \tag{1.33}$$

The constant k_1 must be chosen. The dimension and magnitude of this constant determines the units of the charges. It is possible and consistent to set $k_1 = 1$. If one starts from the fundamental dimensions l (length), t (time) and m (mass), we obtain, in the case where k_1 is dimensionless, the following dimension for the charge:

$$[q] = \left([F]l^2\right)^{1/2} = \left(\frac{ml}{t^2}l^2\right)^{1/2} = \frac{m^{1/2}l^{3/2}}{t}. \tag{1.34}$$

In the system used in particle physics, where t has the same dimension as l and m has the inverse dimension, the charge is dimensionless. It is a pure number, *the coupling constant*, which determines the strength of the electromagnetic interaction:

$$\alpha = \frac{k_1 e^2}{\hbar c} = k_1 e^2 \cong \frac{1}{137} \qquad \text{is the 'fine structure constant'}$$

where e is the electron charge. In units where $k_1 = \hbar = c = 1$, the fine structure constant is simply $\alpha = e^2$.

The electric field is defined by $E = \frac{F}{q'} = \frac{k_1 q}{r^2}$, and thus has the units $[E] = \frac{m^{1/2}}{tl^{1/2}}$ in a system with $k_1 = 1$, something adopted in the Gaussian system. Observe that in this case $[\mathbf{E}^2] = \frac{m}{t^2 l}$, i.e. $\mathbf{E}^2$ has the dimensions of energy density, $\left[\frac{m\ell^2}{t^2\ell^3}\right]$.

Analogously, Ampère's law determines the force between two stationary currents I and I' at a distance d,

$$\frac{dF}{dl} = 2\frac{k_2 II'}{d}. \tag{1.35}$$

The constants k_1 and k_2 are not independent. Since $[I] = [q]/t$, it follows that $[k_1/k_2] = l^2/t^2$. The quantity k_1/k_2 can be measured and one finds $k_1/k_2 = c^2$, where c is the speed of light,

$$\frac{k_1}{k_2} = c^2. \tag{1.36}$$

The magnetic induction $\mathbf{B}$ can be derived from Ampère's law (1.6). A long and straight metallic wire carrying a current I induces a B at a distance d given by:

$$B = 2k_2\beta\frac{I}{d}. \tag{1.37}$$

The constant β determines the difference between the dimensions of E and B. Equations (1.33) and (1.37) lead to:

$$\left[\frac{E}{B}\right] = \left[\frac{k_1 qtl}{l^2 k_2 \beta q}\right] \quad \text{(using } I = \tfrac{q}{t}\text{)}.$$

With $\frac{k_1}{k_2} = \frac{l^2}{t^2}$ this gives

$$\left[\frac{E}{B}\right] = \left[\frac{l}{t\beta}\right].$$

If we now write the law of induction in the form $\nabla \wedge \mathbf{E} + k_3 \partial_t B = 0$, it is clear that k_3 has the same dimension as β^{-1}. Indeed, $k_3 = \beta^{-1}$. The simplest way to verify this is to write Maxwell's equations with our constants:

$$\left.\begin{array}{rcl} \nabla \cdot \mathbf{E} & = & 4\pi k_1 \rho \\ \nabla \wedge \mathbf{B} - \frac{k_2}{k_1}\beta \partial_t \mathbf{E} & = & 4\pi k_2 \beta \mathbf{J} \\ \nabla \wedge \mathbf{E} + k_3 \partial_t \mathbf{B} & = & 0 \\ \nabla \cdot \mathbf{B} & = & 0 \end{array}\right\}. \tag{1.38}$$

In a region where $\rho = 0$ and $\mathbf{J} = 0$, these equations lead to

$$\Delta \mathbf{B} - k_3 \frac{k_2 \beta}{k_1} \partial_t^2 \mathbf{B} = 0. \tag{1.39}$$

Hence $\frac{k_3 k_2 \beta}{k_1} = \frac{1}{c^2}$ and, with (1.36), $k_3 \beta = 1$. (The values of k_1, k_2, β and k_3 in different systems are shown in table I).

Magnitudes and Dimensions of the Electromagnetic Constants for Various Systems of Units

The dimensions are given after the numerical values. The symbol c stands for the velocity of light in vacuum ($c = 2.998 \times 10^{10}$ cm/sec $= 2.998 \times 10^8$ m/sec). The first four systems of units use the centimeter, gram, and second as their fundamental units of length, mass, and time (l, m, t). The MKSA system uses the meter, kilogram, and second, plus current (I) as a fourth dimension, with the ampere as unit.

System	k_1	k_2	β	k_3
Electrostatic (esu)	1	$c^{-2}(t^2 l^{-2})$	1	1
Electromagnetic (emu)	$c^2(l^2 t^{-2})$	1	1	1
Gaussian	1	$c^{-2}(t^2 l^{-2})$	$c(lt^{-1})$	$c^{-1}(tl^{-1})$
Heaviside-Lorentz	$\frac{1}{4\pi}$	$\frac{1}{4\pi c^2}(t^2 l^{-2})$	$c(lt^{-1})$	$c^{-1}(tl^{-1})$
Rationalized MKSA	$\frac{1}{4\pi\varepsilon_0} = 10^{-7}c^2$ $(ml^3 t^{-4} I^{-2})$	$\frac{\mu_0}{4\pi} \equiv 10^{-7}$ $(mlt^{-2} I^{-2})$	1	1

Table I (*Table 1 of Jackson, Appendix)*

Furthermore, for the relations between the fundamental fields ($\mathbf{E}$, $\mathbf{B}$) and the fields ($\mathbf{D}$, $\mathbf{H}$) we can still choose some constants

$$\begin{aligned} \mathbf{D} &= \varepsilon_0 \mathbf{E} + \lambda \mathbf{P}, \\ \mathbf{H} &= \frac{1}{\mu_0}\mathbf{B} - \lambda' \mathbf{M}. \end{aligned}$$

There is no point in assigning different dimensions to $\mathbf{D}$ and $\mathbf{P}$ or to $\mathbf{H}$ and $\mathbf{M}$. Indeed, I do not know of any unit system where λ and λ' are dimension-full. The systems with $\lambda = \lambda' = 1$ are called *rational systems.* In non-rational systems one has $\lambda = \lambda' = 4\pi$. The constants ε_0 and μ_0 are the *dielectric constant* and the *vacuum permeability.* In linear and isotropic media one has

$$\mathbf{D} = \varepsilon \mathbf{E}, \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B},$$

The relative permittivity (relative dielectric constant) is $\frac{\varepsilon}{\varepsilon_0}$ and the relative permeability is $\frac{\mu}{\mu_0}$.

In table II you can find the values of ε_0 and μ_0 in different unit systems as well as the corresponding forms of the macroscopic Maxwell's equations and of the Lorentz force. In table III you can find the names and conversion factors between the rationalised MKSA units and the Gaussian units to be used in this course. Note that in the Gaussian system of units, $\mathbf{E}$ and $\mathbf{B}$ have the same dimension, a fact that will play an important role for the relativistic and explicitly covariant formulation of the electromagnetic field.

Definitions of ε_0, μ_0, D, H, Macroscopic Maxwell Equations, and Lorentz Force Equation in Various Systems of Units

Where necessary the dimensions of quantities are given in parentheses. The symbol c stands for the velocity of light in vacuum with dimensions lt^{-1}.

System	ε_0	μ_0	**D, H**	Macroscopic Maxwell Equations				Lorentz Force per Unit Charge
Electrostatic (esu)	1	c^{-2} (t^2l^{-2})	$\mathbf{D}=\mathbf{E}+4\pi\mathbf{P}$ $\mathbf{H}=c^2\mathbf{B}-4\pi\mathbf{M}$	$\nabla\cdot\mathbf{D}=4\pi\rho$	$\nabla\times\mathbf{H}=4\pi\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}$	$\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0$	$\nabla\cdot\mathbf{B}=0$	$\mathbf{E}+\mathbf{v}\times\mathbf{B}$
Electromagnetic (emu)	c^{-2} (t^2l^{-2})	1	$\mathbf{D}=\frac{1}{c^2}\mathbf{E}+4\pi\mathbf{P}$ $\mathbf{H}=\mathbf{B}-4\pi\mathbf{M}$	$\nabla\cdot\mathbf{D}=4\pi\rho$	$\nabla\times\mathbf{H}=4\pi\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}$	$\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0$	$\nabla\cdot\mathbf{B}=0$	$\mathbf{E}+\mathbf{v}\times\mathbf{B}$
Gaussian	1	1	$\mathbf{D}=\mathbf{E}+4\pi\mathbf{P}$ $\mathbf{H}=\mathbf{B}-4\pi\mathbf{M}$	$\nabla\cdot\mathbf{D}=4\pi\rho$	$\nabla\times\mathbf{H}=\frac{4\pi}{c}\mathbf{J}+\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}$	$\nabla\times\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}=0$	$\nabla\cdot\mathbf{B}=0$	$\mathbf{E}+\frac{\mathbf{v}}{c}\times\mathbf{B}$
Heaviside-Lorentz	1	1	$\mathbf{D}=\mathbf{E}+\mathbf{P}$ $\mathbf{H}=\mathbf{B}-\mathbf{M}$	$\nabla\cdot\mathbf{D}=\rho$	$\nabla\times\mathbf{H}=\frac{1}{c}(\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t})$	$\nabla\times\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}=0$	$\nabla\cdot\mathbf{B}=0$	$\mathbf{E}+\frac{\mathbf{v}}{c}\times\mathbf{B}$
Rationalized MKSA	$\frac{10^7}{4\pi c^2}$ $(I^2t^4m^{-1}l^{-3})$	$4\pi\times10^{-7}$ $(mlI^{-2}t^{-2})$	$\mathbf{D}=\varepsilon_0\mathbf{E}+\mathbf{P}$ $\mathbf{H}=\frac{1}{\mu_0}\mathbf{B}-\mathbf{M}$	$\nabla\cdot\mathbf{D}=\rho$	$\nabla\times\mathbf{H}=\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}$	$\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0$	$\nabla\cdot\mathbf{B}=0$	$\mathbf{E}+\mathbf{v}\times\mathbf{B}$

Conversion Table for Given Amounts of a Physical Quantity

The table is arranged so that a given amount of some physical quantity, expressed as so many MKSA or Gaussian units of that quantity, can be expressed as an equivalent number of units in the other system. Thus the entries in each row stand for the same amount, expressed in different units. All factors of 3 (apart from exponents) should, for accurate work, be replaced by (2.99792456), arising from the numerical value of the velocity of light. For example, in the row for displacement (D), the entry ($12\pi \times 10^5$) is actually ($2.99792 \times 4\pi \times 10^5$). Where a name for a unit has been agreed on or is in common usage, that name is given. Otherwise, one merely reads so many Gaussian units, or MKSA or SI units.

Physical Quantity	Symbol	Rationalized MKSA	Gaussian	
Length	l	1 meter (m)	10^2	centimeters (cm)
Mass	m	1 kilogramm (kg)	10^3	grams (gm)
Time	t	1 second (sec)	1	second (sec)
Frequency	ν	1 hertz (Hz)	1	hertz (Hz)
Force	F	1 newton	10^5	dynes
Work Energy	$\left.\begin{matrix} W \\ U \end{matrix}\right\}$	1 joule	10^7	ergs
Power	P	1 watt	10^7	ergs $\sec^{-1}$
Charge	q	1 coulomb	3×10^9	statcoulombs
Charge density	ρ	1 coul m^{-3}	3×10^3	statcoul cm^{-3}
Current	I	1 ampere (amp)	3×10^9	statamperes
Current density	J	1 amp m^{-2}	3×10^5	statamp cm^{-2}
Electric field	E	1 volt m^{-1}	$\frac{1}{3} \times 10^{-4}$	statvolt cm^{-1}
Potential	Φ, V	1 volt	$\frac{1}{300}$	statvolt
Polarization	P	1 coul m^{-2}	3×10^5	dipole moment cm^{-3}
Displacement	D	1 coul m^{-2}	$12\pi \times 10^5$	statvolt cm^{-1} (statcoul cm^{-2})
Conductivity	σ	1 $ohm^{-1}m$	9×10^9	$\sec^{-1}$
Resistance	R	1 ohm	$\frac{1}{9} \times 10^{-11}$	sec cm^{-1}
Capacitance	C	1 farad	9×10^{11}	cm
Magnetic flux	ϕ, F	1 weber	10^8	gauss cm^2 or maxwells
Magnetic induction	B	1 tesla	10^4	gauss
Magnetic field	H	1 ampere-turn m^{-1}	$4\pi \times 10^{-3}$	oersted
Magnetization	M	1 ampere m^{-1}	10^{-3}	magnetic moment cm^{-3}
Inductance	L	1 henry	$\frac{1}{9} \times 10^{-11}$	

Table III (*Table 4 of Jackson, Appendix*)

1.3 The Green method

For a given linear differential operator (with constant coefficients) $\mathcal{D}$, a solution G of the equation

$$\mathcal{D}G = \delta$$

is said to be a Green function for the operator $\mathcal{D}$. Here δ is the Dirac δ distribution in the corresponding number of dimensions. If G is a Green function for $\mathcal{D}$ and φ is a homogeneous solution – that is to say, if $\mathcal{D}\varphi = 0$ – then $G' = G + \varphi$ is also a Green function for $\mathcal{D}$. And vice-versa, if G and G' are Green functions, their difference is a homogeneous solution. To determine the correct Green function for a given problem, the boundary conditions must be specified.

We are interested in the wave (or D'Alambert) operator, $\mathcal{D} = \partial_t^2 - c^2\Delta$. We look then for a solution of the equation

$$(\partial_t^2 - c^2\Delta)G(\mathbf{x}, t) = \delta^3(\mathbf{x})\delta(t) \ . \tag{1.40}$$

We want a solution that decreases for $r \to \infty$ and vanishes for $t < 0$. The last condition assures causality: the field should not be present before the charge that originates it.

To solve (1.40) we first perform a Fourier transform in the spatial coordinates,

$$\begin{aligned} \hat{G}(\mathbf{k}, t) &= \int d^3x e^{i(\mathbf{k}\cdot\mathbf{x})} G(\mathbf{x}, t) \\ G(\mathbf{x}, t) &= \frac{1}{(2\pi)^3} \int d^3k e^{-i(\mathbf{k}\cdot\mathbf{x})} \hat{G}(\mathbf{k}, t) \end{aligned}$$

With $\int \delta^3(\mathbf{x}) e^{i(\mathbf{k}\cdot\mathbf{x})} = 1$, eq. (1.40) becomes

$$(\partial_t^2 + c^2\mathbf{k}^2)\hat{G}(\mathbf{k}, t) = \delta(t) \ . \tag{1.41}$$

It is easily verified that (see exercices)

$$\hat{G}_R(\mathbf{k}, t) = H(t)\frac{\sin(ckt)}{ck} \tag{1.42}$$

is a solution of (1.41). Here $H(t)$ is the Heaviside function,

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \,. \end{cases}$$

If now we use the fact that the Fourier transform of $\delta(|\mathbf{x}| - R)$ is $4\pi R \sin(kR)/k$, we obtain

$$G_R(\mathbf{x}, t) = \frac{H(t)}{4\pi c^2 t}\delta(ct - |\mathbf{x}|) = \frac{H(t)}{4\pi c|\mathbf{x}|}\delta(ct - |\mathbf{x}|) \ . \tag{1.43}$$

This Green function clearly obeys the required boundary conditions. It is possible to show that it is the only Green function with that property[1].

The solution to the problem $(\partial_t^2 - c^2\Delta)\phi(\mathbf{x},t) = f(\mathbf{x},t)$ is then (see exercises)

$$\phi(\mathbf{x},t) = \int d^3x' dt' G_R(\mathbf{x}-\mathbf{x}', t-t') f(\mathbf{x}',t') = \frac{1}{4\pi}\int d^3x' \frac{f(\mathbf{x}', t-|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}\,. \quad (1.44)$$

1.4 Potentials and Gauge transformations

(See Jackson §6.5)

You already know from magnetostatics that the equation $\nabla\cdot\mathbf{B} = 0$ is automatically satisfied if one sets

$$\mathbf{B} = \nabla\wedge\mathbf{A}, \quad (1.45)$$

since $\nabla\cdot(\nabla\wedge\mathbf{A}) \equiv 0$ for any vector $\mathbf{A}$. One can also show that any vector field $\mathbf{v}$ in three dimensions can be split up into an *irrotational* part and a *rotational* part (*spin-0 component* and *spin-1 component*),

$$\mathbf{v} = -\nabla\phi + \nabla\wedge\mathbf{a}\,. \quad (1.46)$$

ϕ and $\mathbf{a}$ can be chosen to be solutions of the following equations

$$\left.\begin{array}{rcl} \Delta\phi & = & -\nabla\cdot\mathbf{v} \\ \Delta\mathbf{a} & = & -\nabla\wedge\mathbf{v} \end{array}\right\}. \quad (1.47)$$

Notee that $\mathbf{a}$ and ϕ are not unique unless $\nabla\cdot\mathbf{a} = 0$ and some *boundary conditions* are imposed.

The law of induction,

$$\nabla\wedge\mathbf{E} + \frac{1}{c}\partial_t\mathbf{B} = \nabla\wedge(\mathbf{E} + \frac{1}{c}\partial_t\mathbf{A}) = 0$$

implies the existence of a scalar ϕ such that

$$\mathbf{E} + \frac{1}{c}\partial_t\mathbf{A} = -\nabla\phi$$

[1] From this one might be led to think that this one is the only Green function of any relevance in physics. The situation, however, is a bit more intricate in quantum mechanics. The most important Green function in quantum mechanics is Feynman's one, which is a superposition of the retarded Green function, G_R, and the advanced Green function, G_A, the latter being obtainable from the former under the replacement of $t - |\mathbf{x}|$ with $t + |\mathbf{x}|$.

or

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\partial_t\mathbf{A}. \tag{1.48}$$

With this *ansatz*, the homogeneous Maxwell equations are automatically satisfied. In terms of our electromagnetic potentials $(\phi, \mathbf{A})$, the inhomogeneous Maxwell equations (for the case $\epsilon = \mu = 1$) give

$$(\nabla \cdot \mathbf{E}) = -\Delta\phi - \frac{1}{c}\nabla \cdot \dot{\mathbf{A}} = 4\pi\rho \tag{1.49}$$

and

$$(\nabla \wedge \mathbf{B} - \frac{1}{c}\partial_t\mathbf{E}) = \nabla(\nabla \cdot \mathbf{A}) - \Delta\mathbf{A} + \frac{1}{c}\nabla\dot{\phi} + \frac{1}{c^2}\ddot{\mathbf{A}} = \frac{4\pi}{c}\mathbf{J}. \tag{1.50}$$

The potentials ϕ and $\mathbf{A}$ are not uniquely fixed by conditions (1.45) and (1.48). If we perform the transformation

$$\mathbf{A} \to \mathbf{A} + \nabla\chi, \tag{1.51}$$

the field $\mathbf{B}$ remains the same. If we farther replace ϕ with

$$\phi \to \phi - \frac{1}{c}\dot{\chi}, \tag{1.52}$$

the field $\mathbf{E}$ is also invariant. Transformations (1.51) and (1.52) are known as *gauge transformations*. Under a gauge transformation

$$\begin{aligned}(\nabla \cdot \mathbf{A} + \frac{1}{c}\dot{\phi}) \quad &\to \quad \Delta\chi - \frac{1}{c^2}\partial_t^2\chi + \nabla \cdot \mathbf{A} + \frac{1}{c}\dot{\phi} \\ &= \quad (\Delta - \frac{1}{c^2}\partial_t^2)\chi + \nabla \cdot \mathbf{A} + \frac{1}{c}\dot{\phi}.\end{aligned}$$

Thus, if one finds a scalar field χ such that

$$(\Delta - \frac{1}{c^2}\partial_t^2)\chi = -(\nabla \cdot \mathbf{A} + \frac{1}{c}\dot{\phi}), \tag{1.53}$$

then the transformed potentials,

$$\tilde{\mathbf{A}} = \mathbf{A} + \nabla\chi \quad \text{and} \quad \tilde{\phi} = \phi - \frac{1}{c}\dot{\chi},$$

satisfy the equation

$$\nabla \cdot \tilde{\mathbf{A}} + \frac{1}{c}\dot{\tilde{\phi}} = 0, \tag{1.54}$$

which is known as the *Lorentz* gauge condition[2].

[2]Since the wave equation for a given source always has solution (see eq. (1.44)), we can always solve (1.53), at least locally.

In the Lorentz gauge (1.54), Maxwell's equations have a particularly simple form because they are completely decoupled (we omit the tilde $\sim$):

$$(\Delta - \frac{1}{c^2}\partial_t^2)\phi = -4\pi\rho \tag{1.55}$$

$$(\Delta - \frac{1}{c^2}\partial_t^2)\mathbf{A} = -\frac{4\pi}{c}\mathbf{J}. \tag{1.56}$$

Equations (1.55) and (1.56) together with condition (1.54) are equivalent to Maxwell's equations (four 1st order equations have been transformed into two 2nd order equations).

From (1.43), equations (1.55) and (1.56) are solved by

$$\phi(\mathbf{x},\, t) = \int d^3x' dt G_R(\mathbf{x}-\mathbf{x}', t-t')\rho(\mathbf{x}', t') = \int \frac{\rho\left(\mathbf{x}',\, t - |\mathbf{x}-\mathbf{x}'|/c\right)}{|\mathbf{x}-\mathbf{x}'|} d^3x' \tag{1.57}$$

and

$$\mathbf{A}(\mathbf{x},\, t) = \int d^3x' dt G_R(\mathbf{x}-\mathbf{x}', t-t')\mathbf{J}(\mathbf{x}', t') = \frac{1}{c}\int \frac{\mathbf{J}\left(\mathbf{x}',\, t - |\mathbf{x}-\mathbf{x}'|/c\right)}{|\mathbf{x}-\mathbf{x}'|} d^3x'. \tag{1.58}$$

1.5 Conservation laws

(See Jackson, §6.8)

Charge conservation

Maxwell's equations imply

$$\dot\rho + \nabla\cdot\mathbf{J} = \nabla\cdot\left(\frac{1}{4\pi}\dot{\mathbf{D}} + \mathbf{J}\right) = \frac{c}{4\pi}\nabla\cdot(\nabla\wedge\mathbf{H}) = 0. \tag{1.59}$$

For an open set $G \subset \mathbb{R}^3$ we then have

$$0 = \int_G (\dot\rho + \nabla\cdot\mathbf{J})d^3x = \frac{d}{dt}Q(G) + \int_{\partial G}\mathbf{J}\cdot\mathbf{e}\,ds, \tag{1.60}$$

which corresponds to the *conservation of charge.*

Energy conservation

By using

$$\begin{aligned} \nabla \wedge \mathbf{E} + \frac{1}{c}\dot{\mathbf{B}} &= 0 \quad \text{and} \\ \nabla \wedge \mathbf{H} - \frac{1}{c}\dot{\mathbf{D}} &= \frac{4\pi}{c}\mathbf{J}, \end{aligned}$$

one can find

$$\mathbf{E} \cdot (\nabla \wedge \mathbf{H}) - \mathbf{H} \cdot (\nabla \wedge \mathbf{E}) = \frac{4\pi}{c}\mathbf{E} \cdot \mathbf{J} + \frac{1}{c}\mathbf{E} \cdot \dot{\mathbf{D}} + \frac{1}{c}\mathbf{H} \cdot \dot{\mathbf{B}}. \tag{1.61}$$

But, in agreement with a simple identity, one has $-\nabla \cdot (\mathbf{A} \wedge \mathbf{B}) = \mathbf{A} \cdot (\nabla \wedge \mathbf{B}) - \mathbf{B} \cdot (\nabla \wedge \mathbf{A})$ and so the left-hand side of (1.61) corresponds to $-\nabla \cdot (\mathbf{E} \wedge \mathbf{H})$. Equation (1.61) leads finally to

$$\frac{1}{4\pi}(\mathbf{E} \cdot \dot{\mathbf{D}} + \mathbf{H} \cdot \dot{\mathbf{B}}) + \nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E}, \tag{1.62}$$

where

$$\mathbf{S} := \frac{c}{4\pi}\mathbf{E} \wedge \mathbf{H} \tag{1.63}$$

is the *Poynting vector*. The relation (1.62) holds in general. Let us now assume that there is an instantaneous and linear relation between $\mathbf{H}$ and $\mathbf{B}$ and between $\mathbf{D}$ and $\mathbf{E}$ such as, for instance, $\mathbf{D} = \varepsilon\mathbf{E}$ with ε time-independent (that is the case for microscopic fields, where $\mathbf{E} = \mathbf{D}$ and $\mathbf{H} = \mathbf{B}$). In that case we have that $\mathbf{E} \cdot \dot{\mathbf{D}} = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D})^{\cdot}$ and $\mathbf{H} \cdot \dot{\mathbf{B}} = \frac{1}{2}(\mathbf{H} \cdot \mathbf{B})^{\cdot}$, and (1.62) can be written in the form

$$\dot{u} + \nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E} \tag{1.64}$$

where

$$u := \frac{1}{8\pi}(\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B}) \ .$$

If we integrate this identity over an open set $G \subset \mathbb{R}^3$, we obtain

$$\frac{d}{dt}\int_G u d^3x + \int_{\partial G} \mathbf{S} \cdot \mathbf{e}\, ds = -\int_G \mathbf{J} \cdot \mathbf{E}\, d^3x. \tag{1.65}$$

The last term is minus the work done on the charges by the field. If the region G and the electromagnetic fields are such that the surface integral can be neglected, a decrease of the integral of u corresponds then to the work performed by the field. This justifies the interpretation of u as the *energy density of the electromagnetic field.*

The first term of the left-hand side of equation (1.64) is then the time derivative of the electromagnetic energy, and the term to the right is the work per unit time performed by the electromagnetic field on the currents *taken with the minus sign*. If $\mathbf{J}$ is parallel to $\mathbf{E}$, the field loses energy because it performs *work* on the currents. $\mathbf{S}$ must then be interpreted as the *energy flux*: equation (1.65) means therefore that the energy of the electric field in G changes depending on the energy flux towards the outside of G and on the work performed on the currents.

If the nonlinear effects are relevant, (1.65) is no longer valid for the macroscopic fields and one has to turn to equation (1.62).

Conservation of momentum

The force density applied on the charges and currents by the electromagnetic field is

$$\mathbf{k} = \rho\mathbf{E} + \frac{1}{c}\mathbf{J} \wedge \mathbf{B}.$$

We get rid of ρ and $\mathbf{J}$ by means of Maxwell's equations (Coulomb and Ampère's laws) to obtain

$$4\pi\mathbf{k} = \mathbf{E}(\nabla \cdot \mathbf{D}) + (\nabla \wedge \mathbf{H} - \frac{1}{c}\dot{\mathbf{D}}) \wedge \mathbf{B}.$$

To this equation we can add the vanishing term

$$0 = \mathbf{H}(\nabla \cdot \mathbf{B}) - \mathbf{D} \wedge (\nabla \wedge \mathbf{E} + \frac{1}{c}\dot{\mathbf{B}})$$

which gives

$$4\pi\mathbf{k} = [\mathbf{E}(\nabla \cdot \mathbf{D}) - \mathbf{D} \wedge (\nabla \wedge \mathbf{E})] + [\mathbf{H}(\nabla \cdot \mathbf{B}) - \mathbf{B} \wedge (\nabla \wedge \mathbf{H})] - \frac{1}{c}\partial_t(\mathbf{D} \wedge \mathbf{B}).$$

Let us first consider the macroscopic theory, where

$$\mathbf{D} \equiv \mathbf{E} \quad \text{et} \quad \mathbf{H} \equiv \mathbf{B}.$$

With $\mathbf{\Pi} := \frac{1}{c^2}\mathbf{S} = \frac{1}{4\pi c}(\mathbf{E} \wedge \mathbf{B})$ we find that

$$\dot{\mathbf{\Pi}} = -\mathbf{k} + \frac{1}{4\pi}\left[\mathbf{E}(\nabla \cdot \mathbf{E}) - \mathbf{E} \wedge (\nabla \wedge \mathbf{E}) + \mathbf{B}(\nabla \cdot \mathbf{B}) - \mathbf{B} \wedge (\nabla \wedge \mathbf{B})\right]. \tag{1.66}$$

Let us compute the i-th component of the expression for $\mathbf{E}$ inside [ ]. In what follows, we won't write the summation symbol $\sum$ but will use instead the convention according to which a sum from 1 to 3 is implied for any repeated indices in a term (Einstein's convention).
For example:

$$\begin{aligned} \nabla \cdot \mathbf{A} &= \sum_{i=1}^{3} \partial_i A_i \equiv \partial_i A_i \\ (\nabla \wedge \mathbf{A})_i &= \sum_{j,\ell=1}^{3} \varepsilon_{ij\ell}\partial_j A_\ell \equiv \varepsilon_{ij\ell}\partial_j A_\ell. \end{aligned}$$

We will also use the identities (the proof is left as an exercise)

$$\varepsilon_{i\ell m} = \varepsilon_{\ell m i} = \varepsilon_{m i \ell},$$

$$\varepsilon_{i\ell m}\varepsilon_{ijk} = \delta_{\ell j}\delta_{mk} - \delta_{\ell k}\delta_{jm}.$$

With this we have that

$$\begin{aligned}
[\mathbf{E}(\nabla\cdot\mathbf{E}) - \mathbf{E}\wedge(\nabla\wedge\mathbf{E})]_i &= E_i\partial_j E_j - \underbrace{\varepsilon_{ijk}E_j\varepsilon_{k\ell m}}_{(\delta_{\ell i}\delta_{jm}-\delta_{im}\delta_{j\ell})E_j}\partial_\ell E_m \\
&= E_i\partial_j E_j - E_j\partial_i E_j + E_j\partial_j E_i \\
&= \partial_j[E_iE_j - \frac{1}{2}\delta_{ij}\mathbf{E}^2].
\end{aligned}$$

Since the expression for $\mathbf{B}$ is completely analogous, we arrive at the relation

$$\begin{aligned}
\frac{1}{4\pi}[\mathbf{E}\cdot(\nabla\cdot\mathbf{E}) - \mathbf{E}\wedge(\nabla\wedge\mathbf{E}) + \mathbf{B}(\nabla\cdot\mathbf{B}) - \mathbf{B}\wedge(\nabla\wedge\mathbf{B})]_i &= \\
\frac{1}{4\pi}\partial_j(E_iE_j - \frac{1}{2}\delta_{ij}\mathbf{E}^2 + B_iB_j - \frac{1}{2}\delta_{ij}\mathbf{B}^2) &= -\partial_j T_{ij}
\end{aligned}$$

with

$$T_{ij} := \frac{1}{4\pi}[\frac{1}{2}\delta_{ij}(\mathbf{E}^2+\mathbf{B}^2) - E_iE_j - B_iB_j]. \tag{1.67}$$

Finally, equation (1.66) reduces to

$$\dot{\Pi}_i + \partial_j T_{ij} = -k_i\ .$$

Using the fact that $\int_G \mathbf{k} = \frac{d}{dt}\mathbf{P}^{(\mathrm{mec})}$, one finds

$$\int_G (\dot{\mathbf{\Pi}} + \mathbf{k})_i\, d^3x = \frac{d}{dt}\left(P_i^{(\mathrm{field})} + P_i^{(\mathrm{mec})}\right) = \int_{\partial G} T_{ik}e^k ds. \tag{1.68}$$

If this expression is interpreted as the equation of *conservation of momentum* in an open set G, then $\int_G \mathbf{\Pi}\, d^3x$ represents the *momentum of the electromagnetic field.* In the right-hand side, $T_{ik}e^k$ is the flow per unit area of the field's force, and $P_i^{(\mathrm{field})}$ is the i-th component of the momentum of the electromagnetic field in G. From (1.63),

$$\frac{1}{c^2}\mathbf{S} = \frac{1}{4\pi c}\mathbf{E}\wedge\mathbf{H}. \tag{1.69}$$

In a linear and isotropic medium where $\mathbf{D} = \varepsilon\mathbf{E}$ and $\mathbf{H} = \frac{1}{\mu}\mathbf{B}$ with $\varepsilon = $ const. and $\mu = $ const., one arrives at equation (1.68) with $\mathbf{\Pi}$ and T_{ij} given by

$$\mathbf{\Pi} = \frac{1}{4\pi c}\mathbf{D}\wedge\mathbf{B} = \frac{\varepsilon\mu}{c^2}\mathbf{S}$$

and

$$T_{ij} = \frac{1}{4\pi}[\frac{1}{2}\delta_{ij}(\mathbf{E}\cdot\mathbf{D}+\mathbf{B}\cdot\mathbf{H}) - E_i D_j - H_i B_j].$$

If ε and μ are position-dependent (but time-independent!) the supplementary forces $\mathbf{k}_E = -\frac{1}{8\pi}\mathbf{E}^2\nabla\varepsilon$ and $\mathbf{k}_M = \frac{1}{8\pi}\mathbf{H}^2\nabla\mu$ must be added.

The conservation of angular momentum can be discussed along similar lines:

$$\frac{d}{dt}(\mathbf{L}^{(\text{field})} + \mathbf{L}^{(\text{mec})})_i = -\int_{\partial G} M_{ij} e^j \, ds$$

or

$$\partial_t(\mathcal{L}^{(\text{field})} + \mathcal{L}^{(\text{mec})})_i = -\partial_j M_{ji} \; . \tag{1.70}$$

If the electric and magnetic susceptibilities can be neglected, $\epsilon \equiv \mu \equiv 1$, one finds (1.70) with

$$\begin{aligned}
\mathcal{L}_i^{(\text{field})} &= \frac{1}{4\pi c}\left(\mathbf{x}\wedge(\mathbf{E}\wedge\mathbf{B})\right)_i \\
&= (\mathbf{x}\wedge\mathbf{\Pi})_i \\
\mathcal{L}^{(\text{mec})} &= \mathbf{x}\wedge\mathbf{p} \\
L_i^{(\cdot)} &= \int_G \mathcal{L}_i \, d^3x \\
M_{ij} &= \varepsilon_{j\ell m} T_{i\ell} x_m .
\end{aligned}$$

The proof of equation (1.70) is left as an *exercise.* M_{ij} is a rank-2 (pseudo) tensor that describes the flow of angular momentum of the electromagnetic field.

1.6 The relativistic formalism of electrodynamics

(Jackson, chap. 11)

Lorentz transformations

The fundamental principle of special relativity is the equivalence of all laws of physics in every inertial reference frame. A particle that does not experience any force moves with constant velocity with respect to an inertial frame. (This is often used as the definition of inertial frames. However, in order to establish the definition unambiguously, one would have to know a priori all the forces, which are in general defined as via their acceleration of a particle as perceived by an inertial observer...) I do not know of any fully satisfactory definition of inertial frame, but it is clear that if one inertial frame is given, all the others are obtained through boosts (see below), rotations, reflections and time reversal transformations. (In reality, not all physics laws are invariant under the latter discrete transformations. The weak interactions, responsible for the β–decay of unstable isotopes, do not possess the reflection and time reversal invariances!)

If Σ is an inertial reference frame, then Σ', which moves with constant speed along a straight line with respect to Σ (boost) or has an arbitrary but fixed rotation with respect to Σ, is also an inertial frame. A physicist in the reference frame Σ obtains the same results as her colleague in Σ' when they perform the same experiment. This result was originally formulated by Galilei (principle of Galilei), who postulated that the laws of mechanics are invariant under the transformation:

$$\begin{array}{ccc} \Sigma & \rightarrow & \Sigma' \\ \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} & \rightarrow & \begin{pmatrix} t' \\ \mathbf{x}' \end{pmatrix} = \begin{pmatrix} t \\ \mathbf{x} - t\mathbf{v} \end{pmatrix} \end{array} \qquad \text{(Galilei transformation).} \tag{1.71}$$

Here Σ and Σ' are two inertial frames that coincide at $t = 0$, and Σ' moves with velocity $\mathbf{v}$ with respect to Σ. If a particle has the velocity $\mathbf{u} = d\mathbf{x}/dt$ in Σ, then it has the velocity $\mathbf{u}' = d\mathbf{x}'/dt' = \mathbf{u} - \mathbf{v}$ in Σ'.

According to Maxwell's equations, the invariance of the laws of physics implies that the speed of light is the same in every inertial frame. It is precisely this condition that Einstein wanted to implement with new transformations different from (1.71). New transformations are certainly necessary since we want $c' = c \neq c - v$.

A simple but significant consequence of the constancy of the speed of light is the relativeness of simultaneity. To illustrate this, let us consider a lighting flash that takes place halfway between two screens fixed on either side. For an observer at rest with respect to the experiment, both screens will light up at the same time. An observer moving to the right perceives the right screen as moving towards the source of the flash and so, according to her, the photons must travel a shorter path to arrive at the right screen than to do so at the left one. Since the speed of light is the same for her, the right screen lights up before the left screen, and therefore the events are not seen as simultaneous in her reference frame. This indicates that the new transformations will transform not only $\mathbf{x}$ but also the time.

We now want to determine how the coordinates t and $\mathbf{x}$ are modified under a change of the frame of reference. For this we just use the fact that the speed of light c which comes out from Maxwell's equations is the same in every frame. The equation $(ct)^2 - \mathbf{x}^2 = 0$ is then the same independently of the frame of reference. This equation is also invariant if one of the frames is dilated with respect to the other, $t' = \alpha t$, $\mathbf{x}' = \alpha\mathbf{x}$ by an arbitrary constant $\alpha \in \mathbb{R}$. But we know that the physics is not invariant under dilations (for example, the spectral lines of the hydrogen atom are the same in every inertial reference frame). To get rid of these dilations that are in disagreement with phenomena, we assume that $(ct)^2 - \mathbf{x}^2$ is frame-independent for any value of this invariant[3]. We also require transformation to be linear and we consider Σ' to be moving with speed $v = \text{const.}$ in the direction $x = x^1$. We assume

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \alpha(t - c^{-1}\delta x) \\ \gamma(x - c\beta t) \\ y \\ z \end{pmatrix} \tag{1.72}$$

and we demand that $c^2t^2 - x^2 = c^2t'^2 - x'^2$. If we insert the expressions of (1.72) in $(ct)^2 - \mathbf{x}^2$, we find

$$c^2t^2 - x^2 = c^2\alpha^2t^2 - 2c\alpha\delta tx + \alpha^2\delta^2x^2 - \gamma^2x^2 + 2c\gamma\beta xt - c^2\gamma^2\beta^2t^2 \quad \forall x, t\,.$$

Requiring that the coefficients of the quadratic expressions on both sides be the same, we obtain $\alpha = \gamma$, $\beta = \delta$ and

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}\,. \tag{1.73}$$

The dimensionless, discontinuous function $\beta(v)$ remains. At low speeds, $v \ll c$, we want to restore the result of Galilei, which implies $\beta = v/c + \mathcal{O}((v/c)^2)$. A boost of velocity v in direction x then transforms

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \equiv \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \to \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \Lambda \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}, \tag{1.74}$$

with

$$\Lambda = (\Lambda^\mu{}_\nu) = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.75}$$

[3] It is a mathematical exercise to show that the transformations that preserve $(ct)^2 - \mathbf{x}^2 = 0$ are precisely those that keep $(ct)^2 - \mathbf{x}^2 = s^2$ invariant for any value s and dilations. If only linear transformations are considered, the exercise is simple. It is striking that the same can be said even if non-linear (but differentiable) transformations are considered; nevertheless, the proof becomes much more difficult. This is Alexandrov's theorem (1975) (See: W. Benz, 'Geometrische Transformationen', BI Wissenschaftsverlag, 1992).

The inverse boost is the one with velocity $-v$, and it is easy to verify that the matrix $\Lambda^{-1}(\beta) = \Lambda(-\beta)$. Therefore $\beta(-v) = -\beta(v)$ and β cannot have terms quadratic in v/c. It is an experimental fact that β has no corrections, i.e.

$$\beta = v/c\,.$$

We define the four-vector

$$x = (x^\mu) = (ct, \mathbf{x}), \quad x^0 = ct,\ x^i = x^i \text{ for } i = 1, 2, 3$$

and the 'metric'

$$g \equiv (g_{\mu\nu}) \equiv \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = (g_{\mu\nu})^{-1} \equiv (g^{\mu\nu}) \equiv g^{-1}\ . \tag{1.76}$$

In this course we have chosen the metric signature $(-, +, +, +)$, and we denote with $(g^{\mu\nu})$ the inverse of the metric.

One therefore has that $(ct)^2 - \mathbf{x}^2 = -x^\mu g_{\mu\nu} x^\nu$ is independent of the inertial frame[4].

$$x^\mu g_{\mu\nu} y^\nu = [(x+y)^\mu g_{\mu\nu}(x+y)^\nu - (x-y)^\mu g_{\mu\nu}(x-y)^\nu]/4\ ,$$

it follows that $x^\mu g_{\mu\nu} y^\nu$ is frame-invariant for every couple of four-vectors $(x^\mu), (y^\nu)$. For a boost,

$$(x')^\mu = \Lambda^\mu{}_\nu x^\nu\ ,$$

this implies that $\Lambda^\alpha{}_\nu x^\nu g_{\alpha\beta} \Lambda^\beta{}_\mu y^\mu = x^\nu g_{\nu\mu} y^\mu$ for any four-vectors $(x^\mu), (y^\nu) \in \mathbb{R}^4$, and so

$$\Lambda^\alpha{}_\nu g_{\alpha\beta} \Lambda^\beta{}_\mu = g_{\nu\mu}\ . \tag{1.77}$$

In matrix notation (1.77) translates into $\Lambda^T g \Lambda = g$. Apart from the boost in direction x^1, it is clear that rotations and reflections, with

$$\Lambda(R) \equiv \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}$$

where $RR^T = \mathbf{1}$, i.e, $R \in O(3)$, also satisfy condition (1.77).

A boost in an arbitrary direction can be decomposed into a rotation followed by a boost in direction x^1 and the inverse rotation. It is possible to show that a matrix Λ that satisfies $\Lambda^T g \Lambda = g$ with $\det(\Lambda) = 1$ and $\Lambda^0_0 \geq 1$ can be decomposed into a boost and a rotation.

[4]A sum from 0 to 3 must be performed over any Greek index appearing twice.

For all four-vector (v^μ) that transforms as $v^\mu \to \Lambda^\mu{}_\nu v^\nu$, eq (1.77) implies that $v^2 \equiv g_{\mu\nu} v^\mu v^\nu$ is invariant under Lorentz transformations. We require $v_\mu \equiv g_{\mu\nu} v^\nu$, such that $v^2 = v_\mu v^\mu$ is invariant under Lorentz transformations.

Examples
$(v^\mu) = (x^\mu) = (ct, \mathbf{x})$. So $v^2 = -c^2t^2 + \mathbf{x}^2 =$ constant is the equation of motion of a light ray. That it is independent of the reference system is the basis of special relativity.
$(v^\mu) = (p^\mu) = (\varepsilon/c, \mathbf{p})$. So $-\varepsilon^2/c^2 + p^2 = -m^2 =$ constant. Here ε is the energy of the particle (see the Mechanics I course). $(p^\mu) = (\frac{\varepsilon}{c}, \mathbf{p})$ is the energy-momentum four-vector of the particle.

$$\begin{aligned} (p^\mu) &= (mu^\mu) = (m\gamma c, m\gamma \mathbf{v}) \\ p^2 &= -(p^0)^2 + \mathbf{p}^2 = -m^2c^2; \end{aligned}$$

here m denotes the invaiant rest mass of the particle.

Multiplying eq. (1.77) by $(\Lambda^{-1})^\mu{}_\sigma$ from the right and by $g^{\rho\nu}$ from the left one finds

$$(\Lambda^{-1})^\rho{}_\sigma = g^{\rho\beta} \Lambda^\nu{}_\beta g_{\nu\sigma}. \tag{1.78}$$

In matrix notation this is $\Lambda^{-1} = g^{-1} \Lambda^T g$. With eq. (1.78) we find that

$$v'_\mu = g_{\mu\nu} \Lambda^\nu{}_\sigma v^\sigma = g_{\mu\nu} \Lambda^\nu{}_\sigma g^{\sigma\rho} v_\rho = ((\Lambda^{-1})^T)_\mu{}^\rho v_\rho \,.$$

Frequently we denote

$$g_{\mu\nu} \Lambda^\nu{}_\sigma g^{\sigma\rho} \equiv \Lambda_\mu{}^\rho \equiv ((\Lambda^{-1})^T)_\mu{}^\rho \,, \qquad \text{such that} \qquad v'_\mu = \Lambda_\mu{}^\nu v_\nu \,. \tag{1.79}$$

The four-vectors (v^μ) are called *contravariant* vectors; they transform with $\Lambda = (\Lambda^\mu{}_\nu)$ under Lorentz transformations. In contrast, the four-vectors (v_μ) are called *covariant* vectors, and they transform with $(\Lambda^{-1})^T = (\Lambda_\mu{}^\nu)$ under Lorentz transformations.

We consider not only four-vectors but also tensors, for example $T_{\mu\nu}$, such that $T_{\mu\nu} v^\mu w^\nu$ is an scalar, hence invariant under Lorentz transformations for arbitrary four-vectors (v^μ) and (w^ν). With $v'^\mu = \Lambda^\mu{}_\nu v^\nu$ and similarly for w^ν we have

$$T'_{\mu\nu} \Lambda^\mu{}_\alpha v^\alpha \Lambda^\nu{}_\beta w^\beta = T_{\mu\nu} v^\mu w^\nu \qquad \text{therefore} \qquad T'_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = T_{\mu\nu} \,. \tag{1.80}$$

Multiplying by $(\Lambda^{-1})^\alpha{}_\rho (\Lambda^{-1})^\beta{}_\sigma$ using (1.79), we find

$$T'_{\rho\sigma} = \Lambda_\rho{}^\alpha \Lambda_\sigma{}^\beta T_{\alpha\beta} \,. \tag{1.81}$$

For a four-tensor of rank n we define the general way to "raise and lower indices":

$$T_{\mu_1}{}^{\mu_2 \cdots \mu_n} = g_{\mu_1 \nu} T^{\nu \mu_2 \cdots \mu_n} \;,$$

which implies

$$T^{\mu_1\cdots\mu_n} = g^{\mu_1\nu}T_\nu{}^{\mu_2\cdots\mu_n} \ ,$$

and analogously for the other indices. For a four-tensor every lower index transforms then with $(\Lambda_\mu{}^\nu)$ while upper indices transform with $(\Lambda^\mu{}_\nu)$ under a Lorentz transformation.

If we have an equation of the form

$$T^{\mu_1\mu_2\cdots\mu_n} = S^{\mu_1\mu_2\cdots\mu_n}$$

then this equation remains valid in all frames. On the other hand, if we want an equation to keep the same form in any frame, then this equation must be written in terms of four-tensors. According to *special relativity*, the fundamental laws of nature are valid in any frame and so they must be expressed in terms of four-tensors (this is often called *covariantly*).

Proper time We consider a particle or an object that moves with speed $\mathbf{v}$ in Σ. In a time interval dt the particle moves a distance $d\mathbf{x} = \mathbf{v}dt$. The invariance of the speed of light implies that

$$c^2d\tau^2 = ds^2 = c^2dt^2 - d\mathbf{x}^2 = c^2(1-\beta^2)dt^2 = \frac{c^2}{\gamma^2}dt^2 \,, \qquad \beta^2 = \mathbf{v}^2/c^2$$

is the same in all frames. In the reference system in which the particle is (momentarily) at rest this gives $d\tau^2 = c^{-2}ds^2 = dt'^2$, and so one calls τ the proper time of the particle. The proper time is independent of the frame, it is a Lorentz scalar. In a generic frame $dt = \gamma d\tau$. The time that elapses between two proper times τ_1 and τ_2 is then

$$t_2 - t_1 = \int_{t_1}^{t_2} dt = \int_{\tau_1}^{\tau_2} \gamma d\tau \geq \tau_2 - \tau_1 \,. \tag{1.82}$$

So a watch that moves is slower than a watch at rest. This fact has been experimentally tested not only with elementary particles whose lifetime becomes longer in accelerators, but also with very precise watches in aircrafts.

Addition of velocities We consider once again the system Σ' that moves with speed v in direction x^1 with respect to Σ. For coordinate differentials we then have

$$\begin{pmatrix} dx^0 \\ dx^1 \\ dx^2 \\ dx^3 \end{pmatrix} = \begin{pmatrix} \gamma_v(dx'^0 + \beta dx'^1) \\ \gamma_v(dx'^1 + \beta dx'^0) \\ dx'^2 \\ dx'^3 \end{pmatrix} \qquad \gamma_v = \frac{1}{\sqrt{1 - v^2/c^2}} \,. \tag{1.83}$$

Let us consider a particle that moves with speed $\mathbf{u}' = d\mathbf{x}'/dt' = cd\mathbf{x}'/dx'^0$ in Σ'. We assume that the part of the velocity that is normal to x'^1 is parallel to $y' = x'^2$. Its velocity in Σ is then

$$\begin{aligned} u_\parallel &\equiv u^1 = \frac{dx^1}{dt} = c\frac{dx'^1 + \beta dx'^0}{dx'^0 + \beta dx'^1} = \frac{u'^1 + c\beta}{1 + \beta u'^1/c} = \frac{u'_\parallel + v}{1 + (vu'_\parallel)/c^2} = \frac{u'_\parallel + v}{1 + (\mathbf{v}\cdot\mathbf{u}')/c^2} \\ u_\perp &\equiv u^2 = \frac{dx^2}{dt} = \frac{c}{\gamma_v}\frac{dx'^2}{dx'^0 + \beta dx'^1} = \frac{1}{\gamma_v}\frac{u'_\perp}{1 + (vu'_\parallel)/c^2} = \frac{1}{\gamma_v}\frac{u'_\perp}{1 + (\mathbf{v}\cdot\mathbf{u}')/c^2}\,. \end{aligned}$$

This expression does not look symmetrical in v and u'. But it is easy to show that $\gamma_u = \gamma_v\gamma_{u'}\left(1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)$ and so the above equations become

$$\begin{aligned} \gamma_u u_\parallel &= \gamma_v\gamma_{u'}(u'_\parallel + v) \\ \gamma_u u_\perp &= \gamma_{u'}u'_\perp\,, \end{aligned}$$

which is perfectly symmetric in v and u'.

Formulated as an exercise, the following part is addressed to the students of theoretical physics:

Definition: The set of matrices $\left\{\Lambda \in \mathbb{R}^{4\times4} | \Lambda g \Lambda^T = g\right\} \equiv \mathrm{O}(1,3) \equiv L$, with $g = \mathrm{diag}(-1,1,1,1)$ is the Lorentz group.

Show the following statements.

- $\Lambda^T g \Lambda = g$ if Λ is a rotation or a boost of direction x^1 and if $\Lambda = T = \mathrm{diag}(-1,1,1,1)$ or $\Lambda = P = \mathrm{diag}(1,-1,-1,-1)$. (i.e. rotations, boosts of direction x^1, T (time reversal) and P (parity) are all elements of L.)
- L is a genuine group in the mathematical sense of the term.
- For $\Lambda \in \mathrm{O}(1,3)$, $|\det\Lambda| = 1$ and $|\Lambda^0_0| \geqslant 1$.
- $\mathrm{O}(1,3) \subset \mathbb{R}^{4\times4} \cong \mathbb{R}^{16}$ is composed by four disconnected components:

$$\begin{aligned} L^\uparrow_+ &= \{\Lambda \in \mathrm{O}(1,3) |\det\Lambda = 1,\ \Lambda^0_0 \geqslant 1\} \\ L^\uparrow_- &= \{\Lambda \in \mathrm{O}(1,3) |\det\Lambda = -1,\ \Lambda^0_0 \geqslant 1\} \\ L^\downarrow_- &= \{\Lambda \in \mathrm{O}(1,3) |\det\Lambda = -1,\ \Lambda^0_0 \leqslant -1\} \\ L^\downarrow_+ &= \{\Lambda \in \mathrm{O}(1,3) |\det\Lambda = 1,\ \Lambda^0_0 \leqslant -1\}. \end{aligned}$$

- In a neighbourhood of identity, the group $O(1,3)$ is parametrised by six parameters, i.e the Lorentz group is 6-dimensional.

Boost and rotations compose a group of matrices referred to as $L_+^\uparrow$, the *proper Lorentz group.* In terms of *time inversion*, T, and parity, P. The complete Lorentz group is given by

$$\begin{aligned} L &= L_+^\uparrow \oplus PL_+^\uparrow \oplus TL_+^\uparrow \oplus PTL_+^\uparrow \\ &= L_+^\uparrow \oplus L_-^\uparrow \oplus L_-^\downarrow \oplus L_+^\downarrow \end{aligned}$$

Here $\pm$ accounts for the sign of the determinant, and $\uparrow$ or $\downarrow$ denotes ${\Lambda^0}_0 \geqslant 1$ or $\Lambda_0^0 \leqslant -1$, respectively.

Covariant electrodynamics

We then want to find a covariant (or relativistic) formulation of the laws of electrodynamics.

To find a relativistic formulation of the conservation of charge we first write $(j^\mu) = (c\rho,\, \mathbf{J})$. Considering that $(x^\mu) = (ct,\, \mathbf{x})$, then it is clear that the partial derivatives $(\partial_\mu) \equiv (\frac{1}{c}\partial_t,\, \partial_i)$ also constitute a four-vector. With these definitions, the equation for the conservation of charge, eq. (1.59), can be rewritten as

$$\partial_\mu j^\mu = 0 \qquad (\dot\rho + \nabla\cdot\mathbf{J} = 0). \tag{1.84}$$

Since 0 is a Lorentz-invariant scalar, this implies that (j^μ) is also a four-vector.

We would like to find a covariant form for Maxwell's equations in $\mathbf{E}$ and $\mathbf{B}$. With that in mind, let us first consider the static case where ρ is constant and $\mathbf{J}$ vanishes. We will see that requesting *covariance* will completely determine the full Maxwell equations. The equations of electrostatics are

$$\nabla\wedge\mathbf{E} = 0 \;\text{ and }\; \nabla\cdot\mathbf{E} = 4\pi\rho.$$

To promote Coulomb's equation to a covariant equation one has to notice that ρ is the ^{0}-component of the four-vector j^μ. The desired equation must then be of the form

$$-(\partial\cdot F)^\mu = \frac{4\pi}{c} j^\mu.$$

Here $\partial\cdot F$ is the divergence of a rank-2 tensor field that is linear in $\mathbf{E}$. (The sign $-$ is simply a convention.) We therefore have

$$-\partial_\nu F^{\nu\mu} = \frac{4\pi}{c} j^\mu. \tag{1.85}$$

In addition, we want to make sure that charge is conserved, $\partial_\mu j^\mu = 0$, which implies that

$$\partial_\mu \partial_\nu F^{\mu\nu} = 0.$$

This drives us to postulate an antisymmetric tensor $F^{\mu\nu}$,

$$F^{\mu\nu} = -F^{\nu\mu}.$$

(We could add to $F_{\mu\nu}$ a symmetric part $S_{\mu\nu}$ that satisfies $\partial_\mu \partial_\nu S^{\mu\nu} = 0$, but $S^{\mu\nu}$ is entirely decoupled from the antisymmetric part and from the source (j^μ). It is therefore consistent with Maxwell's equations to fix $S_{\mu\nu} \equiv 0$, which we do also for reasons of economy.)

A four dimensional antisymmetric tensor has 6 independent components. Because of (1.85) for $\mu = 0$, the components F^{i0} are to be identified with the electric field. We then write

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}, \tag{1.86}$$

where the interpretation of the components B_i is not clear *a priori*. This translates into the relations

$$\left.\begin{array}{l} F^{i0} = -E^i = -F^{0i} \\ F^{ij} = F_{ij} = \varepsilon_{ij\ell} B_\ell = -F_{ji} \end{array}\right\}. \tag{1.87}$$

With this, $-\partial_\mu F^{\mu 0} = \frac{4\pi}{c} j^0$ is equivalent to Coulomb's law.

By making $\mathbf{B} = (B_1, B_2, B_3)$, equation (1.85) for $\mu = i$ gives

$$\begin{aligned} \partial_0 F^{0i} + \partial_\ell F^{\ell i} &= -\frac{4\pi}{c} j^i \\ \frac{1}{c}\dot{E}^i + \underbrace{\underbrace{\partial_\ell \varepsilon^{\ell i j} B_j}_{-\varepsilon^{i\ell j}\partial_\ell B_j}}_{-(\nabla\wedge\mathbf{B})^i} &= -\frac{4\pi}{c} j^i \\ -\frac{1}{c}\dot{\mathbf{E}} + \nabla \wedge \mathbf{B} &= \frac{4\pi}{c}\mathbf{J}, \end{aligned}$$

that is nothing but Ampère's law if we interpret $\mathbf{B}$ as the induction field. The existence of an induction field $\mathbf{B}$ is then a pure consequence of the *covariance* of electrodynamics! This is a clear example of the power of symmetries in physics: the requirement that equations are to be the same in any inertial frame, i.e. that they are to be covariant, leads us from electrostatics to the dynamic Maxwell equations and implies the existence of magnetic induction. The tensor $F^{\mu\nu}$ is called the electromagnetic field tensor or the 'Faraday tensor'.
(Historically, of course, this is not how things happened!)

To find the generalisation of the homogeneous static equation, $\nabla \wedge \mathbf{E} = 0$, we first define the dual tensor of $F^{\mu\nu}$,

$$*F^{\mu\nu} := \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}, \tag{1.88}$$

where

$$\epsilon^{\mu\nu\alpha\beta} = \begin{cases} \mathrm{sgn}(0,1,2,3 \mapsto \mu\nu\alpha\beta) & \text{if } \mu,\nu,\alpha,\beta \text{ are all different} \\ 0 & \text{otherwise.} \end{cases}$$

$$(*F^{\mu\nu}) = \begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & -E_3 & E_2 \\ -B_2 & E_3 & 0 & -E_1 \\ -B_3 & -E_2 & E_1 & 0 \end{pmatrix}. \tag{1.89}$$

From this representation it follows that $F^{\mu\nu} \to *F^{\mu\nu}$ if $(\mathbf{E},\mathbf{B}) \to (\mathbf{B}, -\mathbf{E})$.

We therefore have

$$(*F)^{ij} = \frac{1}{2}(\epsilon^{ij\ell 0}F_{\ell 0} + \epsilon^{ij0\ell}F_{0\ell}) = -\varepsilon^{ij\ell}E_\ell$$

and $0 = (\nabla \wedge \mathbf{E})_i = \varepsilon_{ij\ell}\partial_j E_\ell$ is equivalent to

$$0 = (\nabla \wedge \mathbf{E})_i = -\partial_j(*F)^{ij}.$$

The covariant generalisation of this equation is, evidently,

$$\partial_\mu(*F)^{\nu\mu} = 0, \tag{1.90}$$

which is equivalent to the homogeneous Maxwell equations (exercise). In conclusion: Maxwell's equations are compatible with special relativity and their explicitly covariant form is

$$\begin{aligned} \partial_\nu(*F^{\nu\mu}) &= 0 && \text{(homogeneous equations),} \\ \partial_\nu F^{\nu\mu} &= -\tfrac{4\pi}{c}j^\mu && \text{(inhomogeneous equations).} \end{aligned}$$

The homogeneous equations, (1.90), are equivalent to

$$F_{\alpha\beta,\nu} + F_{\beta\nu,\alpha} + F_{\nu\alpha,\beta} = 0$$

(show it as an exercise). In 4 dimensions this implies (*Poincaré's Lemma*) the existence of a potential A_μ such that

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha. \tag{1.91}$$

We make $(A^\mu) = (\phi, \mathbf{A})$, which gives

$$\begin{aligned} E_i &= F_{i0} = -\partial_i \phi - \frac{1}{c}\partial_t A_i \\ B_\ell &= \frac{1}{2}\varepsilon_\ell{}^{ij} F_{ij} = \frac{1}{2}\varepsilon_\ell{}^{ij}(\partial_i A_j - \partial_j A_i) \\ &= (\nabla \wedge \mathbf{A})_\ell . \end{aligned}$$

The *ansatz* (1.91) automatically satisfies the homogeneous Maxwell's equations. The inhomogeneous equations become

$$\begin{aligned} \partial^\mu F_{\mu\nu} = \partial^\mu \partial_\mu A_\nu - \partial^\mu \partial_\nu A_\mu &= -\frac{4\pi}{c} j_\nu \\ -\Box A_\nu - \partial^\mu \partial_\nu A_\mu &= -\frac{4\pi}{c} j_\nu . \end{aligned} \tag{1.92}$$

Here we have used

$$\Box = -\partial_\mu \partial^\mu = \frac{1}{c^2}\partial_t^2 - \Delta,$$

the d'Alembertian or wave operator. A gauge transformation is now a transformation of the form:

$$A_\mu \to A_\mu + \partial_\mu \chi$$

and the Lorentz gauge condition is simply (cf. eq. (1.54))

$$\partial_\mu A^\mu = 0. \tag{1.93}$$

Within the Lorentz gauge, the inhomogeneous equations (1.92) give

$$\Box A^\mu = \frac{4\pi}{c} j^\mu . \tag{1.94}$$

In vacuum, $j^\mu = 0$, we have that (within the Lorentz gauge)

$$\Box F_{\mu\nu} = \Box(\partial_\mu A_\nu - \partial_\nu A_\mu) = \partial_\mu \Box A_\nu - \partial_\nu \Box A_\mu = 0.$$

Given that $F_{\mu\nu}$ is independent of the gauge, this result is valid in any gauge whatsoever. Indeed, without having to turn to A_μ, we have that

$$\begin{aligned} 0 &= \partial^\alpha(\partial_\alpha F_{\mu\nu} + \partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu}) \\ &= -\Box F_{\mu\nu} + \partial_\mu \underbrace{\partial^\alpha F_{\nu\alpha}}_{0} + \partial_\nu \underbrace{\partial^\alpha F_{\alpha\mu}}_{0} . \end{aligned}$$

Then

$$\Box F_{\mu\nu} = 0 \quad (\text{for } j^\mu = 0). \tag{1.95}$$

In vacuum, every component of the electromagnetic field satisfies the wave equation.

Lorentz transformations of the electromagnetic field

$F^{\mu\nu}$ is a second rank tensor. Under a Lorentz transformation Λ, it then transforms as

$$(F^{\mu\nu})' = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F^{\alpha\beta}. \tag{1.96}$$

We first consider a rotation,

$$(\Lambda^\mu{}_\nu) = \left(\begin{array}{c|ccc} 1 & & 0 & \\ \hline & & & \\ 0 & & R & \\ & & & \end{array}\right), \tag{1.97}$$

with $R \cdot R^T = 1\!\!1$ and $\det R = 1$ ($R \in SO(3)$). Since **E** and **B** are three-dimensional vectors, (1.96) must imply $(E^i)' = R^i{}_j E^j$, and similarly for **B**. We want to verify this:

$$(E^i)' = (-F^{i0})' = -\Lambda^i{}_\alpha \Lambda^0{}_\beta F^{\alpha\beta} = -\Lambda^i{}_j F^{j0} = R^i{}_j E^j;$$

$$\begin{aligned} F^{ij} &= \varepsilon^{ij}{}_\ell B^\ell \;\Rightarrow\; \varepsilon^k{}_{ij} F^{ij} = \varepsilon^k{}_{ij}\varepsilon^{ij}{}_\ell B^\ell = 2B^k, \\ (B^k)' &= \frac{1}{2}\varepsilon^k{}_{ij}(F^{ij})' = \frac{1}{2}\varepsilon^k{}_{ij}\Lambda^i{}_\alpha \Lambda^j{}_\beta F^{\alpha\beta} \\ &= \frac{1}{2}\varepsilon^k{}_{ij} R^i{}_m R^j{}_n F^{mn} = \frac{1}{2}\varepsilon^k{}_{ij} R^i{}_m R^j{}_n \varepsilon^{mn}_\ell B^\ell. \end{aligned}$$

Let us now use the fact that $\varepsilon_{mn\ell}$ is a tensor invariant under rotations:

$$R_p{}^k R^j{}_m R^i{}_n \varepsilon_k{}^{mn} = \varepsilon_p{}^{ji}$$

$$\Rightarrow \quad R^p{}_\ell \varepsilon_p{}^{ji} = R^j{}_m R^i{}_n R^p{}_k R^p{}_\ell \varepsilon_k{}^{mn} = R^j{}_m R^i{}_n \varepsilon_\ell{}^{mn}\ .$$

For the last equality we have made use of the identity $R \cdot R^T = 1\!\!1$, which is equivalent to $R^p{}_k R^p{}_\ell = \delta_{k\ell}$.

Hence $R^i{}_m R^j{}_n \varepsilon_\ell{}^{mn} = R^p{}_\ell \varepsilon_p{}^{ij}$. Therefore

$$(B^k)' = \frac{1}{2}\underbrace{\varepsilon^k{}_{ij} R^p{}_\ell \varepsilon_p{}^{ij}}_{2\delta^k_p R^p{}_\ell} B^\ell = R^k{}_\ell B^\ell.$$

We have consequently found that **B** and **E** transform as three-dimensional vectors under rotations, and this agrees with the behaviour expected for electrodynamics in its non-covariant formulation.

Let us now study the behaviour of **E** and **B** under a boost of speed v in the direction x_1. This Lorentz transformation is given by

$$(\Lambda^\mu{}_\nu) = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1.98}$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$.

$$\begin{aligned} (F^{\mu\nu})' &= \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F^{\alpha\beta} \\ (E_1)' &= (-F^{10})' = -\Lambda^1{}_0\Lambda^0{}_1 F^{01} - \Lambda^1{}_1\Lambda^0{}_0 F^{10} \\ (E_1)' &= \underbrace{(\Lambda^1{}_1\Lambda^0{}_0 - \Lambda^1{}_0\Lambda^0{}_1)}_{\gamma^2(1-\beta^2)} E_1 = E_1 \\ (E_2)' &= (-F^{20})' = -\Lambda^2_2(\Lambda^0_0 F^{20} + \Lambda^0_1 F^{21}) \end{aligned} \tag{1.99}$$

$$(E_2)' = \gamma E_2 - \gamma\beta B_3 \tag{1.100}$$

and, likewise,

$$(E_3)' = \gamma E_3 + \gamma\beta B_2 \tag{1.101}$$

$$(B_1)' = \frac{1}{2}(\varepsilon_{ij1}F^{ij})' = \frac{1}{2}\varepsilon_{ij1}\Lambda^i{}_\alpha\Lambda^j{}_\beta F^{\alpha\beta} = \frac{1}{2}(\varepsilon_{231}F^{23} + \varepsilon_{321}F^{32}) = B_1$$

$$B_1' = B_1 \tag{1.102}$$

$$(B_2)' = \frac{1}{2}(\varepsilon_{ij2}F^{ij})' = (\varepsilon_{132}F^{13})' = \varepsilon_{132}\Lambda^1{}_\alpha\Lambda^3{}_\beta F^{\alpha\beta} = \varepsilon_{132}(\Lambda^1_0 F^{03} + \Lambda^1_1 F^{13})$$

$$B_2' = \gamma B_2 + \gamma\beta E_3 \tag{1.103}$$

and similarly

$$B_3' = \gamma B_3 - \gamma\beta E_2. \tag{1.104}$$

Example: electromagnetic field of a point charge in uniform motion

We consider a charge that travels with speed v in the direction x^1 in the reference frame K. In the frame K', which moves with speed v in the direction x^1, the charge is at rest. In K' it then produces the field

$$\mathbf{E}'(\mathbf{x}', t') = \frac{e\mathbf{x}'}{|\mathbf{x}'|^3}, \quad \mathbf{B}'(\mathbf{x}', t') = 0.$$

Using the expressions of the transformation of (x^μ) under a boost of speed $-v$ in the direction x^1, we have that

$$\begin{aligned} ct &= \gamma(ct' + \beta x'^1), \quad ct' = \gamma(ct - \beta x^1) &\quad (1.105)\\ x^1 &= \gamma(x'^1 + \beta t'), \quad x'^1 = \gamma(x^1 - \beta ct) &\quad (1.106)\\ x^2 &= x'^2 &\quad (1.107)\\ x^3 &= x'^3 &\quad (1.108)\end{aligned}$$

and applying equations (1.100) to (1.104) we find

$$\begin{aligned} E_1(\mathbf{x},t) &= E'_1(\mathbf{x}',t') = \frac{e\gamma(x^1 - vt)}{\left[\gamma^2(x^1-vt)^2 + \underbrace{(x^2)^2 + (x^3)^2}_{b^2}\right]^{\frac{3}{2}}} \\ &= \frac{e\gamma(x^1 - vt)}{\left[\gamma^2(x^1-vt)^2 + b^2\right]^{\frac{3}{2}}}, \quad b^2 = (x^2)^2 + (x^3)^2 &\quad (1.109)\\ E_2(\mathbf{x},t) &= \frac{e\gamma x^2}{\left[\gamma^2(x^1-vt)^2 + b^2\right]^{\frac{3}{2}}}, &\quad (1.110)\\ E_3(\mathbf{x},t) &= \frac{e\gamma x^3}{\left[\gamma^2(x^1-vt)^2 + b^2\right]^{\frac{3}{2}}}. &\quad (1.111)\end{aligned}$$

At time $t = 0$, we obtain

$$\mathbf{E}(\mathbf{x}, t=0) = \frac{e(1-\beta^2)\mathbf{x}}{\left[r^2 - \beta^2 b^2\right]^{\frac{3}{2}}}, \quad r = |\mathbf{x}|. \quad (1.112)$$

With $\sin\vartheta = \frac{b}{r}$ this gives

$$|\mathbf{E}|(\mathbf{x}, t=0) = \frac{e(1-\beta^2)}{r^2(1-\beta^2\sin\vartheta)^{\frac{3}{2}}}. \quad (1.113)$$

For a fixed value of r, $|\mathbf{E}|$ attains its maximum value in the direction $\vartheta = \frac{\pi}{2}$ (i.e. when $b = r$), which corresponds to the normal plane to the direction of motion. $|\mathbf{E}|$ minimizes in the direction $\pm x^1$, $\vartheta = 0$ and π:

$$\left.\begin{aligned} |\mathbf{E}| &= \frac{e}{r^2(1-\beta^2)^{1/2}} &\quad \vartheta = \frac{\pi}{2} \\ |\mathbf{E}| &= \frac{e(1-\beta^2)}{r^2} &\quad \vartheta = 0,\, \pi \end{aligned}\right\}. \quad (1.114)$$

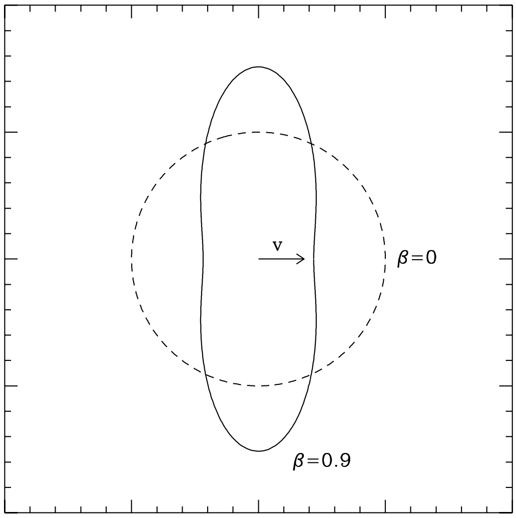


Figure 1.1: Surfaces $|\mathbf{E}|$ = constant for a point charge at rest (dashed line) and for a point charge moving at a constant velocity $\mathbf{v}$ (solid line).

The surfaces $|\mathbf{E}|$ = const. are contracted in the direction of motion and dilated in the orthogonal directions (see figure 1.1).

For the magnetic field we have

$$B_1 = 0,\ B_2 = -\beta\gamma E_3' = -\beta E_3,\ B_3 = \beta\gamma E_2' = \beta E_2,$$

or, equivalently,

$$\mathbf{B} = \boldsymbol{\beta} \wedge \mathbf{E}. \tag{1.115}$$

A moving charge produces a magnetic field perpendicular to $\mathbf{v}$ of magnitude proportional to $\frac{v}{c}$.

The relativistic Lorentz force

In order to complete the relativistic formalism, we look for the relativistic form of the Lorentz force equation. For particles of charge q moving at low speed $v \ll c$ we have the nonrelativistic limit

$$m\frac{dv^i}{dt} = q\left(E^i + \frac{1}{c}(\mathbf{v} \wedge \mathbf{B})^i\right). \tag{1.116}$$

From relativistic mechanics, you know the four-velocity

$$(u^\mu) = (c\gamma, \mathbf{v}\gamma)\ , \quad u^2 = -(u^0)^2 + \sum_i (u^i)^2 = -c^2\ , \qquad p = mu\ .$$

But, due to the fact that ct is the 0-component of a four-vector, $\frac{du^\mu}{dt}$ is not a four-vector. We have to replace dt with the particle's *proper time*, $d\tau = dt/\gamma$ which is, as we have seen, a Lorentz scalar. In terms of proper time we have $u^\mu = dx^\mu/d\tau$. With $dt = \gamma d\tau$ (1.116) becomes

$$m\frac{du^i}{d\tau} = \frac{q}{c}F^{i\alpha}u_\alpha.$$

The four-dimensional formulation of this equation is

$$\frac{dp^\mu}{d\tau} = m\frac{du^\mu}{d\tau} = \frac{q}{c}F^{\mu\nu}u_\nu. \tag{1.117}$$

The zeroth component of (1.117) describes the change of energy ε of a particle in a magnetic field:

$$\frac{dp^0}{d\tau} = \frac{q}{c}F^{0i}v_i\gamma = q\gamma\frac{1}{c}\mathbf{E}\cdot\mathbf{v}$$

$$\varepsilon = p^0c, \quad \frac{d\varepsilon}{d\tau} = q\gamma\mathbf{E}\cdot\mathbf{v}. \tag{1.118}$$

The energy of the particle changes due to the work done by the electric field.

The energy-momentum tensor of the electromagnetic field

We define yet another second rank tensor, the energy-momentum tensor of the electromagnetic field $T_{\mu\nu}$, given by

$$\left.\begin{array}{rcl} T_{00} &=& \frac{1}{8\pi}(E^2+B^2) = u \\ T_{i0} &=& \frac{-1}{4\pi}(\mathbf{E}\wedge\mathbf{B})_i = -c\Pi_i \ = \ T_{0i} \\ T_{ij} &=& \frac{1}{4\pi}\left(\frac{1}{2}\delta_{ij}(E^2+B^2) - E_iE_j - B_iB_j\right) = T_{ji} \end{array}\right\}. \tag{1.119}$$

Exercise: *Show that the expression given for the components of $T_{\mu\nu}$ in eq. (1.119) is equivalent to*

$$T_{\mu\nu} = \frac{1}{4\pi}\left[F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right]. \tag{1.120}$$

The indices of $T_{\mu\nu}$ are, as usual, raised and lowered with the Lorentz metric:

$$T_{\mu\nu} = T_\mu{}^\alpha g_{\alpha\nu}, \quad (g_{\alpha\nu}) = \mathrm{diag}(-1,1,1,1) \tag{1.121}$$

$$T^{\mu\nu} = g^{\mu\alpha}T_\alpha{}^\nu. \tag{1.122}$$

In order to show that $T_{\mu\nu}$ is a true four-tensor it is necessary to show, as for the electromagnetic field tensor, $F^{\mu\nu}$, that under a Lorentz transformation $(\Lambda_\mu{}^\nu)$ one has

$$(T^{\mu\nu})' = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta}. \tag{1.123}$$

We first consider a rotation R,

$$(\Lambda^\nu{}_\mu) = (\Lambda_\nu{}^\mu) = \left(\begin{array}{c|ccc} 1 & & 0 & \\ \hline & & & \\ 0 & & R & \\ & & & \end{array} \right), \tag{1.124}$$

with $R \cdot R^T = 1\!\!1$. We know that $\mathbf{E}$ and $\mathbf{B}$ transform, under rotations, the way three-dimensional vectors do,

$$E'^i = R^i{}_j E^j, \quad B'^i = R^i{}_j B^j,$$

so

$$\begin{aligned} \mathbf{E}'^2 &= E'^i E'^i = R^i_j E^j R^i_\ell E^\ell \\ &= (R^T \cdot R)_{j\ell} E^j E^\ell = \delta_{j\ell} E^j E^\ell = E^j E^j = \mathbf{E}^2 \end{aligned}$$

and similarly for $\mathbf{B}$. $T_{00} = \frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{B}^2)$ is then invariant under rotations. For the components $T_i{}^0$, we use

$$(\mathbf{E}' \wedge \mathbf{B}')_q = \varepsilon_{q\ell m} E'^\ell B'^m = \varepsilon_{q\ell m} R^\ell{}_k R^m_j E^k B^j. \tag{1.125}$$

As for the tensor $F^{\mu\nu}$, the invariance of the tensor ε_{ijl} implies

$$R_q{}^i \varepsilon_{i\ell m} = R^k{}_\ell R^n{}_m \varepsilon_{qkn}.$$

With (1.125) this gives

$$(\mathbf{E}' \wedge \mathbf{B}')_q = R_q{}^i \varepsilon_{i\ell m} E^\ell B^m = R_q{}^i (\mathbf{E} \wedge \mathbf{B})_i.$$

We have therefore shown that $(\mathbf{E} \wedge \mathbf{B})$ transforms as a vector. We now proceed to verify (1.123). For the "$_{00}$" component we have

$$T'_{00} = \Lambda_0{}^\mu \Lambda_0{}^\nu T_{\mu\nu} = T_{00}.$$

Which is as expected because

$$\frac{1}{8\pi}(\mathbf{E}'^2 + \mathbf{B}'^2) = \frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{B}^2).$$

For the "$_{i0}$" components we make use of the fact that

$$T'_{i0} = \Lambda_i{}^\mu \Lambda_0{}^\nu T_{\mu\nu} = \Lambda_i{}^j T_{j0} = \frac{-1}{4\pi} R_i{}^j (\mathbf{E} \wedge \mathbf{B})_j = \frac{-1}{4\pi} (\mathbf{E}' \wedge \mathbf{B}')_i.$$

Correspondingly, for the "$_{ij}$" components we have:

$$\begin{aligned} T'_{ij} &= \Lambda_i{}^\mu \Lambda_j{}^\nu T_{\mu\nu} = \Lambda_i{}^k \Lambda_j{}^\ell T_{k\ell} \\ &= R_i{}^k R_j{}^\ell T_{k\ell} \\ &= R_i{}^k R_j{}^\ell \left(\frac{1}{2}\delta_{k\ell}(\mathbf{E}^2 + \mathbf{B}^2) - E_k E_\ell - B_k B_\ell \right) \end{aligned}$$

but $R_i{}^k E_k = E'_i$, $R_i{}^k B_k = B'_i$, $\mathbf{E}^2 = \mathbf{E}'^2$, $\mathbf{B}^2 = \mathbf{B}'^2$ and $R_i{}^k R_j{}^\ell \delta_{k\ell} = R_i{}^k (R^T)^k{}_j = \delta_{ij}$. We then finally have

$$T'_{ij} = \frac{1}{2}\delta_{ij}(\mathbf{E}'^2 + \mathbf{B}'^2) - E'_i E'_j - B'_i B'_j \ .$$

Thus we have verified that under a rotation $\Lambda^\mu_\nu(R)$, given by (1.124), $T_{\mu\nu}$ transforms according to (1.123).

We now show that $T_{\mu\nu}$ also transforms as a second rank tensor under a *boost* of speed v in the direction x_1. Let again β and γ be given by $\beta = \frac{v}{c}$ and $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$;

$$(\Lambda^\mu{}_\nu) = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(If $\chi := \mathrm{arcth}\beta$, it follows that $\gamma = \cosh\chi$ and $\beta\gamma = \sinh\chi$, where χ is the *speed* of the boost). Let us consider first T_{00}:

$$\begin{aligned} (T^{00})' &= \Lambda^0{}_\alpha \Lambda^0{}_\beta T^{\alpha\beta} = \gamma^2 T^{00} - 2\gamma^2\beta T^{01} + \gamma^2\beta^2 T^{11} \\ &= \frac{1}{8\pi}[\gamma^2(E^2+B^2) - 4\gamma^2\beta(E_2B_3 - B_2E_3) + \gamma^2\beta^2(E^2+B^2 \\ &\quad -2(E_1^2 + B_1^2))] \\ &= \frac{1}{8\pi}[\overbrace{\gamma^2(1-\beta^2)}^{1}(E_1^2+B_1^2) + \gamma^2(1+\beta^2)(E_2^2+E_3^2+B_2^2+B_3^2) \\ &\quad -4\gamma^2\beta(E_2B_3 - B_2E_3)] \\ &= \frac{1}{8\pi}\left[(E_1^2+B_1^2) + \gamma^2(E_2-\beta B_3)^2 + \gamma^2(E_3+\beta B_2)^2 + \gamma^2(B_2+\beta E_3)^2 \right. \\ &\quad \left. +\gamma^2(B_3-\beta E_2)^2\right] \\ &= \frac{1}{8\pi}\left(\mathbf{E}'^2 + \mathbf{B}'^2\right). \end{aligned}$$

The last equality follows from equations (1.100) to (1.104). For T^{01}, we obtain

$$
\begin{aligned}
(T^{01})' &= \Lambda^0{}_\alpha \Lambda^1{}_\beta T^{\alpha\beta} \\
&= \gamma^2 T^{01} - \beta\gamma^2 (\overbrace{T^{00}}^{1/8\pi(E^2+B^2)} + \overbrace{T^{11}}^{1/4\pi(1/2(\mathbf{E}^2+\mathbf{B}^2)-E_1^2-B_1^2)}) + \gamma^2\beta^2 T^{10} \\
&= \frac{1}{4\pi}[\gamma^2(1+\beta^2)(E_2B_3 - B_2E_3) - \beta\gamma^2(E_2^2+E_3^2+B_2^2+B_3^2)] \\
&= \frac{1}{4\pi}[\gamma^2(E_2-\beta B_3)(B_3-\beta E_2) - \gamma^2(B_2+\beta E_3)(E_3+\beta B_2)] \\
&= \frac{1}{4\pi}(E_2'B_3' - B_3'E_2') = \frac{1}{4\pi}(\mathbf{E}'\wedge\mathbf{B}')_1
\end{aligned}
$$

and similarly for the other components.
We illustrate this by computing $(T^{23})'$

$$
\begin{aligned}
(T^{23})' &= \Lambda^2{}_\alpha \Lambda^3{}_\beta T^{\alpha\beta} = T^{23} = \frac{-1}{4\pi}(E_2E_3 + B_2B_3) \\
&= \frac{-1}{4\pi}[\gamma^2(E_2-\beta B_3)(E_3+\beta B_2) + \gamma^2(B_2+\beta E_3)(B_3-\beta E_2)] \\
&= \frac{-1}{4\pi}(E_2'E_3' + B_2'B_3').
\end{aligned}
$$

The computation of the remaining components is left as an exercise.

The conservation of energy and momentum can now be simply written as

$$
\partial_\mu T^{\mu\nu} = -k^\nu, \text{ with } \quad k^0 = \frac{1}{c}\mathbf{J}\cdot\mathbf{E}, \quad k^i = (\rho\mathbf{E} + \frac{1}{c}\mathbf{J}\wedge\mathbf{B})^i\ . \tag{1.126}
$$

k^0 is the work done on the charges and $\mathbf{k}$ is the Lorentz force density. It is easy to see that $k_\mu = F_{\mu\nu}j^\nu$, and therefore (k_μ) is a four-vector.

We have discussed the behaviour of $T_{\mu\nu}$ and $F^{\mu\nu}$ (second rank tensors) under rotations around, and boosts in the direction x^1. A boost in an arbitrary direction $\mathbf{n}$ is always the product of a rotation R_1 that rotates $\mathbf{n}$ in the direction x^1, of a boost $\Lambda(\beta, \mathbf{e}_1)$ in direction x^1 and, finally, of the inverse rotation, R_1^{-1}:

$$
\Lambda(\beta, \mathbf{n}) = R_1^{-1}\Lambda(\beta, \mathbf{e}_1)R_1.
$$

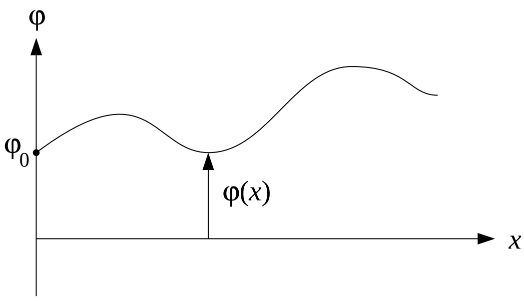
It then follows that $T_{\mu\nu}$ transforms correctly under a boost in an arbitrary direction.

Chapter 2

Electromagnetic waves

2.1 One-dimensional waves

In order to restrict ourselves to a concrete example, let us consider a string in a two-dimensional plane with one of its points fixed:



We look for an equation that describes the wave-like motion of the string, i.e. an equation for the acceleration $d^2\varphi/dt^2$ at every point x, taking into account the tension forces. We make:

$$\frac{\partial^2\varphi}{\partial t^2}(x,\, t) = \sum_{n=0}^{\infty} \alpha_n \frac{\partial^n}{\partial x^n}\varphi(x,\, t).$$

- Since the choice of the position φ_0 is arbitrary, the force is independent of the amplitude φ and, consequently, $\alpha_0 = 0$.

- If $\varphi(x)$ describes a straight line, no tensions are applied, the force does not depend on the slope, and therefore $\alpha_1 = 0$.

- The lowest-order term is α_2: the curvature implies a tension.

For many applications, α_2 turns out to be the dominant term and we then obtain ($\alpha_2 \equiv v^2$):

$$\frac{\partial^2\varphi}{\partial t^2} = v^2\frac{\partial^2\varphi}{\partial x^2}. \tag{2.1}$$

Eq. (2.1) is the equation of a one-dimensional wave. It can also be used to describe the motion of a spring or the air compressions in a tube, as well as the physics of many other one-dimensional problems.

The general solution of (2.1) which satisfies the initial conditions

$$\varphi(x, t=0) = \varphi_0(x) \quad \text{and} \quad \dot{\varphi}(x, t=0) = \varphi_1(x)$$

is

$$\varphi(x,t) = f(x+vt) + g(x-vt) \tag{2.2}$$

such that

$$f(x) = \frac{1}{2}[\varphi_0(x) + \frac{1}{v}\int^x \varphi_1(x')dx'] \text{ and } g(x) = \frac{1}{2}[\varphi_0(x) - \frac{1}{v}\int^x \varphi_1(x')dx'] \ .$$

The function f describes the part of the wave that moves towards the $-x$ direction, the "left mover", and the function g describes the "right mover". v is the speed of propagation of the wave. In all physically realistic cases (with finite energy), the functions f and g can be written as Fourier integrals,

$$f(x+vt) = \int dkA(k)\exp[i(kx+\omega t)] \ , \tag{2.3}$$

with $\omega = vk$. The function $|A(k)|^2$ is the *spectrum* of the wave f.

Normal modes (stationary waves)

The time dependence of the normal modes of a wave go as $\cos(\omega t+\beta)$. We therefore look for a solution to (2.1) of the form

$$\varphi(x,\, t) = A(x)\cos(\omega t + \beta).$$

Using (2.1) it is found that

$$-\omega^2 A(x) = v^2\frac{d^2A}{dx^2},$$

whose solution is

$$A(x) = A_0\cos(kx+\alpha) \tag{2.4}$$

$$k^2v^2 = \omega^2. \tag{2.5}$$

We then have:

$$\begin{aligned}\varphi(x,\,t) &= A_0\cos(\omega t+\beta)\cos(kx+\alpha)\\ &= \frac{A_0}{2}\left[\cos(\omega t+kx+\gamma_1)+\cos(\omega t-kx+\gamma_2)\right],\end{aligned}\qquad(2.6)$$

with $\gamma_1 = \beta+\alpha$ and $\gamma_2 = \beta-\alpha$.

Equation (2.5) is the *dispersion law* for the wave equation (2.1). A dispersion law is a relation between ω and k. We will encounter other dispersion relations that are more complicated than (2.5). The quantity ω is the *angular frequency* (or pulsatance), whose dimension is [1/s]. k is the wavenumber, of dimension [1/cm]. A_0 is the amplitude; its dimension depends on the physical phenomenon under consideration. For the string, for example, it corresponds to [cm]. γ_1 are γ_2 phases determined by the initial conditions. $T = 2\pi/\omega$ is the wave's period, the time needed for one complete oscillation, and $\nu = 1/T$ is the frequency of the wave. The distance $\lambda = 2\pi/k$ is the wavelength, the length of one complete oscillation. From (2.5) we have

$$\lambda\nu = v. \qquad (2.7)$$

Considering that the dispersion relation (2.5) has the solutions $k = \pm\omega/v$, it is possible to add to (2.6) a term $B_0\left[\cos(\omega t-kx+\gamma_1)+\cos(\omega t+kx+\gamma_2)\right]$. This leads to the general solution

$$\begin{aligned}\varphi(x,\,t) &= A\cos(\omega t+kx+\delta_1)+B\cos(\omega t-kx+\delta_2)\\ &= A_1\cos(\omega t{+}kx)+A_2\sin(\omega t{+}kx)+B_1\cos(\omega t{-}kx)+B_2\sin(\omega t{-}kx).\end{aligned}$$

with

$$\begin{aligned}A_1 &= A\cos\delta_1, & A_2 &= -A\sin\delta_1,\\ B_1 &= B\cos\delta_2, & B_2 &= -B\sin\delta_2.\end{aligned}$$

This solution can also be written as follows

$$\varphi(x,\,t) = \mathrm{Re}\left[C_1e^{i(\omega t+kx)}+C_2e^{i(\omega t-kx)}\right] \qquad (2.8)$$

where $C_1 = A(\cos\delta_1{+}i\sin\delta_1)$ and $C_2 = B(\cos\delta_2{+}i\sin\delta_2)$. The first term describes a wave that propagates towards the left ("left mover") and the second term represents a wave propagating towards the right ("right mover"). The position x_c of the crest of an oscillation is given by $\omega t+kx_c+\delta_1 = 2\pi N$, so $dx_c/dt = -\omega/k = -v$ for the first term; and $\omega t-kx_c+\delta_2$, $dx_c/dt = \omega/k = v$ for the second. It is because of this that v is known as the *phase velocity* of the wave.

We now consider a string with fixed ends: $\varphi(0,\,t) = \varphi(L,\,t) = 0$. For $t = 0$ and $x = 0$ this gives $C_2 = -C_1$. For $x = L$ and t arbitrary, we have

$$\mathrm{Re}\left[C_1e^{i\omega t}\left(e^{ikL}-e^{-ikL}\right)\right] = 0 \quad\text{and so}\quad 2iC_1\sin(kL) = 0. \qquad (2.9)$$

This implies that $\sin(kL) = 0$, or, in other words, that k can only take discrete values

$$k_j = \frac{j\pi}{L}, \quad \lambda_j = \frac{2L}{j}, \quad j \in \mathbb{N}. \tag{2.10}$$

From this it follows that $\lambda_0 = \infty$. The angular frequency that corresponds to k_j is given by $\omega_j = vk_j$. The general solution of our problem with boundary conditions $\varphi(0,\, t) = \varphi(L,\, t) = 0$ is finally given by

$$\begin{aligned} \varphi(x,\, t) &= \mathrm{Re}\left[\sum_{j=1}^{\infty} C_j e^{i\omega_j t}\left(e^{ik_j x} - e^{-ik_j x}\right)\right] \\ &= \sum_{j=1}^{\infty} (A_j \cos\omega_j t + B_j \sin\omega_j t)\sin k_j x \end{aligned} \tag{2.11}$$

with $A_j = -2\mathrm{Im}\,[C_j]$ et $B_j = -2\mathrm{Re}\,[C_j]$. A situation like this is said to be of the *discrete* spectrum type. The function A defined in equation (2.3) is zero except for the discrete values $k = j\pi/L$.

Exercise: Determine the coefficients C_j for a given solution $\varphi(x,\, t)$.

Remark: For the string, φ is an amplitude in a direction *transverse* to the direction of propagation. A wave like this is called a *transverse wave*. For a spring that compresses and stretches, the oscillating magnitude (the number of coils per centimetre) is parallel to the direction of propagation. In this case we speak of a longitudinal wave, of which sound waves constitute another example.

2.2 Three-dimensional waves, electromagnetic waves

To study three-dimensional waves, let us forget the case of the string and take instead, for the moment, a compression wave in a gas (an acoustic wave). The quantity φ is no longer the position of the string, but the density of a gas. For a wave that propagates in an arbitrary direction $\mathbf{n} \in \mathbb{R}^3$ ($\mathbf{n}^2 = 1$) we then have

$$\varphi = \mathrm{Re}\left[Ce^{i(\omega t - k\mathbf{n}\cdot\mathbf{x})}\right] = \mathrm{Re}\left[Ce^{i(\omega t - \mathbf{k}\cdot\mathbf{x})}\right] \tag{2.12}$$

where $\mathbf{k} = k\mathbf{n}$ is called the *wave vector*. By means of simple inspection, one can verify that φ satisfies the equation

$$\partial_t^2 \varphi = v^2(\partial_x^2 + \partial_y^2 + \partial_z^2)\varphi$$

with $v^2 = \omega^2/\mathbf{k}^2$, which can be rewritten as

$$\left(\partial_t^2 - v^2 \triangle\right)\varphi = 0. \tag{2.13}$$

Equation (2.13) is the equation of a three-dimensional wave. v is the *phase velocity* of the wave. There is also something known as the group velocity of the wave:

$$v_g = \frac{\partial \omega}{\partial k}.$$

In the examples so far considered, $v_g = v$ because the relation between ω and k has always been linear. We will arrive at situations where this will no longer be the case.

Remark: There are also two-dimensional waves. Examples of this are the waves on the surface of water and the acoustic waves of a metallic disc.

Let us turn to the subject we are interested on: electromagnetic waves. We have already seen [equation (1.32)] that the electric and magnetic fields in vacuum satisfy the wave equations:

$$\left.\begin{array}{rcl} (\partial_t^2 - c^2\triangle)\,\mathbf{E} &=& 0 \\ (\partial_t^2 - c^2\triangle)\,\mathbf{B} &=& 0 \end{array}\right\} \tag{2.14}$$

If we first consider a plane wave-like solution:

$$\begin{aligned} \mathbf{E}(\mathbf{x},\,t) &= \mathrm{Re}\left[\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right] \\ \mathbf{B}(\mathbf{x},\,t) &= \mathrm{Re}\left[\mathbf{B}_0 e^{i(\mathbf{q}\cdot\mathbf{x}-\sigma t)}\right]. \end{aligned}$$

then equation (2.14) demands $\omega^2 = c^2k^2$ and $\sigma^2 = c^2q^2$. Hence c is the phase (and group) velocity of the wave. The equations $\nabla\cdot\mathbf{E} = 0$ and $\nabla\cdot\mathbf{B} = 0$ imply

$$\mathbf{k}\cdot\mathbf{E} = \mathbf{q}\cdot\mathbf{B} = 0.$$

Electromagnetic waves are then *transverse* waves. In addition, the law of induction gives

$$0 = -\partial_t\mathbf{E} + c\nabla\wedge\mathbf{B} = \mathrm{Re}\left[i\omega\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + ci\mathbf{q}\wedge\mathbf{B}_0 e^{i(\mathbf{q}\cdot\mathbf{x}-\sigma t)}\right].$$

For this equation to be satisfied in $\mathbf{x} = 0$, say, for every time t, one must have $\sigma = \omega$. What is more, if it is to be satisfied in any point $\mathbf{x}$ of space, then the equality $\mathbf{q} = \mathbf{k}$ is needed. Maxwell's equations then imply for the fields that

$$\begin{aligned} \mathbf{E}(\mathbf{x},\,t) &= \mathrm{Re}\left[\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right] \\ \mathbf{B}(\mathbf{x},\,t) &= \mathrm{Re}\left[\mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right] \end{aligned} \tag{2.15}$$

with

$$\begin{aligned} \mathbf{k}\cdot\mathbf{E}_0 = \mathbf{k}\cdot\mathbf{B}_0 &= 0 \\ \omega\mathbf{E}_0 + c\mathbf{k}\wedge\mathbf{B}_0 &= 0 \end{aligned} \tag{2.16}$$

or, using $\mathbf{k}^2 = \omega^2/c^2$ and $\hat{\mathbf{k}} = \mathbf{k}/k$,

$$\mathbf{E}_0 = -\hat{\mathbf{k}} \wedge \mathbf{B}_0. \tag{2.17}$$

Likewise, Ampère's law, $\partial_t \mathbf{B} - c\nabla \wedge \mathbf{E} = 0$, gives

$$\mathbf{B}_0 = \hat{\mathbf{k}} \wedge \mathbf{E}_0$$

(something that follows also from (2.17) after multiplying by $\hat{\mathbf{k}}\wedge$...).

The vectors $\mathbf{k}$, $\mathbf{E}_0$, and $\mathbf{B}_0$ are therefore mutually orthogonal, and $\mathbf{E}_0^2 = \mathbf{B}_0^2$. $\mathbf{E}_0$ and $\mathbf{B}_0$ are, in general, *complex* vectors ($\mathbf{E}_0 \in \mathbb{C}^3$, $\mathbf{B}_0 \in \mathbb{C}^3$), and the equations we have derived are valid for both their real and imaginary parts (why are they valid for the imaginary part too?!).

Polarisation, Stokes parameters

Let us decompose $\mathbf{E}_0 = \mathbf{A}_1 - i\mathbf{A}_2$, $\mathbf{A}_{1,2} \in \mathbb{R}^3$. In a fixed position $\mathbf{x}$ ($\mathbf{x} \equiv 0$, say), we have

$$\mathbf{E}(t) = \mathbf{A}_1 \cos\omega t + \mathbf{A}_2 \sin\omega t. \tag{2.18}$$

a) *Linear polarisation*

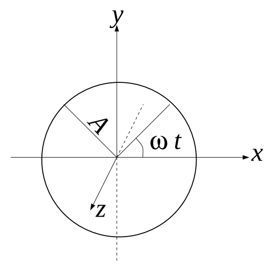
If $\mathbf{A}_1 \parallel \mathbf{A}_2$, then only the amplitude of the electromagnetic field, and not its *direction*, changes in time ($\mathbf{E}(t) \parallel \mathbf{E}(t')$). The same can be stated for $\mathbf{B}$. Such an electromagnetic wave is said to be 'linearly polarized'.

b) *Circular polarisation*

Consider now the opposite case, $\mathbf{A}_1 \perp \mathbf{A}_2$, but with the further condition $\mathbf{A}_1^2 = \mathbf{A}_2^2 = A^2$. We choose a coordinate system in which $\mathbf{A}_1$ points in the direction x and $\mathbf{A}_2$ points in the direction y. Our wave then propagates in the direction z ($\mathbf{k}$ is parallel to the z direction). Thus, from (2.18), we have (at a fixed position $\mathbf{x}$)

$$E_x = A\cos\omega t, \quad E_y = A\sin\omega t, \quad E_z = 0,$$

i.e., $\mathbf{E}$ sweeps out a circle of radius A in the (x, y)-plane.
If $(\mathbf{A}_1, \mathbf{A}_2, \mathbf{k})$ form a system of positive helicity ($(\mathbf{A}_1 \wedge \mathbf{A}_2) \cdot \mathbf{k} > 0$), the sweep goes counterclockwise, i.e. in the trigonometric direction (from the x axis to the y axis). In optics, such a wave is called a *left wave*. On the other hand, if $(\mathbf{A}_1, \mathbf{A}_2, \mathbf{k})$ form a negative helix ($(\mathbf{A}_1 \wedge \mathbf{A}_2) \cdot \mathbf{k} < 0$), we have a *right wave*. (It rotates clockwise for an observer that receives the wave.) Because the direction of $\mathbf{E}$ determines that of $\mathbf{B}$, $\mathbf{B}_0 = \hat{\mathbf{k}} \wedge \mathbf{E}_0$, the polarisation of $\mathbf{B}$ is the same as the one of $\mathbf{E}$.



c) *General case*

Let $(\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2, \hat{\mathbf{k}})$ be an orthonormal system of positive helicity. The general form of $\mathbf{E}$ is

$$\mathbf{E}(\mathbf{x},\, t) = \mathrm{Re}\left[(E_1\boldsymbol{\epsilon}_1 + E_2\boldsymbol{\epsilon}_2)e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right] \tag{2.19}$$

$$= \mathrm{Re}\left[(E_+\boldsymbol{\epsilon}_+ + E_-\boldsymbol{\epsilon}_-)e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right] \tag{2.20}$$

$$\text{where } \boldsymbol{\epsilon}_\pm = \frac{1}{\sqrt{2}}\left(\boldsymbol{\epsilon}_1 \pm i\boldsymbol{\epsilon}_2\right), \quad E_\pm = \frac{1}{\sqrt{2}}\left(E_1 \mp iE_2\right).$$

Equations (2.19) and (2.20) are the explicit decompositions into linearly polarised ($E_1 = 0$ or $E_2 = 0$) and circularly polarised ($E_- = 0$, left wave; $E_+ = 0$, right wave) waves.

We now show that in the general case $\mathbf{E}$ describes an ellipse and we determine its direction and axes lengths. Let us choose $\boldsymbol{\epsilon}_1 = \mathbf{e}_x$ and $\boldsymbol{\epsilon}_2 = \mathbf{e}_y$. The X and Y components of $\mathbf{E}(t) = X\mathbf{e}_x + Y\mathbf{e}_y$ (at the fixed position $\mathbf{x} = 0$) are then given by

$$\begin{aligned}
X &= \frac{1}{2}\left(E_1e^{-i\omega t} + E_1^*e^{+i\omega t}\right)\\
Y &= \frac{1}{2}\left(E_2e^{-i\omega t} + E_2^*e^{+i\omega t}\right)\\
X \pm iY &= \frac{1}{2}\left((E_1 \pm iE_2)e^{-i\omega t} + (E_1^* \pm iE_2^*)e^{i\omega t}\right)\\
&= \frac{1}{\sqrt{2}}\left(E_\mp e^{-i\omega t} + E_\pm^* e^{i\omega t}\right).
\end{aligned}$$

If we rotate counter clockwise the system $(x,\, y)$ by an angle α, we obtain the components

$$(\xi,\, \eta) = (X\cos\alpha - Y\sin\alpha,\, X\sin\alpha + Y\cos\alpha).$$

Then $\xi \pm i\eta = (X \pm iY)e^{\pm i\alpha}$, and similarly for the new $E'_\pm$;

$$(\xi \pm i\eta) = \frac{1}{\sqrt{2}}(E'_\mp e^{-i\omega t} + {E'_\pm}^* e^{i\omega t})$$

with $E'_\pm = e^{\mp i\alpha}E_\pm$. So $E'_+/E'_- = (E_+/E_-)e^{-2i\alpha}$. We can choose α such that $\rho = E'_+/E'_-$ becomes real and positive. The angle α and the radius ρ are then determined by

$$\frac{E_+}{E_-} = \rho e^{2i\alpha}.$$

Let E'_- be given by $E'_- = \sqrt{2}Be^{i\delta}$ with $B,\, \delta \in \mathbb{R}$, $B > 0$ the polar decomposition of the complex number E'_-. We have

$$\begin{aligned}
\xi + i\eta &= Be^{i\delta}e^{-i\omega t} + \rho Be^{-i\delta}e^{i\omega t}\\
\xi - i\eta &= \rho Be^{i\delta}e^{-i\omega t} + Be^{-i\delta}e^{i\omega t}
\end{aligned}$$

and, consequently,

$$\xi = B(1+\rho)\cos(\omega t - \delta)$$
$$\eta = B(1-\rho)\sin(\omega t - \delta)$$
$$\frac{\xi^2}{B^2(1+\rho)^2} + \frac{\eta^2}{B^2(1-\rho)^2} = 1.$$

This means that $\mathbf{E}(t)$ describes an ellipse with semi-axes $B(1+\rho)$ and $B|(1-\rho)|$ rotated an angle α with respect to $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$. Their respective lengths are

$$B(1 \pm \rho) = \frac{|E_-|}{\sqrt{2}}\left|1 \pm \frac{|E_+|}{|E_-|}\right|.$$

If $\rho = 1$, this ellipse is reduced to a line in the ξ direction. In that case the polarisation is linear and has direction ξ. If, in contrast, $\rho = 0$, the polarisation is circular.

Let us make, as before, $\boldsymbol{\epsilon}_1 = \mathbf{e}_x$, $\boldsymbol{\epsilon}_2 = \mathbf{e}_y$ and $\boldsymbol{\epsilon}_\pm = \frac{1}{\sqrt{2}}(\boldsymbol{\epsilon}_1 \pm i\boldsymbol{\epsilon}_2)$. Additionally, we define $\boldsymbol{\epsilon}_3 = \boldsymbol{\epsilon}_1 \wedge \boldsymbol{\epsilon}_2 \equiv \hat{\mathbf{k}}$. It is not difficult to verify that

$$\left.\begin{array}{rcl} \boldsymbol{\epsilon}_\pm^* \boldsymbol{\epsilon}_\mp &=& 0 \\ \boldsymbol{\epsilon}_\pm \boldsymbol{\epsilon}_3 = \boldsymbol{\epsilon}_\pm^* \boldsymbol{\epsilon}_3 &=& 0 \\ \boldsymbol{\epsilon}_\pm^* \boldsymbol{\epsilon}_\pm &=& 1 \\ \boldsymbol{\epsilon}_\pm^* &=& \boldsymbol{\epsilon}_\mp \end{array}\right\}. \tag{2.21}$$

According to what we have shown, the polarisation state of a wave is established if we can write it in the form (2.19) (or 2.20)) with known amplitudes E_1, E_2 (or E_+, E_-). We now ask the opposite question: Given a wave in the form (2.15), $\mathbf{E} = \mathrm{Re}\left[\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right]$, how can we determine the polarisation state? In this regard, a useful formalism was proposed by Stokes: the Stokes parameters.

These are four parameters determined by the measure of intensity and two relatively easy polarisation measures (linear and circular). Let us make

$$\begin{array}{rcl} \tilde{\mathbf{E}} &=& (E_1\boldsymbol{\epsilon}_1 + E_2\boldsymbol{\epsilon}_2)e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \\ \mathbf{E} &=& \mathrm{Re}[\tilde{\mathbf{E}}] \\ E_1 &=& a_1 e^{i\delta_1}, \quad E_2 = a_2 e^{i\delta_2} \\ E_+ &=& a_+ e^{i\delta_+}, \quad E_- = a_- e^{i\delta_-} \\ a_+ e^{i\delta_+} &=& \dfrac{1}{\sqrt{2}}(a_1 e^{i\delta_1} - i a_2 e^{i\delta_2}) \\ a_- e^{i\delta_-} &=& \dfrac{1}{\sqrt{2}}(a_1 e^{i\delta_1} + i a_2 e^{i\delta_2}). \end{array}$$

We define

$$\left.\begin{array}{rcl} I &=& |\boldsymbol{\epsilon}_1\cdot\tilde{\mathbf{E}}|^2+|\boldsymbol{\epsilon}_2\cdot\tilde{\mathbf{E}}|^2 = a_1^2+a_2^2 = |\mathbf{E}_0|^2 \\ Q &=& |\boldsymbol{\epsilon}_1\cdot\tilde{\mathbf{E}}|^2-|\boldsymbol{\epsilon}_2\cdot\tilde{\mathbf{E}}|^2 = a_1^2-a_2^2 \\ U &=& 2\mathrm{Re}\left[(\boldsymbol{\epsilon}_1\cdot\tilde{\mathbf{E}})^*(\boldsymbol{\epsilon}_2\cdot\tilde{\mathbf{E}})\right] = 2a_1a_2\cos(\delta_2-\delta_1) \\ V &=& 2\mathrm{Im}\left[(\boldsymbol{\epsilon}_1\cdot\tilde{\mathbf{E}})^*(\boldsymbol{\epsilon}_2\cdot\tilde{\mathbf{E}})\right] = 2a_1a_2\sin(\delta_2-\delta_1) \end{array}\right\}. \tag{2.22}$$

This four parameters are not independent. Clearly, $I^2 = Q^2+U^2+V^2$. The Stokes parameters can also be written in the base of circular polarisations, $\boldsymbol{\epsilon}_\pm$:

$$\left.\begin{array}{rcl} I &=& |\boldsymbol{\epsilon}_+^*\cdot\tilde{\mathbf{E}}|^2+|\boldsymbol{\epsilon}_-^*\cdot\tilde{\mathbf{E}}|^2 = a_+^2+a_-^2 \\ Q &=& 2\mathrm{Re}\left[(\boldsymbol{\epsilon}_+^*\cdot\ \tilde{\mathbf{E}})^*(\boldsymbol{\epsilon}_-^*\cdot\tilde{\mathbf{E}})\right] = 2a_+a_-\cos(\delta_--\delta_+) \\ U &=& 2\mathrm{Im}\left[(\boldsymbol{\epsilon}_+^*\cdot\ \tilde{\mathbf{E}})^*(\boldsymbol{\epsilon}_-^*\cdot\tilde{\mathbf{E}})\right] = 2a_+a_-\sin(\delta_--\delta_+) \\ V &=& |\boldsymbol{\epsilon}_+^*\cdot\tilde{\mathbf{E}}|^2-|\boldsymbol{\epsilon}_-^*\cdot\tilde{\mathbf{E}}|^2 = a_+^2-a_-^2 \end{array}\right\}. \tag{2.23}$$

$I = 2\langle|\mathbf{E}|^2\rangle$ measures the "intensity" $|\mathbf{E}|^2$ of the wave averaged over a period. Since $\mathbf{E}^2 = \mathbf{B}^2$, we have $I = \langle\mathbf{E}^2+\mathbf{B}^2\rangle$, where $\langle\ldots\rangle$ denotes average over a period $T = 2\pi/\omega$. The energy density averaged over this interval of time is then $u = I/(8\pi)$.

For a linearly polarised wave ($\delta_1 = \delta_2$), $V = 0$, whereas for a circularly polarised wave ($a_1 = a_2$) one has $Q = 0$. In general, Q measures the difference of intensity in directions $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$, and V measures the difference of intensity of helicities $+$ and $-$.

Exercise: Show that under a rotation of $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$ by an angle α around $\hat{\mathbf{k}}$, the Stokes parameters transform as

$$\begin{array}{rcl} I &\to& I \\ Q &\to& Q\cos(2\alpha)+U\sin(2\alpha) \\ U &\to& U\cos(2\alpha)-Q\sin(2\alpha) \\ V &\to& V\ . \end{array}$$

Doppler effect, aberration

For a plane wave, the electromagnetic field is of the form

$$\tilde{F}_{\mu\nu}(x) = f_{\mu\nu}e^{-i(k_\mu x^\mu)}\ , \qquad F_{\mu\nu}(x) = \mathrm{Re}[\tilde{F}_{\mu\nu}(x)]$$

where we define $k^\mu = (\omega/c, \mathbf{k})$. Let $F_{\mu\nu}$ be specified in an inertial frame Σ. In a different inertial frame Σ', related to Σ through the Lorentz transformation $\Lambda_\mu{}^\alpha$, it becomes

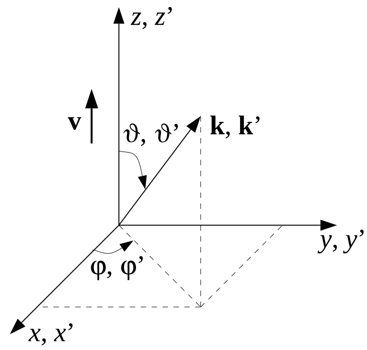
$$\tilde{F}'_{\mu\nu}(x') = \Lambda^\alpha_\mu \Lambda^\beta_\nu \tilde{F}_{\alpha\beta}(x) = \Lambda^\alpha_\mu \Lambda^\beta_\nu f_{\alpha\beta} e^{-i(k_\mu x^\mu)}.$$

For $k'^\mu = \Lambda^\mu{}_\nu k^\nu$ one has $(k_\mu x^\mu) = (k'_\mu x'^\mu)$, and so

$$F'_{\mu\nu}(x') = f'_{\mu\nu} e^{-i(k'_\mu x'^\mu)}, \tag{2.24}$$

with $f'_{\mu\nu} = \Lambda^\alpha_\mu \Lambda^\beta_\nu f_{\alpha\beta}$. Equation (2.24) shows that (k^μ) is really a four-vector and that a plane wave that propagates with the four-vector (k^μ) in Σ does the same in Σ' with four-vector (k'^μ) (and is also a plane wave in this transformed system). We thus have $\mathbf{k}' \cdot \mathbf{x}' - \omega' t' = \mathbf{k} \cdot \mathbf{x} - \omega t$.

We consider a light source in an inertial system Σ and an observer in the system Σ' that moves with speed v along the z direction:



We have, $\beta = v/c$,

$$\begin{aligned}
\Lambda^\mu{}_\alpha &= \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \\
z' &= \gamma(z - vt), \quad x' = x, \quad y' = y \\
t' &= \gamma(t - \frac{\beta}{c} z)
\end{aligned}$$

and

$$\left.\begin{aligned}
k'_x &= k_x, \quad k'_y = k_y \\
k'_z &= \gamma(k_z - \frac{\beta}{c}\omega) \\
\omega' &= \gamma(\omega - v k_z)
\end{aligned}\right\} . \tag{2.25}$$

Since $|k| = \omega/c$, $|k'| = \omega'/c$, it follows that

$$\omega' = \gamma\omega(1 - \beta\cos\vartheta) \; . \tag{2.26}$$

This is the formula of the relativistic *Doppler effect*.

We also want to determine the direction and amplitude of the transformed wave vector, $\mathbf{k}'$. To determine ϑ' we use

$$\cot\vartheta' = \frac{k'_z}{(k'^2_x + k'^2_y)^{\frac{1}{2}}} = \frac{k'_z}{(k^2_x + k^2_y)^{\frac{1}{2}}} \; .$$

With (2.25), this gives

$$\begin{aligned}
\cot\vartheta' &= \frac{\gamma(k_z-\beta\frac{\omega}{c})}{(k_x^2+k_y^2)^{\frac{1}{2}}} = \gamma\cot\vartheta-\gamma\beta\frac{k}{(k_x^2+k_y^2)^{\frac{1}{2}}}\\
\cot\vartheta' &= \gamma\left[\cot\vartheta-\beta\frac{1}{\sin\vartheta}\right]=\gamma\frac{\cos\vartheta-\beta}{\sin\vartheta}\\
\tan\vartheta' &= \frac{1}{\gamma}\frac{\sin\vartheta}{\cos\vartheta-\beta}. \qquad (2.27)
\end{aligned}$$

This equation describes the relativistic *aberration*. Already in the base of symmetry arguments one sees that the angle φ is not modified, and since $k_x = k_x'$, $k_y = k_y'$, we have

$$\varphi' = \varphi. \qquad (2.28)$$

Relation (2.27) is equivalent to (exercise)

$$\left.\begin{aligned}
\sin\vartheta' &= \frac{1}{\gamma}\frac{\sin\vartheta}{1-\beta\cos\vartheta}\\
\cos\vartheta' &= \frac{\cos\vartheta-\beta}{1-\beta\cos\vartheta}\\
\tan\frac{\vartheta'}{2} &= \sqrt{\frac{1+\beta}{1-\beta}}\tan\frac{\vartheta}{2}\ .
\end{aligned}\right\} \qquad (2.29)$$

Equation (2.26) is the formula for the *relativistic Doppler effect.* Equations (2.28) and (2.29) describe the *aberration.*

For $\vartheta = 0$ ($\mathbf{k}$ parallel to $\mathbf{v}$), the **longitudinal** Doppler effect, one has

$$\omega' = \gamma\omega(1-\beta) = \omega\left(\frac{(1-\beta)^2}{1-\beta^2}\right)^{\frac{1}{2}} = \omega\left(\frac{1-\beta}{1+\beta}\right)^{\frac{1}{2}}$$

or, with $\lambda = c/\nu = 2\pi c/\omega$

$$\lambda' = \sqrt{\frac{1+\beta}{1-\beta}}\lambda. \qquad (2.30)$$

We define the redshift Z through the relation

$$\begin{aligned}
1+Z &= \frac{\lambda'}{\lambda} = 1+\frac{\Delta\lambda}{\lambda}, \quad \Delta\lambda = \lambda'-\lambda \qquad (2.31)\\
&= \left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}, \qquad (2.32)
\end{aligned}$$

for small redshifts, $Z \ll 1$, we have $Z \simeq \beta = v/c$, the same formula obtained when analysing the non-relativistic case. The largest redshift observed for distant galaxies is $Z \approx 10$. To which speed does this value correspond?

For $\vartheta = \pi/2$, **transversal** Doppler effect, (2.26) implies

$$\left.\begin{aligned} \omega' &= \gamma\omega = \tfrac{\omega}{\sqrt{1-\beta^2}} \\ 1+Z &= \sqrt{1-\beta^2}; \end{aligned}\right\} \tag{2.33}$$

for small redshifts, $Z \ll 1$, this gives $Z \simeq -\frac{1}{2}\beta^2$. The transversal effect is a purely relativistic effect and is much harder to perceive for low speeds.

Equations (2.28) and (2.29) determine the aberration of light. Let, for example, Σ be the rest frame of a star and let ϑ be the angle of incidence with respect to the Earth's surface. An observer on Earth moves with velocity $\mathbf{v}$ parallel to the planet's surface. For $\vartheta = \pi/2$ (vertical incidence) we have

$$\tan\vartheta' = \pm\frac{1}{\beta\gamma}$$

and for the aberration angle, $\alpha = |\pi/2 - \vartheta'|$,

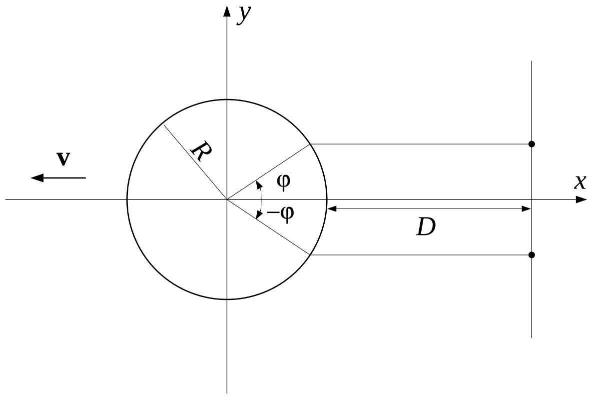
$$\tan\alpha = \beta\gamma. \tag{2.34}$$

In the non-relativistic limit one obtains $\tan\alpha \simeq \beta = v/c$, which is corrected by the γ factor.

Already in the base of symmetry arguments it is clear that there is no aberration if $\vartheta = 0$ or π. For the Earth's speed, $\beta \approx 10^{-4}$, the relativistic correction in (2.34) is not measurable $[O(10^{-8})]$. However, for the case of emission of elementary particles in relativistic motion, this factor γ becomes relevant (for example for the position of counters in the decay process $\pi^0 \to \gamma + \gamma$ for fast π^0).

Apparent superluminal velocity

In the '70s, some radio astronomers discovered that the composing elements of certain "quasars" move away at speeds four to six times the speed of light c. The same effect was observed in the nineties in galactic objects. We will show with an example that the effects of *retardation* can simulate superluminal relative velocities of moving objects. To simplify our notation, we remove the third dimension, the axis z, which does not play a role in our analysis. We consider a disk of gas centred, at $t = 0$, in the $(x,\, y)$ plane of system Σ, and travelling at speed v along $-x$ direction:
In system Σ', which moves at speed v along the $-x$ direction (comoving with the disk), the disk emits a light flash at time $t' = 0$. The observer located at a distance D receives first the light emitted from point $\varphi = 0$. He then observes two rays coming from positions $\pm\varphi$ that move at first away from each other (for



$0 < |\varphi| < \pi/2$) and then come closer together (for $\pi/2 < |\varphi| < \pi$). We will see that (for φ small enough) those two rays appear to be moving at superluminal speed.

With respect to Σ' the ray event has the coordinates

$$t' = 0, \quad x' = R\cos\varphi, \quad y' = R\sin\varphi \quad (-\pi < \varphi \leqslant \pi).$$

In system Σ these coordinates correspond to

$$\begin{aligned} t &= \gamma(t' - \frac{v}{c^2}x') = -\frac{\gamma v}{c^2}R\cos\varphi \\ x &= \gamma(x' - vt') = \gamma R\cos\varphi \\ y &= y' = R\sin\varphi. \end{aligned}$$

The instant of flashing, t, depends on the position, i.e, on φ. The time t_a of arrival at distance D from the center $x = y = 0$ is given by

$$c(t_a - t) = \left[(D - \gamma R\cos\varphi)^2 + R^2\sin^2\varphi\right]^{\frac{1}{2}}.$$

Since $R \ll D$,

$$ct_a = -\frac{\gamma v}{c}R\cos\varphi + [\ldots]^{\frac{1}{2}} \approx -\frac{\gamma v}{c}R\cos\varphi + D - \gamma R\cos\varphi + O\left(\frac{R^2}{D}\right). \quad (2.35)$$

The distance between the rays is $2y = 2R\sin\varphi$. With (2.35) one finds

$$\begin{aligned} y(t_a) &= \sqrt{R^2(1-\cos^2\varphi)} = R\left[1 - \left(\frac{ct_a - D}{R\gamma(1+\beta)}\right)^2\right]^{\frac{1}{2}} \\ &= \left[R^2 - (D - ct_a)^2\frac{1-\beta}{1+\beta}\right]^{\frac{1}{2}}. \end{aligned}$$

The relative speed is then

$$
\begin{aligned}
2\frac{dy}{dt_a} &= \underbrace{2 \times \frac{1}{2}\frac{1}{y}}_{\frac{1}{R\sin\varphi}} 2c \underbrace{(D - ct_a)}_{\gamma R(1+\beta)\cos\varphi} \frac{1-\beta}{1+\beta} = 2c\gamma(1-\beta)\cot\varphi \\
&= 2c\sqrt{\frac{1-\beta}{1+\beta}}\cot\varphi + \mathcal{O}(R/D)\ . \qquad (2.36)
\end{aligned}
$$

This is proportional to $\cot\varphi$ and *diverges* for $\varphi \to 0$, even for low speeds. For an observer, this gives the impression of an object that explodes into two fragments that move away at superluminal speed (as long as φ is small enough).

Chapter 3

Propagation of electromagnetic waves

3.1 Introduction, ponderable media

We consider a medium with *dielectric constant* ε and *permeability* μ, but with neither currents nor charges; ε are μ assumed to be independent of both position and time. We have

$$\mathbf{D} = \varepsilon \mathbf{E} \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B},$$

and

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 \qquad & \nabla \wedge \mathbf{E} + \frac{1}{c}\partial_t \mathbf{B} &= 0 \\ \nabla \cdot \mathbf{E} &= 0 \qquad & \nabla \wedge \mathbf{B} - \frac{\mu\varepsilon}{c}\partial_t \mathbf{E} &= 0, \end{aligned}$$

and therefore

$$\begin{aligned} \left(\Delta - \frac{1}{v^2}\partial_t^2\right)\mathbf{E} &= 0 \\ \left(\Delta - \frac{1}{v^2}\partial_t^2\right)\mathbf{B} &= 0 \end{aligned} \tag{3.1}$$

with

$$v = \frac{c}{\sqrt{\mu\varepsilon}}. \tag{3.2}$$

v is the speed of light in the medium. For plane wave-like solutions we obtain

$$\begin{aligned} \mathbf{E}(\mathbf{x},\, t) &= \mathrm{Re}[\mathbf{E}_0\; e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}] \\ \mathbf{B}(\mathbf{x},\, t) &= \mathrm{Re}[\mathbf{B}_0\; e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}], \end{aligned} \tag{3.3}$$

with $\mathbf{E}_0 = \mathbf{A}_1 + i\mathbf{A}_2$, $\mathbf{E}_0 \in \mathbb{C}^3$, $\mathbf{A}_1$, $\mathbf{A}_2 \in \mathbb{R}^3$, and the dispersion relation

$$k^2 = \frac{\omega^2}{v^2} = \frac{\mu\varepsilon}{c^2}\,\omega^2.$$

Equations $\nabla\cdot\mathbf{E} = \nabla\cdot\mathbf{B} = 0$ lead to

$$\mathbf{k}\cdot\mathbf{E} = \mathbf{k}\cdot\mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{k}\cdot\mathbf{E}_0 = \mathbf{k}\cdot\mathbf{B}_0 = 0.$$

The law of induction gives

$$\mathbf{k}\wedge\mathbf{E}_0 - \frac{\omega}{c}\mathbf{B}_0 = 0.$$

Using $\omega = v|\mathbf{k}| = c/\sqrt{\mu\varepsilon}\,|\mathbf{k}|$ this gives

$$\mathbf{B}_0 = \sqrt{\mu\varepsilon}\,\hat{\mathbf{k}}\wedge\mathbf{E}_0, \quad \mathbf{B} = \sqrt{\mu\varepsilon}\,\hat{\mathbf{k}}\wedge\mathbf{E}. \tag{3.4}$$

The energy flux is given by the Poynting vector,

$$\mathbf{S} = \frac{c}{4\pi}[\mathbf{E}\wedge\mathbf{H}] = \frac{c}{4\pi\mu}\mathbf{E}\wedge\mathbf{B} = \frac{c}{4\pi}\sqrt{\frac{\varepsilon}{\mu}}\,\mathbf{E}\wedge(\hat{\mathbf{k}}\wedge\mathbf{E}).$$

Let us consider $\mathbf{S}$ at point $\mathbf{x} = 0$; we have

$$\mathbf{E} = \mathrm{Re}[(\mathbf{A}_1 + i\mathbf{A}_2)e^{-i\omega t}] = \mathbf{A}_1\cos\omega t + \mathbf{A}_2\sin\omega t$$

and

$$\begin{aligned}
\mathbf{E}\wedge\mathbf{B} = \quad & \sqrt{\mu\varepsilon}\Big\{\mathbf{A}_1\wedge(\hat{\mathbf{k}}\wedge\mathbf{A}_1)\cos^2\omega t + \mathbf{A}_2\wedge(\hat{\mathbf{k}}\wedge\mathbf{A}_2)\sin^2\omega t \\
& + \Big[\mathbf{A}_1\wedge(\hat{\mathbf{k}}\wedge\mathbf{A}_2) + \mathbf{A}_2\wedge(\hat{\mathbf{k}}\wedge\mathbf{A}_1)\Big]\sin\omega t\cos\omega t\Big\} \\
= \quad & \sqrt{\mu\varepsilon}\left(\mathbf{A}_1^2\cos^2\omega t + \mathbf{A}_2^2\sin^2\omega t + 2\mathbf{A}_1\cdot\mathbf{A}_2\sin\omega t\cos\omega t\right)\hat{\mathbf{k}}
\end{aligned}$$

where we have used the relations $\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}$ and $\mathbf{A}_1\cdot\mathbf{k} = \mathbf{A}_2\cdot\mathbf{k} = 0$. The time average of the Poynting vector over a period $T = 2\pi/\omega$ is then

$$\langle\mathbf{S}\rangle = \frac{1}{2}\frac{c}{4\pi}\sqrt{\frac{\varepsilon}{\mu}}\,(\mathbf{A}_1^2 + \mathbf{A}_2^2)\hat{\mathbf{k}} = \frac{c}{8\pi}\sqrt{\frac{\varepsilon}{\mu}}\,|\mathbf{E}_0|^2\hat{\mathbf{k}}. \tag{3.5}$$

In the same way, one finds for the energy density

$$\begin{aligned}
u \quad &= \quad \frac{1}{8\pi}\left(\varepsilon\mathbf{E}^2 + \frac{1}{\mu}\mathbf{B}^2\right) \\
\langle u\rangle \quad &= \quad \frac{1}{16\pi}\left(\varepsilon|\mathbf{E}_0|^2 + \frac{1}{\mu}|\mathbf{B}_0|^2\right) = \frac{\varepsilon}{8\pi}|\mathbf{E}_0|^2
\end{aligned} \tag{3.6}$$

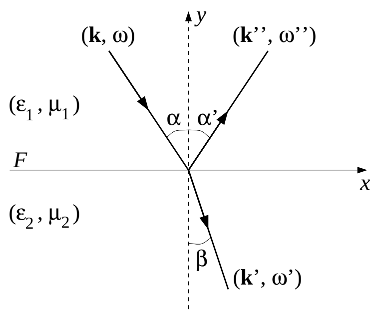
where we have used (3.4). The time average of $|\mathbf{S}|$ is called the intensity I of the wave:

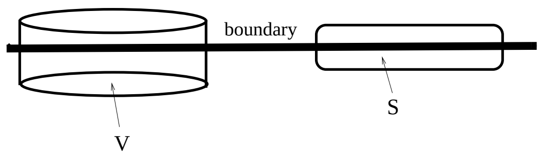
$$I = \frac{c}{8\pi}\sqrt{\frac{\varepsilon}{\mu}}\,\tilde{\mathbf{E}}\cdot\tilde{\mathbf{E}}^* = \frac{c}{8\pi}\sqrt{\frac{\mu}{\varepsilon}}\,\tilde{\mathbf{H}}\cdot\tilde{\mathbf{H}}^* = \sqrt{\frac{1}{\mu\varepsilon}}\,c\,\langle u\rangle = v\langle u\rangle; \tag{3.7}$$

and has the units of energy/(area $\times$ time), [erg/(cm^2 s)].

3.2 Reflection and refraction

As before, we consider electromagnetic waves in homogeneous and isotropic ponderable media but we now add interfaces where μ or ε change their value discontinuously. In the absence of charges ρ and currents $\mathbf{J}$, we analyse what happens when an electromagnetic wave crosses the border between two ponderable media with different dielectric constants ε_1 and ε_2 and different permeabilities μ_1 and μ_2. We consider the following situation with the incident and reflected plane waves and a third plane wave that has crossed the bordering surface F between the media:





Coulomb's law, $\int_{\partial V}(\mathbf{D}\cdot\mathbf{e})ds = 0$, and $\int_{\partial V}(\mathbf{B}\cdot\mathbf{e})ds = 0$, integrated over the boundary surface ∂V of the volume as indicated in the figure, imply that $\mathbf{D}_\perp = \varepsilon\mathbf{E}_\perp$ and $\mathbf{B}_\perp = \mu\mathbf{H}_\perp$ are continuous in the interface. Similarly, Ampère's law gives $\oint_{\partial S}\mathbf{H}\cdot\mathbf{n}dl = c^{-1}\partial_t\int_S\mathbf{D}\cdot\mathbf{e}ds \to 0$ for the limit where the thickness of the surface S indicated in the figure tends to 0. Last but not least, the law of induction, $\nabla\wedge\mathbf{E} + c^{-1}\partial_t\mathbf{B} = 0$, implies $\oint_{\partial S}\mathbf{E}\cdot\mathbf{n}dl \to 0$. From the two last equations we conclude that $\mathbf{E}_\parallel$, $\mathbf{H}_\parallel$ are continuous in the interface.

For a continuous component we have ($A \neq 0$):

$$Ae^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + A''e^{i(\mathbf{k}''\cdot\mathbf{x}-\omega'' t)} = A'e^{i(\mathbf{k}'\cdot\mathbf{x}-\omega' t)}, \quad \forall\,\mathbf{x}\in F.$$

Denoting by $\tilde{\mathbf{k}}$, $\tilde{\mathbf{k}}'$ and $\tilde{\mathbf{k}}''$ the projections of $\mathbf{k}$, $\mathbf{k}'$ and $\mathbf{k}''$ on the plane F, the last equation can be equivalently written as:

$$Ae^{i(\tilde{\mathbf{k}}\cdot\mathbf{x}-\omega t)} + A''e^{i(\tilde{\mathbf{k}}''\cdot\mathbf{x}-\omega'' t)} = A'e^{i(\tilde{\mathbf{k}}'\cdot\mathbf{x}-\omega' t)}, \quad \forall\,\mathbf{x}\in\mathbb{R}^3.$$

For $\mathbf{x} = 0$, this gives $\omega = \omega'' = \omega'$, whereas for $t = 0$ we obtain $\tilde{\mathbf{k}} = \tilde{\mathbf{k}}' = \tilde{\mathbf{k}}''$. Hence, the projections of the three vectors on the plane F are equal to each other. The three vectors $\mathbf{k}$, $\mathbf{k}'$, and $\mathbf{k}''$ thus live in a single common plane (the incidence plane),

$$\mathbf{k},\, \mathbf{k}',\, \mathbf{k}'' \in \mathbb{R}\tilde{\mathbf{k}} \oplus \mathbb{R}\mathbf{e}_y.$$

Considering that

$$|\tilde{\mathbf{k}}| = \frac{\omega}{v_1}\sin\alpha, \quad |\tilde{\mathbf{k}}'| = \frac{\omega}{v_2}\sin\beta \quad \text{and} \quad |\tilde{\mathbf{k}}''| = \frac{\omega}{v_1}\sin\alpha',$$

the fact that all the three vectors are equal leads then to

$$\alpha = \alpha' \quad \text{and} \quad \frac{1}{v_1}\sin\alpha = \frac{1}{v_2}\sin\beta.$$

Let us now go a step further and use $v_1 = c/\sqrt{\varepsilon_1\mu_1}$ and $v_2 = c/\sqrt{\varepsilon_2\mu_2}$, which gives

$$\sqrt{\varepsilon_1\mu_1}\,\sin\alpha = \sqrt{\varepsilon_2\mu_2}\,\sin\beta,$$

or, more insightfully,

$$n_1\,\sin\alpha = n_2\,\sin\beta \quad (\textit{Snell's law}), \tag{3.8}$$

where we have introduced the *refractive index*:

$$n := \sqrt{\varepsilon\mu} \quad (\textit{Maxwell's relation}). \tag{3.9}$$

As you can see, the requirement of continuity of a component has enabled us to find the reflection and refractive angles. However, this is not all we can do; electrodynamics enables us to compute also the intensities and polarisations. To this end, we distinguish two different cases:

(i) $\mathbf{E} = (0,\, 0,\, E_z)$ is orthogonal to the plane where the vectors $\mathbf{k}$,, $\mathbf{k}'$, $\mathbf{k}''$ lye [$(x,\, y)$ plane in the figure, or plane of incidence].

The directions $\hat{\mathbf{k}}$, $\hat{\mathbf{k}}'$, $\hat{\mathbf{k}}''$ are given by

$$\begin{aligned} \hat{\mathbf{k}} &= (\sin\alpha, -\cos\alpha, 0) \\ \hat{\mathbf{k}}'' &= (\sin\alpha, \cos\alpha, 0) \\ \hat{\mathbf{k}}' &= (\sin\beta, -\cos\beta, 0). \end{aligned}$$

Additionally, $\mathbf{H} = \sqrt{\varepsilon_1/\mu_1}\,\hat{\mathbf{k}} \wedge \mathbf{E}$ gives

$$H_x = \sqrt{\frac{\varepsilon_1}{\mu_1}}\,\hat{k}_y E_z, \quad H_y = -\sqrt{\frac{\varepsilon_1}{\mu_1}}\,\hat{k}_x E_z, \quad \text{etc.},$$

which leads to:

“E_z”	“H_x”
E_z	$-(\varepsilon_1/\mu_1)^{1/2}E_z\cos\alpha$
E'_z	$-(\varepsilon_2/\mu_2)^{1/2}E'_z\cos\beta$
E''_z	$+(\varepsilon_1/\mu_1)^{1/2}E''_z\cos\alpha$

Because E_z and H_x are continuous in the interface, we have $E_z + E''_z = E'_z$ and $H_x + H''_x = H'_x$. Thus,

$$\begin{aligned}
E_z + E''_z &= E'_z \\
E_z - E''_z &= E'_z \left(\frac{\varepsilon_2\mu_1}{\varepsilon_1\mu_2}\right)^{\frac{1}{2}} \frac{\cos\beta}{\cos\alpha} \\
E_z &= \frac{E'_z}{2}\left(1 + \left(\frac{\varepsilon_2\mu_1}{\varepsilon_1\mu_2}\right)^{\frac{1}{2}} \frac{\cos\beta}{\cos\alpha}\right) \\
E''_z &= \frac{E'_z}{2}\left(1 - \left(\frac{\varepsilon_2\mu_1}{\varepsilon_1\mu_2}\right)^{\frac{1}{2}} \frac{\cos\beta}{\cos\alpha}\right).
\end{aligned}$$

(ii) **H** is orthogonal to the plane of incidence.

In this case we find (simply by substituting $\mathbf{E} \to \mathbf{H}$, $\varepsilon \to \mu$ and $\mu \to \varepsilon$)

$$\begin{aligned}
H_z &= \frac{H'_z}{2}\left(1 + \left(\frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}\right)^{\frac{1}{2}} \frac{\cos\beta}{\cos\alpha}\right) \\
H''_z &= \frac{H'_z}{2}\left(1 - \left(\frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}\right)^{\frac{1}{2}} \frac{\cos\beta}{\cos\alpha}\right).
\end{aligned}$$

In the important case in which $\mu_1 = \mu_2 = 1$, one has

$$\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^{\frac{1}{2}} = \frac{n_1}{n_2} = \frac{\sin\beta}{\sin\alpha}.$$

All this leads to the following relations for the cases (i) and (ii):

(i) **E** normal to the plane of incidence:

$$\left.\begin{aligned}
E'_z &= E_z\left(\frac{2}{1+\frac{\sin\alpha\cos\beta}{\sin\beta\cos\alpha}}\right) &&= 2E_z\frac{\sin\beta\cos\alpha}{\sin(\alpha+\beta)} \\
E''_z &= E_z\frac{\sin\beta\cos\alpha}{\sin(\alpha+\beta)}\left(1-\frac{\sin\alpha\cos\beta}{\sin\beta\cos\alpha}\right) &&= E_z\frac{\sin(\beta-\alpha)}{\sin(\beta+\alpha)}\ .
\end{aligned}\right\} \tag{3.10}$$

(*ii*) **H** normal to the plane of incidence:

$$\left.\begin{aligned}
H_z' &= H_z\left(\frac{2}{1+\frac{\sin\beta\cos\beta}{\sin\alpha\cos\alpha}}\right) = 2H_z\frac{\sin 2\alpha}{\sin 2\alpha+\sin 2\beta} \\
H_z'' &= H_z\frac{\sin 2\alpha}{\sin 2\alpha+\sin 2\beta}\underbrace{\left(1-\frac{\sin\beta\cos\beta}{\sin\alpha\cos\alpha}\right)}_{1-\frac{\sin 2\beta}{\sin 2\alpha}} \\
&= H_z\frac{\sin 2\alpha-\sin 2\beta}{\sin 2\alpha+\sin 2\beta} \\
&= H_z\frac{\sin\alpha\cos\alpha-\sin\beta\cos\beta}{\sin\alpha\cos\alpha+\sin\beta\cos\beta} = H_z\frac{\tan(\alpha-\beta)}{\tan(\alpha+\beta)}\,.
\end{aligned}\right\} \tag{3.11}$$

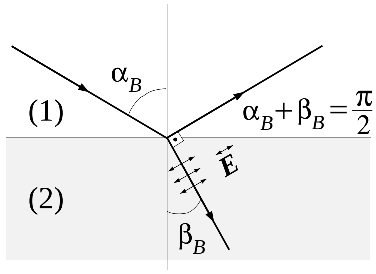
To arrive at the last equation we have used the trigonometric identities $\sin 2\alpha = 2\frac{\tan\alpha}{1+\tan^2\alpha}$ and $\tan(\alpha\pm\beta)=\frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$.

Relations (3.10) and (3.11) are the *Fresnel equations* (which were derived from Maxwell's theory by Lorentz for the first time in 1875).

Remarks:

- We have not used the conditions $\varepsilon\mathbf{E}_\perp = \mathbf{D}_\perp$ and $\mu\mathbf{H}_\perp = \mathbf{B}_\perp$, which should also be continuous at the interfaces, but it is easy to verify that they also hold.

- In case (*i*), there is a reflected ray (if $n_1 \neq n_2$) for all values $0 < \alpha < \pi/2$, whereas in case (*ii*) $\mathbf{H}''$ and, consequently, the intensity of the reflected ray vanish for $\alpha = \alpha_B$, defined through $\alpha_B + \beta_B = \frac{\pi}{2}$. For $\alpha_B + \beta_B = \frac{\pi}{2}$, $\sin\beta_B = \cos\alpha_B$, and
$$\frac{n_2}{n_1} = \frac{\sin\alpha_B}{\sin\beta_B} = \frac{\sin\alpha_B}{\cos\alpha_B}.$$
The angle α_B is known as *Brewster's angle*:
$$\tan\alpha_B = \frac{n_2}{n_1}. \tag{3.12}$$
If unpolarised light reach, with an angle α_B, the surface that divides two media, the reflected ray is then linearly polarised (its electric field is normal to the plane of incidence). When the medium (1) is the vacuum ($\varepsilon_1 = 1$), the microscopic explanation of this phenomenon goes as follows: the reflected ray is created by the oscillations of the dipoles (atoms, molecules) present in medium (2), parallel to the transmitted ray. If the electric field $\mathbf{E}$ thus produced is parallel to $\mathbf{k}''$, the intensity of the reflected wave (which must, when propagating in direction $\mathbf{k}''$, be normal to $\mathbf{k}''$) vanishes.



Total internal reflection

This phenomenon occurs when $n_1 > n_2$. In this case the equation $\sin\beta = (n_1/n_2)\sin\alpha$ has no real solutions for β if $\alpha > \alpha_T$, where

$$\sin\alpha_T = \frac{n_2}{n_1}.$$

Since $\sin\beta$ must be real, only the complex values of β that are of the form $\beta = \pi/2 + i\gamma$ are allowed. Thus

$$\begin{aligned}
\sin\beta &= \frac{e^{i\beta} - e^{-i\beta}}{2i} = \frac{e^{i\frac{\pi}{2}-\gamma} - e^{-i\frac{\pi}{2}+\gamma}}{2i} = \frac{e^{-\gamma}+e^{\gamma}}{2} = \cosh\gamma \\
\cosh\gamma &= \frac{n_1}{n_2}\sin\alpha. \qquad (3.13)
\end{aligned}$$

The absolute value of γ can then be determined with eq. (3.13). The wave vector of the transmitted ray is now complex:

$$\mathbf{k}' = \frac{n_2\omega}{c}(\sin\beta, -\cos\beta, 0) = \frac{n_2\omega}{c}(\cosh\gamma, i\sinh\gamma, 0).$$

The phase of the transmitted wave is

$$e^{i(\mathbf{k}'\cdot\mathbf{x}-\omega t)} = e^{\omega(\frac{in_2}{c}x\cosh\gamma - \frac{n_2}{c}y\sinh\gamma - it)}. \qquad (3.14)$$

For the amplitude of this wave to decrease when $y \to -\infty$, it is necessary to set γ as a *negative* quantity. The refracted wave (3.14) propagates then in direction $\mathbf{e}_x$, along the limit surface, and it is attenuated exponentially for $y < 0$ (evanescent wave).

For case (i) we find

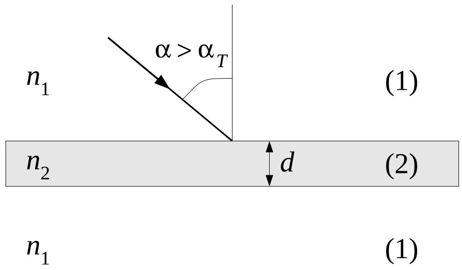
$$\frac{E''_z}{E_z} = \frac{\sin(\frac{\pi}{2} - \alpha + i\gamma)}{\sin(\frac{\pi}{2} + \alpha + i\gamma)} = \frac{\cos\alpha\cosh\gamma + i\sin\alpha\sinh\gamma}{\cos\alpha\cosh\gamma - i\sin\alpha\sinh\gamma}.$$

Since this is a ratio between two complex numbers conjugates of each other, we find that the intensities are equal:

$$\frac{I''}{I} = \frac{E''_z E''^{\star}_z}{E_z E^{\star}_z} = 1. \qquad (3.15)$$

The same is true for case (ii).

The presence of the wave in medium (2) can be verified experimentally. If one picks a thin sheet of material (2) and continues in the other side with medium (1) as shown below,



one finds that the wave is being attenuated by a factor:

$$\begin{aligned} e^{-\frac{n_2\omega}{c} d \sinh\gamma} &= e^{-\frac{d}{d_2}}; \\ d_2(\omega) &:= \frac{c}{n_2\omega\sinh\gamma} \quad \text{is the length of the attenuation.} \end{aligned}$$

Intensity

The intensity is given by equation Eq. (3.7), $I = |\mathbf{S}| = (c/8\pi)n|\mathbf{E}|^2$. To study the intensity of waves, let us consider first two extreme cases.

(a) *Grazing incidence*, $\alpha = \pi/2$

From (3.8)

$$\begin{aligned} \sin\beta &= \frac{n_1}{n_2} \\ \sin(\alpha+\beta) &= \cos\beta\,, \qquad \sin(\beta-\alpha) = -\cos\beta. \end{aligned}$$

From (3.10) and (3.11), we therefore have for cases (i) and (ii)

$$\frac{I''}{I} = 1, \quad I' = 0. \tag{3.16}$$

So, in the grazing incidence case, the reflection is always total (the mirror image of mountains at the opposite shore in a lake, or the mirror image of the setting sun in the sea are examples of this).

(b) *Perpendicular incidence*, $\alpha \approx 0$

In this case, $\sin\alpha \approx \alpha$ and $\sin\beta \approx \beta$, and so $n_1\alpha \approx n_2\beta$. For case (i), equation (3.10) gives

$$\frac{E_z''}{E_z} \approx \frac{n_1 n_2(\beta-\alpha)}{n_1 n_2(\beta+\alpha)} = \frac{n_1-n_2}{n_1+n_2}; \quad \frac{I''}{I} = \frac{(n_1-n_2)^2}{(n_1+n_2)^2}. \tag{3.17}$$

Analogously

$$\begin{aligned} \left|\frac{E_z'}{E_z}\right|^2 &\approx \left(\frac{2\beta}{\alpha+\beta}\right)^2 \approx \frac{4n_1^2}{(n_1+n_2)^2} \\ I' &= \frac{c}{8\pi} n_2 {E_z'}^2 \\ I &= \frac{c}{8\pi} n_1 E_z^2 \end{aligned}$$

and then

$$\frac{I'}{I} = \frac{4n_1 n_2}{(n_1+n_2)^2}. \tag{3.18}$$

The same is obtained for case (ii).

The *reflexion and transmission coefficients* are defined by the ratios

$$R = \left|\frac{\mathbf{e}_y \cdot \mathbf{S}''}{\mathbf{e}_y \cdot \mathbf{S}}\right| = \frac{E_z''^2}{E_z^2} = \frac{I''}{I}, \quad T = \left|\frac{\mathbf{e}_y \cdot \mathbf{S}'}{\mathbf{e}_y \cdot \mathbf{S}}\right| = \frac{n_2}{n_1}\frac{\cos\beta E_z'^2}{\cos\alpha E_z^2} = \frac{I'\cos\beta}{I\cos\alpha}.$$

In the cases of grazing and perpendicular incidences, these coefficients do not depend on the polarisation; they are equal for the two linear polarisations (i) and (ii). This is no longer the case in the most general situation. Indeed, in general, because the reflected intensity depends on the polarisation, the reflected ray is partially polarised even if the incident ray is not (see figure and exercises).

Clearly, for all values of the incidence angle α, $R+T=1$ in the light of energy conservation. The behaviour of the transmitted and reflected intensities as a function of the incidence angle is illustrated in figure 3.1.

Exercise: Show that $R+T=1$ in all cases.

3.3 Optics of metals

In this paragraph we consider good conductors (such as metals). Because the electromagnetic field performs work on the currents, its magnitude decreases quickly by producing heat if the conductivity σ is high (*dissipation*). This is why metals are not transparent but are good reflective materials.

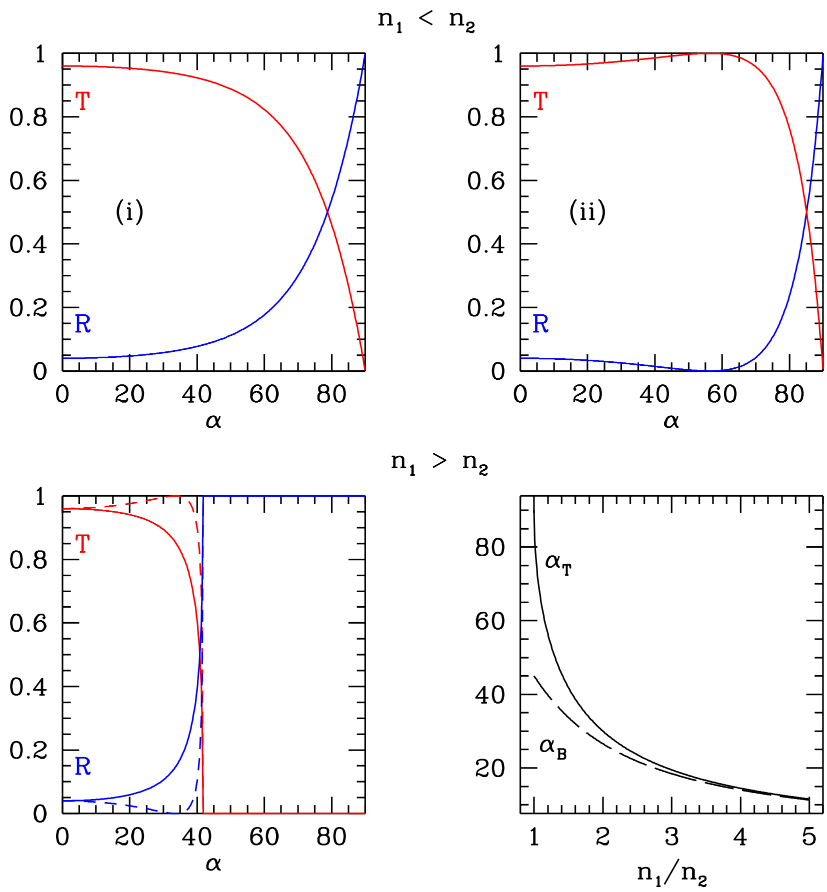


Figure 3.1: The transmission and reflection coefficients are represented for $n_1 < n_2$ (top) and for $n_1 > n_2$ (bottom left, case (ii) in dashed line). The angles α_B and α_T for $n_1 > n_2$ are also indicated (bottom right).

To make our discussion quantitatively concrete, let us make $\mu = 1$ and let us neglect the dispersion of ε and σ – i.e., the dependence of ε and σ on the frequency. (This is not an accurate approximation for visible electromagnetic waves, but it is approximately valid in the infrared regime). In any case, we can assume consider a wave of a given frequency (monochromatic wave), $\mathbf{E} \propto e^{-i\omega t}$. We use Ohm's law, which is valid for good conductors:

$$\mathbf{J} = \sigma \mathbf{E}.$$

Maxwell's equations give

$$\begin{aligned} \nabla \cdot \mathbf{H} &= 0 & \qquad \nabla \wedge \mathbf{E} + \frac{1}{c}\dot{\mathbf{H}} &= 0 \\ \nabla \cdot \mathbf{E} &= \frac{4\pi}{\varepsilon}\rho & \qquad \nabla \wedge \mathbf{H} - \frac{\varepsilon}{c}\dot{\mathbf{E}} &= \frac{4\pi}{c}\sigma \mathbf{E}. \end{aligned}$$

Taking the divergence of the last equation we obtain

$$\frac{\varepsilon}{c}\nabla\cdot\dot{\mathbf{E}}+\frac{4\pi\sigma}{c}\nabla\cdot\mathbf{E}=0,$$

which leads to

$$\dot\rho+\frac{4\pi\sigma}{\varepsilon}\rho=0,$$

an expression that can be also deduced from charge conservation $\dot\rho+\nabla\cdot\mathbf{J}=0$. Its solution is

$$\rho=\rho_0e^{-\frac{t}{\tau}},\quad \tau=\frac{\varepsilon}{4\pi\sigma},\tag{3.19}$$

where τ is the relaxation time. For high values of the conductivity, τ is a very short period of time, the density of interior charges vanishes quickly, and we recover $\nabla\cdot\mathbf{E}=0$, which will be used from now on.

The difference of the curl of the law of induction and the time derivative of Ampère's law leads to:

$$\left(\Delta-\frac{\varepsilon}{c^2}\partial_t^2\right)\mathbf{E}=\frac{4\pi\sigma}{c^2}\dot{\mathbf{E}}.\tag{3.20}$$

The right-hand side of the above equation acts as an attenuation term. In the monochromatic case, $\mathbf{E}$ and $\mathbf{H}\propto e^{-i\omega t}$, eq. (3.20) gives

$$\left(\Delta+\frac{\omega^2\varepsilon'}{c^2}\right)\mathbf{E}=0.\tag{3.21}$$

with

$$\varepsilon'=\varepsilon+i\frac{4\pi\sigma}{\omega},\quad (n_{old}=\sqrt{\varepsilon}).\tag{3.22}$$

We define

$$\left.\begin{array}{rcl} n'^2 &=& \varepsilon',\quad n'=n(1+i\kappa)\\ \mathrm{Re}(n'^2) &=& \varepsilon=n^2(1-\kappa^2)=n_{old}^2\\ \mathrm{Im}(n'^2) &=& \dfrac{4\pi\sigma}{\omega}=2n^2\kappa \end{array}\right\}.\tag{3.23}$$

We have in addition that

$$\nabla\cdot\mathbf{E}=\nabla\cdot\mathbf{H}=0 \text{ and } \quad \frac{i\omega}{c}\mathbf{H}=\nabla\wedge\mathbf{E}.\tag{3.24}$$

Equations (3.21) and (3.24) are formally identical to Maxwell's equations for insulators, except for the fact that here ε', *the dielectric constant, is complex.* Thus, the solutions for insulators are valid in this case. For a plane wave the wave vector is now a complex quantity:

$$\begin{aligned} \mathbf{k}^2 &= \frac{\omega^2}{c^2}\varepsilon'=\frac{\omega^2}{c^2}n'^2, &(3.25)\\ \mathbf{k} &= \frac{\omega}{c}n'\hat{\mathbf{k}}. \end{aligned}$$

The electric field is given by

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} = \mathbf{E}_0 e^{-\frac{\kappa n\omega}{c}(\hat{\mathbf{k}}\cdot\mathbf{x})} e^{i(\frac{n\omega}{c}\hat{\mathbf{k}}\cdot\mathbf{x}-\omega t)}.$$

The attenuation of a plane wave is then measured by the attenuation constant:

$$\chi = 2\frac{\kappa n\omega}{c}, \quad I = I_0 e^{-\chi\hat{\mathbf{k}}\cdot\mathbf{x}}. \tag{3.26}$$

$d = 1/\chi$ is the penetration depth. For copper, for example, one has:

$c\frac{2\pi}{\omega} = \lambda_0$ (vacuum)	1 Å$= 10^{-8}$ cm	$1\ \mu = 10^{-4}$ cm	1 cm
d	0.18×10^{-4} cm	0.18×10^{-2} cm	0.18 cm

An electromagnetic wave cannot penetrate through metal. It is because of this that automobiles can actually protect you from lightings! If the conductivity is very high, $4\pi\sigma/\omega \gg \varepsilon$, we find, from eq. (3.23), that

$$\kappa \approx 1 \quad \text{and} \quad n \approx \sqrt{\frac{2\pi\sigma}{\omega}}.$$

Reflection of light off a metallic surface

Given that the equations for conductors are formally identical to those for insulators, we can, by substituting the dielectric constant ε by the complex constant ε' and n by n', repeat the arguments that led us to the reflection and refraction equations, and in particular we can reproduce Fresnel equations. We should, however, take care of the interpretation and physical meaning of our results!

From Snell's law we have

$$\sin\alpha = n'\sin\beta.$$

n' and hence β are complex numbers and this has some interesting consequences. First, remember that $n'^2 = \varepsilon + i4\pi\sigma/\omega$ is very large for good conductors (because σ is very large) and so:

$$\cos\beta = \sqrt{1-\sin^2\beta} = \sqrt{1-\frac{\sin^2\alpha}{n'^2}} \approx 1.$$

For a refracted wave we obtain

$$\mathbf{k}' = \frac{\omega}{c}n'\hat{\mathbf{k}}', \quad \hat{\mathbf{k}}' = (\sin\beta,\, -\cos\beta,\, 0) \approx (\sin\beta,\, -1,\, 0),$$

which can be rewritten as

$$\mathbf{k}' = k_0 n' \left(\frac{\sin\alpha}{n'},\, -1,\, 0 \right)$$

where $k_0 = \omega/c$ is the wave number in vacuum. With $n' = n(1 + i\kappa)$ this gives

$$e^{i(\mathbf{k}'\cdot\mathbf{x}-\omega t)} = \exp\{i[k_0(x\sin\alpha - ny) - \omega t] + n\kappa k_0 y\}. \tag{3.27}$$

The transmitted wave is then exponentially attenuated for $y < 0$. Fresnel equations give in case (i) (eq. (3.10))

$$\begin{aligned} E'_z &= E_z \frac{2\sin\beta\cos\alpha}{\sin(\alpha+\beta)} = E_z \frac{2\sin\beta\cos\alpha}{\sin\beta\cos\alpha + \sin\alpha\cos\beta} \\ &\approx E_z \frac{(2/n')\sin\alpha\cos\alpha}{\sin\alpha\cos\alpha/n' + \sin\alpha} \approx E_z \frac{2\cos\alpha}{n'} \qquad (|n'| \gg 1). \end{aligned} \tag{3.28}$$

For H'_x we find (from $H'_x = -\sqrt{\varepsilon'} E'_z \cos\beta$) that

$$H'_x \approx -2E_z \cos\alpha. \tag{3.29}$$

The component H'_x remains then finite in $y = 0$ in the limit $|n'| \to \infty$, $(\sigma \to \infty)$. On the other hand, it is easy to verify that the component H'_y decreases. The refracted wave does not penetrate through the material and so it is not of our interest. The reflected wave is more interesting: since β is complex, a phase difference between the incident and the reflected wave appears (see exercise).

Intensity relations. As in (3.17), we restrict ourselves to the case of normal incidence ($\alpha \approx 0$), with $n_1 = 1$:

$$\frac{I''}{I} = \left|\frac{E''_z}{E_z}\right|^2 = \left|\frac{n'-1}{n'+1}\right|^2 = \frac{(n-1)^2 + n^2\kappa^2}{(n+1)^2 + n^2\kappa^2}$$

and then

$$1 - \frac{I''}{I} = \frac{4n}{(n+1)^2 + n^2\kappa^2} \approx \frac{2}{n} \approx 2\sqrt{\frac{\omega}{2\pi\sigma}} = 2\sqrt{\frac{c}{\lambda\sigma}} \tag{3.30}$$

where we have used the fact that $\frac{\sigma}{\omega}$ is very large and so $\kappa^2 \approx 1$, $n \approx 2\pi\sigma/\omega \gg 1$. It is for this reason that metals are ideal reflectors. For copper, for example, one finds, for $\lambda = 12\mu$ (= 12000 Å, infrared),

$$\begin{aligned} 1 - \left.\frac{I''}{I}\right|_{\exp} &= 1.6 \times 10^{-2} \\ 1 - \left.\frac{I''}{I}\right|_{(3.30)} &= 1.4 \times 10^{-2}. \end{aligned}$$

Exercise: At which value of its wavelength does metal with conductivity σ lose its good reflectiveness properties? What happens at this point?

3.4 Dispersion

Until now we have described the properties of materials for which ε, μ and σ are constants. This approximation is often inadequate, specially for electromagnetic waves, which represent fast changing fields: a certain time will elapse before ε, μ and σ react to the variations of the field. An everyday life example of this is the decomposition of white light by a prism, a phenomenon non compatible with a constant (frequency-independent) index of refraction. For monochromatic light (composed by a single constant frequency ω) the results of the previous section remain valid.

To study the effects resulting from the dependence of ε and μ on ω, we first discuss a simple model.

Simple model for $\varepsilon(\omega)$

Let us assume that the motion of an electron in matter can be seen as a damped harmonic oscillator,

$$m[\ddot{\mathbf{x}} + \gamma\dot{\mathbf{x}} + \omega_0^2\mathbf{x}] = -e\mathbf{E}(\mathbf{x},\, t). \tag{3.31}$$

The left-hand side of the above equation is the most general linear approximation, and so it is always valid for sufficiently small deviations $\mathbf{x}$ from equilibrium. For small amplitude oscillations $\mathbf{x}$, the field $\mathbf{E}$ can be evaluated at the electron's average position, $\mathbf{x} = 0$. Let us consider a harmonically oscillating field, $E \propto e^{-i\omega t}$. The same *ansatz* for $\mathbf{x}(t)$ leads to

$$m\mathbf{x}(-\omega^2 - i\omega\gamma + \omega_0^2) = -e\mathbf{E}$$

and so the contribution of this electron to the dipole moment is given by

$$\mathbf{p} = -e\mathbf{x} = \frac{e^2}{m}(\omega_0^2 - \omega^2 - i\omega\gamma)^{-1}\mathbf{E}.$$

Let us consider a substance composed by N molecules per unit volume with Z electrons each. In each molecule we have f_j electrons of natural frequency ω_j (*binding energy* $= \hbar\omega_j$) and friction coefficient (damping) γ_j. It follows that $\sum_j f_j = Z$, and the polarisation per unit volume is

$$\mathbf{P} = \frac{Ne^2}{m}\sum_j f_j(\omega_j^2 - \omega^2 - i\omega\gamma_j)^{-1}\mathbf{E} = \chi_e\mathbf{E}. \tag{3.32}$$

Thus, we obtain the dielectric constant

$$\varepsilon(\omega) = 1 + 4\pi\chi_e = 1 + \frac{4\pi N e^2}{m} \sum_j f_j (\omega_j^2 - \omega^2 - i\omega\gamma_j)^{-1}. \tag{3.33}$$

Clearly, the constants f_j, ω_j and γ_j are determined only after having a quantitative description of the substance. Being incapable of computing these constants in detail, we limit ourselves to the discussion of some general properties.

Anomalous dispersion and resonant absorption

The constants γ_j are in general very small with respect to the resonant frequencies, $\gamma_j/\omega_j \ll 1$, and so $\varepsilon(\omega)$ is almost entirely real (its imaginary component becomes very small) for most frequencies. For $\omega < \omega_j$, the factor $(\omega_j^2 - \omega^2)$ is positive whereas for $\omega > \omega_j$ it is negative (see Jackson §7).

At low values of the frequency, $\omega < \min_j\{\omega_j\}$, all the terms in the summation (3.33) are positive, and therefore

$$\varepsilon(\omega) > 1 \quad \text{at low frequency.}$$

The higher the frequency, the larger is the number of terms in the summation that are negative, making the summation become eventually negative:

$$\varepsilon(\omega) < 1 \quad \text{at high frequency}$$

The behaviour is very interesting for values close to a resonant frequency ω_j: the real part of the denominator vanishes and the term becomes large and imaginary. (see figure 3.2).

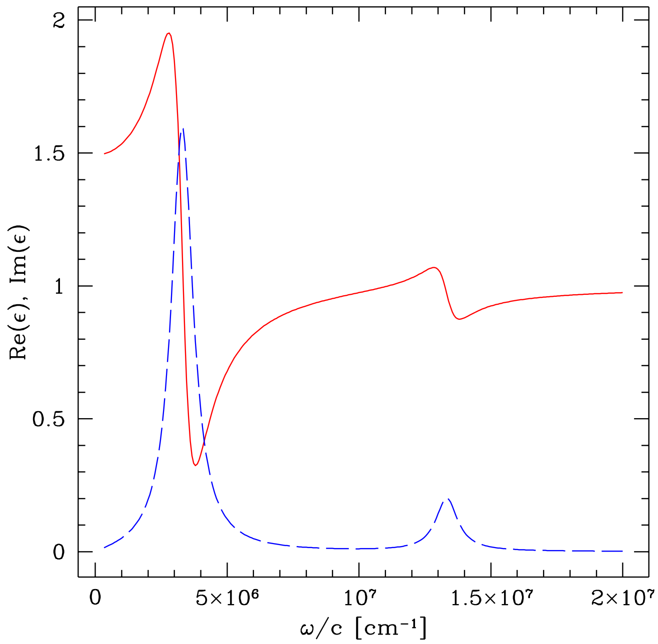


Figure 3.2: The real (solid line) and imaginary (dashed line) parts of the dielectric constant as a function of frequency.

If $\mathrm{Re}[\varepsilon(\omega)]$ increases for increasing values of the frequency ω we then speak of *normal dispersion*. The opposite case, that in which $\mathrm{Re}[\varepsilon(\omega)]$ vanishes as the frequency increases, goes by the name of *anomalous dispersion*. Close to a resonant frequency we then experience the phenomenon of anomalous dispersion. Correspondingly, $\mathrm{Im}[\varepsilon(\omega)]$ conduces to resonant absorption. As we have already seen, this is easily expressed in terms of the real and imaginary parts of the wave number:

$$k = \beta + i\alpha$$

$$\left.\begin{array}{rcl} \beta^2 - \alpha^2 & = & \dfrac{\omega^2}{c^2}\mathrm{Re}(\varepsilon) \\ 2\beta\alpha & = & \dfrac{\omega^2}{c^2}\mathrm{Im}(\varepsilon) \end{array}\right\} k^2 = \frac{\omega^2}{c^2}\varepsilon.$$

If the absorption is not so high ($\alpha \ll \beta$), one finds that

$$\alpha \approx \frac{1}{2}\frac{\mathrm{Im}(\varepsilon)}{\mathrm{Re}(\varepsilon)}\beta; \quad \beta = \sqrt{\mathrm{Re}(\varepsilon)}\,\frac{\omega}{c}.$$

The intensity of the wave decreases with the distance d as

$$I = I_0 e^{-2\alpha d} = I_0 e^{-\left(\frac{\mathrm{Im}(\varepsilon)}{\mathrm{Re}(\varepsilon)}\right)\beta d} = I_0 e^{-\frac{\omega}{c}d\frac{\mathrm{Im}(\varepsilon)}{\sqrt{\mathrm{Re}(\varepsilon)}}}.$$

Electric conductivity at low frequency

In the limit of low frequencies, $\omega \to 0$, the behaviour of $\varepsilon(\omega)$ is qualitatively different if a fraction f_0 of the electrons in each molecule are free, i.e., if they possess the resonant frequency $\omega_0 = 0$. In this case, which corresponds precisely to metals, the dielectric constant at low frequency has a non-negligible imaginary part:

$$\varepsilon(\omega) = \varepsilon_0(\omega) + i\frac{4\pi N e^2 f_0}{m\omega(\gamma_0 - i\omega)}, \tag{3.34}$$

where $\varepsilon_0(\omega)$ accounts for the contribution of bound electrons to the dielectric constant. If we compare this result with our discussion of metals in the previous section, it is clear that the conductivity is given by

$$\sigma = \frac{f_0 N e^2}{m(\gamma_0 - i\omega)}. \tag{3.35}$$

This corresponds roughly to Drude's conductivity model (1900). $N f_0$ is the number of free electrons per unit volume; γ_0 is the friction coefficient that depends on the collision processes of the free electrons with themselves and with the ions of the metal. For copper, for instance, one has

$$\begin{aligned} N &\approx 8 \times 10^{22}\,\text{atomes/cm}^3 \\ \sigma &\approx 5 \times 10^{17}\,\text{sec}^{-1} \quad \text{at low frequency,} \end{aligned}$$

and consequently, for $\omega < 10^{11}\,\text{sec}^{-1}$ (microwaves)

$$\frac{\gamma_0}{f_0} \approx \gamma_0 \approx 3 \times 10^{13}\,\text{sec}^{-1}.$$

High frequency limit, plasma frequency

In the domain where $\omega \gg \max_j\{\omega_j\}$, equation (3.33) takes the form:

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2} \tag{3.36}$$

with

$$\omega_p^2 = \frac{4\pi N Z e^2}{m}. \tag{3.37}$$

The value ω_p^2, which depends uniquely on the electronic density NZ, is the material's *plasma frequency*. The dispersion relation reduces in this situation to

$$ck = \sqrt{\omega^2 - \omega_p^2} \quad \text{or} \quad \omega^2 = \omega_p^2 + c^2 k^2. \tag{3.38}$$

In dielectric media, the limit (3.36) holds only for values $\omega \gg \omega_p$. The dielectric constant is then very close to, but always smaller than 1. It converges to 1 for $\omega \to \infty$. In other situations, such as in our atmosphere's ionosphere or in the interior of stars (as well as in certain laboratory plasmas), the electrons are all free and friction is then negligible. In this last case, the relation (3.36) is valid even for frequencies $\omega \ll \omega_p$.

For $\omega < \omega_p$ the wave number k becomes purely imaginary. This means that any incident electromagnetic wave will be completely reflected by such a plasma. The intensity decreases exponentially at the interior of the plasma with an attenuation constant given by

$$\alpha_{\text{plasma}} \approx \frac{2\omega_p}{c} \quad \text{for} \quad \omega \ll \omega_p.$$

At very high frequencies the metallic dielectric constant (3.34) is of the form

$$\begin{aligned} \varepsilon(\omega) &= \varepsilon_0(\omega) - \frac{\omega_p^2}{\omega^2}, \quad \omega \gg \gamma_0 \\ \text{with} \quad \omega_p^2 &= \frac{4\pi N f_0 e^2}{m}. \end{aligned}$$

For $\omega \ll \omega_p$ this gives a negative dielectric constant and light does not penetrate through the material. But for sufficiently high values of the frequency, $\varepsilon(\omega) > 0$ and the metal becomes transparent! This happens typically in the regime of ultraviolet waves (a phenomenon known as the *ultraviolet transparency of metals*).

3.5 General properties of the dielectric "constant"

We proceed under the hypothesis that the polarisation density $\mathbf{P}$ depends linearly on $\mathbf{E}$ (linear response) (something untrue in the case of very strong fields). The most general relation between $\mathbf{P}$ and $\mathbf{E}$ is then of the form:

$$P_i(\mathbf{x}, t) = \int d^3x' dt' \, K_{ij}(\mathbf{x}, \mathbf{x}'; t, t') E_j(\mathbf{x}', t').$$

If the properties of the medium are time-independent, K_{ij} does not depend on the difference $t - t'$. Moreover, if the medium is homogeneous and isotropic, K_{ij} becomes of the form $K_{ij}(\mathbf{x}, \mathbf{x}'; t - t') = \delta_{ij} K(\mathbf{x} - \mathbf{x}'; t - t')$. Furthermore, the spatial dependence is usually *local*,

$$\begin{aligned} K(\mathbf{x} - \mathbf{x}'; t - t') &= \delta(\mathbf{x} - \mathbf{x}') \, \chi(t - t') \\ \mathbf{P}(\mathbf{x}, t) &= \int dt' \, \chi(t - t') \, \mathbf{E}(\mathbf{x}, t'), \quad \textit{i.e.} \qquad (3.39) \\ \mathbf{P} &= \chi * \mathbf{E}. \qquad (3.40) \end{aligned}$$

Here "$*$" represents the convolution with respect to time[1].

We restrict ourselves to the case $\mu = 1$, $\mathbf{H} = \mathbf{B}$, and we consider dielectrics only.

We perform the Fourier transform with respect to time:

$$\begin{aligned} \mathbf{E}(\mathbf{x},\, t) &= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, \hat{\mathbf{E}}(\mathbf{x},\, \omega)\, e^{-i\omega t} \\ \hat{\mathbf{E}}(\mathbf{x},\, \omega) &= \int_{-\infty}^{\infty} dt\, \mathbf{E}(\mathbf{x},\, t)\, e^{i\omega t}. \end{aligned}$$

Let us assume that $\chi(t) \in L^1(\mathbb{R})$, i.e.

$$\int |\chi(t)|\, dt < \infty.$$

We now use the mathematical result that tells us that for f, $g \in L^1(\mathbb{R})$, one has $\hat{f}$, $\hat{g} \in C_0(\mathbb{R}) \subset L^\infty(\mathbb{R}) \subset L^1(\mathbb{R})$ and $\widehat{f * g} = \hat{f}\hat{g}$. For the Fourier transforms we find therefore the relations

$$\begin{aligned} \hat{\mathbf{P}}(\mathbf{x},\, \omega) &= \hat{\chi}(\omega)\, \hat{\mathbf{E}}(\mathbf{x},\, \omega) && (3.41) \\ \hat{\mathbf{D}}(\mathbf{x},\, \omega) &= \hat{\varepsilon}(\omega)\, \hat{\mathbf{E}}(\mathbf{x},\, \omega) && (3.42) \\ \hat{\varepsilon}(\omega) &= 1 + 4\pi\hat{\chi}(\omega). && (3.43) \end{aligned}$$

In principle, how $\hat{\varepsilon}(\omega)$ depends on the frequency is something determined by quantum mechanics. We have discussed in the previous section a simple and classical model. We now want to discuss some general conditions that one can deduce and that hold even when the quantum effects are not negligible.

(a) *Dissipativity*: Matter polarisation absorbs energy; the work performed on the medium is *positive*. In a medium where $\mu = 1$, $\mathbf{J} = \rho = 0$, with

$$\frac{1}{4\pi}(\mathbf{E}\cdot\dot{\mathbf{D}} + \mathbf{H}\cdot\dot{\mathbf{B}}) = \frac{1}{8\pi}\frac{d}{dt}(\mathbf{E}^2 + \mathbf{H}^2) + \mathbf{E}\cdot\dot{\mathbf{P}},$$

energy conservation gives

$$\nabla\cdot\mathbf{S} + \frac{1}{8\pi}\frac{d}{dt}(\mathbf{E}^2 + \mathbf{H}^2) = -\mathbf{E}\cdot\dot{\mathbf{P}}$$

where we have used $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$. Hence, the work performed on the medium by the electromagnetic field is

$$\begin{aligned} \frac{dW}{dt} &= \int d^3x\, \mathbf{E}\cdot\dot{\mathbf{P}}, \\ W &= \int d^3x\, dt\, \mathbf{E}\cdot\dot{\mathbf{P}} > 0. && (3.44) \end{aligned}$$

[1]The convolution of two functions, $f, g \in L^1(\mathbb{R})$ is defined by $(f * g)(t) = \int f(t-t')g(t')dt'$ One can easily show that $f * g = g * f$; $(f * g)' = f' * g = f * g'$.

Parseval's equation ($\int |f|^2\, dt = \int |\hat{f}|^2\, d\omega$) leads to $\int f^\star g\, dt = \frac{1}{2\pi}\int \hat{f}^\star \hat{g}\, d\omega$.

$$\begin{aligned} 0 < W &= \frac{1}{2\pi}\int d^3x\, d\omega\, \hat{\mathbf{E}}^\star(\mathbf{x},\,\omega)(-i\omega)\hat{\mathbf{P}}(\mathbf{x},\,\omega) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\,(-i\omega)\int d^3x\, \hat{\mathbf{E}}^\star(\mathbf{x},\,\omega)\cdot\hat{\mathbf{E}}(\mathbf{x},\,\omega)\,\hat{\chi}(\omega). \end{aligned}$$

Since $\chi(t)$ is real, $\chi^*(\omega) = \chi(-\omega)$ and similarly for $\mathbf{E}$. Thus,

$$\begin{aligned} 2\pi W &= \int d^3x \int_0^\infty d\omega\,[-i\omega\hat{\chi}(\omega) + i\omega\hat{\chi}(-\omega)]\,|\mathbf{E}(\mathbf{x},\,\omega)|^2 \\ &= \int d^3x \int_0^\infty d\omega\, 2\omega \mathrm{Im}\,[\hat{\chi}(\omega)]\,|\mathbf{E}(\mathbf{x},\,\omega)|^2 \\ &= \int d^3x \int_0^\infty d\omega\, 2\omega \mathrm{Im}\,[\hat{\varepsilon}(\omega)]\,|\mathbf{E}(\mathbf{x},\,\omega)|^2 > 0. \end{aligned}$$

As this has to be true for any value of the electromagnetic field $\mathbf{E}(\mathbf{x},\,\omega)$, we conclude that

$$\mathrm{Im}\,[\hat{\chi}(\omega)] = \frac{\mathrm{Im}\,[\hat{\varepsilon}(\omega)]}{4\pi} > 0 \quad \forall\, \omega > 0\,. \tag{3.45}$$

As we had already seen in our simple model, the imaginary part of $\hat{\varepsilon}$ determines the dissipation (absorption). Any non-stationary process occurring in a realistic medium is irreversible at some point. This is why there are always losses and dissipation, that is to say, $\mathrm{Im}\,[\hat{\varepsilon}(\omega)] > 0$ for all $\omega > 0$. The set of frequencies for which $\mathrm{Im}\,[\hat{\varepsilon}(\omega)]$ is very small constitute the domain of transparency of the medium.

(b) *Causality*: The polarisation $\mathbf{P}(\mathbf{x},\,t)$ cannot depend on the future value the electromagnetic field. Consequently,

$$\chi(t) = 0 \quad \text{for} \quad t < 0. \tag{3.46}$$

3.6 Kramers-Kronig dispersion relations

(Jackson, 7.10)

From the causality condition ($\chi(t) = 0$ for $t < 0$) we deduce here some properties of the Fourier transform of $\chi(t)$,

$$\hat{\chi}(\omega) = \int_0^\infty dt\,\chi(t)\,e^{i\omega t}. \tag{3.47}$$

In this equation, we consider ω a complex variable and we assume that $\int_0^\infty |\chi(t)|\,dt < \infty$. In this case, (3.47) is well defined for all $\omega \in \mathbb{C}$ with $\mathrm{Im}(\omega) \geq 0$ because

$|e^{i\omega t}| = e^{-\mathrm{Im}(\omega)t} \leq 1$ for $t \geq 0$. For $\mathrm{Im}(\omega) > 0$, the function in (3.47) is analytic (holomorphic). Besides, $\hat{\chi}(\omega)$ is continuous and bounded in the plane $\mathrm{Im}(\omega) \geqslant 0$ and $\hat{\chi}(\omega) \to 0$ for $|\omega| \to \infty$ uniformly in all directions of the complex plane where $\mathrm{Im}(\omega) \geqslant 0$. (This is an elementary consequence of the *Riemann-Lebesgue lemma*:[2] $\chi \in L^1(\mathbb{R}) \Rightarrow \hat{\chi} \in C_0(\mathbb{R})$ and for all $\mathrm{Im}(\omega) > 0$ the convergence is even faster.)

Let us summarize the properties of the susceptibility $\hat{\chi}(\omega)$:

$$
\begin{array}{rl}
(i) & \hat{\chi}(\omega) \text{ is analytic in the region of the complex plane where } \\
 & \mathrm{Im}(\omega) > 0. \\
(ii) & \hat{\chi} \text{ is continuous and bounded in the } \mathrm{Im}(\omega) \geqslant 0 \text{ half-plane} \\
 & \text{and } |\hat{\chi}(\omega)| \to 0 \text{ for } |\omega| \to \infty \text{ uniformly in all directions} \\
 & \text{for which } \mathrm{Im}(\omega) \geqslant 0. \\
(iii) & \hat{\chi}^\star(\omega) = \hat{\chi}(-\omega^\star). \\
(iv) & \mathrm{Im}\,[\hat{\chi}(\omega)] > 0 \text{ for } 0 < \omega < \infty. \\
(v) & \hat{\chi}(0) = \int_0^\infty \chi(t)\,dt > 0.
\end{array}
\tag{3.48}
$$

The last property follows from the fact that the static polarisation "costs" a certain amount of work. For $\mathbf{E}(\mathbf{x}, t) = H(t)\mathbf{E}(\mathbf{x})$, H being the Heavyside function, that work is given by (3.44)

$$
\int d^3x \int_0^{\mathbf{P}(\infty)} \mathbf{E}(\mathbf{x}) \cdot d\mathbf{P} = \int_0^\infty dt\,\chi(t) \int d^3x\,\mathbf{E}^2(\mathbf{x}) = \hat{\chi}(0) \int d^3x\,\mathbf{E}^2(\mathbf{x}) > 0.
$$

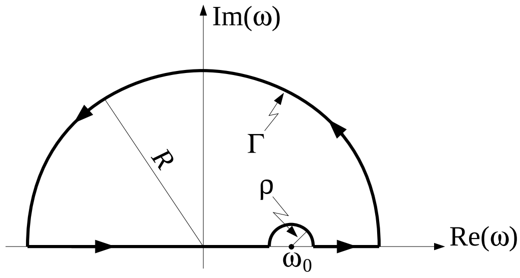
The Kramers-Kronig relations link the real and imaginary parts of $\hat{\chi}(\omega)$. In order to derive those relations we need a number of results of the theory of complex functions that you have already worked out in your math courses (I hope):

Cauchy theorem — Let $f : D \to \mathbb{C}, D \subset \mathbb{C}$ be an analytic (holomorphic) function and let $\Gamma : [0, 1] \to D$ be a closed path whose interior belongs to D. For $\omega_0 \in D$, we have

$$
\int_\Gamma \frac{f(\omega)\,d\omega}{\omega - \omega_0} = \begin{cases} 0 & \text{if } \omega_0 \text{ lies outside } \Gamma \\ 2\pi i f(\omega_0) & \text{if } \omega_0 \text{ lies inside } \Gamma. \end{cases}
$$

We consider the following path

[2] See compl. de math.



According to the Cauchy theorem,

$$\int_\Gamma \frac{\hat{\chi}(\omega)\, d\omega}{\omega - \omega_0} = 0. \tag{3.49}$$

In the limit $\rho \to 0$, the contribution of the semicircle $\omega = \omega_0 + \rho e^{i\phi}, \quad \pi \geq \phi \geq 0$ gives

$$\lim_{\rho \to 0} \int_\frown \frac{\hat{\chi}(\omega)\, d\omega}{\omega - \omega_0} = i \lim_{\rho \to 0} \int_\pi^0 \hat{\chi}(\omega) d\phi = -i\pi\hat{\chi}(\omega_0). \tag{3.50}$$

On the other hand, the principal value of integral (3.49) is defined by

$$\lim_{\rho \to 0} \left\{ \int_{-\infty}^{\omega_0 - \rho} \frac{\hat{\chi}(\omega)\, d\omega}{\omega - \omega_0} + \int_{\omega_0 + \rho}^{\infty} \frac{\hat{\chi}(\omega)\, d\omega}{\omega - \omega_0} \right\} \equiv P.V. \int_{-\infty}^{\infty} \frac{\hat{\chi}(\omega)\, d\omega}{\omega - \omega_0}.$$

In the limit $R \to \infty$ the circle at infinity *does not contribute* because of property (*ii*) of (3.48) and because of the exponential decreasing for $\mathrm{Im}(\omega) > 0$. (3.49) then gives

$$P.V. \int_{-\infty}^{\infty} \frac{\hat{\chi}(\omega)\, d\omega}{\omega - \omega_0} - i\pi\hat{\chi}(\omega_0) = 0.$$

The real and imaginary parts of this equation are the Kramers-Kronig relations:

$$\left.\begin{array}{rcl} \mathrm{Re}\,[\hat{\chi}(\omega_0)] &=& \dfrac{1}{\pi}\, P.V. \displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{Im}\,[\hat{\chi}(\omega)]}{\omega - \omega_0}\, d\omega \\ \mathrm{Im}\,[\hat{\chi}(\omega_0)] &=& -\dfrac{1}{\pi}\, P.V. \displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{Re}\,[\hat{\chi}(\omega)]}{\omega - \omega_0}\, d\omega \end{array}\right\} . \tag{3.51}$$

From property (*iii*) we conclude (for $\omega \in \mathbb{R}$)

$$\begin{array}{rcl} \mathrm{Re}\,[\hat{\chi}(\omega)] &=& \mathrm{Re}\,[\hat{\chi}(-\omega)] \\ \mathrm{Im}\,[\hat{\chi}(\omega)] &=& -\mathrm{Im}\,[\hat{\chi}(-\omega)]\ . \end{array}$$

We can go further and write (3.51) in the form

$$\left.\begin{aligned} \mathrm{Re}\,[\hat\chi(\omega_0)] &= \frac{2}{\pi}\,P.V.\int_0^\infty \frac{\omega \mathrm{Im}\,[\hat\chi(\omega)]}{\omega^2-\omega_0^2}\,d\omega \\ \mathrm{Im}\,[\hat\chi(\omega_0)] &= -\frac{2\omega_0}{\pi}\,P.V.\int_0^\infty \frac{\mathrm{Re}\,[\hat\chi(\omega)]}{\omega^2-\omega_0^2}\,d\omega\ . \end{aligned}\right\} \tag{3.52}$$

The first equation implies for the dielectric function $\hat\varepsilon(\omega) = 1 + 4\pi\hat\chi(\omega)$ that:

$$\mathrm{Re}\,[\hat\varepsilon(\omega_0)] = 1 + \frac{2}{\pi}\,P.V.\int_0^\infty \frac{\omega \mathrm{Im}\,[\hat\varepsilon(\omega)]}{\omega^2-\omega_0^2}\,d\omega. \tag{3.53}$$

Equations (3.51) to (3.53) are the Kramers-Kronig dispersion relations. They show that the absorption properties ($\mathrm{Im}\,[\hat\chi]$) together with causality determine entirely the dispersion $\mathrm{Re}\,[\hat\chi]$ and vice-versa. These relations also play an important role in particle physics: diffusion and absorption are intimately related via the causality condition. Equations (3.52) were first derived by Kramers (1927) and Kronig (1926), whom wrote them as relations for the refractive index $n(\omega) = \sqrt{\hat\varepsilon(\omega)}$. (Relation (3.39) can also be justified in the case in which $\chi(t)$ is a distribution.)

Theorem — The properties (3.48) of the susceptibility lead to the following results:

- $\hat\chi(\omega)$ does not take real values in the half-plane $\mathrm{Im}(\omega) \geqslant 0$ apart from the imaginary axis.
- Along the imaginary axis $\hat\chi(\omega)$ decreases monotonically from $\chi_0 \equiv \hat\chi(0) > 0$ to $\lim_{s\to\infty}\hat\chi(is) = 0$.

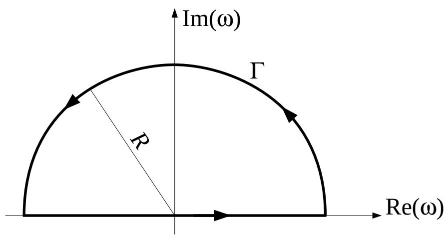
In particular, $\hat\chi$ has no zeros in the upper half-plane.

Proof — We use a result of the theory of complex functions: let $D \subset \mathbb{C}$ be an open set and $f : D \to \mathbb{C}$ a meromorphic function (analytic in all its domain except in a set of isolated points z_i in which the function diverges as $(z - z_i)^{-n_i}$, z_i being then called a pole of order n_i). Let Γ be the boundary of a compact subset of D, $K \subset D$, such that f has no poles on Γ and $f(z) \neq a\ \forall z \in \Gamma$. In this situation it is possible to show that

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)\,dz}{f(z)-a} = Z - P$$

where Z is the number of zeros (multiplied by their order) and P is the number of poles (multiplied by their order) of $f - a$ in K. (This result is a simple corollary of the residue theorem.)

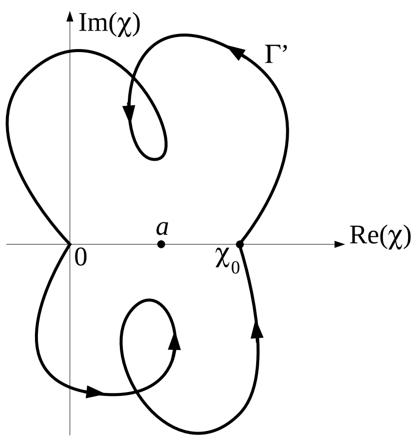
We apply this result to the function $\hat\chi(\omega)$ by choosing Γ as shown below in the figure:



In the upper half-plane $\hat{\chi}$ has no poles. Hence

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{\hat{\chi}'(\omega)\,d\omega}{\hat{\chi}(\omega)-a}=Z.$$

Let Γ' be the image of Γ under the application $\omega \mapsto \hat{\chi}(\omega)$. We take the limit $R \to \infty$. Since $\mathrm{Im}\,[\hat{\chi}(\omega)] > 0$ for $0 < \omega < \infty$ [property(iv)] and thus $\mathrm{Im}\,[\hat{\chi}(\omega)] < 0$ for $-\infty < \omega < 0$, $\Gamma' = \hat{\chi}(\Gamma)$ intersects the real axis only at $\chi_0 = \hat{\chi}(0)$ and $0 = \hat{\chi}(\infty)$. Γ' looks then as follows



We conclude that

$$Z = \frac{1}{2\pi i}\int_{\Gamma}\frac{\hat{\chi}'(\omega)\,d\omega}{\hat{\chi}(\omega)-a} = \frac{1}{2\pi i}\int_{\Gamma'}\frac{d\hat{\chi}}{\hat{\chi}-a} = \begin{cases} 1 & \text{if } 0 < a < \chi_0 \\ 0 & \text{if } a \in \mathbb{R}\backslash[0,\, \chi_0]. \end{cases}$$

Hence, every value $a \in [0, \chi_0]$ is reached exactly once by the function $\hat{\chi}$. But since $[\chi_0,\, 0]$ is the image of the imaginary half-plane $i\mathbb{R}_+$, there exists $x \in \mathbb{R}_+$ such that $\hat{\chi}(ix) = a$. Therefore $\hat{\chi}$ (which is real on the imaginary axis) decreases *monotonically* along the imaginary axis. The real values of $\hat{\chi}$ are just the images of $i\mathbb{R}_+$ (on the upper half-plane).

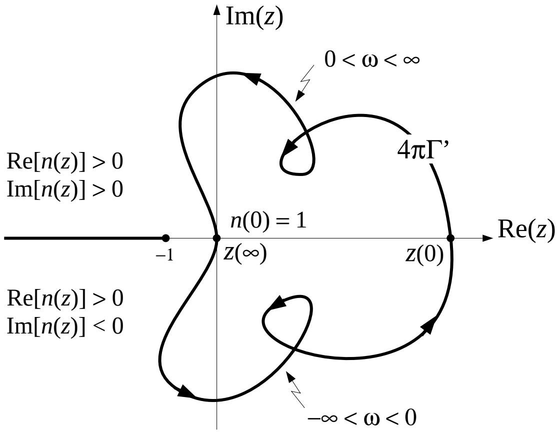
This theorem allows us to unambiguously define the refractive index as

$$n(\omega) = \sqrt{\hat{\varepsilon}(\omega)}. \tag{3.54}$$

For $z \equiv 4\pi\hat{\chi}(\omega)$ we then have

$$n(z) = \sqrt{1+z}.$$

We choose the root by limiting the z plane from -1 to $-\infty$



In the reduced plane, $D = \mathbb{C}\backslash\{z \in \mathbb{R},\, z \leqslant -1\}$ we uniquely define the root by $n(0) = 1$. This gives $\mathrm{Re}(n) > 0$ and $\mathrm{Im}(n) \geqslant 0$ for $\mathrm{Im}(z) \geqslant 0$, $\mathrm{Re}(n) > 0$ and $\mathrm{Im}(n) \leqslant 0$ for $\mathrm{Im}(z) \leqslant 0$.

If we choose the path Γ in the plane ω as in the theorem, we find its image in the plane $z = 4\pi\hat{\chi}(\omega)$. The upper circle $|\omega| = R$ is applied on 0 in the limit $R \to \infty$. The imaginary axis, $i\mathbb{R}_+$ is applied on the real interval $[0,\, z(0)]$. The image of the upper half-plane $\mathrm{Im}(\omega) > 0$ is the interior of the path $4\pi\Gamma'$.

From these facts we can draw the following conclusions:

$$\begin{array}{rl}
(i) & n(\omega) \text{ is analytic for } \mathrm{Im}(\omega) > 0. \\
(ii) & n(\omega) \text{ is continuous and bounded for } \mathrm{Im}(\omega) \geqslant 0 \text{ and } |n(\omega) - 1| \to 0 \text{ for } |\omega| \to \infty \text{ uniformly in all the directions for which } \mathrm{Im}(\omega) \geqslant 0. \\
(iii) & n^\star(\omega) = n(-\omega^\star). \\
(iv) & 1 \leqslant n(0) < \infty. \\
(v) & \mathrm{Im}\,[n(\omega)] > 0 \text{ for } \omega \in \mathbb{R}_+ \text{ and } \mathrm{Im}\,[n(\omega)] < 0 \text{ for } \omega < 0 \\
 & \mathrm{Re}\,[n(\omega)] > 0 \text{ for all } \omega \text{ with } \mathrm{Im}(\omega) \geqslant 0.
\end{array} \tag{3.55}$$

Along the same lines as those leading to (3.52), we can derive a dispersion relation for n:

$$\mathrm{Re}\,[n(\omega)] \;=\; 1 + \frac{2}{\pi}\, P.V. \int_0^\infty \frac{\omega' \mathrm{Im}\,[n(\omega')]}{\omega'^2 - \omega^2}\, d\omega'\,, \tag{3.56}$$

$$\mathrm{Im}\,[n(\omega)] \;=\; -\frac{2\omega}{\pi}\,P.V.\int_0^\infty \frac{\mathrm{Re}\,[n(\omega')-1]}{\omega'^2-\omega^2}\,d\omega'\,. \tag{3.57}$$

3.7 Electromagnetic waves in dispersive media

(Jackson, 7.11)

We first consider a homogeneous and isotropic medium with a dynamic dielectric function $\varepsilon(\omega)$, with $\mu = 1$ and with neither charges nor currents ($\rho = \mathbf{J} = 0$). We decompose the fields $\mathbf{E}(\mathbf{x},\, t)$ and $\mathbf{B}(\mathbf{x},\, t)$ into modes of frequency ω, $\hat{\mathbf{E}}(\mathbf{x},\, \omega)$ and $\hat{\mathbf{B}}(\mathbf{x},\, \omega)$ (the Fourier transforms of $\mathbf{E}$ and $\mathbf{B}$ with respect to time). $\hat{\mathbf{E}}$ and $\hat{\mathbf{B}}$ are complex but they satisfy the reality condition $\hat{\mathbf{E}}^\star(\omega) = \hat{\mathbf{E}}(-\omega)$ (and similarly for $\hat{\mathbf{B}}$).

In the Maxwell equations for $\hat{\mathbf{E}}$ and $\hat{\mathbf{B}}$ the derivatives are replaced by factors $-i\omega$, i.e.

$$\begin{aligned} \nabla\cdot\hat{\mathbf{E}} &= 0 \qquad & \nabla\wedge\hat{\mathbf{E}} - \frac{i\omega}{c}\hat{\mathbf{B}} &= 0 \\ \nabla\cdot\hat{\mathbf{B}} &= 0 \qquad & \nabla\wedge\hat{\mathbf{B}} + \frac{i\omega}{c}\varepsilon(\omega)\hat{\mathbf{E}} &= 0. \end{aligned}$$

As before, these equations lead to the wave equations for $\hat{\mathbf{E}}$ and $\hat{\mathbf{B}}$:

$$\left(\Delta + k^2\right)\hat{\mathbf{E}} = \left(\Delta + k^2\right)\hat{\mathbf{B}} = 0 \tag{3.58}$$

with

$$k^2 = \frac{\omega^2}{c^2}n^2(\omega). \tag{3.59}$$

Given that $\varepsilon = n^2$, then k^2 is also complex. Equations (3.58) have as solutions the following plane waves:

$$\hat{\mathbf{E}}(\mathbf{x},\, \omega) = \mathbf{E}_0(\omega)\, e^{i\mathbf{k}\cdot\mathbf{x}},$$

with $\mathbf{k} = (\omega/c)n(\omega)\hat{\mathbf{k}}$, $\hat{\mathbf{k}} \in \mathbb{R}^3$, $|\hat{\mathbf{k}}| = 1$ and $n(\omega) = \sqrt{\varepsilon(\omega)}$ where this square root is defined as in the previous paragraph:

$$\begin{aligned} &\mathrm{Re}\,[n(\omega)] > 0 \quad \text{and} \quad \mathrm{Im}\,[n(\omega)] > 0 \quad \text{for} \quad \omega > 0 \\ &\mathrm{Re}\,[n(\omega)] > 0 \quad \text{and} \quad \mathrm{Im}\,[n(\omega)] < 0 \quad \text{for} \quad \omega < 0. \end{aligned}$$

$$\mathbf{E}(\mathbf{x},\, t) = \frac{1}{2\pi}\int d\omega\, \mathbf{E}_0(\omega)\, e^{i\left(\frac{\omega}{c}\mathrm{Re}[n(\omega)]\hat{\mathbf{k}}\cdot\mathbf{x} - \omega t\right)}\, e^{-\frac{\omega}{c}\mathrm{Im}[n(\omega)]\hat{\mathbf{k}}\cdot\mathbf{x}}.$$

Since $\omega\,\mathrm{Im}\,[n(\omega)]$ is *always* positive, the last factor always represents an *attenuation* in the direction in which the wave propagates. As in the optics of metals, we thus obtain an exponentially damped wave that propagates in direction $\hat{\mathbf{k}}$. The wavelength that corresponds to a given frequency ω is determined by $\mathrm{Re}\,[n(\omega)]$

(*dispersion*). The attenuation is determined by $\mathrm{Im}\,[n(\omega)]$ (*damping*). If $\mathrm{Re}\,[n(\omega)]$ is an increasing function, the dispersion is said to be "normal". Accordingly, if $\mathrm{Re}\,[n(\omega)]$ is a decreasing function, we speak of "anomalous" dispersion.

Several different velocities play a role in the description of the propagation of electromagnetic waves in dispersive media.

Phase velocity

As we have seen, the crest (surface of constant phase) of a wave of the form $\mathbf{E}(\mathbf{x},\, t) = \mathbf{E}_0 \exp\{i(\mathbf{k}\cdot\mathbf{x} - \omega t)\}$ with $\mathbf{k} = (\omega/c)\, n(\omega)\hat{\mathbf{k}}$ propagates in the direction $\hat{\mathbf{k}}$ with the velocity

$$v_{\rm phase} = \frac{\omega}{|\mathrm{Re}[\mathbf{k}]|} = \frac{c}{\mathrm{Re}[n(\omega)]}.$$

For a refractive index $\mathrm{Re}[n(\omega)] < 1$, the phase velocity is greater than c (example: high frequency plasmas, $\omega > \omega_p$, or $ck = \sqrt{\omega^2 - \omega_p^2}$).

Group velocity $v_{\rm g} \equiv \partial\omega/\partial k$

We consider ω as a function of $k = |\mathbf{k}|$: $\omega = \omega(k)$. We study, for simplicity, a scalar wave (the E_1 component of the electric field, say). In particular, we consider a wave packet of the form

$$\begin{aligned} \phi(\mathbf{x},\, t) &= \mathrm{Re}\,[\psi(\mathbf{x},\, t)] \\ \psi(\mathbf{x},\, t) &= \frac{1}{(2\pi)^3}\int d^3k\, a(\mathbf{k})\, e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}, \end{aligned}$$

where $a(\mathbf{k})$ is focused around a value $\mathbf{k}_0$, i.e. $a(\mathbf{k})$ is different from zero only for values of $\mathbf{k}$ sufficiently close to $\mathbf{k}_0$. The energy density is proportional to ϕ^2. We define the *energy centre*, $\langle \mathbf{x}(t)\rangle$, through

$$\langle \mathbf{x}(t)\rangle \equiv \frac{\int d^3x\, \mathbf{x}\, \phi^2(\mathbf{x},\, t)}{\int d^3x\, \phi^2(\mathbf{x},\, t)}.$$

$\phi = \frac{1}{2}(\psi + \psi^\star)$ and so $\phi^2 = \frac{1}{4}\psi^2 + \frac{1}{4}(\psi^\star)^2 + \frac{1}{2}\psi\psi^\star$. The first two terms oscillate rapidly at frequencies close to $2\omega(k_0)$ (*Zitterbewegung*, "trembling motion") whereas the term $|\psi|^2$ varies slowly with respect to time. We neglect the first two terms, which vanish in time averages taken over an interval much longer than the wave's period $T_0 \approx 2\pi/\omega(k_0)$. The oscillating contributions taken away, we are left with:

$$\langle \mathbf{x}(t)\rangle = \frac{\int d^3x\, \mathbf{x}\, |\psi(\mathbf{x},\, t)|^2}{\int d^3x\, |\psi(\mathbf{x},\, t)|^2}.$$

To compute the denominator we use

$$\int d^3x\, x_j |\psi(\mathbf{x},\, t)|^2 = \frac{1}{(2\pi)^6} \int d^3x\, d^3k\, d^3k'\, \underbrace{x_j\, e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}}_{-i\frac{d}{dk_j} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}} a(\mathbf{k}) a^\star(\mathbf{k}')\, e^{-i(\omega-\omega')t}.$$

Integration by parts conduces to

$$\begin{aligned}\langle x_j(t)\rangle &= \frac{1}{(2\pi)^6} \int d^3x\, d^3k\, d^3k'\, e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \left[\left(a^\star(\mathbf{k}') i \frac{\partial a}{\partial k_j}\right) e^{-i(\omega-\omega')t} + \right. \\ &\left. + t\, a^\star(\mathbf{k}') a(\mathbf{k}) \frac{\partial \omega}{\partial k_j} e^{-i(\omega-\omega')t} \right] \Big/ \int d^3x\, |\psi(\mathbf{x},\, t)|^2.\end{aligned}$$

The integration over d^3x gives the distribution $(2\pi)^3 \delta(\mathbf{k}-\mathbf{k}')$ and so we find, with $\int d^3x\, |\psi|^2 = \frac{1}{(2\pi)^3} \int d^3k\, |a|^2$, that:

$$\langle x_j(t)\rangle = \frac{\int d^3k\, a^\star(\mathbf{k}) i \frac{\partial a}{\partial k_j}}{\int d^3k\, |a|^2} + t \frac{\int d^3k\, \frac{\partial \omega}{\partial k_j} |a|^2}{\int d^3k\, |a|^2}.$$

For the speed of propagation of the centre of energy, we obtain

$$\begin{aligned}\frac{d}{dt}\langle \mathbf{x}(t)\rangle &= \langle \frac{\partial \omega}{\partial \mathbf{k}} \rangle \\ \langle \frac{\partial \omega}{\partial \mathbf{k}} \rangle &\equiv \frac{\int d^3k\, \frac{\partial \omega}{\partial \mathbf{k}} |a|^2}{\int d^3k\, |a|^2},\end{aligned}$$

the average of the group velocity, $\mathbf{v}_{\mathrm{g}}$, defined by $\mathbf{v}_{\mathrm{g}} = \frac{\partial \omega}{\partial \mathbf{k}}$. In the isotropic case, $\omega = \omega(|\mathbf{k}|)$, $\mathbf{v}_{\mathrm{g}} = \hat{\mathbf{k}} \frac{d\omega}{dk}$, $v_{\mathrm{g}} = |\mathbf{v}_{\mathrm{g}}| = \frac{d\omega}{dk}$ or, with $\omega = v_{\mathrm{phase}}\, k$, $v_{\mathrm{g}} = v_{\mathrm{phase}} + k \frac{dv_{\mathrm{phase}}}{dk}$. The relation $\omega = \frac{ck}{\mathrm{Re}[n(\omega)]}$ gives

$$v_{\mathrm{g}} = \frac{d\omega}{dk} = \frac{1}{dk/d\omega} = \frac{c}{\mathrm{Re}(n) + \omega \frac{d[\mathrm{Re}(n)]}{d\omega}}. \tag{3.60}$$

For normal dispersion $\left(\frac{d[\mathrm{Re}(n)]}{d\omega} > 0 \right)$ and $\mathrm{Re}(n) > 1$ that leads to

$$v_{\mathrm{g}} < v_{\mathrm{phase}} < c.$$

For anomalous dispersion, $\frac{d[\mathrm{Re}(n)]}{d\omega}$ can become negative. In this scenario, it is possible that $v_{\mathrm{g}} \gg v_{\mathrm{phase}}$ and, moreover, that $v_{\mathrm{g}} > c$. Large values of $\left|\frac{d[\mathrm{Re}(n)]}{d\omega}\right|$ traduce to a fast variation of ω as a function of $\mathbf{k}$. In this case, the oscillating terms in the expression for $\langle \mathbf{x}(t)\rangle$ cannot be neglected and the center of energy moves in a very complicated way!

Speed of propagation

Here we show that the true speed of propagation is always $\leqslant c$, even if $v_{\rm g}$ and $v_{\rm phase}$ can be greater than c. This is a consequence of the analytic properties of $n(\omega)$ (which are themselves a consequence of causality).

We consider a linearly polarised wave that represents the superposition of plane waves of direction $\mathbf{e} = (1, 0, 0)$, such that the vector $\mathbf{E}$ has non-zero component only along y, $E(\mathbf{x},\, t) = E(x, t)$:

$$\begin{aligned}
\mathbf{E} &= (0,\, E(x,\, t),\, 0) \\
E(x,\, t) &= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, A(\omega)\, e^{i(kx-\omega t)} \\
A(\omega) &= \int_{-\infty}^{\infty} dt\, E(0,\, t) e^{i\omega t}
\end{aligned}$$

We assume that this wave reaches $x = 0$ at $t = 0$, and not before, i.e.:

$$E(0,\, t) = 0 \quad \text{for} \quad t < 0,$$

hence

$$A(\omega) = \int_0^{\infty} dt\, E(0,\, t) e^{i\omega t}.$$

We now show that in this case $E(x,\, t) = 0$ for $t < \frac{x}{c}$, which means that $v_{\rm signal} \leqslant c$). For simplicity, we assume in our proof that $|E(0,\, t)|$, $|\frac{d}{dt}E(0,\, t)|$ and $|\frac{d^2}{dt^2}E(0,\, t)|$ are integrable over $0 \leqslant t < \infty$. This is the case if, for example, $E(0, t)$ describes a pulse of finite duration. Under these considerations, $A(\omega)$ is analytic for $\mathrm{Im}(\omega) > 0$, and bounded and continues for $\mathrm{Im}(\omega) \geqslant 0$. Since the time derivative of $E(0,\, t)$ corresponds to a multiplication of A by a factor $-i\omega$, the same properties hold for ωA and $\omega^2 A$. Since A and $\omega^2 A$ are bounded for $\mathrm{Im}(\omega) \geqslant 0$, there exists a constant $a > 0$ such that

$$|A(\omega)| \leqslant \frac{a}{1 + |\omega|^2} \quad \text{for} \quad \mathrm{Im}(\omega) \geqslant 0.$$

$n(\omega)$ is analytic for $\mathrm{Im}(\omega) > 0$ and so $e^{i(kx-\omega t)} = e^{(\frac{i\omega n(\omega)}{c}x - i\omega t)}$ is also analytic for $\mathrm{Im}(\omega) > 0$.

$$\begin{aligned}
\left|e^{i(kx-\omega t)}\right| &= e^{-\mathrm{Im}(\omega)(\frac{x}{c}-t)} e^{-\mathrm{Im}[\omega(n(\omega)-1)]\frac{x}{c}} \\
\mathrm{Im}\left[\omega(n(\omega)-1)\right] &= \mathrm{Im}(\omega)(\mathrm{Re}(n)-1) + \mathrm{Re}(\omega)\mathrm{Im}(n).
\end{aligned}$$

The last term is always non-negative, according to property (v) of (3.55). We thus find

$$\left|A(\omega)\, e^{i(kx-\omega t)}\right| \leqslant \frac{a}{1+\omega^2} \exp\left[-\mathrm{Im}(\omega)\left(\frac{x}{c} - t + (\mathrm{Re}\left[n(\omega)\right] - 1)\frac{x}{c}\right)\right].$$

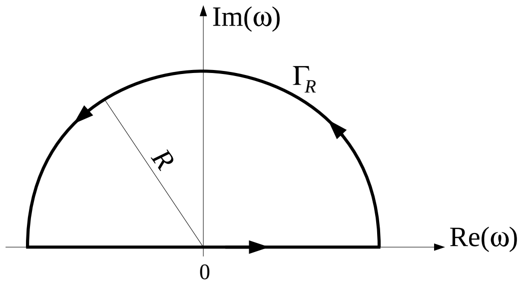
For $|\omega| \to \infty$, $\mathrm{Re}\,[n(\omega)] \to 1$. If $\frac{x}{c} - t > 0$ the inequality written below is satisfied for sufficiently large values of $|\omega|$:

$$\left|A(\omega)\, e^{i(kx-\omega t)}\right| \leqslant \frac{a}{1+|\omega|^2} e^{-\alpha \mathrm{Im}(\omega)}$$

with $\alpha = \frac{x}{c} - t + (\mathrm{Re}\,[n(\omega)] - 1)\frac{x}{c} > 0$. If we substitute the integral

$$E(x,\, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, A(\omega)\, e^{i(kx-\omega t)}$$

by the integral along the path Γ_R,



In the limit $R \to \infty$, the large circle does not contribute and

$$E(x,\, t) = \frac{1}{2\pi} \lim_{R\to\infty} \int_{\Gamma_R} d\omega\, A(\omega)\, e^{i(kx-\omega t)} \quad \text{for} \quad \frac{x}{c} - t > 0.$$

But $A(\omega)\, e^{i(kx-\omega t)}$ is analytic in the upper half-plane and so, according to Cauchy's theorem, the integral of this function along the path Γ_R vanishes, which gives

$$E(x,\, t) = 0 \quad \text{for} \quad \frac{x}{c} - t > 0.$$

One can show by the same reasoning that for a wave such that $\mathbf{E}(\mathbf{x},\, t) = 0$ for $|x| > L$ at $t < 0$, $\mathbf{E}(\mathbf{y},\, t) = 0$ for $|\mathbf{y}| > L + ct$, independently of the specific form of the dispersion law.

Even if the phase and group velocities can be greater than c, an electromagnetic signal does not propagate at a speed higher than c.

3.8 The optical limit

The wavelength of *visible* light is within the range $(4-7) \times 10^{-5}$ cm. This is, from a macroscopic viewpoint, a very small length, and the limit $\lambda \to 0$ is often a good approximation for the propagation of light. In this limit one obtains the geometrical optics (ray optics), which we derive in this section. The following considerations are completely equivalent to those that lead to classical mechanics as a limit of quantum mechanics.

The Eikonal equation

In addition to the wavelength λ, we consider a macroscopic distance L along which the amplitude and polarisation of the wave change in a significant manner. Geometrical optics is a good approximation if

$$\lambda \ll L.$$

We analyse the case of an isotropic and insulating medium that is overall inhomogeneous with respect to the scale L. We neglect the imaginary part of ε (absorption). A quasi-plane wave is an electromagnetic field of the form

$$\left.\begin{array}{rcl} \mathbf{E}(\mathbf{x},\, t) &=& \mathbf{E}_0(\mathbf{x},\, t)\, e^{i\phi(\mathbf{x},\, t)} \\ \mathbf{B}(\mathbf{x},\, t) &=& \mathbf{B}_0(\mathbf{x},\, t)\, e^{i\phi(\mathbf{x},\, t)} \end{array}\right\}, \tag{3.61}$$

where $\mathbf{E}_0$, $\mathbf{B}_0$, $\nabla\phi$ and $\partial_t\phi$ vary much more slowly than ϕ. For an authentic plane wave, all those magnitudes are constant: $\mathbf{E}_0$ and $\mathbf{B}_0$ are constants; $\phi = \mathbf{k}\cdot\mathbf{x} - \omega t$ and therefore $\nabla\phi = \mathbf{k}$ and $\partial_t\phi = \omega$.

$$L^{-1} = \max\left\{\partial_i\varepsilon,\, \partial_i\mu,\, \frac{\partial_i E_{0j}}{E_{0j}},\, \frac{\partial_i B_{0j}}{B_{0j}},\, \frac{\partial_j\partial_i\phi}{\partial_i\phi},\, \frac{\partial_i\partial_t\phi}{\partial_t\phi},\right\}.$$

Maxwell's equations (in the absence of charges and currents) are

$$\begin{array}{rclcrcl} \nabla\cdot\mathbf{B} &=& 0 && \nabla\wedge\mathbf{E} + \frac{1}{c}\dot{\mathbf{B}} &=& 0 \\ \nabla\cdot\mathbf{D} &=& 0 && \nabla\wedge\mathbf{H} - \frac{1}{c}\dot{\mathbf{D}} &=& 0. \end{array}$$

We insert in the above relations the fields (3.61) and neglect all the derivatives of ε, μ, $\mathbf{E}_0$ and $\mathbf{B}_0$. This leads to

$$\left.\begin{array}{rclcrcl} \nabla\phi\cdot\mathbf{B}_0 &=& 0 && \nabla\phi\wedge\mathbf{E}_0 + \frac{1}{c}\,\dot{\phi}\,\mathbf{B}_0 &=& 0 \\ \nabla\phi\cdot\mathbf{E}_0 &=& 0 && \nabla\phi\wedge\mathbf{B}_0 - \frac{\mu\varepsilon}{c}\,\dot{\phi}\,\mathbf{E}_0 &=& 0 \end{array}\right\}. \tag{3.62}$$

We conclude that $\mathbf{E}_0$, $\nabla\phi$ and $\mathbf{B}_0$ form an orthogonal system in the trigonometric sense. From (3.62) we obtain $-\mathbf{B}_0 = \frac{c}{\dot{\phi}}\,(\nabla\phi\wedge\mathbf{E}_0)$ and so

$$\begin{aligned} \nabla\phi\wedge\left(\frac{c}{\dot{\phi}}\nabla\phi\wedge\mathbf{E}_0\right) &= -\frac{\mu\varepsilon}{c}\,\dot{\phi}\,\mathbf{E}_0, \\ -\nabla\phi\wedge(\nabla\phi\wedge\mathbf{E}_0) &= \frac{\mu\varepsilon}{c^2}\,\dot{\phi}^2\,\mathbf{E}_0, \\ \mathbf{E}_0\,(\nabla\phi)^2 &= \frac{\mu\varepsilon}{c^2}\,\dot{\phi}^2\,\mathbf{E}_0 \end{aligned}$$

where

$$(\nabla\phi)^2 = \frac{\mu\varepsilon}{c^2}\,\dot{\phi}^2 \tag{3.63}$$

(3.63) is the *eikonal equation* (ϕ is the eikonal of geometrical optics).

In most applications, ε and μ are time-independent. In this case, the wave has a constant frequency,

$$\phi(\mathbf{x},\, t) = \chi(\mathbf{x}) - \omega t.$$

With $n^2 = \varepsilon\mu$, the eikonal equation leads then to

$$(\nabla\chi)^2 = \frac{n^2(\omega)\omega^2}{c^2}. \tag{3.64}$$

This is the core equation of geometrical optics. The surfaces defined by $\chi(\mathbf{x}) =$ *constant* are the constant-phase surfaces or wavefronts.

Light rays

The time averages of the electric and magnetic energy densities are

$$\begin{aligned} \langle u_\mathrm{e}\rangle &= \frac{\varepsilon}{16\pi}\mathbf{E}_0\cdot\mathbf{E}_0^\star \\ \langle u_\mathrm{m}\rangle &= \frac{1}{16\pi\mu}\mathbf{B}_0\cdot\mathbf{B}_0^\star. \end{aligned}$$

With (3.62) one finds [by using $\mathbf{B}_0^\star\cdot(\nabla\chi\wedge\mathbf{E}_0) = \mathbf{E}_0\cdot(\mathbf{B}_0^\star\wedge\nabla\chi)$]

$$\langle u_\mathrm{e}\rangle = \frac{c/\omega}{16\pi\mu}\mathbf{E}_0\cdot(\mathbf{B}_0^\star\wedge\nabla\chi) = \langle u_\mathrm{m}\rangle. \tag{3.65}$$

For the time average of the Poynting vector we obtain

$$\begin{aligned} \langle\mathbf{S}\rangle &= \frac{c}{8\pi\mu}\mathrm{Re}\,[\mathbf{E}_0\wedge\mathbf{B}_0^\star] \\ &= \frac{c^2}{8\pi\mu\omega}\mathrm{Re}\,[\mathbf{E}_0\wedge(\nabla\chi\wedge\mathbf{E}_0^\star)] \\ &= \frac{c^2}{8\pi\mu\omega}(\mathbf{E}_0\cdot\mathbf{E}_0^\star)\nabla\chi \\ &= \frac{2c^2}{n^2\omega}\langle u_\mathrm{e}\rangle\nabla\chi \end{aligned}$$

or, with $\langle u\rangle \equiv \langle u_\mathrm{e}\rangle + \langle u_\mathrm{m}\rangle = 2\langle u_\mathrm{e}\rangle$,

$$\langle\mathbf{S}\rangle = \frac{c}{n}\langle u\rangle\frac{\nabla\chi}{n\omega/c}.$$

But from the eikonal equation (3.64)

$$\hat{\mathbf{s}} \equiv \frac{\nabla\chi}{(n\omega/c)} \tag{3.66}$$

is a unit vector and

$$\langle \mathbf{S}\rangle = v_{\rm phase}\langle u\rangle\hat{\mathbf{s}} \tag{3.67}$$

where $v_{\rm phase} = \frac{c}{n}$ is the phase velocity in the medium. The time average of the Poynting vector has the same direction as $\hat{\mathbf{s}}$ and its amplitude is the average energy density multiplied by the phase velocity.

The *rays of light* are integral curves of the vectorial field $\hat{\mathbf{s}}$; they are therefore trajectories orthogonal to the surfaces $\chi =$ constant. These integral curves satisfy the equation

$$\frac{d\mathbf{x}}{ds} = \mathbf{s}\Big(\mathbf{x}(s)\Big) \quad \text{or} \quad \frac{n\omega}{c}\frac{d\mathbf{x}}{ds} = \nabla\chi\Big(\mathbf{x}(s)\Big). \tag{3.68}$$

Since $(d\mathbf{x}/ds)^2 = \hat{\mathbf{s}}^2 = 1$, s is the arclength parameter. We can derive, from (3.68) and (3.64), a differential equation for $\mathbf{x}(s)$ containing only $n(\mathbf{x})$, the refractive index determined by the properties of the medium:

$$\begin{aligned} \frac{d}{ds}\left[n\frac{dx_j}{ds}\right] &= \frac{c}{\omega}\frac{d}{ds}\left[\partial_j\chi\Big(\mathbf{x}(s)\Big)\right] = \frac{c}{\omega}\frac{dx_i}{ds}\cdot\partial_i\partial_j\chi = \\ &= \frac{(c/\omega)^2}{n}\partial_i\chi\cdot\partial_i\partial_j\chi = \frac{(c/\omega)^2}{2n}\partial_j\left[\nabla\chi\right]^2 = \frac{1}{2n}\partial_j n^2 = \\ &= \partial_j n. \end{aligned}$$

We have then found the following equation for the rays:

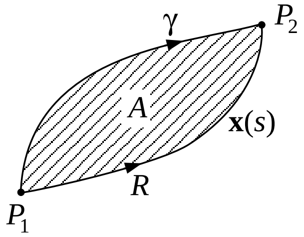
$$\frac{d}{ds}\left[n\frac{d\mathbf{x}(s)}{ds}\right] = \nabla n. \tag{3.69}$$

Fermat's principle

Fermat's principle states that the light that arrives at point P_2 coming from point P_1 takes the path along which the integral

$$\int_{P_1}^{P_2} n\,ds \tag{3.70}$$

attains its minimum value as compared to the values it would have along neighbouring paths with the same ending points. Let us consider a path γ neighbour of the path R taken by light:



Since $(\omega/c)\, n\hat{\mathbf{s}} = \nabla\chi$, $\nabla\wedge(n\hat{\mathbf{s}}) = (c/\omega)\nabla\wedge(\nabla\chi) = 0$. So we have $\int_A \nabla\wedge(n\hat{\mathbf{s}})\, da = 0$. But, after Stokes' theorem,

$$\int_A \nabla \wedge (n\hat{\mathbf{s}})\, da = \int_\gamma n\hat{\mathbf{s}} \cdot d\mathbf{l} - \int_R n\, ds = 0.$$

With $\left|\int_\gamma n\hat{\mathbf{s}} \cdot d\mathbf{l}\right| \leqslant \int_\gamma n\, dl$ we then obtain

$$\int_R n\, ds \leqslant \int_\gamma n\, dl.$$

In the exercises you will show that (3.69) is nothing but the Euler-Lagrange equation for the Lagrangian given by (3.70)

Hamiltonian formulation

We now show that the eikonal equation, $(\nabla\chi)^2 = n^2\omega^2/c^2$, can be interpreted as a Hamilton-Jacobi equation with $S^* = \chi - \omega t$. (See the analytical mechanics course.)

This means that the function

$$H(\mathbf{k}, \mathbf{x}) = \omega$$

where $\mathbf{k} = \nabla\chi$ is a Hamilton function and H is determined by the solution of

$$\mathbf{k}^2 = n^2\omega^2/c^2 \tag{3.71}$$

for the variable ω.

We want to show that the eikonal equation implies the canonical equations for $(\mathbf{k}, \mathbf{x})$. With this in mind, we first consider a focused (tenuously-varying) wave group with wave vector $\mathbf{k}$. For the wave's centre of energy, $\mathbf{x}(t)$, the vector $\dot{\mathbf{x}}(t)$ is the group velocity,

$$\dot{\mathbf{x}}(t) = \frac{\partial\omega}{\partial\mathbf{k}} = \frac{\partial H}{\partial\mathbf{k}}\ ; \tag{3.72}$$

which constitutes the first canonical equation. To arrive at the second canonical equation we use the fact that $\mathbf{k} = \nabla\chi$ and so

$$\dot{k}_j = \frac{d}{dt}\partial_j\chi = \frac{\partial^2\chi}{\partial x^j \partial x^i}\dot{x}^i = \frac{\partial^2\chi}{\partial x^j \partial x^i}\frac{\partial H}{\partial k_i}\ . \tag{3.73}$$

We still need to prove that $\frac{\partial^2\chi}{\partial x^j\partial x^i}\frac{\partial H}{\partial k_i} = -\frac{\partial H}{\partial x^j}$. To arrive at this result, we differentiate eq. (3.71) with respect to k_j and x^i. This gives

$$2k^j = \frac{1}{c^2}\frac{\partial}{\partial\omega}\left(n^2(\omega)\omega^2\right)\frac{\partial H}{\partial k_j} \tag{3.74}$$

$$0 = \frac{2n\omega^2}{c^2}\frac{\partial n}{\partial x^j} + \frac{1}{c^2}\frac{\partial}{\partial\omega}\left(n^2(\omega)\omega^2\right)\frac{\partial H}{\partial x^j} . \tag{3.75}$$

Here we have used the fact that $(\mathbf{k},\mathbf{x})$ are independent variables and $\omega = H(\mathbf{k},\mathbf{x})$. We now differentiate the eikonal equation with respect to x^j. (Note that in the eikonal equation ω is just a parameter and χ and n are not functions of $\mathbf{x}$.):

$$2\frac{\partial\chi}{\partial x^j}\frac{\partial^2\chi}{\partial x^i\partial x^j} = \frac{2n\omega^2}{c^2}\frac{\partial n}{\partial x^j} .$$

With eq. (3.75) this gives

$$2\frac{\partial\chi}{\partial x^j}\frac{\partial^2\chi}{\partial x^i\partial x^j} = -\frac{1}{c^2}\frac{\partial}{\partial\omega}\left(n^2(\omega)\omega^2\right)\frac{\partial H}{\partial x^j} .$$

Substituting $\frac{\partial\chi}{\partial x^j}$ by k_j we find, with (3.74), that

$$\frac{1}{c^2}\frac{\partial}{\partial\omega}\left(n^2(\omega)\omega^2\right)\frac{\partial H}{\partial k_j}\frac{\partial^2\chi}{\partial x^i\partial x^j} = -\frac{1}{c^2}\frac{\partial}{\partial\omega}\left(n^2(\omega)\omega^2\right)\frac{\partial H}{\partial x^j} .$$

Under the hypothesis that $\frac{\partial(n^2(\omega)\omega^2)}{\partial\omega} \neq 0$, this implies

$$\frac{\partial^2\chi}{\partial x^i\partial x^j}\frac{\partial H}{\partial k_j} = -\frac{\partial H}{\partial x^j},$$

which is precisely what we wanted to show. With (3.73) we thus arrive at the second canonical equation,

$$\dot{k}_j = -\frac{\partial H}{\partial x^j} . \tag{3.76}$$

For $\mathbf{x}(t)$ and $\mathbf{k}(t)$ we then have the canonical equations

$$\dot{x}^j = \frac{\partial H}{\partial k_j} , \qquad \dot{k}_j = -\frac{\partial H}{\partial x^j} .$$

Exercise: By using (3.74) and (3.75) one can re-derive equation (3.69) of light rays.

Chapter 4

Emission of electromagnetic waves

So far we have studied the *propagation* of electromagnetic waves without being concerned about how they are originated. However, there is something we already know from Maxwell's equations: the electromagnetic fields are produced by charges and currents. How the electromagnetic waves are created by charges and currents that vary in time is precisely the subject of this chapter.

4.1 Wave zone, multipole expansion

In the Lorentz gauge ($\partial_\mu A^\mu = 0$), Maxwell's equations (with $\mu = \varepsilon = 1$) take the form [(1.55) and (1.56)]:

$$\Box\phi = 4\pi\rho \quad \text{and} \quad \Box\mathbf{A} = \frac{4\pi}{c}\mathbf{J},$$

with the retarded solution (see chap. I)

$$\phi(\mathbf{x},\, t) = \int \frac{\rho\,(\mathbf{x}',\, t - |\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|} d^3x' \tag{4.1}$$

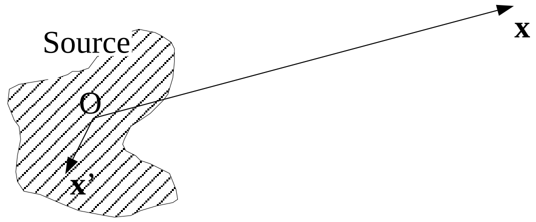
and

$$\mathbf{A}(\mathbf{x},\, t) = \frac{1}{c}\int \frac{\mathbf{J}\,(\mathbf{x}',\, t - |\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|} d^3x'. \tag{4.2}$$

These expressions represent *retarded* potentials, which are the unique solutions of equations (1.55) and (1.56) and satisfy the condition that

$$\begin{aligned} &\phi(\mathbf{x},\, t) = 0, \quad \mathbf{A}(\mathbf{x},\, t) = 0 && \text{if} \\ &\rho(\mathbf{x}',\, t') = 0, \quad \mathbf{J}(\mathbf{x}',\, t') = 0 && \forall\, \mathbf{x}' \in \mathbb{R}^3 \;\text{ with }\; |\mathbf{x}-\mathbf{x}'| < c(t-t')\,. \end{aligned}$$

Consider a finite region of size d that contains charges and currents. Let us determine the energy emitted by this region (which we will call the source) at a large distance:



We place the origin of the coordinate system at a point of the source. Since $|\mathbf{x}| \gg |\mathbf{x}'|$, we can perform the expansion ($\mathbf{n} = \mathbf{x}/|\mathbf{x}|$, $r = |\mathbf{x}|$)

$$|\mathbf{x} - \mathbf{x}'| = r - \mathbf{n} \cdot \mathbf{x}' + O\left(\mathbf{x}'^2/r\right). \tag{4.3}$$

The dominant terms in (4.1) and (4.2) decrease as $1/r$. In order to get them, we can replace the denominator with $1/r$. For the retarded time we use eq. (4.3):

$$t_{\rm ret} = t - \frac{|\mathbf{x} - \mathbf{x}'|}{c} = t - \frac{r}{c} + \frac{\mathbf{n} \cdot \mathbf{x}'}{c} + O\left(\mathbf{x}'^2/rc\right). \tag{4.4}$$

The last term, $O(\mathbf{x}'^2/rc)$, is negligible only if the temporal variation of ρ and $\mathbf{J}$ during the interval d^2/rc is small (d is the diameter of the surface). In the case of harmonic oscillations ($\sim e^{i\omega t}$), this translates into the condition

$$\frac{d^2}{rc} \ll \frac{1}{\omega}.$$

In the general case, ω represents a typical frequency of the source. The *wave zone* is defined by the conditions

$$r \gg d \quad \text{and} \quad r \gg \frac{d^2\omega}{c}. \tag{4.5}$$

Within the wave zone we can then approximate the electromagnetic potentials by

$$\phi(\mathbf{x}, t) \simeq \frac{1}{r}\int \rho\left(\mathbf{x}', t - \frac{r}{c} + \frac{\mathbf{n} \cdot \mathbf{x}'}{c}\right) d^3x' \tag{4.6}$$

$$\mathbf{A}(\mathbf{x}, t) \simeq \frac{1}{rc}\int \mathbf{J}\left(\mathbf{x}', t - \frac{r}{c} + \frac{\mathbf{n} \cdot \mathbf{x}'}{c}\right) d^3x'. \tag{4.7}$$

Let us now compute the contributions $O(1/r)$ of the electromagnetic field, keeping in mind that in the limit $r \to \infty$ the higher order terms do not contribute to the energy emission in the cone of given aperture angle $\Delta\Omega$:

$$-\nabla\phi(\mathbf{x}, t) = \frac{\mathbf{n}}{rc}\int \dot{\rho}\left(\mathbf{x}', t - \frac{r}{c} + \frac{\mathbf{n} \cdot \mathbf{x}'}{c}\right) d^3x' + O\left(\frac{1}{r^2}\right).$$

The continuity equation gives $[t' \equiv t - r/c + \mathbf{n}\cdot\mathbf{x}'/c]$

$$\dot\rho(\mathbf{x}',\, t') = -\nabla'\cdot\mathbf{J}(\mathbf{x}',\, t')|_{t'} = -\nabla'\cdot\mathbf{J}\left(\mathbf{x}',\, t - \frac{r}{c} + \frac{\mathbf{n}\cdot\mathbf{x}'}{c}\right)\bigg|_t + \frac{1}{c}\mathbf{n}\cdot\dot{\mathbf{J}}(\mathbf{x}',\, t')|_t,$$

where $(\ldots)|_{t'}$ implies that the derivative is taken at t' constant. Gauss's theorem gives

$$\int \nabla'\cdot\mathbf{J}\left(\mathbf{x}',\, t - \frac{r}{c} + \frac{\mathbf{n}\cdot\mathbf{x}'}{c}\right) d^3x' = \int_{\mathcal{S}_\infty} \mathbf{J}\cdot\mathbf{n}' d\Omega = 0\,.$$

Neglecting terms of order $1/r^2$ we obtain

$$-\nabla\phi(\mathbf{x},\, t) = \frac{1}{rc^2}\mathbf{n}\int \mathbf{n}\cdot\dot{\mathbf{J}}\left(\mathbf{x}',\; t - \frac{r}{c} + \frac{\mathbf{n}\cdot\mathbf{x}'}{c}\right) d^3x'; \tag{4.8}$$

and

$$-\frac{1}{c}\dot{\mathbf{A}}(\mathbf{x},\, t) = -\frac{1}{rc^2}\int \dot{\mathbf{J}}\left(\mathbf{x}',\, t - \frac{r}{c} + \frac{\mathbf{n}\cdot\mathbf{x}'}{c}\right) d^3x'.$$

Which leads finally to

$$\begin{aligned}
\mathbf{E}(\mathbf{x},\, t) &= -\nabla\phi - \frac{1}{c}\dot{\mathbf{A}} = \frac{1}{rc^2}\left[\mathbf{n}\cdot\left(\mathbf{n}\cdot\int \dot{\mathbf{J}}(\mathbf{x}',\, t')\, d^3x\right) - (\mathbf{n}\cdot\mathbf{n})\int \dot{\mathbf{J}}(\mathbf{x}',\, t')\, d^3x\right] \\
&= \frac{1}{rc^2}\mathbf{n}\wedge\left[\mathbf{n}\wedge\int \dot{\mathbf{J}}\left(\mathbf{x}',\, t - \frac{r}{c} + \frac{\mathbf{n}\cdot\mathbf{x}'}{c}\right) d^3x'\right], \qquad (4.9)
\end{aligned}$$

where the relation $\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$ has been used. For $\mathbf{B}$, one concludes from $\mathbf{B} = \nabla\wedge\mathbf{A}$ that:

$$\mathbf{B}(\mathbf{x},\, t) = -\frac{1}{rc^2}\mathbf{n}\wedge\int \dot{\mathbf{J}}\left(\mathbf{x}',\, t - \frac{r}{c} + \frac{\mathbf{n}\cdot\mathbf{x}'}{c}\right) d^3x' + \mathcal{O}(1/r^2)\,,$$

and so

$$\begin{aligned}
\mathbf{B} &= -\tfrac{1}{c}\mathbf{n}\wedge\dot{\mathbf{A}} \\
\mathbf{E} &= \mathbf{B}\wedge\mathbf{n} = \tfrac{1}{c}\mathbf{n}\wedge(\mathbf{n}\wedge\dot{\mathbf{A}})
\end{aligned} \tag{4.10}$$

and

$$\dot{\mathbf{A}}(\mathbf{x},\, t) = \frac{1}{rc}\int \dot{\mathbf{J}}\left(\mathbf{x}',\, t - \frac{r}{c} + \frac{\mathbf{n}\cdot\mathbf{x}'}{c}\right) d^3x'. \tag{4.11}$$

We see then that a current $\mathbf{J}$ that is time-independent does not emit electromagnetic waves!

First, let us consider the monochromatic case (harmonic oscillation):

$$\begin{aligned}
\mathbf{J}(\mathbf{x},\, t) &= \mathrm{Re}\left[\mathbf{J}(\mathbf{x})e^{-i\omega t}\right] \\
\tilde{\mathbf{J}}(\mathbf{k}) &= \int \mathbf{J}(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}} d^3x.
\end{aligned}$$

We have

$$A(\mathbf{x},\, t) = \frac{1}{rc}\mathrm{Re}\left[\int \mathbf{J}(\mathbf{x}')e^{-i\omega(t-r/c+\mathbf{n}\cdot\mathbf{x}'/c)}d^3x'\right] = \mathrm{Re}\left[\frac{e^{-i\omega(t-r/c)}}{rc}\tilde{\mathbf{J}}(\mathbf{k})\right]$$

with $\mathbf{k} = (\omega/c)\mathbf{n}$. This gives $\mathbf{A}(\mathbf{x},\, t) = \mathrm{Re}\left[\boldsymbol{\epsilon}(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right]$ with $\boldsymbol{\epsilon}(\mathbf{k}) = \tilde{\mathbf{J}}(\mathbf{k})/rc$ and so

$$\mathbf{B} = -\frac{1}{c}\mathbf{n}\wedge\dot{\mathbf{A}} = \mathrm{Re}\left[\frac{i\omega}{c}\mathbf{n}\wedge\boldsymbol{\epsilon}(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right], \quad \mathbf{E} = \mathbf{B}\wedge\mathbf{n}.$$

The time average of the Poynting vector is

$$\langle\mathbf{S}\rangle = \frac{c}{4\pi}\langle\mathbf{E}\wedge\mathbf{B}\rangle = \mathbf{n}\frac{c}{8\pi}\left(\frac{\omega}{c}\right)^2|\mathbf{n}\wedge\boldsymbol{\epsilon}(\mathbf{k})|^2. \tag{4.12}$$

To arrive at this result we have used the fact that $\langle\mathbf{E}\wedge\mathbf{B}\rangle = \langle(\mathbf{B}\wedge\mathbf{n})\wedge\mathbf{B}\rangle = \mathbf{n}\langle B^2\rangle$ and for $A(t) = \mathrm{Re}\,[Ae^{i\omega t}]$, $\langle A(t)^2\rangle = |A|^2/2$.

The average energy emitted per unit time within the solid angle $d\Omega$ in direction $\mathbf{n}$ is

$$\frac{d\overline{P}}{d\Omega}(\mathbf{n}) = (\langle\mathbf{S}\rangle\cdot\mathbf{n})r^2 = \frac{1}{8\pi c^3}|\mathbf{n}\wedge\tilde{\mathbf{J}}(\mathbf{k})|^2\omega^2.$$

Multipole expansion

(Jackson, §§9.2 and 9.3)

We assume here that eq. (4.5) holds as well as the condition

$$\lambda \gg d \;\Rightarrow T = \lambda/c \gg d/c. \tag{4.13}$$

Under these circumstances, the temporal variation of ρ and $\mathbf{J}$ is again small in the time interval d/c within which light to traverses the source. This means that in the interior of the source, the retardation is a small effect. We can then expand

$$\mathbf{J}\left(\mathbf{x}',\, t-\frac{r}{c}+\frac{\mathbf{n}\cdot\mathbf{x}'}{c}\right) = \mathbf{J}\left(\mathbf{x}',\, t-\frac{r}{c}\right) + \frac{\mathbf{n}\cdot\mathbf{x}'}{c}\dot{\mathbf{J}}\left(\mathbf{x}',\, t-\frac{r}{c}\right) + O(d^2/\lambda^2). \tag{4.14}$$

The first term gives the following contribution to $\mathbf{A}$ [see eq. (4.7)]: (Given that the extension of the source is finite, surface terms vanish and $\int x_\ell(\nabla\cdot\mathbf{J}) = -\int(\nabla x_\ell)\cdot\mathbf{J} = -\int J_\ell$.)

$$\begin{aligned}
\frac{1}{rc}\int\mathbf{J}\left(\mathbf{x}',\, t-\frac{r}{c}\right)d^3x' &= -\frac{1}{rc}\int\mathbf{x}'\left[\nabla'\cdot\mathbf{J}\left(\mathbf{x}',\, t-\frac{r}{c}\right)\right]d^3x' \\
&= \frac{1}{rc}\int\mathbf{x}'\dot{\rho}\left(\mathbf{x}',\, t-\frac{r}{c}\right)d^3x' \\
&= \frac{1}{rc}\dot{\mathbf{P}}\left(t-\frac{r}{c}\right),
\end{aligned}$$

where $\mathbf{P}(t) = \int \mathbf{x}\,\rho(\mathbf{x},\, t)d^3x$ is the dipole moment of the charge distribution ρ. The second term of (4.14) gives to the component A_j the following contribution:

$$\begin{aligned} \frac{1}{rc^2}n_\ell \int x'_\ell \dot{J}_j\, d^3x' &= \frac{1}{2rc^2}n_\ell \left[\int (x'_\ell \dot{J}_j + x'_j \dot{J}_\ell)d^3x' + \int (x'_\ell \dot{J}_j - x'_j \dot{J}_\ell)d^3x'\right] \\ &= \frac{1}{2rc^2}\ddot{I}_{\ell j}n_\ell + \frac{1}{rc}\dot{\mu}_{\ell j}n_\ell \end{aligned} \tag{4.15}$$

with

$$\begin{aligned} \dot{\mu}_{\ell j} &= \frac{1}{2c}\int \left[x'_\ell \dot{J}_j\left(\mathbf{x}',\, t - \frac{r}{c}\right) - x'_j \dot{J}_\ell \left(\mathbf{x}',\, t - \frac{r}{c}\right)\right] d^3x' \\ \dot{I}_{\ell j} &= \int \dot{\rho}\left(\mathbf{x}',\, t - \frac{r}{c}\right) x'_\ell x'_j d^3x' = -\int \left(\nabla' \cdot \mathbf{J}\right) x'_\ell x'_j d^3x' \\ &= \int J_m \partial_m (x'_\ell x'_j) d^3x' \;=\; \int (J_\ell x'_j + J_j x'_\ell) d^3x'. \end{aligned}$$

We have then found (neglecting higher order terms)

$$A_j(\mathbf{x},\, t) = \frac{1}{rc}\dot{P}_j\left(t - \frac{r}{c}\right) + \frac{1}{rc}\dot{\mu}_{\ell j}\left(t - \frac{r}{c}\right)n_\ell + \frac{1}{2rc^2}\ddot{I}_{\ell j}\left(t - \frac{r}{c}\right)n_\ell. \tag{4.16}$$

If we define the magnetic dipole as

$$\boldsymbol{\mu} = \frac{1}{2c}\int (\mathbf{x} \wedge \mathbf{J})d^3x,$$

the dipole contributions [the first two terms of(4.16)] give

$$\mathbf{A}^{\text{dipole}} = \frac{1}{rc}\dot{\mathbf{P}}\left(t - \frac{r}{c}\right) - \frac{1}{rc}\mathbf{n} \wedge \dot{\boldsymbol{\mu}}\left(t - \frac{r}{c}\right). \tag{4.17}$$

In the quadrupole contribution, $\ddot{I}_{\ell j}$, we can remove the trace because it ends up being a term parallel to $\mathbf{n}$ in $\mathbf{A}$, and so it does not contribute to the fields $\mathbf{E}$ and $\mathbf{B}$ (see 4.10). We make

$$Q_{\ell j} := I_{\ell j} - \frac{1}{3}\delta_{\ell j} I_{mm}.$$

The quadrupole term then gives

$$A_m^{\text{quadrupole}}(\mathbf{x},\, t) = \frac{1}{2rc^2}\ddot{Q}_{m\ell}\left(t - \frac{r}{c}\right)n_\ell \equiv \frac{1}{2rc^2}\left(\ddot{\mathbf{Q}}\mathbf{n}\right)_m. \tag{4.18}$$

This contribution is in general lower with respect to the electric dipole by a factor $\approx d\omega/c \approx d/\lambda$. Denoting with $\mathbf{Qn}$ the vector whose components are $Q_{m\ell}n_\ell$, we find for the fields [see eqs. (4.10)]:

$$\begin{aligned} \mathbf{B} = -\tfrac{1}{c}\mathbf{n} \wedge \dot{\mathbf{A}} &= \frac{1}{rc^2}\left[\ddot{\mathbf{P}} \wedge \mathbf{n} + (\ddot{\boldsymbol{\mu}} \wedge \mathbf{n}) \wedge \mathbf{n} + \frac{1}{2c}\dddot{\mathbf{Q}}\mathbf{n} \wedge \mathbf{n}\right] \\ \mathbf{E} = \mathbf{B} \wedge \mathbf{n} &= \frac{1}{rc^2}\left[(\ddot{\mathbf{P}} \wedge \mathbf{n}) \wedge \mathbf{n} + (\mathbf{n} \wedge \ddot{\boldsymbol{\mu}}) + \frac{1}{2c}\left(\dddot{\mathbf{Q}}\mathbf{n} \wedge \mathbf{n}\right) \wedge \mathbf{n}\right]. \end{aligned} \tag{4.19}$$

The moments $\mathbf{P}$, $\boldsymbol{\mu}$ and $\mathbf{Q}$ must be evaluated at the retarded time $t - r/c$. The magnetic dipole radiation is obtained from the electric dipole radiation by replacing $\mathbf{E}$ with $-\mathbf{B}$, $\mathbf{B}$ with $\mathbf{E}$ and $\mathbf{P}$ with $\boldsymbol{\mu}$.

Emitted power

As before, the Poynting vector is ($\mathbf{E} = \mathbf{B} \wedge \mathbf{n}$, $\mathbf{B} = \mathbf{n} \wedge \mathbf{E}$)

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \wedge \mathbf{B} = \frac{c}{4\pi}|\mathbf{E}|^2\mathbf{n} = \frac{c}{4\pi}|\mathbf{B}|^2\mathbf{n};$$

and so the emitted power (= energy per unit time) per unit solid angle in a given direction $\mathbf{n}$ is

$$\frac{dP}{d\Omega} = (\mathbf{S}\cdot\mathbf{n})r^2 = \frac{1}{4\pi c^3}\,|[\ldots]|^2\,. \tag{4.20}$$

where $[\ldots]$ corresponds to the terms in brackets in (4.19). If the electric dipole dominates, we have

$$\left(\frac{dP}{d\Omega}\right)_{\text{el. dip.}} = \frac{1}{4\pi c^3}|\ddot{\mathbf{P}} \wedge \mathbf{n}|^2 = \frac{1}{4\pi c^3}|\ddot{\mathbf{P}}|^2 \sin^2\vartheta, \tag{4.21}$$

where ϑ is the angle between $\ddot{\mathbf{P}}$ and $\mathbf{n}$. After integrating over the angles $d\Omega = \sin\vartheta d\vartheta d\varphi$, we obtain the total power (energy emitted per unit time)

$$P_{\text{el. dip.}} = \frac{2}{3c^3}|\ddot{\mathbf{P}}|^2. \tag{4.22}$$

Similarly, one finds for the magnetic dipole

$$P_{\text{magn. dip.}} = \frac{2}{3c^3}|\ddot{\boldsymbol{\mu}}|^2. \tag{4.23}$$

A simple calculation leads to the following result for the electric quadrupole radiation (see exercise):

$$\left(\frac{dP}{d\Omega}\right)_{\text{el. quad.}} = \frac{1}{16\pi c^5}\left[\dddot{Q}_{k\ell}\dddot{Q}_{km}\; n_\ell n_m - \dddot{Q}_{k\ell}\dddot{Q}_{sm}\; n_s n_\ell n_k n_m\right];$$

and, integrating over the angles,

$$P_{\text{el. quad.}} = \frac{1}{20c^5}\;\dddot{Q}_{k\ell}\dddot{Q}_{k\ell} \equiv \frac{1}{20c^5}|\;\dddot{\mathbf{Q}}\;|^2. \tag{4.24}$$

If one integrates (4.20) over the angles, all the mixed terms disappear and one obtains the sum of (4.22), (4.23) and (4.24):

$$P = \frac{2}{3c^3}|\ddot{\mathbf{P}}|^2 + \frac{2}{3c^3}|\ddot{\boldsymbol{\mu}}|^2 + \frac{1}{20c^5}|\;\dddot{\mathbf{Q}}\;|^2. \tag{4.25}$$

4.2 Dipole fields

(See Jackson 9.2,9.3)

Let us consider a small source of diameter d; we assume that

$$r \gg d \quad \text{et} \quad \lambda \gg d \tag{4.26}$$

without making any hypothesis on the relation between λ and r as in (4.5) and (4.13). The second among conditions (4.26) coincides with (4.13) and therefore we can expand the retardation in the source as

$$\rho\left(\mathbf{x}',\, t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right) \approx \rho\left(\mathbf{x}',\, t - \frac{r}{c} + \frac{\mathbf{n}\cdot\mathbf{x}'}{c}\right) \approx \rho\left(\mathbf{x}',\, t - \frac{r}{c}\right) + \frac{\mathbf{n}\cdot\mathbf{x}'}{c}\dot{\rho}\left(\mathbf{x}',\, t - \frac{r}{c}\right).$$

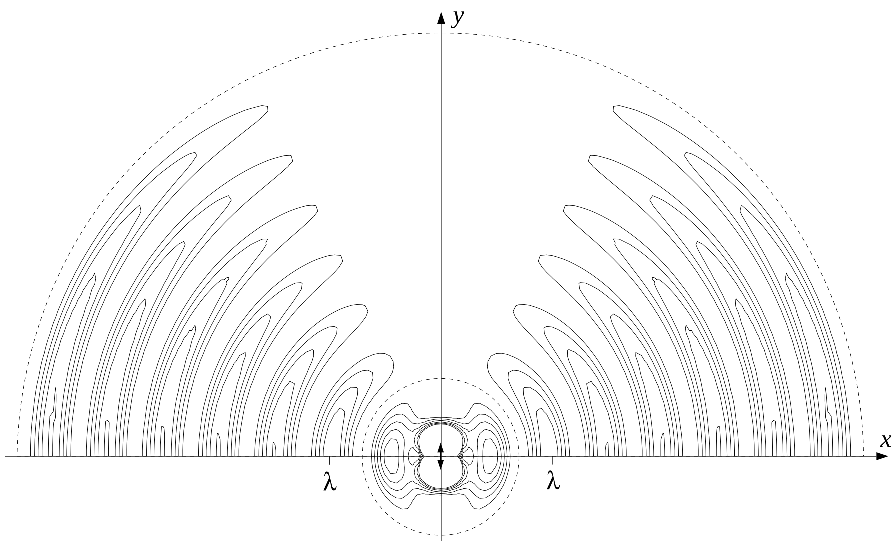
To first order in d/r and $d/\lambda \approx (d/c)\partial_t$, we obtain for the integrand of (4.1):

$$\frac{\rho\left(\mathbf{x}',\, t - |\mathbf{x}-\mathbf{x}'|/c\right)}{|\mathbf{x}-\mathbf{x}'|} \approx \frac{1}{r}\rho\left(\mathbf{x}',\, t - \frac{r}{c}\right) + \frac{\mathbf{n}\cdot\mathbf{x}'}{r^2}\rho\left(\mathbf{x}',\, t - \frac{r}{c}\right) + \frac{\mathbf{n}\cdot\mathbf{x}'}{rc}\dot{\rho}\left(\mathbf{x}',\, t - \frac{r}{c}\right). \tag{4.27}$$

The second term of (4.27) is new. It does not contribute to the emission of radiation. At large distances this and the third term are smaller than the first one by a factor of d/r and of d/λ, respectively.

If Q is the total charge and $\mathbf{P}$ is, as before, the dipole moment, we find the following for the scalar potential [with (4.1) and (4.27)]

$$\phi(\mathbf{x},\, t) = \frac{Q}{r} + \frac{1}{r^2}\mathbf{n}\cdot\mathbf{P}\left(t - \frac{r}{c}\right) + \frac{1}{rc}\mathbf{n}\cdot\dot{\mathbf{P}}\left(t - \frac{r}{c}\right). \tag{4.28}$$



Curves $r^2\langle\mathbf{E}^2(\mathbf{x})\rangle = \text{cst}$ represented in the $(x,\, y)$ plane for an electric dipole oriented in the y direction.

The vector potential is obtained by replacing in (4.27) ρ with $-\mathbf{J}/c$ and by using (4.15)

$$\mathbf{A}(\mathbf{x},\, t) = \frac{1}{rc}\dot{\mathbf{P}}\left(t-\frac{r}{c}\right) - \frac{1}{r^2}\mathbf{n}\wedge\boldsymbol{\mu}\left(t-\frac{r}{c}\right) - \frac{1}{rc}\mathbf{n}\wedge\dot{\boldsymbol{\mu}}\left(t-\frac{r}{c}\right). \tag{4.29}$$

(The same tricks that led to (4.15) have been applied to obtain (4.29)).

We now compute the fields $\mathbf{E}$ and $\mathbf{B}$ in the case where $Q=0$ and $\boldsymbol{\mu}$ is negligible. Such a configuration is called a *Hertzian dipole*. Defining the *Hertz vector*,

$$\mathbf{Z} = \frac{1}{r}\mathbf{P}\left(t-\frac{r}{c}\right) \tag{4.30}$$

we find, for a Hertzian dipole,

$$\phi(\mathbf{x},\, t) = -\nabla\cdot\mathbf{Z} \quad \text{and} \quad \mathbf{A}(\mathbf{x},\, t) = \frac{1}{c}\dot{\mathbf{Z}}. \tag{4.31}$$

One can now determine the fields $\mathbf{B}=\nabla\wedge\mathbf{A}$ and $\mathbf{E}=-\nabla\phi-\frac{1}{c}\dot{\mathbf{A}}$. After a brief calculation one finds:

$$\begin{aligned}
\mathbf{B} &= -\mathbf{n}\wedge\left(\frac{1}{rc^2}\ddot{\mathbf{P}}+\frac{1}{r^2c}\dot{\mathbf{P}}\right)\\
\mathbf{E} &= \frac{1}{rc^2}\left[(\ddot{\mathbf{P}}\cdot\mathbf{n})\mathbf{n}-\ddot{\mathbf{P}}\right]+\frac{1}{r^2c}\left[3(\dot{\mathbf{P}}\cdot\mathbf{n})\mathbf{n}-\dot{\mathbf{P}}\right]+\frac{1}{r^3}\left[3(\mathbf{P}\cdot\mathbf{n})\mathbf{n}-\mathbf{P}\right]
\end{aligned} \tag{4.32}$$

For all $\mathbf{P}$, $\dot{\mathbf{P}}$ and $\ddot{\mathbf{P}}$ must be evaluated at the retarded time $t-r/c$. The last term is the field of a *static* electric dipole that you know from electrostatics. If we consider a harmonic time dependence ($\sim e^{i\omega t}$), we have

$$\left|\frac{1}{c}\dot{\mathbf{P}}\right| \approx \frac{\omega}{c}|\mathbf{P}| \approx \frac{1}{\lambda}|\mathbf{P}|.$$

The three terms of the electric field (4.32) are then the quotients

$$\frac{1}{r\lambda^2} \;:\; \frac{1}{r^2\lambda} \;:\; \frac{1}{r^3}.$$

In the *near zone*, $r\ll\lambda$, the last term dominates and $|\mathbf{B}|\ll|\mathbf{E}|$: this is the field of an *static dipole*. In the *induction zone*, $r\approx\lambda$, $\mathbf{B}$ is of the same order of magnitude as $\mathbf{E}$. In this zone $\mathbf{B}$ is generated by the displacement current $\dot{\mathbf{E}}$. In the *wave zone*, $r\gg\lambda$, the terms of order $1/r$ of the previous paragraph dominate (see figure).

4.3 The field of a moving point charge

(See Jackson Chap. 14)

Here we determine the field of a moving point charge. The result is important for particle accelerators and astrophysics (synchrotron radiation). We consider a point charge e that travels along the path $\mathbf{z}(t)$. The charge and current densities are given by

$$\left.\begin{aligned} \rho(\mathbf{x},\, t) &= e\delta^{(3)}\left(\mathbf{x}-\mathbf{z}(t)\right) \\ \mathbf{J}(\mathbf{x},\, ,t) &= e\,\dot{\mathbf{z}}(t)\delta^{(3)}\left(\mathbf{x}-\mathbf{z}(t)\right) \end{aligned}\right\}. \tag{4.33}$$

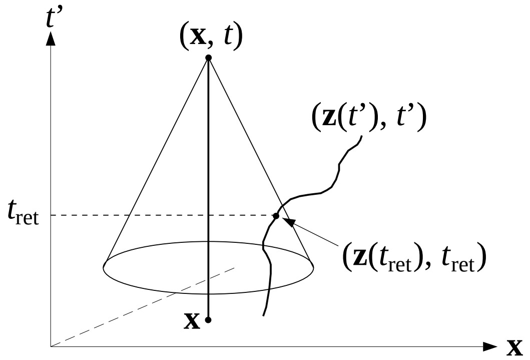
Since its sources are distributions, the potentials (4.1) and (4.2) are also distributions and one can prove mathematically and rigorously that the convolutions (4.1) and (4.2) exist. The following is a formal calculation.

$$\begin{aligned} \phi(\mathbf{x},\, t) &= e\int \frac{\delta\left(t-|\mathbf{x}-\mathbf{x}'|/c-t'\right)}{|\mathbf{x}-\mathbf{x}'|}\delta^{(3)}\left(\mathbf{x}'-\mathbf{z}(t')\right)d^3x'dt' \\ &= e\int \frac{\delta\left(t-|\mathbf{x}-\mathbf{z}(t')|/c-t'\right)}{|\mathbf{x}-\mathbf{z}(t')|}dt', \end{aligned} \tag{4.34}$$

and similarly

$$\mathbf{A}(\mathbf{x},\, t) = \frac{e}{c}\int \frac{\dot{\mathbf{z}}(t')}{|\mathbf{x}-\mathbf{z}(t')|}\delta\left(t-|\mathbf{x}-\mathbf{z}(t')|/c-t'\right)dt'. \tag{4.35}$$

The function $f(t') = t' + |\mathbf{x}-\mathbf{z}(t')|/c - t$ has just one zero, $t_{\text{ret}}(\mathbf{x}, t)$, a fact that follows from the figure below:



$c(t-t_{\text{ret}}) = |\mathbf{x}-\mathbf{z}(t_{\text{ret}})|$ So $\mathbf{z}(t)$ intersects the light cone of $(\mathbf{x},\, t)$ only once. Otherwise the particle would travel at a speed larger than c!

Using the relation (see math. methods II)

$$\int g(y)\delta(f(y))dy = \frac{g(y_0)}{|f'(y_0)|}$$

for a function with a unique zero y_0. When $|\dot{\mathbf{z}}| < c$, the derivative

$$f'(t')|_{t'=t_{\text{ret}}} = 1 - \frac{\dot{\mathbf{z}}_{\text{ret}}\cdot(\mathbf{x}-\mathbf{z}_{\text{ret}})}{c|\mathbf{x}-\mathbf{z}_{\text{ret}}|} > 0,$$

and the function f is then monotone, which proves mathematically that it has only one zero. For the potentials we thus obtain

$$\phi(\mathbf{x},\, t) = \frac{e}{|\mathbf{x}-\mathbf{z}_{\rm ret}|}\frac{1}{1-\frac{\dot{\mathbf{z}}_{\rm ret}\cdot(\mathbf{x}-\mathbf{z}_{\rm ret})}{c|\mathbf{x}-\mathbf{z}_{\rm ret}|}};$$

and so

$$\phi(\mathbf{x},\, t) = \frac{e}{|\mathbf{x}-\mathbf{z}_{\rm ret}| - \frac{1}{c}\dot{\mathbf{z}}_{\rm ret}\cdot(\mathbf{x}-\mathbf{z}_{\rm ret})} \tag{4.36}$$

$$\mathbf{A}(\mathbf{x},\, t) = \frac{e\dot{\mathbf{z}}_{\rm ret}}{c|\mathbf{x}-\mathbf{z}_{\rm ret}| - \dot{\mathbf{z}}_{\rm ret}\cdot(\mathbf{x}-\mathbf{z}_{\rm ret})}. \tag{4.37}$$

Here $\mathbf{z}_{\rm ret}$ designates $\mathbf{z}(t_{\rm ret})$ where $t_{\rm ret}$ is *implicitly* determined by $f(t_{\rm ret}) = 0$, i.e. by the equation

$$t_{\rm ret} = t - \frac{1}{c}|\mathbf{x}-\mathbf{z}(t_{\rm ret})|.$$

(4.36) and (4.37) are the *Liénard and Wiechert potentials.*

One could now compute the fields $\mathbf{E}$ and $\mathbf{B}$ from (4.36) and (4.37), but it is easier to use the original expressions, (4.34) and (4.35). We make

$$\boldsymbol{\beta} = \frac{\dot{\mathbf{z}}}{c}, \quad R = |\mathbf{x}-\mathbf{z}(t')| \quad \text{and} \quad \mathbf{n} = \frac{\mathbf{x}-\mathbf{z}(t')}{R}.$$

For $f(t') = t' - t + R(\mathbf{x},\, t')/c$, we use the fact that $|\boldsymbol{\beta}| < 1$, $|\mathbf{n}| = 1$, and so

$$\kappa := \frac{df}{dt'} = 1 - \mathbf{n}\cdot\boldsymbol{\beta} > 0.$$

With these definitions we have

$$\begin{bmatrix}\phi(\mathbf{x},\, t)\\ \mathbf{A}(\mathbf{x},\, t)\end{bmatrix} = e\int \begin{bmatrix}1\\ \boldsymbol{\beta}(t')\end{bmatrix}\frac{1}{R(t')}\delta\left(f(t')\right)dt'. \tag{4.38}$$

The right-hand side of (4.38) depends on $\mathbf{x}$ only through R and on t only through f. Using $\nabla = \mathbf{n}\frac{\partial}{\partial R}$, we then find

$$\begin{aligned}
\mathbf{E}(\mathbf{x},\, t) &= e\int\left[\frac{1}{R^2}\mathbf{n}\delta\left(t'+\frac{R}{c}-t\right) + \frac{(\boldsymbol{\beta}-\mathbf{n})}{Rc}\delta'\left(t'+\frac{R}{c}-t\right)\right]dt' \\
\mathbf{B}(\mathbf{x},\, t) &= e\int(\mathbf{n}\wedge\boldsymbol{\beta})\left[-\frac{1}{R^2}\delta\left(t'+\frac{R}{c}-t\right) + \frac{1}{Rc}\delta'\left(t'+\frac{R}{c}-t\right)\right]dt'.
\end{aligned}$$

Using the relations

$$\delta\left(f(t')\right) = \frac{1}{|f'(t_{\rm ret})|}\delta(t'-t_{\rm ret}) = \frac{1}{1-\mathbf{n}\cdot\boldsymbol{\beta}}\delta(t'-t_{\rm ret}) = \frac{1}{\kappa}\delta(t'-t_{\rm ret})$$

$$\begin{aligned}
\int \delta'(\underbrace{f(t')}_{\tau})g(t')dt' &= \int \delta'(\tau)g((t'(\tau))\,\frac{d\tau}{f'\left(t'(\tau)\right)}\,, \qquad dt' = \frac{d\tau}{d\tau/dt'} = \frac{d\tau}{f'} \\
&= -\int \delta(\tau)\frac{d}{d\tau}\left(\frac{g\left(t'(\tau)\right)}{f'\left(t'(\tau)\right)}\right)d\tau = -\frac{d}{d\tau}\left(\frac{g\left(t'(\tau)\right)}{f'\left(t'(\tau)\right)}\right)\Bigg|_{\tau=0} \\
&= -\left[\frac{1}{f'(t')}\frac{d}{dt'}\left(\frac{g(t')}{f'(t')}\right)\right]_{t'=t_{\mathrm{ret}}} = -\left[\frac{1}{\kappa(t')}\frac{d}{dt'}\left(\frac{g(t')}{\kappa(t')}\right)\right]_{t'=t_{\mathrm{ret}}},
\end{aligned}$$

we find

$$\begin{aligned}
\mathbf{E} &= e\left[\frac{1}{\kappa R^2}\mathbf{n} + \frac{1}{\kappa c}\frac{d}{dt'}\left(\frac{\mathbf{n}-\boldsymbol{\beta}}{\kappa R}\right)\right]_{\mathrm{ret}} \\
\mathbf{B} &= e\left[\frac{1}{\kappa R^2}\boldsymbol{\beta}\wedge\mathbf{n} + \frac{1}{\kappa c}\frac{d}{dt'}\left(\frac{\boldsymbol{\beta}\wedge\mathbf{n}}{\kappa R}\right)\right]_{\mathrm{ret}}.
\end{aligned} \tag{4.39}$$

To evaluate the derivatives with respect to t' we use

$$\frac{dR}{dt'} = -\dot{\mathbf{z}}\cdot\mathbf{n} = -c\boldsymbol{\beta}\cdot\mathbf{n},$$

and

$$\frac{1}{c}\frac{d}{dt'}\mathbf{n} = -\frac{\boldsymbol{\beta}}{R} - \frac{\mathbf{n}}{cR}\frac{dR}{dt'} = \frac{1}{R}\left[-\boldsymbol{\beta}+\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\beta})\right] = \frac{1}{R}\mathbf{n}\wedge(\mathbf{n}\wedge\boldsymbol{\beta}).$$

This conduces to

$$\mathbf{E}(\mathbf{x},\,t) = e\left[\frac{\mathbf{n}}{\kappa R^2} + \frac{\mathbf{n}}{c\kappa}\frac{d}{dt'}\left(\frac{1}{\kappa R}\right) + \frac{1}{\kappa^2 R^2}\underbrace{\mathbf{n}\wedge(\mathbf{n}\wedge\boldsymbol{\beta})}_{-\boldsymbol{\beta}+\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\beta})} - \frac{1}{c\kappa}\frac{d}{dt'}\left(\frac{\boldsymbol{\beta}}{\kappa R}\right)\right].$$

Using

$$\frac{\mathbf{n}}{\kappa R^2} = \frac{\kappa\mathbf{n}}{\kappa^2 R^2} = \frac{\mathbf{n}-\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\beta})}{\kappa^2 R^2},$$

the sum of the first and third terms can be written more simply as:

$$\mathbf{E}(\mathbf{x},\,t) = e\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\kappa^2 R^2} + \frac{\mathbf{n}}{c\kappa}\frac{d}{dt'}\left(\frac{1}{\kappa R}\right) - \frac{1}{c\kappa}\frac{d}{dt'}\left(\frac{\boldsymbol{\beta}}{\kappa R}\right)\right]_{\mathrm{ret}}. \tag{4.40}$$

Along similar lines, one can find $\mathbf{B}$:

$$\mathbf{B}(\mathbf{x},\,t) = e\left[\left(\frac{\boldsymbol{\beta}}{\kappa^2 R^2} + \frac{1}{c\kappa}\frac{d}{dt'}\left(\frac{\boldsymbol{\beta}}{\kappa R}\right)\right)\wedge\mathbf{n}\right]_{\mathrm{ret}}; \tag{4.41}$$

so

$$\mathbf{B} = \mathbf{n}\wedge\mathbf{E}. \tag{4.42}$$

Writing $\dot{\boldsymbol\beta} = d\boldsymbol\beta/dt'$, we have

$$\begin{aligned}
\frac{1}{c}\frac{d}{dt'}(\kappa R) &= \underbrace{\frac{\kappa}{c}\frac{dR}{dt'}}_{-\kappa\boldsymbol\beta\cdot\mathbf{n}} - \underbrace{\frac{1}{c}R\frac{d}{dt'}(\mathbf{n}\cdot\boldsymbol\beta)}_{R\mathbf{n}\cdot\dot{\boldsymbol\beta}/c+[-\boldsymbol\beta^2+(\mathbf{n}\cdot\boldsymbol\beta)^2]} \\
&= -\mathbf{n}\cdot\boldsymbol\beta + (\mathbf{n}\cdot\boldsymbol\beta)^2 - \frac{R}{c}(\mathbf{n}\cdot\dot{\boldsymbol\beta}) + \boldsymbol\beta^2 - (\mathbf{n}\cdot\boldsymbol\beta)^2 \\
&= \boldsymbol\beta^2 - (\mathbf{n}\cdot\boldsymbol\beta) - \frac{R}{c}(\mathbf{n}\cdot\dot{\boldsymbol\beta}).
\end{aligned}$$

The substitution of this result in (4.40) gives after a brief calculation:

$$\mathbf{E}(\mathbf{x},\, t) = e\left[(\mathbf{n}-\boldsymbol\beta)\frac{(1-\boldsymbol\beta^2)}{\kappa^3R^2}\right]_{\rm ret} + \frac{e}{c}\left[\frac{1}{\kappa^3 R}\mathbf{n}\wedge\Big((\mathbf{n}-\boldsymbol\beta)\wedge\dot{\boldsymbol\beta}\Big)\right]_{\rm ret}. \qquad (4.43)$$

The fields $\mathbf{E}$ and $\mathbf{B}$ are now explicitly determined by (4.43) and (4.42).

The first term of (4.43) is a deformed Coulomb field, a purely kinematic effect. This field is obtained through the Lorentz transformation of a Coulomb field and it decreases as $1/R^2$. The second term, which contains a factor $\dot{\boldsymbol\beta}$, is new. It decreases as $1/R$ and its contribution to the Poynting vector gives the emission (which decreases as $1/R^2$ and so $(\mathbf{S}\cdot\mathbf{n})R^2$ remains finite for $R\to\infty$). The contribution to $\mathbf{S}\cdot\mathbf{n}$ that dominates at a very large distance from the particle is then

$$(\mathbf{S}\cdot\mathbf{n})R^2 = \frac{c}{4\pi}|\mathbf{E}|^2R^2 = \frac{e^2}{4\pi c}\left[\frac{1}{\kappa^6}\left|\mathbf{n}\wedge\Big((\mathbf{n}-\boldsymbol\beta)\wedge\dot{\boldsymbol\beta}\Big)\right|^2\right]_{\rm ret}. \qquad (4.44)$$

The vector inside $|\dots|$ indicates the direction of the field $\mathbf{E}$ (at large distance).

We consider first the *small velocity limit*, $|\boldsymbol\beta|\ll 1$. We then have $\mathbf{n}-\boldsymbol\beta\approx\mathbf{n}$ and $\kappa\approx 1$. For the emitted power in a solid angle $d\Omega$, we obtain

$$\begin{aligned}
\frac{dP}{d\Omega} &= (\mathbf{S}\cdot\mathbf{n})R^2 = \frac{e^2}{4\pi c}|\mathbf{n}\wedge(\mathbf{n}\wedge\dot{\boldsymbol\beta})|^2 \\
&= \frac{e^2}{4\pi c^3}\dot v^2\sin^2\vartheta \qquad (|\boldsymbol\beta|\ll 1). \qquad (4.45)
\end{aligned}$$

The total power is then the integral of (4.45) over the angles $d\Omega = \sin\vartheta d\vartheta d\varphi$,

$$P = \frac{2e^2}{3c^3}|\dot{\mathbf{v}}|^2. \qquad (4.46)$$

This is the *Larmor formula.*

For the discussion of the general case ($v\lesssim c$), one should observe that (4.44) gives the energy flow per unit time in the direction $\mathbf{n}$. This radiation was emitted at time $t_{\rm ret} = t - R(t_{\rm ret})/c$. If we consider the radiation *emitted* during the time interval $T_1 \leqslant t_{\rm ret} \leqslant T_2$, we obtain for the energy received by the observer

$$W = \int_{T_1+R(T_1)/c}^{T_2+R(T_2)/c} (\mathbf{S}\cdot\mathbf{n})R^2 dt = \int_{T_1}^{T_2} (\mathbf{S}\cdot\mathbf{n})R^2 \frac{dt}{dt_{\rm ret}} dt_{\rm ret}.$$

The angular power *emitted* is then

$$\frac{dP}{d\Omega}(t_{\rm ret}) = (\mathbf{S}\cdot\mathbf{n})R^2 \frac{dt}{dt_{\rm ret}};$$

with $t = t_{\rm ret} + R(t_{\rm ret})/c$, we obtain

$$\frac{dt}{dt_{\rm ret}} = 1 - \boldsymbol{\beta}\cdot\mathbf{n} = \kappa$$

and using (4.44) we arrive at the expression

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c}\frac{1}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^5}\left|\mathbf{n}\wedge\left((\mathbf{n}-\boldsymbol{\beta})\wedge\dot{\boldsymbol{\beta}}\right)\right|^2. \tag{4.47}$$

It is possible (but a bit annoying) to integrate this relation over the direction, an operation that conduces to the result

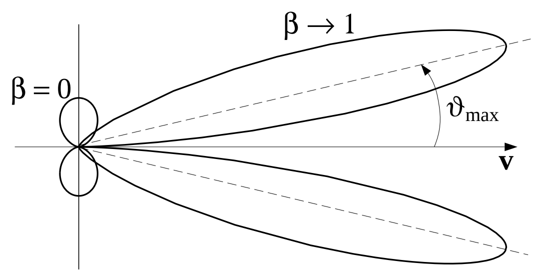
$$\begin{aligned} P &= \frac{2e^2}{3c}\gamma^6\left[|\dot{\boldsymbol{\beta}}|^2 - (\boldsymbol{\beta}\wedge\dot{\boldsymbol{\beta}})^2\right] \\ \gamma &= \frac{1}{\sqrt{1-\boldsymbol{\beta}^2}}. \end{aligned} \tag{4.48}$$

This is the *relativistic Larmor formula*. It is this effect that produces the enormous losses present in accelerators at CERN, for example.

The simplest example that one could consider is the *rectilinear* motion in which $\dot{\boldsymbol{\beta}}$ is parallel to $\boldsymbol{\beta}$. In this scenario, $\cos\vartheta = \boldsymbol{\beta}\cdot\mathbf{n}$ is constant and $\boldsymbol{\beta}\wedge\dot{\boldsymbol{\beta}} = 0$. Equation (4.47) then gives

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3}|\dot{\mathbf{v}}|^2\frac{\sin^2\vartheta}{(1-|\boldsymbol{\beta}|\cos\vartheta)^5}. \tag{4.49}$$

Because of the power 5 at the denominator, the radiation is highly focused in the forward direction for $|\boldsymbol{\beta}|$ close to 1.



The angle for which the radiation maximizes is given by ($\beta \equiv |\boldsymbol{\beta}|$)

$$\cos\vartheta_{\max} = \left[\frac{1}{3\beta}\left(\sqrt{1+15\beta^2}-1\right)\right] \qquad \vartheta_{\max} \approx \frac{1}{2\gamma} \quad \text{for} \quad \beta \to 1.$$

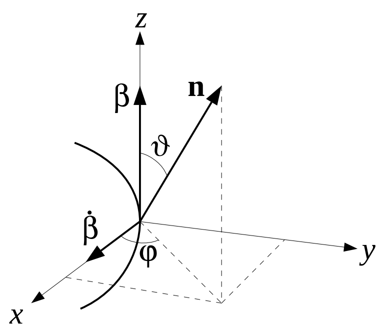
For small angles, $\vartheta \ll 1$, one obtains for $\beta \to 1$:

$$\frac{dP}{d\Omega} \approx \frac{8}{\pi}\frac{e^2}{c^3}|\dot{\mathbf{v}}|^2\gamma^8\frac{(\gamma\vartheta)^2}{\left[1+(\gamma\vartheta)^2\right]^5}. \tag{4.50}$$

For the power, the integral of (4.49) gives

$$P = \frac{2e^2}{3c^3}|\dot{\mathbf{v}}|^2\gamma^6 \quad \text{(rectilinear motion)}. \tag{4.51}$$

As a second example, we consider a particle in circular motion, i.e. with $\dot{\boldsymbol{\beta}} \perp \boldsymbol{\beta}$. Let $\boldsymbol{\beta}$ be parallel to the axis z and $\dot{\boldsymbol{\beta}}$ parallel to x as shown on the right.
Equation (4.47) gives (after a trigonometric exercise)



$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3}|\dot{\mathbf{v}}|^2\frac{1}{(1-\beta\cos\vartheta)^3}\left[1-\frac{\sin^2\vartheta\cos^2\varphi}{\gamma^2(1-\beta\cos\vartheta)^2}\right]. \tag{4.52}$$

For $\beta \to 1$, we obtain once again radiation focused around the forward direction (see figure). For small angles, $\vartheta \ll 1$, and high speeds, $\beta \approx 1$, we obtain

$$\frac{dP}{d\Omega} \approx \frac{2e^2}{\pi c^3}|\dot{\mathbf{v}}|^2\frac{\gamma^6}{(1+\gamma^2\vartheta^2)^3}\left[1-\frac{4\gamma^2\vartheta^2\cos^2\varphi}{(1+\gamma^2\vartheta^2)^2}\right]. \tag{4.53}$$

For the total power eq. (4.48) gives

$$P = \frac{2e^2}{3c^3}|\dot{\mathbf{v}}|^2\gamma^4 \quad \text{(circular motion)}. \tag{4.54}$$

For circular motion of radius ρ and of angular frequency ω, one has $|\dot{\mathbf{v}}| = v^2/\rho = v\omega$ and so

$$P = \frac{2e^2}{3}\frac{c}{\rho^2}\beta^4\gamma^4 = \frac{2e^2}{3c}\omega^2\beta^2\gamma^4. \tag{4.55}$$

The factor γ^4 leads the huge synchrotron losses.

Numerical example: For the LEP at CERN, $\rho \approx 5$ km. The final energy of an electron was about

$$E_{\text{él.}} \approx 100 \text{ GeV} \Rightarrow \gamma_{LEP} = \frac{E_{\text{él.}}}{\underbrace{mc^2}_{0.5 \text{ MeV}}} \approx 2 \times 10^5.$$

Therefore $v \approx c \approx 3\times 10^{10}$ cm/s and $\beta \approx 1$. The frequency is $\omega = v/\rho \approx 6\times 10^4$ s^{-1}. The electron charge can be expressed as $e^2 = m_e r_e c^2 \approx 0.5$ MeV $\times\, 2.8\times 10^{-13}$ cm, which gives

$$\frac{2e^2}{3c} \approx 3\times 10^{-24} \text{ MeV s}.$$

For the total power (4.55) then gives

$$\begin{aligned} P &= 3\times 10^{-24}\text{ MeV s}\times(6\times 10^4\text{ s}^{-1})^2\times(2\times 10^5)^4 \approx 2\times 10^7\text{ MeV/s} \\ &\approx 3\times 10^{-6}\text{ J/s} \quad \text{per electron!} \end{aligned}$$

(1 J $= 6.242\times 10^{12}$ MeV.) Since $\gamma = E/mc^2$, for a fixed energy, the losses are less important if the accelerated particle is heavier. This is why the particles that are accelerated in the new machine, the LHC (that reaches energies of 7TeV), are protons ($m_p c^2 \simeq 938$MeV) and not electrons. With this, $\gamma_{LHC} \simeq 7.5\times 10^3$ is around 27 times smaller than γ_{LEP} and the losses are reduced by a factor 5×10^5 per particle.

4.4 Cherenkov radiation

(Jackson §13.5)

A free particle in vacuum (non accelerated) does not radiate ($\dot{\boldsymbol{\beta}} = 0$). However, a particle of constant velocity inside a medium does radiate if its speed is higher than the speed of light in that medium. This form of radiation was discovered by Cherenkov in 1934 and was explained theoretically by Frank and Tamm in 1937. (In 1958, these three physicists obtained the Nobel prize for these works,) The Cherenkov radiation is of practical importance, particularly for the identification and counting of elementary particles of high energy (Cherenkov detector).

Here we restrict ourselves to the elementary aspects of the theory of Frank and Tamm (for an extensive discussion, see Jackson). We consider an homogeneous and isotropic medium with $\mu = 1$. We neglect the dispersion, $\varepsilon(\omega) = \varepsilon =$ constant. Consider a small source in rectilinear motion at constant speed v in the direction x. The charge and current densities are:

$$\begin{aligned} \rho(\mathbf{x},\, t) &= \rho_0(\mathbf{x}-\mathbf{v}t) \\ \mathbf{J}(\mathbf{x},\, t) &= \rho_0(\mathbf{x}-\mathbf{v}t)\mathbf{v}, \quad \mathbf{v} = (v,\, 0,\, 0). \end{aligned}$$

Maxwell equations become

$$\begin{aligned} \nabla\cdot\mathbf{B} &= 0, \qquad \nabla\wedge\mathbf{E} + \frac{1}{c}\partial_t\mathbf{B} = 0 \\ \nabla\cdot\mathbf{E} &= \frac{4\pi}{\varepsilon}\rho, \qquad \nabla\wedge\mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{\varepsilon}{c}\dot{\mathbf{E}}. \end{aligned} \tag{4.56}$$

With $\mathbf{B} = \nabla \wedge \mathbf{A}$ and $\mathbf{E} = -\nabla\phi - \frac{1}{c}\dot{\mathbf{A}}$, we obtain from (4.56)

$$\Delta\phi + \frac{1}{c}\nabla\cdot\dot{\mathbf{A}} = -\frac{4\pi}{\varepsilon}\rho \quad \text{et} \quad \Delta\mathbf{A} - \frac{\varepsilon}{c^2}\ddot{\mathbf{A}} - \nabla(\nabla\cdot\mathbf{A} + \frac{\varepsilon}{c}\dot{\phi}) = -\frac{4\pi}{c}\mathbf{J}.$$

We use the "Lorentz gauge", $\nabla\cdot\mathbf{A} + \frac{\varepsilon}{c}\dot{\phi} = 0$, which gives

$$\left(\Delta - \frac{\varepsilon}{c^2}\partial_t^2\right)\phi = -\frac{4\pi}{\varepsilon}\rho; \quad \left(\Delta - \frac{\varepsilon}{c^2}\partial_t^2\right)\mathbf{A} = -\frac{4\pi}{c}\mathbf{J}. \tag{4.57}$$

The phase velocity is then $c/n = c/\sqrt{\varepsilon}$ where n is the index of refraction. Since the sources are function of the variables $x - vt$, y and z, we propose the same Ansatz for ϕ and $\mathbf{A}$:

$$\phi,\ \mathbf{A} \sim f(x - vt,\ y,\ z).$$

The derivatives with respect to time are $\partial_t f = -v\partial_x f$ and $\partial_t^2 f = v^2\partial_x^2 f$. With $J_y = J_z = 0$, we obtain

$$\left(\Delta - \frac{n^2v^2}{c^2}\partial_x^2\right)A_{y,z} = 0.$$

Looking for solutions that decrease at infinity, $|\mathbf{A}(\mathbf{x})| \overset{|x|\to\infty}{\to} 0$, we make $A_y = A_z = 0$. For A_x and ϕ we have

$$\left[\left(1 - \frac{n^2v^2}{c^2}\right)\partial_x^2 + \partial_y^2 + \partial_z^2\right]A_x = -\frac{4\pi}{c}\rho v \tag{4.58}$$

$$\left[\left(1 - \frac{n^2v^2}{c^2}\right)\partial_x^2 + \partial_y^2 + \partial_z^2\right]\phi = -\frac{4\pi}{n^2}\rho. \tag{4.59}$$

The Lorentz condition, $\partial_x A_x - \frac{n^2}{c}v\partial_x\phi = 0$ and (4.58) are satisfied with (4.59) and

$$A_x = \frac{n^2}{c}v\phi. \tag{4.60}$$

Equation (4.59) remains. With $\beta := nv/c = v/v_{\text{phase}}$, we write it as

$$\left[(1-\beta^2)\partial_x^2 + \partial_y^2 + \partial_z^2\right]\phi = -\frac{4\pi}{n^2}\rho. \tag{4.61}$$

For $\beta < 1$, this is an elliptic equation. Its solution is a Coulomb potential contracted in the direction x. At $\beta = 1$, the nature of the equation changes and it becomes hyperbolic for $\beta > 1$. This last case is the one we want to examine.

Let $\beta > 1$. When making the variable transformation $x \to \tau = (-x+vt)/\sqrt{\beta^2-1}$; $\partial_x = -1/(\sqrt{\beta^2-1})\partial_\tau$, the equation (4.61) becomes

$$(\partial_y^2 + \partial_z^2 - \partial_\tau^2)\phi = -\frac{4\pi}{n^2}\rho_0\left(-\tau\sqrt{\beta^2-1},\ y,\ z\right). \tag{4.62}$$

This is the two-dimensional wave equation. In the mathematical complements, we have determined its retarded Green function which satisfies $(\partial_y^2 + \partial_z^2 - \partial_\tau^2)G = \delta$ and $G = 0$ for $\tau < 0$:

$$G(y,\, z,\, \tau) = \begin{cases} \frac{1}{2\pi\sqrt{\tau^2 - y^2 - z^2}} & \text{for } \tau^2 > y^2 + z^2,\ \tau > 0 \\ 0 & \text{otherwise.} \end{cases}$$

For $\rho_0(\mathbf{x}) = e\delta(\mathbf{x})$, we have

$$\begin{aligned} -\frac{4\pi}{n^2}\rho_0\left(-\tau\sqrt{\beta^2 - 1},\, y,\, z\right) &= -\frac{4\pi e}{n^2}\delta\left(\tau\sqrt{\beta^2 - 1}\right)\delta(y)\delta(z) \\ &= -\frac{4\pi e}{n^2\sqrt{\beta^2 - 1}}\delta(\tau)\delta(y)\delta(z). \end{aligned}$$

If we convolve with G, this gives

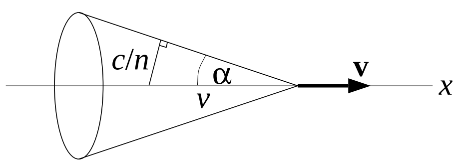
$$\phi = -\frac{4\pi e}{n^2\sqrt{\beta^2 - 1}}G,$$

that is to say

$$\phi(x,\, y,\, z,\, t) = \begin{cases} \dfrac{2e}{n^2}\dfrac{1}{\sqrt{(x - vt)^2 - (\beta^2 - 1)(y^2 + z^2)}} \\ \qquad \text{if } (x - vt)^2 > (\beta^2 - 1)(y^2 + z^2) \\ 0 \qquad \text{otherwise,} \end{cases} \tag{4.63}$$

$$A_x = n\beta\phi, \quad A_y = 0, \quad A_z = 0. \tag{4.64}$$

We then obtain a discontinuity surface where ϕ diverges. It forms a cone, the *Mach cone*, with an aperture angle $\tan\alpha = 1/\sqrt{\beta^2 - 1}$



$$\sin\alpha = \frac{1}{\beta} = \frac{c/n}{v} = \frac{\text{phase velocity of the waves}}{\text{speed of the particle}}.$$

As for supersonic motion, α is called the Mach angle. In the exterior of a Mach cone the fields vanish and on the cone they diverge. This is not a physical singularity. It disappears if one takes into account the *dispersion*. In vacuum, this last remark does not apply. (A further indication that v cannot be greater than c!)

Chapter 5

Scattering of electromagnetic waves

If an electromagnetic wave enters a medium, the charges in the medium are accelerated by the field of the wave. This acceleration creates new waves, the *scattered* waves. This process is at the origin of the redness of the sunset and the blueness of the sky, for example.

5.1 Thomson scattering

(Jackson §14.7)

If a plane wave reaches a free point charge, the latter is accelerated. The particle then absorbs part of the energy of the incident wave and emits a new wave. We want to compute the cross section of this process:

$$\frac{d\sigma}{d\Omega}(\mathbf{n}) = \frac{\text{(emitted energy)/(time}\times\text{angle)}}{\text{(incident energy)/(time}\times\text{area)}} \tag{5.1}$$

is the differential cross section in the direction $\mathbf{n}$.

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega$$

is the total cross section. Its dimension is cm^2. We now consider a plane wave scattered by a non-relativistic particle of mass m and charge e. The electric field of the incident plane wave (in complex representation) is given by

$$\mathbf{E}(\mathbf{x},\, t) = E_0 \boldsymbol{\epsilon} e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$$

where $|E_0|$ is the amplitude, $\boldsymbol{\epsilon}$ is the polarisation vector and ω is the angular frequency. Let $\mathbf{z}(t)$ be the position of the particle at time t. We have

$$m\ddot{\mathbf{z}}(t) = e\mathbf{E}(\mathbf{z},\, t) = eE_0\boldsymbol{\epsilon}e^{i(\mathbf{k}\cdot\mathbf{z}(t)-\omega t)}.$$

(We have assumed that $v \ll c$ and so we can neglect the Lorentz force, $\frac{e}{c}\mathbf{v} \wedge \mathbf{B}$.) In order to compute the energy emitted per unit time using the Larmor formula (4.45), $\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3}\dot{\mathbf{v}}^2 \sin^2\vartheta$, we need the time average $\langle\dot{\mathbf{v}}^2\rangle$. Under the hypothesis that the particle does not move too much during a period as compared to the wave length (which is equivalent to the condition $v \ll c$), we obtain

$$\langle\dot{\mathbf{v}}^2\rangle = \langle|\mathrm{Re}(\ddot{\mathbf{z}})|^2\rangle = \frac{1}{2}|E_0|^2\left(\frac{e}{m}\right)^2.$$

Using the Larmor formula this gives

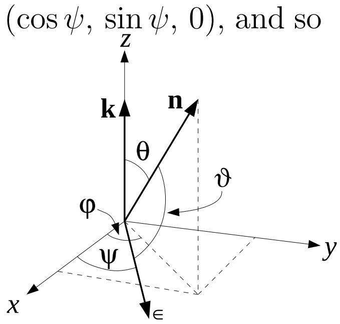
$$\frac{dP}{d\Omega}(\mathbf{n}) = \frac{c}{8\pi}|E_0|^2\left(\frac{e^2}{mc^2}\right)^2\sin^2\vartheta$$

where ϑ is the angle between the direction of observation $\mathbf{n}$ and the direction of acceleration $\ddot{\mathbf{z}} = \dot{\mathbf{v}}$. The flux of incident energy (energy per unit time and unit area) for a plane wave is the average amplitude of the Poynting vector, $|\mathbf{S}| = \frac{c}{8\pi}|E_0|^2$. This gives

$$\frac{d\sigma}{d\Omega}(\mathbf{n}) = \left(\frac{e^2}{mc^2}\right)^2\sin^2\vartheta, \tag{5.2}$$

which is the Thomson cross section. For a wave coming from direction z ($\hat{\mathbf{k}} = \mathbf{e}_z$) and with a polarisation $\boldsymbol{\epsilon}$ in the $(x,\, y)$ plane that forms an angle ψ with the x axis, we have (see figure) $\mathbf{n} = (\sin\theta\cos\varphi,\, \sin\theta\sin\varphi,\, \cos\theta)$, $\boldsymbol{\epsilon} = (\cos\psi,\, \sin\psi,\, 0)$, and so

$$\begin{aligned}
\cos\vartheta &= \mathbf{n}\cdot\boldsymbol{\epsilon} \\
&= \sin\theta(\cos\varphi\cos\psi + \sin\varphi\sin\psi) \\
&= \sin\theta\cos(\varphi-\psi) \\
\cos^2\vartheta &= \sin^2\theta\cos^2(\varphi-\psi).
\end{aligned}$$



For unpolarised incident radiation, we can average (integrate) over ψ by using the identity $\frac{1}{2\pi}\int_0^{2\pi}\cos^2(\varphi-\psi)d\psi = \frac{1}{2}$. We obtain

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2\left(1-\frac{1}{2}\sin^2\theta\right) = \left(\frac{e^2}{mc^2}\right)^2\frac{1}{2}(1+\cos^2\theta) \tag{5.3}$$

where θ is now the angle between $\mathbf{n}$ and $\mathbf{k}$, i.e. between the direction of the incident wave and the direction of the scattered wave. The total Thomson scattering cross section is

$$\sigma_T = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 = 0.665 \times 10^{-24}\text{cm}^2 = 0.665 \text{ barn} \quad \textit{for an electron.} \tag{5.4}$$

The magnitude $r_e = \frac{e^2}{mc^2} = 2.82 \times 10^{-13}$ cm is called the "classical electron radius". In quantum electrodynamics one can show that the equation (5.3) is valid only at low frequencies, $\hbar\omega \ll mc^2$.

5.2 Elastic and inelastic scattering by quasi-free charges

We study here in a more general way an incident wave on a system of quasi-free charges $\{e_j\}$ located at positions $\{\mathbf{x}_j\}$. As a concrete example, one can think of the scattering of X rays by the electrons in a solid. The energy of a photon in the X ray is much higher than the binding energy of the electrons in the solid, which is why the latter can be considered as free. As in the previous section, the incident wave is given by

$$\mathbf{E}(\mathbf{x}, t) = E_0 \boldsymbol{\epsilon} e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}.$$

According to (4.43), the radiation of the sources $\{e_j\}$ is ($v_j \ll c$)

$$\mathbf{E}_s(\mathbf{x}, t) = \frac{1}{c}\sum_j \frac{e_j}{R_j}\left[\mathbf{n} \wedge (\mathbf{n} \wedge \dot{\boldsymbol{\beta}}_j)\right]_{ret}. \tag{5.5}$$

The acceleration is given by

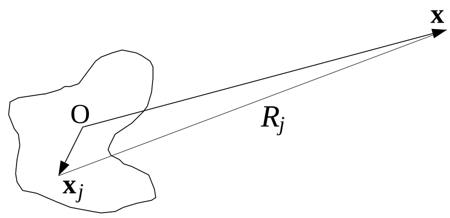
$$\dot{\boldsymbol{\beta}}_j = \frac{1}{c}\dot{\mathbf{v}}_j = \frac{e_j}{m_j c} E_0 \boldsymbol{\epsilon} e^{i(\mathbf{k}\cdot\mathbf{x}_j - \omega t)}. \tag{5.6}$$

This leads to

$$\mathbf{E}_s(\mathbf{x}, t) = E_0\left[\mathbf{n} \wedge (\mathbf{n} \wedge \boldsymbol{\epsilon})\right]\sum_j \frac{e_j^2}{m_j c^2}\frac{1}{R_j}\exp[i\mathbf{k}\cdot\mathbf{x}_j - i\omega t_{ret}] \tag{5.7}$$

with $t_{ret} = t - \frac{R_j}{c}$. With enough precision ($|\mathbf{x}_j| \ll |\mathbf{x}|$), we have

$$\begin{aligned} R_j &= |\mathbf{x} - \mathbf{x}_j| \approx |\mathbf{x}| - \mathbf{n}\cdot\mathbf{x}_j \\ \mathbf{n} &= \frac{\mathbf{x}}{r}, \quad r = |\mathbf{x}|. \end{aligned}$$



We then obtain

$$\mathbf{E}_s(\mathbf{x},\, t) = E_0 \left[\mathbf{n} \wedge (\mathbf{n} \wedge \boldsymbol{\epsilon})\right] \frac{e^{-i\omega\left(t-\frac{r}{c}\right)}}{r} \sum_j \frac{e_j^2}{m_j c^2} e^{-i\mathbf{q}\cdot\mathbf{x}_j} \tag{5.8}$$

where $\mathbf{q} = (\omega/c)\mathbf{n} - \mathbf{k}$. As in the previous paragraph, we find for the cross section

$$\frac{d\sigma}{d\Omega} = \frac{r^2|\mathbf{E}_s|^2}{|E_0|^2} = \left|\sum_j \frac{e_j^2}{m_j c^2} e^{-i\mathbf{q}\cdot\mathbf{x}_j}\right|^2 \sin^2\vartheta, \tag{5.9}$$

where ϑ is the angle between $\mathbf{n}$ and $\boldsymbol{\epsilon}$. This equation is valid only if the electrons are quasi free, i.e. if the frequency of the incident wave is much higher than the frequencies of the atomic transitions. To use (5.9), we yet have to take the average of the positions $\mathbf{x}_j$.

Coherent and incoherent scattering

We discuss here the equation (5.9). The considerations to follow are also relevant for other situations beyond ours, and they can be applied to any scattering process, in particular to the scattering of a beam of particles, because matter particles also possess a wave-like nature. The cross section (5.9) depends markedly on $|\mathbf{q}|$. Let $a = \langle|\mathbf{x}_j|\rangle$ be the dimension of the system of particles. The cross section will be very different for $qa \ll 1$ and $qa \gg 1$. Defining the scattering angle θ, $\cos\theta = (\mathbf{k}\cdot\mathbf{n})/k$, and use $\omega/c = k$ we obtain

$$\begin{aligned} q^2 &= \left(\frac{\omega}{c}\mathbf{n} - \mathbf{k}\right)^2 = 2k^2(1-\cos\theta) = 4k^2\sin^2\frac{\theta}{2} \\ q &= 2k\sin\frac{\theta}{2}. \end{aligned}$$

If $ka \ll 1$, $qa \ll 1$ for every angle θ. In the opposite case in which $ka \gg 1$, qa is small ($qa \ll 1$) only if $\theta \ll \theta_c = 1/ka$ and $qa \gg 1$ for large angles.

For $qa \ll 1$, all the exponentials $e^{-i\mathbf{q}\cdot\mathbf{x_j}}$ are very close to 1 and (5.9) gives

$$\left.\frac{d\sigma}{d\Omega}\right|_{qa\ll 1} \approx \left|\sum_j \frac{e_j^2}{m_j c^2}\right|^2 \sin^2\vartheta. \tag{5.10}$$

For an atom with Z electrons, this leads to

$$\left.\frac{d\sigma}{d\Omega}\right|_{qa\ll 1} \approx Z^2\left(\frac{e^2}{mc^2}\right)^2 \sin^2\vartheta. \tag{5.11}$$

The Z electrons behave in a *coherent* manner, as a particle of radius $R = Zr_e = Ze^2/(mc^2)$. The cross section is Z^2 times the cross section of a single electron!

In the opposite limit, $qa \gg 1$, the exponentials in (5.9) oscillate rapidly and have very different phases. The mixed terms vanish and only the diagonal terms contribute to the summation. Thus,

$$\left.\frac{d\sigma}{d\Omega}\right|_{qa\gg 1} \approx \sum_j \left(\frac{e_j^2}{m_j c^2}\right)^2 \sin^2\vartheta. \tag{5.12}$$

For Z identical particles of mass m and change e, instead of (5.11), this leads to

$$\left.\frac{d\sigma}{d\Omega}\right|_{qa\gg 1} \approx Z \left(\frac{e^2}{mc^2}\right)^2 \sin^2\vartheta. \tag{5.13}$$

In this case, the contributions of the particles superpose in an *incoherent* way. In quantum mechanics, one calculates (Thomas-Fermi model) $a \approx 1.4 a_0 Z^{-1/3}$ where $a_0 = \hbar^2/(me^2) = r_e/\alpha^2 \simeq 5.3 \times 10^{-9}$cm is the Bohr radius of the hydrogen atom and $\alpha = e^2/(\hbar c) \approx 1/137$ is the fine structure constant.

We now consider in more detail the situation of identical particles, $e_j \equiv e$ and $m_j \equiv m \ \forall j$. The charges considered can be the electrons of an atom of total charge Z, for example. In this case, (5.9) reduces to

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2 \sin^2\vartheta \left|\sum_{j=1}^{Z} e^{-i\mathbf{q}\cdot\mathbf{x}_j}\right|^2. \tag{5.14}$$

We take the statistical average of the last factor. Let $W(\mathbf{x}_1, \ldots, \mathbf{x}_Z)$ be the distribution of the probability of presence of the electrons. We define the *form factor*

$$F^2(\mathbf{q}) = \int \left|\sum_{j=1}^{Z} e^{-i\mathbf{q}\cdot\mathbf{x}_j}\right|^2 W(\mathbf{x}_1, \ldots, \mathbf{x}_Z) d^{3Z}x. \tag{5.15}$$

We have therefore

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Thomson}} F^2(\mathbf{q}). \tag{5.16}$$

From (5.15), one has (W is a probability, so $\int W(\mathbf{x}_1, \ldots, \mathbf{x}_Z) d^{3Z}x = 1$)

$$F^2(0) = Z^2. \tag{5.17}$$

We split (5.15) into the diagonal and non-diagonal terms:

$$\left|\sum_{j=1}^{Z} e^{-i\mathbf{q}\cdot\mathbf{x}_j}\right|^2 = Z + \sum_{j\neq m} e^{i\mathbf{q}\cdot(\mathbf{x}_j - \mathbf{x}_m)}.$$

Therefore

$$F^2(\mathbf{q}) = Z + \sum_{j\neq m} \int e^{i\mathbf{q}\cdot(\mathbf{x}_j-\mathbf{x}_m)} W(\mathbf{x}_1, \ldots, \mathbf{x}_Z) d^{3Z}x.$$

We call

$$W_{mj}(\mathbf{x}_m, \mathbf{x}_j) := \int W(\mathbf{x}_1, \ldots, \mathbf{x}_Z) \prod_{k\neq j,m} d^3x_k \tag{5.18}$$

the distribution of probability of presence of the electrons m and j. We define additionally $P(\mathbf{q})$ through

$$Z(Z-1)P(\mathbf{q}) = \sum_{j\neq m} \int e^{i\mathbf{q}\cdot(\mathbf{x}_j-\mathbf{x}_m)} W_{mj}(\mathbf{x}_m, \mathbf{x}_j) d^3x_m d^3x_j. \tag{5.19}$$

One then has

$$F^2(\mathbf{q}) = Z + Z(Z-1)P(\mathbf{q}).$$

Evidently $P(0) = 1$, which agrees with (5.17).
The integral in (5.19) is proportional to the Fourier transform of $W_{mj}(\mathbf{x}_m, \mathbf{x}_j)$ at the position $(\mathbf{q}, -\mathbf{q}) \in \mathbb{R}^6$. The Riemann-Lebesgue Lemma tells us that the Fourier transform of a continuous function decreases to 0 at infinity: $\widehat{W}_{mj} \in L^\infty$, so $\lim_{|\mathbf{q}|\to\infty} \widehat{W}_{mj}(\mathbf{q}, -\mathbf{q}) = 0$. It follows that

$$\begin{aligned} \lim_{|\mathbf{q}|\to\infty} \mathbf{P}(\mathbf{q}) &= 0 \\ \lim_{|\mathbf{q}|\to\infty} F^2(\mathbf{q}) &= Z. \end{aligned}$$

For large values of the momentum, the cross section is then constant and its value is Z times the cross section of a single particle. This is known as the "deeply inelastic" regime.

The comparison between the elastic and the total cross sections allows to determine the number of particles in the scattering target,

$$Z = \frac{\left.\frac{d\sigma}{d\Omega}\right|_{qa\ll 1}}{\left.\frac{d\sigma}{d\Omega}\right|_{qa\gg 1}}. \tag{5.20}$$

This is why the measurements of the cross section of nucleons were interpreted as evidence of the composition of the nucleon by three "partons" (quarks).

This means that with scattering experiments one can study the structure of an object. Using the fact that $Z = F^2(0)/F^2(\infty)$ one can in principle determine whether an object is composed or not. For this, one needs waves with a wave number $k > 1/a$, if a is the dimension of the object (i.e., for the study of an object of size a, "photons" of energy $E = \hbar\omega = \hbar kc > \hbar c/a$ are required.)

Elastic scattering

The *elastic* part of the scattered wave is the one with the same frequency ω as the incident wave. The incident wave depends on time as $e^{i\omega t}$. The scattered wave depends additionally on the factor $\sum_j e^{-i\mathbf{q}\cdot\mathbf{x}_j}$, which depends on time through the positions $\mathbf{x}_j$. This is why it contains some other frequencies apart from ω. We obtain the part of the field with frequency ω by taking the time average of the factor $\sum_j e^{-i\mathbf{q}\cdot\mathbf{x}_j}$. With (5.14) we obtain

$$\left(\frac{d\sigma}{d\Omega}\right)_{elast.} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Thomson}} \left|\overline{\sum_j e^{-i\mathbf{q}\cdot\mathbf{x}_j}}\right|^2 . \tag{5.21}$$

We substitute the time average by the spacial average:

$$\overline{\sum_j e^{-i\mathbf{q}\cdot\mathbf{x}_j}} = \sum_{j=1}^{Z} \int e^{-i\mathbf{q}\cdot\mathbf{x}_j} W(\mathbf{x}_1, \dots, \mathbf{x}_Z) d^{3Z}x = \sum_{j=1}^{Z} F_j^\star(\mathbf{q}).$$

Here $F_j(\mathbf{q})$ is the Fourier transform of the probability density of a single electron,

$$W_j(\mathbf{x}_j) = \int W(\mathbf{x}_1, \dots, \mathbf{x}_Z) \prod_{k\neq j} d^3x_k \,,$$

and its Fourier transform or 'form' factor is defined by

$$F_j(\mathbf{q}) = \int W_j(\mathbf{x}) e^{i\mathbf{q}\cdot\mathbf{x}} d^3x \,. \tag{5.22}$$

With (5.21) this gives

$$\left(\frac{d\sigma}{d\Omega}\right)_{elast.} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Thomson}} F^2_{elast.}(\mathbf{q}) \,, \tag{5.23}$$

$$F^2_{elast.} = \left|\sum_{j=1}^{Z} F_j(\mathbf{q})\right|^2 . \tag{5.24}$$

The "elastic form factor" can be interpreted as follows: the density of charge of the Z electrons is

$$\rho(\mathbf{x}) = e\sum_{j=1}^{Z} \delta(\mathbf{x}-\mathbf{x}_j)$$

and its statistical average is

$$\langle\rho\rangle(\mathbf{x}) = e\sum_j \int \delta(\mathbf{x}-\mathbf{x}_j) W(\mathbf{x}_1, \dots, \mathbf{x}_Z) d^{3Z}x = e\sum_j W_j(\mathbf{x}) \,, \tag{5.25}$$

From (5.22), the Fourier transform of the charge distribution is then $\langle\hat{\rho}\rangle(\mathbf{q}) = e\sum_j F_j(\mathbf{q})$, and so

$$e^2 F^2_{elast.}(\mathbf{q}) = |\langle\hat{\rho}\rangle(\mathbf{q})|^2. \tag{5.26}$$

The elastic form factor is the square of the absolute value of the Fourier transform of the average charge density (ignoring the extra e^2 factor). In the forward scatter direction ($\theta = 0$, hence $\mathbf{q} = 0$), one has from (5.22) $F_{elast.}(0) = Z$: the forward scattering is coherent

$$\left(\frac{d\sigma}{d\Omega}\right)_{\acute{e}last.}(\theta = 0) \;= Z^2 \left(\frac{d\sigma}{d\Omega}\right)_{\text{Thomson}} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{total}}(\theta = 0). \tag{5.27}$$

From the Riemann-Lebesgue Lemma,

$$\lim_{|\mathbf{q}|\to\infty} F_{elast.}(\mathbf{q}) = 0.$$

Qualitatively this conduces to the situation sketched in figure 5.1.

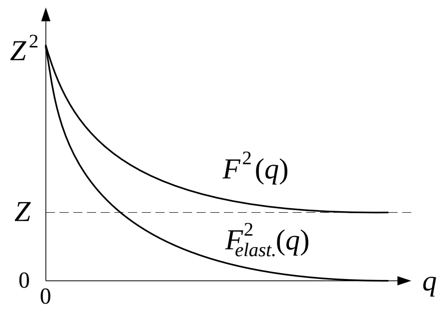


Figure 5.1: The qualitative behaviour of the elastic and total form factors as a function of $q = |k\mathbf{n} - \mathbf{k}| = 2k\sin(\theta/2)$.

5.3 Scattering in gases and liquids

(Jackson §9.7)

Here we analyse the following situation: we look upon an homogeneous and isotropic dielectric medium with a dielectric "constant" $\varepsilon_0(\omega)$ that does not depend on the position $\mathbf{x}$. We consider a wave that propagates in this ponderable medium and that finds in its way a precisely localised inhomogeneity of the dielectric. A scattered wave is emitted from this inhomogeneity. We want to determine the energy emitted and compute the cross section of this process. As important applications let us remark:

i) The scattering of light by particles of dust.

ii) The scattering of light by the inhomogeneities of a gas (Lorentz theory).

Generalities

Let $\mathbf{E}(\mathbf{x},\,\omega)$ and $\mathbf{B}(\mathbf{x},\,\omega)$ be the Fourier transforms:

$$\begin{aligned}\mathbf{E}(\mathbf{x},\,\omega) &= \int \mathbf{E}(\mathbf{x},\,t)e^{i\omega t}dt \\ \mathbf{B}(\mathbf{x},\,\omega) &= \int \mathbf{B}(\mathbf{x},\,t)e^{i\omega t}dt.\end{aligned}$$

The Maxwell equations in the absence of charges and currents and without magnetisation ($\mu = 1$) are

$$\left.\begin{array}{rclrcl}\nabla\cdot(\varepsilon\mathbf{E}) &=& 0 & \nabla\wedge\mathbf{E}-\dfrac{i\omega}{c}\mathbf{B} &=& 0\\ \nabla\cdot\mathbf{B} &=& 0 & \nabla\wedge\mathbf{B}+\dfrac{i\omega}{c}\varepsilon\mathbf{E} &=& 0\end{array}\right\}. \tag{5.28}$$

In our situation we have

$$\varepsilon(\mathbf{x},\,\omega) = \varepsilon_0(\omega) + \varepsilon_1(\mathbf{x},\,\omega) = 1 + 4\pi(\chi_0 + \chi_1) \tag{5.29}$$

where ε_1 is a *localised* contribution to the dielectric constant. We represent the fields $\mathbf{B}$ and $\mathbf{E}$ by the potentials:

$$\mathbf{B} = \nabla\wedge\mathbf{A}, \quad \mathbf{E} = -\frac{i\omega}{c}\mathbf{A} - \nabla\varphi$$

with the gauge conditions

$$\nabla\cdot\mathbf{A} - \frac{i\omega}{c}\varepsilon_0\varphi = 0. \tag{5.30}$$

Maxwell's equations lead to

$$\nabla\wedge(\nabla\wedge\mathbf{A}) = -\frac{i\omega}{c}\varepsilon\mathbf{E} = -\frac{i\omega}{c}\varepsilon_0\left(-\frac{i\omega}{c}\mathbf{A} - \nabla\varphi\right) - \frac{i\omega}{c}\underbrace{\varepsilon_1\mathbf{E}}_{4\pi\mathbf{P}_1}$$

Using our gauge condition this gives

$$\left[\Delta + \frac{\omega^2}{c^2}\varepsilon_0(\omega)\right]\mathbf{A} = \frac{4\pi i\omega}{c}\mathbf{P}_1. \tag{5.31}$$

The equation for φ is obtained by taking the divergence of (5.31). With (5.30) this gives

$$\left[\Delta + \frac{\omega^2}{c^2}\varepsilon_0(\omega)\right]\varphi = \frac{4\pi}{\varepsilon_0}\nabla\cdot\mathbf{P}_1. \tag{5.32}$$

If we make

$$\mathbf{Z} = \frac{ic}{\omega\varepsilon_0}\mathbf{A}, \tag{5.33}$$

then we have $\varphi = -\nabla\cdot\mathbf{Z}$ and

$$-(\Delta + k^2)\mathbf{Z} = \frac{4\pi}{\varepsilon_0}\mathbf{P}_1, \tag{5.34}$$

with

$$k^2 = \frac{\omega^2 n_0^2(\omega)}{c^2}, \quad n_0^2(\omega) = \varepsilon_0(\omega).$$

The fields $\mathbf{B}$ and $\mathbf{E}$ are easy to find from $\mathbf{Z}$:

$$\mathbf{B} = -\frac{i\omega}{c}\varepsilon_0\nabla\wedge\mathbf{Z}, \quad \mathbf{E} = k^2\mathbf{Z} + \nabla(\nabla\cdot\mathbf{Z}). \tag{5.35}$$

Observe that in the right-hand side of (5.34), $\mathbf{P}_1 = \frac{\varepsilon_1}{4\pi}\mathbf{E}$ is a function of $\mathbf{Z}$. The Green function of the Helmholtz operator, $-(\Delta + k^2)$,

$$G(\mathbf{x}) = \frac{1}{4\pi}\frac{e^{ikr}}{r}\ . \tag{5.36}$$

That is to say, $-(\Delta + k^2)G(\mathbf{x}) = \delta^3(\mathbf{x})$ and so for

$$\varphi_s(\mathbf{x}) = G * s(\mathbf{x}) \equiv \int d^3x' G(\mathbf{x}-\mathbf{x}')s(\mathbf{x}') \tag{5.37}$$

one has

$$-(\Delta+k^2)\varphi_s(\mathbf{x}) = -\int d^3x'(\Delta_{\mathbf{x}}+k^2)G(\mathbf{x}-\mathbf{x}')s(\mathbf{x}') = \int d^3x'\delta(\mathbf{x}-\mathbf{x}')s(\mathbf{x}') = s(\mathbf{x})\ . \tag{5.38}$$

The general solution of the equation $-(\Delta + k^2)\varphi(x) = s(\mathbf{x})$ is then $\varphi = \varphi_0 + \varphi_s$ where φ_0 is a homogeneous solution, $(\Delta + k^2)\varphi_0 = 0$.

This allows us to transform (5.34) into the following integral equation:

$$\mathbf{Z}(\mathbf{x}) = \mathbf{Z}^{(0)} + \frac{1}{\varepsilon_0}\int \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\mathbf{P}_1(\mathbf{x}')d^3x'. \tag{5.39}$$

Here $\mathbf{Z}^{(0)}$ is a homogeneous solution of (5.34) [i.e. $-(\Delta+k^2)\mathbf{Z}^{(0)} = 0$], and describes the incident wave. The second term describes the scattered wave. We denote it by $\mathbf{Z}^{(d)}$. At large distances $|\mathbf{x}| = R$, we approximate $\mathbf{Z}^{(d)}$ by

$$\mathbf{Z}^{(d)} = \frac{e^{ikR}}{\varepsilon_0 R}\int \mathbf{P}_1(\mathbf{x}')e^{-i\mathbf{k}'\cdot\mathbf{x}'}d^3x' = \frac{e^{ikR}}{\varepsilon_0 R}\tilde{\mathbf{P}}_1(\mathbf{k}') \tag{5.40}$$

with $\mathbf{n} = \mathbf{R}/R$ and $\mathbf{k}' = k\mathbf{n}$. Here $\tilde{\mathbf{P}}_1$ is the Fourier transform of $\mathbf{P}_1$,

$$\tilde{\mathbf{P}}_1(\mathbf{k}') = \int \mathbf{P}_1(\mathbf{x}') e^{-i\mathbf{k}'\cdot\mathbf{x}'} d^3x' \ . \tag{5.41}$$

Using the relations $\nabla\left(e^{ikR}/R\right) \approx ik\mathbf{n}e^{ikR}/R + \mathcal{O}(R^{-2})$, et $\nabla \wedge \left(e^{ikR}\mathbf{V}/R\right) \approx ik\mathbf{n} \wedge \mathbf{V}e^{ikR}/R + \mathcal{O}(R^{-2})$ for an arbitrary vector $\mathbf{V}$, we obtain the scattered fields

$$\left.\begin{aligned} \mathbf{B}^{(d)} &= \frac{k^2}{n_0}\frac{e^{ikR}}{R}\mathbf{n} \wedge \tilde{\mathbf{P}}_1(\mathbf{k}') \\ \mathbf{E}^{(d)} &= \frac{k^2}{n_0^2}\frac{e^{ikR}}{R}\left(\mathbf{n} \wedge \tilde{\mathbf{P}}_1(\mathbf{k}')\right) \wedge \mathbf{n} = \frac{1}{n_0}\left(\mathbf{B}^{(d)} \wedge \mathbf{n}\right) \end{aligned}\right\}. \tag{5.42}$$

We now consider the fields $\mathbf{E}(\mathbf{x}, \omega)$ and $\mathbf{B}(\mathbf{x}, \omega)$ as the amplitudes of waves of a given frequency ω. Adding the factor $e^{-i\omega t}$, we find with $t_{ret} := t - R/(\frac{c}{n_0})$

$$\left.\begin{aligned} \mathbf{B}^{(d)} &= \frac{k^2}{Rn_0}\mathbf{n} \wedge \tilde{\mathbf{P}}_1(\mathbf{k}', t_{ret}) \\ \mathbf{E}^{(d)} &= \frac{1}{n_0}\mathbf{B}^{(d)} \wedge \mathbf{n} \end{aligned}\right\}. \tag{5.43}$$

We want to determine the intensity of the scattered wave polarised according to $\boldsymbol{\epsilon}'$, with $\boldsymbol{\epsilon}' \cdot \mathbf{k}' = 0$. This gives

$$\boldsymbol{\epsilon}' \cdot \mathbf{E}^{(d)} = \frac{1}{R}\frac{k^2}{n_0^2}\boldsymbol{\epsilon}' \cdot \tilde{\mathbf{P}}_1. \tag{5.44}$$

The Poynting vector of the part of the wave with polarisation $\boldsymbol{\epsilon}'$ is

$$\mathbf{S} = \frac{c}{4\pi}n_0\left|\boldsymbol{\epsilon}' \cdot \mathbf{E}^{(d)}\right|^2\mathbf{n} = \frac{1}{R^2}\frac{c}{4\pi}n_0\left(\frac{k^2}{n_0^2}\right)^2|\boldsymbol{\epsilon}' \cdot \tilde{\mathbf{P}}_1|^2\mathbf{n}.$$

The intensity of the scattered wave is then (time average taken!)

$$\frac{dI}{d\Omega}(\mathbf{n}) = \frac{c}{4\pi}\frac{k^4}{n_0^3}\left\langle|\boldsymbol{\epsilon}' \cdot \tilde{\mathbf{P}}_1|^2\right\rangle = \frac{n_0}{4\pi c^3}\omega^4\left\langle|\boldsymbol{\epsilon}' \cdot \tilde{\mathbf{P}}_1|^2\right\rangle. \tag{5.45}$$

Scattering at long wavelengths

Let us assume that the wavelength λ is much larger than the size of the inhomogeneity. In this situation we can replace with 1 the exponential factor in the definition of $\tilde{P}_1$. This is the *dipole approximation*,

$$\begin{aligned} \frac{dI}{d\Omega} &= \frac{n_0}{4\pi c^3}\omega^4\left\langle|\boldsymbol{\epsilon}' \cdot \mathcal{P}_1|^2\right\rangle \\ \mathcal{P}_1 &= \int \mathbf{P}_1(\mathbf{x}')\, d^3x'. \end{aligned} \tag{5.46}$$

The intensity is then proportional to $k^4 \propto \lambda^{-4}$.

Consequences: Blue light is much more scattered than red light. This is why the sky appears blue and the setting sun, that has lost all its components of wavelength below red, appears red.

Born approximation

It is an accurate enough approximation to substitute in (5.41)

$$\mathbf{P}_1(\mathbf{x}') = \frac{\varepsilon_1(\mathbf{x}')}{4\pi}\,\mathbf{E}(\mathbf{x}') \approx \frac{\varepsilon_1(\mathbf{x}')}{4\pi}\,\mathbf{E}_0(\mathbf{x}') \tag{5.47}$$

where $\mathbf{E}_0(\mathbf{x}')$ is the incident field. Taking $\mathbf{E}_0(\mathbf{x}') = E_0\boldsymbol{\epsilon} e^{i(\mathbf{k}\cdot\mathbf{x}'-\omega t)}$, a plane wave linearly polarised in the direction $\boldsymbol{\epsilon}$ ($\mathbf{k}\cdot\boldsymbol{\epsilon} = 0$), we obtain

$$\tilde{\mathbf{P}}_1(\mathbf{k}') = \int \frac{\varepsilon_1(\mathbf{x}')}{4\pi}\,E_0\boldsymbol{\epsilon} e^{i(\mathbf{k}\cdot\mathbf{x}'-\omega t)} e^{-i\mathbf{k}'\cdot\mathbf{x}'}\,d^3x' = \frac{E_0}{4\pi}\boldsymbol{\epsilon}\,\tilde{\varepsilon}_1(\mathbf{k}'-\mathbf{k})e^{-i\omega t}, \tag{5.48}$$

and for the cross section of emission with polarisation $\boldsymbol{\epsilon}'$:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Born}} = \frac{dI}{d\Omega}\left(\frac{1}{\frac{c}{4\pi}n_0|E_0|^2}\right) = \frac{1}{(4\pi)^2}\left(\frac{\omega}{c}\right)^4 |\boldsymbol{\epsilon}'\cdot\boldsymbol{\epsilon}|^2\,|\tilde{\varepsilon}_1(\mathbf{k}'-\mathbf{k})|^2\,.$$

In the dipole approximation, $\mathbf{k}-\mathbf{k}'| \ll 1/d$, this gives

$$\left(\frac{d\sigma}{d\Omega}\right)_{dip.} = \frac{1}{16\pi^2}\left(\frac{\omega}{c}\right)^4 |\boldsymbol{\epsilon}'\cdot\boldsymbol{\epsilon}|^2 \left|\int \varepsilon_1(\mathbf{x}')\,d^3x'\right|^2. \tag{5.49}$$

After integrating over all the directions $\boldsymbol{\epsilon}$ (which gives a factor $4\pi/3$) and adding over the two final polarisations (factor 2), we obtain the total cross section (within the dipole approximation):

$$\sigma = \frac{1}{6\pi}\left(\frac{\omega}{c}\right)^4 \left|\int \varepsilon_1(\mathbf{x}')d^3x'\right|^2. \tag{5.50}$$

We now apply this general result to a gas with small statistical fluctuations of density. For a gas with N_V atoms in a small volume V ($V^{1/3} \ll \lambda$), one has

$$\varepsilon(\omega) - 1 \approx 4\pi\alpha(\omega)\frac{N_V}{V}$$

where $\alpha(\omega)$ is the molecular polarisability. For a fluctuation ΔN_V of the number of atoms, this gives (from 5.50) a cross section

$$\sigma \approx \frac{1}{6\pi}(4\pi\alpha)^2\left(\frac{\omega}{c}\right)^4 \langle(\Delta N_V)^2\rangle. \tag{5.51}$$

This scattering due to the inhomogeneities in a gas is known as the *Rayleigh* scattering. For a random distribution of particles (uncorrelated), the statistical average, $\langle(\Delta N_V)^2\rangle$ is proportional to V and $\sigma/V = S$, the *scattering coefficient*, is independent of the volume. Einstein used this result to determine the Avogadro number (see A. Einstein, Collected Papers Vol 3, p. 287, 1910).

5.4 Diffraction

(Jackson §9.8 ff)

Any deviation of a ray of light from its optical path is called *diffraction*. In the geometrical optic limit, an illuminated body creates a precise shadow. At sufficiently large wavelengths, or for sufficiently small bodies, this approximation is no longer valid. The diffraction phenomena are entirely explained by the wave-like nature of light, but in this case the boundary conditions are not trivial. It is necessary in this case to solve Maxwell's equations for some given sources and boundary conditions at the screens and at infinity.

Here we will discuss only the Kirchhoff approximation. Additionally, we make the following (non-essential) simplifications:

i) We make use of a *scalar* wave equation

$$\left(\Delta - \frac{1}{c^2}\partial_t^2\right) u = 0, \tag{5.52}$$

that is assumed to be valid everywhere outside the source and the screens. (Every component of the electric and magnetic fields satisfy this equation). We also assume that the dispersive effects of the medium are negligible: $\varepsilon = \mu = 1$.

ii) We consider monochromatic light,

$$u = u(\mathbf{x})e^{-i\omega t}.$$

Equation (5.52) then becomes

$$(\Delta + k^2)u = 0 \quad \text{with} \quad k = \frac{\omega}{c}. \tag{5.53}$$

Kirchhoff approximation

The 2^{nd} Green formula is (D, an open set, $u, v \in C^2(D)$)

$$\int_D (v\Delta u - u\Delta v)d^3x = \int_{\partial D}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) d\sigma.$$

(This is a simple consequence of Gauss's law:

$$\int_D \nabla \cdot \mathbf{W} = \int_{\partial D} \mathbf{W} \cdot \mathbf{n} d\sigma \quad \text{with} \quad W_i = v\partial_i u - u\partial_i v. \text{)}$$

If we apply this formula to a solution u of equation (5.53) and for

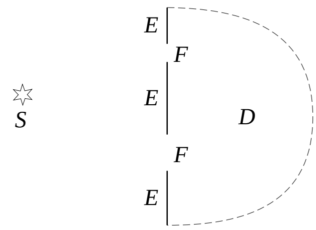
$$\begin{aligned} v(\mathbf{x}) &= G(\mathbf{x}-\mathbf{y}) = -\frac{1}{4\pi}\frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}, \\ (\Delta_{\mathbf{x}} + k^2)G(\mathbf{x}-\mathbf{y}) &= \delta(\mathbf{x}-\mathbf{y}) \ , \end{aligned}$$

the Green function of the Helmholtz operator, $\Delta + k^2$, we obtain

$$\begin{aligned} \int_{\partial D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) d\sigma &= \int_D \left[vk^2 u + u\delta(\mathbf{x}-\mathbf{y}) - k^2 vu \right] d^3x \\ &= \begin{cases} u(\mathbf{y}) & \mathbf{y} \in D \\ 0 & \mathbf{y} \notin D \end{cases} \end{aligned} \tag{5.54}$$

Should we know the boundary conditions $u(\mathbf{x})$ and $\frac{\partial u}{\partial n}(\mathbf{x})$, $\mathbf{x} \in \partial D$, we could then find the solution $u(\mathbf{y})$ in D.

In diffraction problems, one determines the influence of non-transparent screens (E) and of apertures (slits, F) on the propagation of light.



The Kirchhoff approximation consists in assuming that

$$\left. \begin{array}{lll} u = 0, & \dfrac{\partial u}{\partial n} = 0 & \text{on the screens } E \\ u = u_0, & \dfrac{\partial u}{\partial n} = \dfrac{\partial u_0}{\partial n} & \text{in the slits } F \end{array} \right\} , \tag{5.55}$$

where $u_0 = Ae^{ikr_1}/r_1$ is the solution without screens (r_1 is the distance to the source). In the Kirchhoff approximation, we obtain

$$u(\mathbf{y}) = \int_F \left(u_0 \frac{\partial v}{\partial n} - v\frac{\partial u_0}{\partial n} \right) d\sigma \, . \tag{5.56}$$

If $u = u_0$ and $\frac{\partial u}{\partial n} = \frac{\partial u_0}{\partial n}$ on both, the screens and the slits, we should find the solution u_0 [according to (5.54)]. Therefore (5.56) is equivalent to

$$u(\mathbf{y}) = u_0(\mathbf{y}) - \int_E \left(u_0 \frac{\partial v}{\partial n} - v\frac{\partial u_0}{\partial n} \right) d\sigma. \tag{5.57}$$

This approximation is useful if the apertures F are relatively large relative to the wavelength λ. From the theory of partial differential equations, it is known that in general it is not possible to solve (5.53) by specifying the two conditions, u and $\frac{\partial u}{\partial n}$, at the boundaries: the solution $u(\mathbf{x})$ found with (5.56) and (5.57) does not satisfy in general the conditions (5.55). The mathematical contradiction of giving both, u and $\frac{\partial u}{\partial n}$ at the boundaries can be eliminated if we choose the correct Green function satisfying the correct boundary conditions

$$\textit{Dirichlet}: \quad G_D = 0 \quad \text{on the screen and apertures,} \tag{5.58}$$

$$\textit{Neumann}: \quad \frac{\partial G_N}{\partial n} = 0 \quad \text{on the screens and apertures.} \tag{5.59}$$

In addition, we demand that G decreases at infinity, as the emitted wave:

$$r\left(\frac{\partial G}{\partial n} - ikG\right) \overset{r\to\infty}{\to} 0, \quad G \overset{r\to\infty}{\to} \frac{e^{ikr}}{r}.$$

With (5.58) and (5.59), we obtain

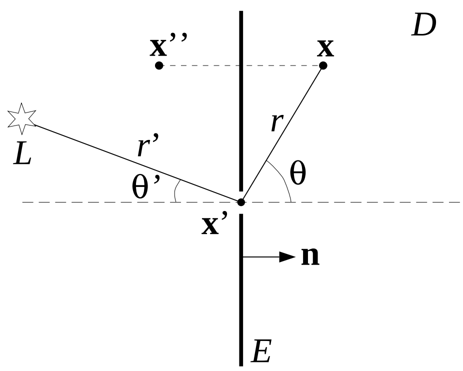
$$u(\mathbf{x}) = \int_{E\cup F} u(\mathbf{x}')\frac{\partial G_D}{\partial n}(\mathbf{x},\, \mathbf{x}')d\sigma(\mathbf{x}') \tag{5.60}$$

$$u(\mathbf{x}) = -\int_{E\cup F} \frac{\partial u}{\partial n}(\mathbf{x}')G_N(\mathbf{x},\, \mathbf{x}')d\sigma(\mathbf{x}'). \tag{5.61}$$

This approximation in not contradictory mathematically speaking. For a *plane screen*, the Green functions G_N and G_D are easy to determine. One has

$$G_{D,N}(\mathbf{x},\mathbf{x}') = -\frac{1}{4\pi}\left(\frac{e^{ikr}}{r} \mp \frac{e^{ikr''}}{r''}.\right) \tag{5.62}$$

The sign $-$ corresponds to G_D and the sign $+$ corresponds to G_N, and $r = |\mathbf{x}-\mathbf{x}'|$ and $r'' = |\mathbf{x}'' - \mathbf{x}'|$. Here $\mathbf{x}''$ is the *mirror* position of $\mathbf{x}$ with respect to the screen (see figure)



For $\mathbf{x}' \in E \cup F$, we have $r = r''$ and

$$\frac{\partial G_D}{\partial n} = \frac{k}{2\pi i}\frac{e^{ikr}}{r}\left(1 + \frac{i}{kr}\right)\frac{\mathbf{n}\cdot\mathbf{r}}{r}$$

where $\mathbf{r} = \mathbf{x} - \mathbf{x}'$. In the Kirchhoff approximation we then obtain

$$u(\mathbf{x}) = \frac{k}{2\pi i}\int_F \frac{e^{ikr}}{r}\left(1 + \frac{i}{kr}\right)\frac{\mathbf{n}\cdot\mathbf{r}}{r}u_0(\mathbf{x}')d\sigma. \tag{5.63}$$

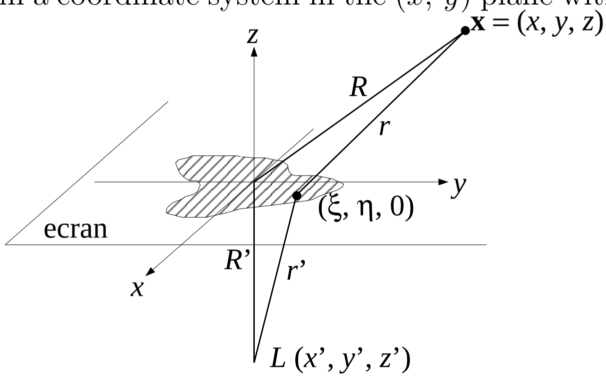
Neglecting the term $1/r^2$, this gives for $u_0 = Ae^{ikr'}/r'$ (r' is the distance between $\mathbf{x}'$ and the source)

$$u(\mathbf{x}) = \frac{kA}{2\pi i}\int_F \frac{e^{ikr}}{r}\frac{e^{ikr'}}{r'}\cos\theta\, d\sigma. \tag{5.64}$$

From the approximative Neumann formula, we obtain the same result with $\cos\theta$ replaced with $\cos\theta'$ and from the original formula (5.56) we obtain (5.64) with $\cos\theta$ replaced with $\frac{1}{2}(\cos\theta + \cos\theta')$. Next, we substitute $\cos\theta$ by an average value $\cos\vartheta$ and $1/rr'$ by an average value $1/RR'$:

$$u(\mathbf{x}) = A\frac{k}{2\pi i}\frac{\cos\vartheta}{RR'}\int_F e^{ik(r+r')}d\sigma$$

We place the split in a coordinate system in the $(x,\, y)$ plane with O in the aperture



$$\begin{aligned}
r^2 &= (x-\xi)^2 + (y-\eta)^2 + z^2 \\
r'^2 &= (x'-\xi)^2 + (y'-\eta)^2 + z'^2 \\
R^2 &= x^2 + y^2 + z^2, \quad R'^2 = x'^2 + y'^2 + z'^2 \\
r &= R - \frac{x\xi + y\eta}{R} + \frac{\xi^2+\eta^2}{2R} + \frac{(x\xi+y\eta)^2}{2R^3} + \ldots \\
r' &= R' - \frac{x'\xi + y'\eta}{R'} + \frac{\xi^2+\eta^2}{2R'} + \frac{(x'\xi+y'\eta)^2}{2R'^3} + \ldots
\end{aligned}$$

This gives

$$u(\mathbf{x}) = A\frac{k}{2\pi i}\cos\vartheta\,\frac{e^{ik(R+R')}}{RR'}\int_F e^{ik\phi(\xi,\,\eta)}d\xi d\eta \tag{5.65}$$

with

$$\begin{aligned}\phi(\xi,\eta) = & -\frac{x\xi + y\eta}{R} + \frac{\xi^2+\eta^2}{2R} + \frac{(x\xi+y\eta)^2}{2R^3} + \dots \\ & -\frac{x'\xi + y'\eta}{R'} + \frac{\xi^2+\eta^2}{2R'} + \frac{(x'\xi+y'\eta)^2}{2R'^2} + \dots\end{aligned}$$

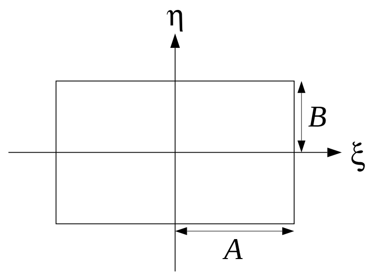
Frauenhofer diffraction

We now assume that R and R' are much larger than the dimension d of the aperture, so that we can neglect the terms of order d/R and d/R'. With the directional cosines,

$$\alpha = \frac{x}{R}, \quad \beta = \frac{y}{R}; \quad \alpha' = \frac{x'}{R'}, \quad \beta' = \frac{y'}{R'},$$

and defining

$$a = \alpha + \alpha', \quad b = \beta + \beta',$$



we obtain

$$u(\mathbf{x}) = \frac{\text{const.}}{\lambda}\int_F \exp(-ik(a\xi + b\eta))d\xi d\eta.$$

For an aperture of rectangular shape, this integral gives an intensity

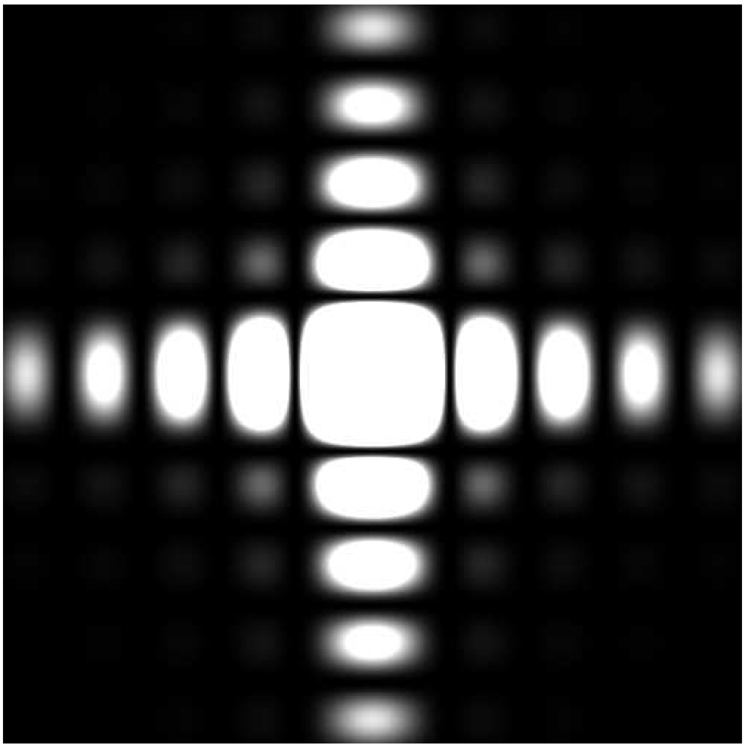
$$I(\mathbf{x}) = |u(\mathbf{x})|^2 = \frac{\text{const}}{\lambda^2}(4AB)^2\left(\frac{\sin kaA}{kaA}\right)^2\left(\frac{\sin kbB}{kbB}\right)^2.$$

For a source at position (α', β') to be distinguished from a source at position (α'', β''), it is necessary that the first zeros in the intensity of the image through the "aperture" of size AB be well distinct. That is to say

$$\begin{aligned}\alpha' - \alpha'' & \geqslant \frac{\pi}{kA} = \frac{\lambda}{2A} \\ \beta' - \beta'' & \geqslant \frac{\pi}{kB} = \frac{\lambda}{2B}.\end{aligned}$$

This *diffraction limit* is important in astronomy. A telescope of diameter D that observes light of wavelength λ is able to distinguish two stars only if their angular separation δ is large enough:

$$\delta > \frac{\lambda}{D}.$$



Exercise: What is the diffraction limit of an optical telescope with a mirror of 4 m (take red light)?

Answer: 0.036 arc seconds .

THE END