Advanced Topics in General Relativity

Doctoral course for theoretical physics
1 semester, 2 hours of course and 1 hour of exercises per week

Most of the material of this course is from the two excellent textbooks [13] and [16] which however contain much more than the topics presented in this course.

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Chapter 1

Exterior Algebra and Differential Forms

We first develop some algebraic preparation which I assume to be more or less known. I also use this occasion to fix the notation. I also assume that you know what an m-dimensional differentiable manifold is, we shall denote it by $\mathcal{M}$ and its tangent space at some point $x \in \mathcal{M}$ is denoted by $T_x\mathcal{M}$. The tangent bundle is $T\mathcal{M} = \cup_{\{x \in \mathcal{M}\}} T_x\mathcal{M}$.

1.1 Exterior algebra

Let A be a commutative, associative, unitary algebra[1] over $\mathbb{R}$ and let E be a module (vector space) on A:

- **Commutative:** $a, b \in A \Rightarrow ab = ba$
- **Associative:** $a(bc) = (ab)c$
- **Unitary:** $\exists\ e \in A$ tel que $ea = a$, $\forall a \in A$, which we shall call 1 in what follows.

We are interested mainly in the case $A = \mathbb{R}$ ou $A = \mathcal{F}(\mathcal{M})$ (the (smooth, i.e. C^∞) functions on $\mathcal{M}$), where $\mathcal{M}$ is a differentiable manifold and E either a real finite dimensional vector space or the infinite dimensional vector space of (smooth) vector fields on $\mathcal{M}$ denoted by $E = \mathcal{X}(\mathcal{M})$.

[1] An algebra is a vector space with a multiplication.

We consider the space of p-linear forms with values in A. Denoting the space of covariant tensors on E of rank p by $T_p(E)$ we define:

Definition 1.1

1. *$\Lambda^p(E) \subset T_p(E)$ is the space of totally antisymmetric covariant p–tensors, called p–forms on E:*

$$\alpha(\cdots v, \cdots w, \cdots) = -\alpha(\cdots w, \cdots v, \cdots)$$

   for all $\alpha \in \Lambda^p(E)$ and $v, w \in E$.

2. *For $t \in T_p(E)$ we define the* **alternation operator** *$\mathcal{A}$ by*

$$(\mathcal{A}t)(v_1, \dots, v_p) = \frac{1}{p!} \sum_{\sigma \in \mathcal{S}_p} (\operatorname{sgn} \sigma) t(v_{\sigma(1)}, \dots, v_{\sigma(p)}) \tag{1.1}$$

   where $v_1, \dots, v_p \in E$, $\mathcal{S}_p$ is the group of permutations of p elements and $\operatorname{sgn}\sigma$ is the signature of the permutation σ.

Proposition 1.1 *$\mathcal{A}$ is the projection from $T_p(E)$ to $\Lambda^p(E)$, i.e., $\mathcal{A}$ is a linear operator on $T_p(E)$ with $\mathcal{A}(T_p(E)) = \Lambda^p(E)$ and $\mathcal{A} \circ \mathcal{A} = \mathcal{A}$.*

Proof: Exercice.

Definition 1.2 (exterior product) *For $\omega \in \Lambda^p(E)$, $\eta \in \Lambda^q(E)$, we define the* **exterior product**

$$\Lambda^{p+q}(E) \ni \omega \wedge \eta := \frac{(p+q)!}{p!q!} \mathcal{A}(\omega \otimes \eta)\,, \tag{1.2}$$

where $\otimes$ denotes the usual tensor product.

Example 1 : Be $(\theta^i)_{i=1}^n$ a basis of $E^* = \Lambda^1(E)$ and $(e_i)_{i=1}^n$ the dual basis of E, i.e. $\theta^i(e_j) = \delta^i_j$. Show that for two vectors $v = v^i e_i$ and $w = w^i e_i$ we obtain

$$\theta^i \wedge \theta^j(v, w) = v^i w^j - v^j w^i\,.$$

Note that we denote basis vectors with low indices and 1-forms or co-vectors (elements of the dual vector space E^*) with upper indices, while for components we use the opposite convention.
We also use Einstein's summation convention: Double indices are summed over,
$v^i e_i \equiv \sum_{i=1}^n v^i e_i$.

Proposition 1.2 *The exterior product has the following properties:*

1. $(\omega_1 + \omega_2) \wedge \eta = \omega_1 \wedge \eta + \omega_2 \wedge \eta$, *for* $\omega_1,\ \omega_2 \in \Lambda^p(E)$ *and* $\eta \in \Lambda^q(E)$.
2. $a(\omega \wedge \eta) = (a\omega) \wedge \eta = \omega \wedge (a\eta)$ *for* $a \in A$
3. $\omega \wedge \eta = (-1)^{pq}\eta \wedge \omega$
4. $(\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3)$, *for* $\omega_1 \in \Lambda^p(E)$, $\omega_2 \in \Lambda^q(E)$ *and* $\omega_3 \in \Lambda^k(E)$.

The $\wedge$*-product is thus bilinear and associative.*

Proof: Exercise.

Proposition 1.3 *Let* $(\theta^i)_{i=1}^n$ *be a basis of* $E^\star = \Lambda^1(E)$. *Then the products*

$$(\theta^{i_1} \wedge \theta^{i_2} \wedge \cdots \wedge \theta^{i_p}); \quad 1 \leq i_1 < i_2 < \cdots < i_p \leq n$$

form a basis of $\Lambda^p(E)$.
Consequently the dimension of $\Lambda^p(E)$, $p \leq n$, *is*

$$\dim(\Lambda^p(E)) = \binom{n}{p} = \frac{n!}{p!(n-p)!}.$$

For $p > n$, $\Lambda^p(E) = \{0\}$.

Proof: Exercise.

Definition 1.3 (Grassmann algebra) *The Grassmann algebra (or* **exterior algebra***) over the vector space* E *is the direct sum*

$$\Lambda(E) = \bigoplus_{p=0}^{n} \Lambda^p(E)$$

According to proposition 1.3, $\dim \Lambda(E) = 2^n$.
$\Lambda(E)$ *is a graded algebra (associative and unitary). We set* $\Lambda^0(E) = A$.

Definition 1.4 (interior product) *The* **interior product** *is the map* $(p > 0)$

$$\begin{array}{l} E \times \Lambda^p(E) \to \Lambda^{p-1}(E) \\ (v, \omega) \mapsto i_v\omega \end{array}$$

where $(i_v\omega)(v_1, \ldots, v_{p-1}) := \omega(v, v_1, \ldots, v_{p-1})$.
For $\omega \in \Lambda^0(E)$ *we define* $i_v\omega \equiv 0$. *The interior product allows us to define the map*

$$i : E \times \Lambda(E) \to \Lambda(E) : (v, \omega) \mapsto i_v\omega.$$

Proposition 1.4

1. *i_v is A-linear*
2. $i_v(\Lambda^p(E)) \subseteq \Lambda^{p-1}(E)$
3. $i_v(\omega \wedge \eta) = (i_v\omega) \wedge \eta + (-1)^p \omega \wedge (i_v\eta)$ *for* $\omega \in \Lambda^p(E)$ *and* $\eta \in \Lambda^q(E)$.

In other words, i_v is an anti-derivation of degree -1 on $\Lambda(E)$ (see def. 1.5 below).

Proof: Exercice.

For this we make use of the definition

Definition 1.5 (derivation, anti-derivation)
A map $\theta : \Lambda(\mathcal{M}) \to \Lambda(\mathcal{M})$ *is a* **derivation** *(respectively* **anti-derivation***) of degree* $k \in \mathbb{Z}$, *if*

1. *θ is A-linear*
2. $\theta(\omega \wedge \eta) = \theta\omega \wedge \eta + \omega \wedge \theta\eta$, *for* $\omega, \eta \in \Lambda(\mathcal{M})$
   (anti-derivation if: $\theta(\omega \wedge \eta) = \theta\omega \wedge \eta + (-1)^p \omega \wedge \theta\eta$, $\omega \in \Lambda^p(\mathcal{M})$, $\eta \in \Lambda(\mathcal{M})$*).*
   This is called the Leibnitz rule.
3. $\theta(\Lambda^p(\mathcal{M})) \subset \Lambda^{p+k}(\mathcal{M})$, $0 \leq p \leq n$.
   (For $p + k > n$ *or* $p + k < 0$ *we set* $\Lambda^{p+k}(\mathcal{M}) = \{0\}$*.)*

1.2 Differential forms

Let $\mathcal{M}$ be a differentiable manifold of dimension m. For $p = 0, 1, \ldots, m$ and $x \in \mathcal{M}$ we consider the spaces

$$\Lambda^p(T_x\mathcal{M}) \subset T_x(\mathcal{M})^0_p \quad \forall p \geq 1$$
$$\Lambda^0(T_x\mathcal{M}) = \mathbb{R};\ \Lambda^1(T_x\mathcal{M}) = (T_x\mathcal{M})^\star$$
$$\Lambda(T_x\mathcal{M}) = \bigoplus_{p=0}^{n} \Lambda^p(T_x\mathcal{M}).$$

Here $T_x(\mathcal{M})^0_p$ denotes the space of p-fold covariant (and 0-fold contravariant) tensors over the tangent space of $\mathcal{M}$ at x, $T_x(\mathcal{M})$.

Definition 1.6 (differential forms) *A* **differential form** *of degree p is a (smooth) covariant tensor field of degree p, called* ω, *such that* $\omega(x) \in \Lambda^p(T_x\mathcal{M})$ *for all*

$x \in \mathcal{M}$. Sometimes we denote $\omega(x)$ by ω_x or we simply suppress the argument x. Often we call differential form of degree p simply a p-form.
$\Lambda^p(\mathcal{M})$ is the module of p–forms on $\mathcal{F}(\mathcal{M})$.
$\Lambda(\mathcal{M}) = \bigoplus_{p=0}^{n} \Lambda^p(\mathcal{M})$ *is the exterior algebra of differential forms on $\mathcal{M}$.*

As all the elements of $\Lambda(\mathcal{M})$ are tensor fields, all results on tensor fields are also valid for differential forms.

The algebraic operations introduced in the previous section are defined point by point for differential forms, also the exterior product. For $\omega \in \Lambda^p(\mathcal{M})$, $X_1, \ldots, X_p \in \mathcal{X}(\mathcal{M})$, the mapping

$$x \mapsto \omega_x(X_1(x), \ldots, X_p(x))$$

is a smooth function on $\mathcal{M}$. The map

$$\underbrace{\mathcal{X}(\mathcal{M}) \times \cdots \times \mathcal{X}(\mathcal{M})}_{p \text{ times}} \to \mathcal{F}(\mathcal{M}) : (X_1, \ldots, X_p) \mapsto \omega_x(X_1, \ldots, X_p)$$

is $\mathcal{F}(\mathcal{M})$-linear and completly anti-symmetric.
For a vector field X we define the interior product

$$(i_X\omega)_x \equiv i_{X(x)}\omega_x$$

In a local coordinate system, $(x^1, \ldots, x^m; \mathcal{U})$, $\omega \in \Lambda^p(\mathcal{M})$ can be written in the basis dx^i as

$$\begin{aligned} \omega &= \sum_{1 \le i_1 < \cdots < i_p \le m} \omega_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \\ &= \frac{1}{p!} \sum_{i_1, \cdots, i_p = 1}^{m} \omega_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \ , \end{aligned}$$

where the $\omega_{i_1 \cdots i_p}$ with arbitrary index positions are obtained from those with $i_1 < i_2 \ldots < i_p$ by anti-symmetry.

Remark: Here dx^i is the 1-form which assigns to a vector field of the form $X = \sum_{j=1}^m v^j \partial_j$ the component v^i. (For a function $df = \sum_{j=1}^m \partial_j f \, dx^j$.)

Example 2: Consider $\mathcal{M} = \mathbb{R}^2$ with Cartesian coordinates (x, y) and the vector fields $X = f(x, y)\partial_x$, $Y = g(x, y)\partial_y$ and $Z = h_1(x, y)\partial_x + h_2(x, y)\partial_y$ as well as the 2-form $\omega = dx \wedge dy$. Here f, g, h_1 and h_2 are arbitrary C^∞ functions of $\mathbb{R}^2$. Determine the 1-forms $i_X\omega$, $i_Y\omega$ and $i_Z\omega$.

Let $\varphi : \mathcal{M} \to \mathcal{N}$ be a differentiable map from the manifold $\mathcal{M}$ to the manifold $\mathcal{N}$. The fact that the pull-back is linear and respects the tensor product $\otimes$ implies that it also commutes with the wedge (exterior) product[2],

$$\varphi^\star : \Lambda(\mathcal{N}) \to \Lambda(\mathcal{M})\,,$$

$\varphi^\star(\omega \wedge \eta) = \varphi^\star\omega \wedge \varphi^\star\eta$. It is therefore an algebra homomorphism from $\Lambda(\mathcal{N})$ into $\Lambda(\mathcal{M})$. If φ is a diffeomorphism, $\varphi^\star$ is even an isomorphism with $(\varphi^\star)^{-1} = (\varphi^{-1})^\star$.

Exercise: Show that the Lie derivative, L_X for an arbitrary vector field X is a derivation of degree 0.
Hint: You just have to show that for $\omega \in \Lambda^p(\mathcal{M})$ also $L_X\omega \in \Lambda^p(\mathcal{M})$, the rest is clear from the general properties of the Lie derivative on tensor fields. These are assumed known. One can also show that for an arbitrary p-form the Lie derivative is given by

$$\begin{aligned}(L_X\omega)(v_1,\cdots,v_p) \;\;=\;\; & X((\omega(v_1,\cdots,v_p)) - \omega([X,v_1],v_2,\cdots,v_p) \\ & \cdots\;\; - \omega(v_1,\cdots,v_{p-1},[X,v_p])\,. \end{aligned} \tag{1.3}$$

This is actually the formula for the Lie derivative of an arbitrary covariant tensor field.

Proposition 1.5

- *For anti-derivations θ, θ' of degrees k, k', $\theta\circ\theta' + \theta'\circ\theta$ is a derivation of degree $k + k'$, if k and k' are both odd.*
- *For anti-derivations θ, θ' of degrees k, k', $\theta\circ\theta' - \theta'\circ\theta \equiv [\theta, \theta']$ is a derivation of degree $k + k'$, if k and k' are both even.*
- *For derivations θ, θ' of degrees k, k', $[\theta, \theta']$ is a derivation of degree $k + k'$.*

Proof: Simple calculation.

Proposition 1.6 *The (anti-)derivations of $\Lambda(\mathcal{M})$ are* **local**, *i.e. for an open set $\mathcal{U} \subset \mathcal{M}$ and $\omega \in \Lambda(\mathcal{M})$ such that $\omega|_{\mathcal{U}} = 0$ we have $\theta\omega|_{\mathcal{U}} = 0$, for an arbitrary (anti-)derivation θ.*

Proof: Let us consider $\omega \in \Lambda(\mathcal{M})$ such that $\omega|_{\mathcal{U}} = 0$. For any given $x \in \mathcal{U}$ there exists a function $h \in \mathcal{F}(\mathcal{M})$ such that $h(x) = 1$ and $h|_{\mathcal{M}\setminus\mathcal{U}} = 0$. Hence $h \cdot \omega \equiv 0$. Linearity then implies, $\theta(h\omega) = 0$, and therefore $\theta h \wedge \omega + h \cdot \theta\omega = 0$ in x which implies $(\theta\omega)_x = 0$. □

[2]For a covariant tensor field t on $\mathcal{N}$ the pullback $\varphi^\star$ is a covariant tensor field on $\mathcal{M}$ defined by $(\varphi^\star t)_x(v_1,\cdots,v_p) = t_{\varphi(x)}(T\varphi v_1,\cdots,T\varphi v_p)$.

Consequently, for $\omega = \omega'$ in $\mathcal{U} \subset \mathcal{M}$ we have $\theta\omega = \theta\omega'$ in $\mathcal{U}$ for every (anti-)derivation θ. We can therefore uniquely define $\theta|_{\mathcal{U}}$ on $\Lambda(\mathcal{U})$:
for $x \in \mathcal{U}$ and $\alpha \in \Lambda(\mathcal{U})$ we choose $\tilde{\alpha} \in \Lambda(\mathcal{M})$ such that $\tilde{\alpha} = \alpha$ in a neighborhood of x and we set

$$\left(\theta\Big|_{\mathcal{U}}\right)\alpha(x) = (\theta\tilde{\alpha})(x).$$

According to proposition 1.6, this definition is independent of the choice of $\tilde{\alpha}$. The existence of such an extension $\tilde{\alpha}$ is a consequence of the continuation lemma:

Lemma 1.1 (continuation lemma) *Let $\mathcal{U} \subset \mathcal{M}$ be an open set and $K \subset \mathcal{U}$ a compact set. For all $\beta \in \Lambda(\mathcal{U})$ there existe an $\alpha \in \Lambda(\mathcal{M})$ such that*

$$\beta\Big|_K = \alpha\Big|_K \quad et \quad \alpha\Big|_{\mathcal{M}\setminus\mathcal{U}} = 0.$$

Proof: There exists a function $h \in \mathcal{F}(\mathcal{M})$ with $h(x) = 1\ \forall x \in K$ and $h(x) = 0\,\forall x \in \mathcal{M} \setminus \mathcal{U}$. We can thus choose

$$\alpha(x) = \begin{cases} h(x)\beta(x), & x \in \mathcal{U} \\ 0, & x \in \mathcal{M} \setminus \mathcal{U} \end{cases}$$

□

We hence have the following result:

Proposition 1.7 (localisation theorem) *Let θ be an (anti-)derivation on $\Lambda(\mathcal{M})$, $\mathcal{U} \subset \mathcal{M}$ an open set. There exists a unique (anti-)derivation $\theta_{\mathcal{U}}$ on $\Lambda(\mathcal{U})$ such that*

$$(\theta\alpha)\Big|_{\mathcal{U}} = \theta_{\mathcal{U}}\left(\alpha\Big|_{\mathcal{U}}\right) \quad \text{for all } \alpha \in \Lambda(\mathcal{M})$$

We also need a globalisation theorem:

Proposition 1.8 (globalisation theorem) *Let $(\mathcal{U}_i)_{i\in I}$ be an open covering of $\mathcal{M}$. For $i \in I$, let θ_i be an (anti-)derivation on $\Lambda(\mathcal{U}_i)$ and θ_{ij} its restriction to $\mathcal{U}_i \cap \mathcal{U}_j$. If $\theta_{ij} = \theta_{ji}$ for every pair $(i,j) \in I \times I$, then there exists a unique (anti-)derivation $\theta \in \Lambda(\mathcal{M})$ such that $\theta_i = \theta|_{\mathcal{U}_i}$.*

Proof: For $\alpha \in \Lambda(\mathcal{M})$ and $x \in \mathcal{U}_i$ we define

$$(\theta\alpha)_x = \theta_i\left(\alpha\Big|_{\mathcal{U}_i}\right)_x . \tag{1.4}$$

Since

$$\left((\theta\alpha)\Big|_{\mathcal{U}_j}\right)\Big|_{\mathcal{U}_i} = \theta_{ji}\left(\alpha\Big|_{\mathcal{U}_i\cap\mathcal{U}_j}\right) = \theta_{ij}\left(\alpha\Big|_{\mathcal{U}_i\cap\mathcal{U}_j}\right)$$

$$= \left((\theta\alpha)\Big|_{\mathcal{U}_i} \right) \Big|_{\mathcal{U}_j}$$

Eq. (1.4) is independent of the choice of $\mathcal{U}_i$ as long as $x \in \mathcal{U}_i$, and hence $\theta\alpha$ is well defined. □

We can now also show the following very useful property of (anti-)derivations:

Proposition 1.9 *Let θ be an (anti-)derivation of degree k and $\theta f = \theta df = 0$ (*) for all $f \in \mathcal{F}(\mathcal{M})$. Then*

$$\theta \equiv 0.$$

Proof: We choose an atlas $(h_i, \mathcal{U}_i)$ of $\mathcal{M}$. We then set $\theta_i := \theta|_{\mathcal{U}_i}$. It is thus enough to show that $\theta_i \equiv 0$ for all i. But in a local coordinate system $(x^1, \ldots, x^n)$ on $\mathcal{U}_i$ an arbitrary p-form $\alpha \in \Lambda^p(\mathcal{M})$ can be written as

$$\alpha\Big|_{\mathcal{U}_i} = \sum \alpha_{j_1 \cdots j_p} dx^{j_1} \wedge \cdots \wedge dx^{j_p}.$$

And because of Leibnitz's rule (point 2 of definition 1.5) and the condition (*)

$$(\theta\alpha)\Big|_{\mathcal{U}_i} = \theta_i \left(\alpha\Big|_{\mathcal{U}_i} \right) = 0.$$

□

Consequence 1.10 *An (anti-)derivation on $\Lambda(\mathcal{M})$ is uniquely determined by its values on the functions ($= \Lambda^0(\mathcal{M})$) and on the "gradients", $\{df \mid f \in \mathcal{F}(\mathcal{M})\} \subset \Lambda^1(\mathcal{M})$.*

1.3 The exterior derivative

Theorem 1.1 *There exists a unique map*

$$d : \Lambda(\mathcal{M}) \to \Lambda(\mathcal{M})$$

with the following properties:

1. *d is an anti-derivation of degree 1*
2. *$d \circ d = 0$*
3. *df is the gradient of f for all $f \in \mathcal{F}(\mathcal{M})$, i.e., $df(X) = Xf$, for $f \in \mathcal{F}(\mathcal{M})$, $X \in \mathcal{X}(\mathcal{M})$.*

Proof: The uniqueness is a consequence of 1.10. (Point 3 determines d on the functions and 2 determines it on the gradients.) Hence we just have to show the existence.
Since a form $\alpha \in \Lambda^p(\mathcal{M})$ on a chart $(x^1, \ldots, x^m; \mathcal{U})$ can be written in the form

$$\alpha\Big|_{\mathcal{U}} = \sum_{1 \le i_1 < \cdots < i_p \le m} \alpha_{i_1 \cdots i_p} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}, \quad \alpha_{i_1 \cdots i_p} \in \mathcal{F}(\mathcal{M})$$

Points 2 & 3 and the Leibniz rule determine

$$d\alpha\Big|_{\mathcal{U}} \overset{2,Leibn.}{=\!=} \sum_{1 \le i_1 < \cdots < i_p \le m} d\alpha_{i_1 \cdots i_p} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} \in \Lambda^{p+1}(\mathcal{M}) \tag{1.5}$$

$$\overset{3}{=\!=} \sum_{1 \le i_1 < \cdots i_k < \cdots < i_{p+1} \le m} \sum_{k=1}^{p+1} (-1)^{k+1} \frac{\partial}{\partial x^{i_k}} \alpha_{i_1 \cdots \check{i}_k \cdots i_{p+1}} dx^{i_1} \wedge \cdots \wedge dx^{i_{p+1}}.$$

(The notation $i_1 \cdots \check{i}_k \cdots i_{p+1}$ means: "leave out the index i_k".) Derive the equality 3 in detail! The globalisation theorem 1.8 implies then the existence of the operator d on $\Lambda(\mathcal{M})$. □

The components of $d\alpha$ are given by

$$(d\alpha)_{i_1 \cdots i_{p+1}} = \sum_{k=1}^{p+1} (-1)^{k+1} \frac{\partial}{\partial x^{i_k}} \alpha_{i_1 \cdots \check{i}_k \cdots i_{p+1}}, \quad i_1 < i_2 < \cdots < i_{p+1} \tag{1.6}$$

Definition 1.7 (exact and closed differential forms)
A differential form $\alpha \in \Lambda(\mathcal{M})$ *is called* **exact** *if there exists a form* β *such that* $\alpha = d\beta$*;* α *is called* **closed** *if* $d\alpha = 0$.

Since $d \circ d = 0$, every exact form is closed. Locally, the inverse is also true:

Lemma 1.2 (Poincaré Lemma) *Let* $\alpha \in \Lambda(\mathcal{M})$ *be closed. For all* $x \in \mathcal{M}$ *exists an open set* $\mathcal{U} \subset \mathcal{M}$*, with* $x \in \mathcal{U}$ *such that* $\alpha|_{\mathcal{U}}$ *is exact.*

Proof: See e.g. Spivak [14], "Calculus on manifolds".

Proposition 1.11 *Let* $\varphi : \mathcal{M} \to \mathcal{N}$ *be a differentiable map from the differentiable manifold* $\mathcal{M}$ *to the differentiable manifold* $\mathcal{N}$*. The following diagram is commutative, in other words* $d \circ \varphi^\star = \varphi^\star \circ d$.

Proof: This property is well known for functions, here we show that it holds generically for the exterior derivative on forms. For an arbitrary covariant tensor field $t \in \mathcal{T}^0_s(\mathcal{N})$ and $v_i \in T_x\mathcal{M}$, the pullback, $\varphi^\star t \in \mathcal{T}^0_s(\mathcal{M})$ is defined by

$$(\varphi^\star t)_x(v_1, \ldots, v_s) := t_{\varphi(x)}(T_x\varphi v_1, \ldots, T_x\varphi v_s)\,, \tag{1.7}$$

$$\begin{array}{ccc}
\Lambda(\mathcal{M}) & \xleftarrow{\varphi^\star} & \Lambda(\mathcal{N}) \\
\Big\downarrow d & & \Big\downarrow d \\
\Lambda(\mathcal{M}) & \xleftarrow{\varphi^\star} & \Lambda(\mathcal{N})
\end{array}$$

where $T_x\varphi$ is the tangent map of φ at x. For a function $f \in \mathcal{F}(\mathcal{N})$ and a vector $v \in T_x\mathcal{M}$

$$(\varphi^\star df)_x(v) = (df)_{\varphi(x)}(T_x\varphi v) = T_x\varphi(v)f = v(f \circ \varphi) = v(\varphi^\star f) = d(\varphi^\star f)(v)\,, \quad (1.8)$$

i.e. $\varphi^\star df = d(\varphi^\star f)$. Applying d on this we have thus

$$d(\varphi^\star df) = d(d\varphi^\star f) = d \circ d(\varphi^\star f) = 0 = \varphi^\star((d \circ d)f)\,.$$

Our statement now follows with prop. 1.9. □

1.4 Relations between d, i_X and L_X

As a reminder let us write the explicite formula for Lie derivative of a p-form, $\omega \in \Lambda^p(\mathcal{M})$:

$$(L_X\omega)(v_1, \cdots v_p) = L_X\left(\omega(v_1, \cdots v_p)\right) - \omega([X, v_1], v_2, \cdots v_p) - \cdots - \omega(v_1, \cdots, [X, v_p])\,. \quad (1.9)$$

Here we have used that for a vector field v we have $L_X v = [X, v]$.

According to Def. 1.5, d is an anti-derivation of degree 1, i_X, $X \in \mathcal{X}(\mathcal{M})$ is an anti-derivation of degree -1 and L_X is a derivation of degree 0 on $\Lambda(\mathcal{M})$.

Proposition 1.12 (Cartan's magic formula)
For $X \in \mathcal{X}(\mathcal{M})$ and $\omega \in \Lambda^p(\mathcal{M})$ we have

$$L_X\omega = (d \circ i_X + i_X \circ d)\omega \quad (1.10)$$

Proof: According to proposition 1.5, $\theta = d \circ i_X + i_X \circ d$ is a derivation of degree 0. Hence if $\theta f = L_X f$ and $\theta(df) = L_X df$ for all $f \in \mathcal{F}(\mathcal{M})$, Eq. (1.10) is shown. But for $f \in \mathcal{F}(\mathcal{M})$

$$\theta(f) = i_X df = df(X) = Xf = L_X f,$$

and

$$\theta(df) = d \circ i_X df = d(Xf).$$

On the other hand

$$(L_X df)(Y) = L_X(df(Y)) - df(L_X Y) = L_X(Yf) - df([X,Y])$$
$$= X(Yf) - [X,Y]f = Y(Xf) = (d(Xf))(Y) \ .$$

□

With $d \circ d = 0$, Eq. (1.10) implies

$$L_X \circ d = d \circ L_X = d \circ i_X \circ d \ . \tag{1.11}$$

Another very useful identity is (exercise!)

$$i_{[X,Y]} = [L_X, i_Y]. \tag{1.12}$$

Proposition 1.13 *For $\omega \in \Lambda^{p-1}(\mathcal{M})$ and $X_i \in \mathcal{X}(\mathcal{M})$,*

$$d\omega(X_1,\ldots,X_p) = \sum_{1\le i\le p} (-1)^{i+1} X_i \omega(X_1,\ldots,\breve{X_i},\ldots,X_p)$$
$$+ \sum_{1\le i<j\le p} (-1)^{i+j} \omega([X_i,X_j],X_1,\ldots,\breve{X_i},\ldots,\breve{X_j},\ldots,X_p) \tag{1.13}$$

(again $\breve{X_i}$ denote omission of X_i).

Proof: For $p=1$, Eq. (1.13) reduces to $df(X) = Xf$.
For $\omega \in \Lambda^1(\mathcal{M})$, (1.10) gives

$$(L_X\omega)(Y) = (i_X d\omega)(Y) + d(i_X\omega)(Y) = d\omega(X,Y) + Y(\omega(X)).$$

With $(L_X\omega)(Y) = X(\omega(Y)) - \omega([X,Y])$ it follows that

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]),$$

i.e., Eq. (1.13).
By induction one can now show the step from p to $p+1$ using Eq. (1.10) and the explicit formula for $L_X\omega$. □

Proposition 1.14 *Let ∇ be a covariant derivative for a symmetric connection. For $\omega \in \Lambda^p(\mathcal{M})$ we find*

$$\mathcal{A}(\nabla\omega) = \frac{(-1)^p}{p+1} d\omega \tag{1.14}$$

Proof: For $\omega \in \Lambda^p(\mathcal{M})$

$$\nabla\omega(X_2, \ldots, X_{p+1}, X_1) = (\nabla_{X_1}\omega)(X_2, \ldots, X_{p+1})$$

$$= X_1(\omega(X_2, \ldots, X_{p+1})) - \sum_{i=2}^{p+1} \omega(X_2, \ldots, \nabla_{X_1}X_i, \ldots, X_{p+1})$$

$$\mathcal{A}(\nabla\omega)(X_2, \ldots, X_{p+1}, X_1) = \frac{1}{p+1}\left[\sum_{i=1}^{p+1}(-1)^{i+1}X_i\omega(X_1, \ldots, \check{X_i}, \ldots, X_{p+1})\right.$$

$$\left. + \sum_{i<j}(-1)^{i+j}\omega(\nabla_{X_i}X_j - \nabla_{X_j}X_i, X_1, \ldots, \check{X_i}, \ldots, \check{X_j}, \ldots, X_p)\right] .$$

But since the torsion vanishes $\nabla_{X_i}X_j - \nabla_{X_j}X_i = [X_i, X_j]$. Hence

$$\mathcal{A}(\nabla\omega)(X_2, \ldots, X_{p+1}, X_1) = \frac{1}{p+1}d\omega(X_1, X_2, \ldots, X_{p+1}) \tag{1.15}$$

$$\mathcal{A}(\nabla\omega)(X_1, X_2, \ldots, X_{p+1}) = \frac{(-1)^p}{p+1}d\omega(X_1, \ldots, X_{p+1}) . \tag{1.16}$$

□

Proposition 1.15 *Be $X = X^\ell\partial_\ell$ a vector field on a (pseudo-)Riemannian manifold $(\mathcal{M}, g)$ with Levi-Civita connection ∇, and be $X^\flat = X_\ell dx^\ell$ the associated 1-form ($X_\ell = g_{\ell m}X^m$). Then*

$$\nabla X^\flat = \frac{1}{2}\left(L_X g - dX^\flat\right) . \tag{1.17}$$

Proof: By prop. 1.14, twice the antisymmetric part of the covariant derivative of a 1-formis is the the exterior derivative (with opposite order). Hence, for two arbitrary vector fields Y and Z,

$$\nabla X^\flat(Y, Z) - \nabla X^\flat(Z, Y) = dX^\flat(Z, Y) . \tag{1.18}$$

For the symmetric part we write

$$\nabla X^\flat(Y, Z) + \nabla X^\flat(Z, Y) = \nabla_Z(X^\flat(Y)) - X^\flat(\nabla_Z Y) + \nabla_Y(X^\flat(Z)) - X^\flat(\nabla_Y Z) . \tag{1.19}$$

As $X^\flat(Y) = X_\ell Y^\ell = g_{\ell m}X^m Y^\ell = g(X, Y)$ this gives

$$\begin{aligned}\nabla X^\flat(Y, Z) + \nabla X^\flat(Z, Y) &= \nabla_Z(g(X, Y)) - g(X, \nabla_Z Y) + \nabla_Y(g(X, Z)) - g(X, \nabla_Y Z) \\ &= g(\nabla_Z X, Y) + g(\nabla_Y X, Z) . \end{aligned} \tag{1.20}$$

For the second equal sign we used the Ricci identity, for any three vector fields X, Y, Z,

$$\nabla_Z(g(X,Y)) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y) = 0\,, \tag{1.21}$$

which is just $\nabla g \equiv 0$. But with $[X,Y] = \nabla_X Y - \nabla_Y X$ we also have

$$\begin{aligned}(L_X)g(Y,Z) &= X(g(Y,Z)) - g([X,Y],Z) - g(Y,[X,Z]) \\ &= X(g(Y,Z)) - g(\nabla_X Y, Z) + g(\nabla_Y X, Z) - g(Y, \nabla_X Z) + g(Y, \nabla_Z, X) \\ &= g(\nabla_Y X, Z) + g(Y, \nabla_Z, X)\,,\end{aligned} \tag{1.22}$$

where we have again used the Ricci identity. Hence

$$\nabla X^\flat(Y,Z) + \nabla X^\flat(Z,Y) = (L_X)g(Y,Z)\,. \tag{1.23}$$

Summing up the symmetric part given above and the anti-symmetric part given in (1.18) yields (1.17). □

An important application of Eq. (1.17) is the Killing equation. A Killing field is a vector field b under the flow of which, denoted by Φ_s, the metric is invariant, hence $\Phi_s^* g = g$. The flow of a Killing field is therefore a 1-parameter group of symmetries of the metric. Invariance implies

$$0 = \lim_{s\to 0} \frac{\Phi_s^* g - g}{s} = L_b g\,. \tag{1.24}$$

Definition 1.8 (Killing field) *A vector field b that satisfies*

$$L_b g = 0 \tag{1.25}$$

is called a **Killing field** *for the metric g.*

According to Eq. (1.17), the symmetric part of the covariant derivative of b vanishes for a Killing field and $\nabla b^\flat = -db^\flat/2$. In coordinates,

$$b_{\mu;\nu} + b_{\nu;\mu} = 0\,. \tag{1.26}$$

This equation is called the Killing equation.

1.5 The $*$-Operation and the Codifferential

Definition 1.9 *A differentiable manifold $\mathcal{M}$ is called 'orientable' if it admits an atlas $\mathcal{A}$ such that for arbitrary charts $(h,\mathcal{U})$ and $(k,\mathcal{V})$ the determinant of the Jacobian of the coordinate change $h\circ k^{-1}|_{h(\mathcal{U})\cap k(\mathcal{V})}$ is positive. The atlas $\mathcal{A}$ is then called an oriented atlas of $\mathcal{M}$ and $(\mathcal{M},\mathcal{A})$ form an oriented manifold.*
An equivalence class of oriented atlases (two oriented atlases are called equivalent if their union is also oriented) is called an orientation of the manifold $\mathcal{M}$.

There exist manifolds which are not orientable. Examples are the Möbius strip or the Klein bottle. More precisely: one can show that every non-orientable manifold contains the möbius strip as a submanifold.

Exercice: Show that the Möbius strip is not ortientable. The Möbius strip is the manifold obtained from the the square $[0,1]\times[0,1]$ where we identify $(0,x)\equiv(1,1-x)$, see Fig. 1.1.

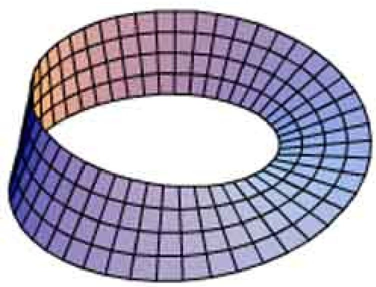

Figure 1.1: The Möbius strip.

A chart $(h,\mathcal{U})$ of an oriented manifold$(\mathcal{M},\mathcal{A}$ is said to be positive (negative) if for every chart $(k,\mathcal{V})\in\mathcal{A}$, the determinant of the Jacobian of $k\circ h^{-1}$ is positive (negative).

We state the following theorem without proof:

Theorem 1.2
Let $\mathcal{M}$ be an m-dimensional, paracompact, orientable manifold. There exists on $\mathcal{M}$ an m-form which does not vanish anywhere on $\mathcal{M}$. Such an m-form is called a volume form of $\mathcal{M}$. The converse is also true.

It is clear that this can be achieved locally, on a coordinate patch, by introducing $dx^i\wedge\cdots dx^m$, but that such a form exists globally on an orientable manifold is non-trivial.

We now consider an orientable (pseudo-)Riemannian manifold $(\mathcal{M},g)$. In a coordinate patch $(x^1,\cdots,x^m)$, the components of the metric are $g_{ij}(x)$ and we introduce

$$|g(x)|=|\det(g_{ij}(x))|\,. \tag{1.27}$$

Let $\bar{g}_{ij}$ be the metric components wrt new coordinates $(y^1,\cdots y^m)$ within an oriented atlas. Then

$$|\bar{g}|=|\det(\bar{g}_{ij})|=|\det(g_{ij})|\left[\det\left(\frac{\partial x^k}{\partial y^l}\right)\right]^2\,. \tag{1.28}$$

We now assume that $\mathcal{M}$ is orientable and the coordinates $(x^1, \cdots, x^m)$ and $(y^1, \cdots y^m)$ have the same orientation so that det $\left(\partial x^k/\partial y^l\right) > 0$. Then

$$\sqrt{|\bar{g}|} = \sqrt{|g|}\,\det\left(\frac{\partial x^k}{\partial y^l}\right) . \tag{1.29}$$

On the other hand, if a m-form ω is given in the two coordinate systems by

$$\omega = a(x)dx^1 \wedge \cdots \wedge dx^m = \bar{a}(y)dy^1 \wedge \cdots \wedge dy^m ,$$

the functions a and $\bar{a}$ are related by $\bar{a} = a\det(\partial x/\partial y)$ (Exercise!). Hence, for positive coordinate systems,

$$\eta = \sqrt{|g|}dx^1 \wedge \cdots \wedge dx^m \tag{1.30}$$

defines an m-form on $\mathcal{M}$. (The function $\sqrt{|g|}$ has the correct behavior under coordinate transformations.) Since g never vanishes, η is a volume form. It is called the canonical volume form on $(\mathcal{M}, g)$.

1.5.1 The $*$-operation

Let $(\mathcal{M}, g)$ be an m-dimensional oriented (pseudo-)Riemannian manifold and let $\eta \in \Lambda^m(\mathcal{M})$be the canonical volume form defined in Eq. (1.30). We now use η to associate to each form $\omega \in \Lambda^p(\mathcal{M})$ an $(m-p)$-form $*\omega$. To do so we consider a positive local coordinate system $(x^1, \ldots, x^m)$ and write ω in the form

$$\omega = \frac{1}{p!}\sum_{i_1\cdots i_p} \omega_{i_1\cdots i_p} dx^{i_1} \wedge \cdots \wedge x^{i_p} , \tag{1.31}$$

where the coefficients $\omega_{i_1\cdots i_p}$ are totally antisymmetric. We can write η in the same form by introducing the totally antisymmetric symbol

$$\epsilon_{i_1\cdots i_m} = \begin{cases} \mathrm{sign}(1,\ldots,m \mapsto i_1,\ldots,i_m) & \text{if } 1,\ldots,m \mapsto i_1,\ldots,i_m \text{ is a permutation,} \\ 0 & \text{if two indices are equal.} \end{cases} \tag{1.32}$$

With this definition we have for

$$\eta = \sqrt{|g|}dx^1 \wedge \cdots \wedge dx^m = \frac{1}{m!}\sum_{i_1\cdots i_m} \eta_{i_1\cdots i_m} dx^{i_1} \wedge \cdots \wedge x^{i_m}$$

with

$$\eta_{i_1\cdots i_m} = \sqrt{|g|}\epsilon_{i_1\cdots i_m} .$$

Definition 1.10 *For a p-form given in local coordinates by (1.31) we set* $\omega^{i_1\cdots i_p} \equiv g^{i_1j_1}\cdots g^{i_pj_p}\omega_{j_1\cdots j_p}$*. The Hodge dual of* ω *is then defined by*

$$(*\omega)_{i_{p+1}\cdots i_m} = \frac{1}{p!}\eta_{i_1\cdots i_m}\omega^{i_1\cdots i_p} \,. \tag{1.33}$$

Here (and in the following) summation over all index pairs is assumed (Einstein's summation convention).

This definition is independent of the chosen coordinate system (it consists in raising indices and then tracing over the indices i_1 to i_p which are both tensor operations). The correspondence $\omega \mapsto *\omega$ defines an isomorphism from Λ^p to Λ^{m-p}. A simple calculations (exercise!) shows that

$$*(*\omega) = (-1)^{p(m-p)}\mathrm{sgn}(g)\omega \,, \tag{1.34}$$

where sgn(g) is the signature of the metric g. Hence up to a sign $*$ it its own inverse. Note that in the physical case with $m = 4$ and sgn$(g) = -1$, 1- and 3-forms do not acquire a sign after two $*$ operations while 0- 2- and 4-forms do.

Exercises:

- consider $\mathbb{R}^3$ with the Euclidean metric. For a 1-form $A = A_i dx^i$ show that $(*dA)_i = (\mathrm{rot}A)_i$, where rot indicates the usual curl in 3-dimensions.
- Consider two p-forms, $\alpha,\ \beta \in \Lambda^p(\mathcal{M})$. Show that

  $$\alpha \wedge *\beta = *\alpha \wedge \beta = \langle \alpha, \beta\rangle \eta \,, \tag{1.35}$$

  where we have introduced a scalar product on Λ^p, $\quad \langle \alpha, \beta\rangle \equiv \frac{1}{p!}\alpha_{i_1\cdots i_p}\beta^{i_1\cdots i_p}$.
- Show also that $\langle *\alpha, *\beta\rangle = \mathrm{sgn}(g)\langle \alpha, \beta\rangle$ and that for $\alpha \in \Lambda^p(\mathcal{M})$ and $\beta \in \Lambda^{m-p}(\mathcal{M})$, $\langle \alpha \wedge \beta, \eta\rangle = \langle *\alpha, \beta\rangle$.

1.5.2 The co-differential

We now introduce an important differential operator which generalizes the notion of 'divergence' to differential forms.

Definition 1.11 *The co-differential* $\delta : \Lambda^p \to \Lambda^{p-1}$ *is defined as*

$$\delta = \mathrm{sgn}(g)(-1)^{m(p+1)} * d * \,. \tag{1.36}$$

The sign chosen here is not a standard convention. We have chosen this sign in order not to have any signs in the coordinate expression given in Eq. (1.37). One can show that the co-differential is an anti-derivation of degree -1.

Since $d \circ d = 0$ and $** = \pm$identity, also $\delta \circ \delta = 0$. Furthermore, $\delta\omega = 0$ is equivalent to $d * \omega = 0$. Therefore, the Poincaré Lemma implies that if $\delta\omega = 0$ there exists locally a form ϕ such that $*\omega = d\phi$. Hence, $\omega = \pm * d\phi = \delta\psi$ where $\psi = \pm * \phi$. We therefore have the result

Proposition 1.16 *If $\delta\omega = 0$, then there exists locally a $p+1$ form ψ with $\omega = \delta\psi$.*

Local coordinate expression for the co-differential

For $\omega \in \Lambda^p$ the coordinate expression for $\delta\omega$ is given by

$$\delta\omega^{i_1\cdots i_{p-1}} = \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}\omega^{k\,i_1\cdots i_{p-1}}\right)_{,k} . \tag{1.37}$$

Proof: We write

$$(*d*\omega)^{k_1\cdots k_{p-1}} = \frac{1}{(m-p+1)!}\eta^{i_1\cdots i_{m-p+1}\,k_1\cdots k_{p-1}}(d*\omega)_{i_1\cdots i_{m-p+1}} . \tag{1.38}$$

This follows directly from the definition 1.10 of the $*$ operator. Now, for an arbitrary s-form α we have in local coordinates

$$\alpha = \frac{1}{s!}\alpha_{i_1\cdots i_s,}dx^{i_1}\wedge\cdots\wedge dx^{i_s} \tag{1.39}$$

$$d\alpha = \frac{1}{s!}\alpha_{i_1\cdots i_s,i_{s+1}}dx^{i_{s+1}}\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_s} \tag{1.40}$$

$$= \frac{(-1)^s}{s!}\alpha_{i_1\cdots i_s,i_{s+1}}dx^{i_1}\wedge\cdots\wedge dx^{i_{s+1}} \tag{1.41}$$

$$= \frac{(-1)^s(s+1)}{(s+1)!}\alpha_{[i_1\cdots i_s,i_{s+1}]}dx^{i_1}\wedge\cdots\wedge dx^{i_{s+1}} \tag{1.42}$$

where $[i_1\cdots i_s, i_{s+1}]$ denotes anti-symmetrisation. Hence

$$(d\alpha)_{i_1\cdots i_{s+1}} = (-1)^s(s+1)\alpha_{[i_1\cdots i_s,i_{s+1}]} .$$

Using this in (1.38) we obtain

$$(*d*\omega)^{k_1\cdots k_{p-1}} = \frac{(-1)^{m-p}(m-p+1)}{(m-p+1)!}\eta^{i_1\cdots i_{m-p+1}\,k_1\cdots k_{p-1}}(*\omega)_{[i_1\cdots i_{m-p},i_{m-p+1}]} .$$

Since η is already anti-symmetric we can ignore anti-symmetrization in $d{*}\omega$. Using once again the definition 1.10 we obtain

$$(*d*\omega)^{k_1\cdots k_{p-1}} = \frac{(-1)^{m-p}}{(m-p)!p!}\eta^{i_1\cdots i_{m-p+1}\,k_1\cdots k_{p-1}}(\eta_{j_1\cdots j_p i_1\cdots i_{m-p}}\omega^{j_1\cdots j_p})_{,i_{m-p+1}}\,.$$

We now insert

$$\eta_{i_1\cdots i_m} = \sqrt{|g|}\epsilon_{i_1\cdots i_m} \quad \text{and} \quad \eta^{i_1\cdots i_m} = \sqrt{|g|}\det\left((g^{-1}_{ij}\right)\epsilon^{i_1\cdots i_m} = \frac{\operatorname{sgn}(g)}{\sqrt{|g|}}\epsilon^{i_1\cdots i_m}\,.$$

(The totally anti-symmetric symbol ϵ is identical for arbitrary index positions.) Furthermore

$$\epsilon_{i_1\cdots i_{m-p}\,j_1\cdots j_p}\epsilon^{i_1\cdots i_{m-p}\,i_{m-p+1}k_1\cdots k_{p-1}} = p!(m-p)!\delta^{i_{m-p+1}}_{[j_1}\delta^{k_1}_{j_2}\cdots\delta^{k_{p-1}}_{j_p]}$$

where again $[\cdots]$ denotes anti-symmetrisation. Calling the index i_{m-p+1} k then yields

$$(*d*\omega)^{k_1\cdots k_{p-1}} = (-1)^{m-p-p(m-p)}\operatorname{sgn}(g)\frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}\omega^{k\,k_1\cdots k_{p-1}}\right)_{,k}\,.$$

The sign comes from ordering the indices in the two ϵ tensors in a convenient way. Finally, using the definition $\delta = \operatorname{sgn}(g)(-1)^{m(p+1)} * d*$ we obtain (1.37). □

Exercise: Show that for a one form $a = A_i dx^i$ in flat space, the co-differential is the usual divergence, $\delta a = A^i{}_{,i} = \operatorname{div} A$.

Exercise: Show that the $*$-operation commutes with the pull-back by an orientation preserving isometry $\phi : \mathcal{M} \to \mathcal{M}\,,\ \phi^* g = g$. Conclude that the same is true for the co-differential. What happens if the orientation is reversed?

Exercise: Maxwell's equation. We introduce the 1-form related to the electromagnetic 4-potential $A = A_\mu dx^\mu$. Show that Maxwell's field strength tensor $F_{\mu\nu}$ is then given by

$$dA = F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$$

and Maxwell's equation become

$$dF = 0\,, \qquad \delta F = -4\pi J\,. \tag{1.43}$$

where J is the current 1-form $J = j_\mu dx^\mu = \rho dt + j_i dx^i$. Show that in terms of forms, gauge-invariance of electromagnetism is simply the statement that under a change $A \to A + d\chi$ the field tensor F remains invariant, which is a consequence of $d \circ d = 0$. Show also that charge conservation, $\delta J = \partial_\mu j^\mu = 0$ follows from Maxwell's equation (1.43).

We finally introduce the wave operator (Laplace-Beltrami operator) on forms

$$\Box = d \circ \delta + \delta \circ d\,. \tag{1.44}$$

The Laplace-Beltrami operator is a derivation of degree 0.

In Lorentz gauge defined by $\delta A = 0$, we have $\Box A = -4\pi J$.

1.6 Theorems of Stokes and Gauss

To define an integral over an m-dimensional manifold one first considers a m-form which is non-vanishing only in one chart $(U_i,\, h_i)$ where it takes the form $\omega = f(x)dx^1 \wedge \cdots \wedge dx^m$ and one defines

$$\int_{\mathcal{M}} \omega = \int_{U_i} f(x) dx^m\,, \tag{1.45}$$

where the right hand side is the usual Lebesgue integral. This definition is independent of the chosen chart since the Lebesgue measure and ω transform in the same way under coordinate transformations. To extend this definition one uses a partition of unity, i.e. a set of functions $\{\varphi_j\}$ on $\mathcal{M}$ which are such that $\varphi_j(x) \neq 0$ only in one chart, $U_{i(j)}$, and $\varphi_j(x) \geq 0$ and such that each φ_j has compact support and, especially,

$$\sum_j \varphi_j(x) = 1 \quad \forall x \in \mathcal{M}\,.$$

For an arbitrary m-form ω therefore

$$\omega = \sum_j \varphi_j \omega = \sum_j \omega_j$$

Since each ω_j has support only in one chart we can now define

$$\int_{\mathcal{M}} \omega := \sum_j \int_{U_{i(j)}} \omega_j\,. \tag{1.46}$$

We now consider a region (a connected open subset) $D \subset \mathcal{M}$ with a smooth boundary ∂D. A boundary ∂D is called smooth if it is a C^∞ submanifold of $\mathcal{M}$ of dimension $m-1$. For an $m-$form ω we define the integral of ω over D by $\int_D \omega = \int_{\mathcal{M}} \chi_D \cdot \omega$ where χ_D is the indicator function of the set D.

The theorem of Stokes then says

Theorem 1.3 Stokes
Be ω an $m-1$ form on $\mathcal{M}$. Then via the inclusion map $\iota : \partial D \to \mathcal{M}$ this defines a $m-1$ form $\iota^*\omega$ on ∂D which we also denote by ω. We then have

$$\int_D d\omega = \int_{\partial D} \omega\,. \tag{1.47}$$

Proof: See e.g. [14].

Be now Ω an arbitrary m form which does not vanish anywhere on $\mathcal{M}$ (such volume forms exist if $\mathcal{M}$ is orientable) and X a vector field on $\mathcal{M}$. We define the divergence of X wrt. Ω, a function denoted $\mathrm{div}_\Omega X$, by

$$L_X\Omega = (\mathrm{div}_\Omega X)\Omega\,. \tag{1.48}$$

Since $L_X\Omega$ is an m-form, there exists a function f such that $L_X\Omega = f\Omega$ and this function is called $\mathrm{div}_\Omega X$, i.e. the divergence of X wrt Ω.

Cartan's formula gives

$$L_X\Omega = (d \circ i_X + i_X \circ d)\Omega = d(i_X\Omega)\,. \tag{1.49}$$

Stoke's theorem now implies Gauss' theorem,

Theorem 1.4 Gauss
Be X a vector field on $\mathcal{M}$ and $D \subset \mathcal{M}$ a region. Then

$$\int_{\partial D} i_X\Omega = \int_D (\mathrm{div}_\Omega X)\Omega\,. \tag{1.50}$$

Exercise: Apply Gauss' theorem to a region in $\mathbb{R}^3$ with $\Omega = d^3x = dx^1 \wedge dx^2 \wedge dx^3$.

1.7 Frobenius' theorem

Definition 1.12 *A (smooth)* **distribution of dimension** k *on a differential manifold is a map*

$E : \mathcal{M} \to \{\text{subspaces of } T\mathcal{M}\} : p \mapsto E_p \in \{\text{linear subspaces of } T_p\mathcal{M} \textit{ of dimension } k\}$

such that E_p is a subspace of $T_p\mathcal{M}$ of dimension k and in a neighborhood V of every point $p \in \mathcal{M}$ there exist smooth vector fields $X_1, \cdots X_k$ such that $X_1(q), \cdots X_k(q)$ form a basis of E_q for all $q \in V$.

Definition 1.13 *A k-dimensional submanifold $\mathcal{N} \subset \mathcal{M}$ is called an* **integral manifold of** *E if $E_q = T_q\mathcal{N}$ for all $q \in \mathcal{N}$. (For notational simplicity we omit the inclusion map and simply consider $T_q\mathcal{N} \subset T_q\mathcal{M}$.)*
A distribution E is called integrable if through each point $p \in \mathcal{M}$ there passes a submanifold $\mathcal{N}$ with $p \in \mathcal{N}$ such that $E_q = T_q\mathcal{N}, \quad \forall q \in \mathcal{N}$.

We want to study under which circumstances a distribution is integrable. Be X and Y vector fields on $\mathcal{M}$ such that $X_p,\ Y_p \in E_p$ for all $p \in \mathcal{M}$. In this case we say $X,\ Y$ belong to E. If E is integrable, then clearly also the commutator, $[X, Y]$ needs to belong to E, since X and Y are vector fields on a submanifold $\mathcal{N}$. Frobenius' theorem states that also the converse is true, at least locally.

Definition 1.14 *A distribution is called* involutive *if for each pair of vector fields X and Y which belong to E also the commutator $[X, Y]$ belongs to E.*

Theorem 1.5 (Frobenius' Theorem (first version))
Let E be a smooth k-dimensional distribution on $\mathcal{M}$ which is involutive. Then E is (locally) integrable.
More precisely the following holds: For each point $p \in \mathcal{M}$ there exists a coordinate system $\{x^1, ..., x^m\}$ in a neighborhood U of p with $x^i(p) = 0$ and an $\epsilon > 0$, such that for each $(a^{k+1}, \cdots a^m)$ with all $a^j \in (-\epsilon, \epsilon)$ the points $\{q \in U | x^{k+1}(q) = a^{k+1}, \ldots, x^m(q) = a^m\}$ form an integral manifold of E. Furthermore, every connected integral manifold of E restricted to U is contained in one of these sets.

In such a coordinate system the integral manifolds of E are just the subsets of U with coordinates x^{k+1} to x^m fixed.

We postpone the proof of this important theorem and first reformulate it in a more useful manner in terms of differential forms. For this we need to introduce some concepts.

To each k-dimensional distribution E we associate the set $I(E) \subset \Lambda(\mathcal{M})$ of differential forms ω with the property that each homogeneous component $\omega^\ell \in \Lambda^\ell(\mathcal{M})$ of ω, $\quad \omega = \sum_{\ell=0}^m \omega^\ell$, annihilates the sets $(X_1, ..., X_\ell)$ of vector fields belonging to E: $\omega^\ell(X_1, ..., X_\ell) = 0$ for all sets vector fields belonging to E.

The set $I(E)$ is called the annihilator of E. Clearly, $I(E)$ is an ideal of the algebra $\Lambda(\mathcal{M})$. (This means it is a sub-algebra of $\Lambda(\mathcal{M})$ such that for each $\omega \in I(E)$ and $\theta \in \Lambda(\mathcal{M})$ also $\omega \wedge \theta \in I(\mathcal{M})$.) Locally, the ideal $I(E)$ is generated by $m - k$ independent 1-forms $\omega^{k+1}, \ldots, \omega^m$.

In the general case we complete the linearly independent vector fields $X_1, ..., X_k$ to a basis $X_1, ..., X_m$ and consider the dual basis $\omega^1, \ldots \omega^m$. By definition $\omega^\beta(X_\alpha) = \delta^\beta_\alpha$ hence the $\omega^{k+1}, \ldots \omega^m$ annihilate E.

We now consider an ℓ-form $\omega \in I(E)$ expanded with respect to the basis $\omega^1, \dots, \omega^m$,

$$\omega = \sum_{i_1 < i_2 < \cdots i_\ell} c_{i_1 \cdots i_\ell} \omega^{i_1} \wedge \cdots \wedge \omega^{i_\ell} .$$

If it contains one (non-vanishing) term $c_{i_1 \cdots i_\ell} \omega^{i_1} \wedge \cdots \wedge \omega^{i_\ell}$ which does not include any $i_j > k$, then $\omega(X_{i_1}, \cdots X_{i_\ell}) \neq 0$ which contradicts $\omega \neq I(E)$. This proves the following:

Proposition 1.17 *Let $I(E)$ be the ideal of $\Lambda(\mathcal{M})$ belonging to the k-dimensional distribution E. Then $I(E)$ is locally generated by $m-k$ linearly independent 1-forms: For each point of $\mathcal{M}$ there is a neighborhood U and $m-k$ pointwise linearly independent 1-forms $\omega^{k+1}, \dots, \omega^m \in \Lambda^1(U)$ such that each $\omega \in I(E)$*

$$\omega|_U = \sum_{i=k+1}^{m} \theta_i \wedge \omega_i$$

for some $\theta_i \in \Lambda(U)$.

Now we can reformulate the condition in the Frobenius theorem.

Proposition 1.18 *A distribution E on $\mathcal{M}$ is involutive if and only if $I(E)$ is a differential ideal, i.e. $dI(E) \subset I(E)$. (For $\omega \in I(E)$ also $d\omega \in I(E)$.)*

Proof: We use the same notation as above. It is easy to see that E is involutive if and only if there exist smooth functions C_{ij}^ℓ such that

$$[X_i, X_j] = \sum_{\ell=1}^{k} C_{ij}^\ell X_\ell \qquad \text{for } i, j = 1, \dots, k .$$

Now we have, see (1.13)

$$d\omega^\alpha(X_j, X_j) = X_i \omega^\alpha(X_j) - X_j \omega^\alpha(X_i) - \omega^\alpha([X_i, X_j]) .$$

For $i,\ j \leq k$ and $\alpha > k$ the first two terms on the right vanish. So $d\omega^\alpha(X_i, X_j) = 0$ if and only if $\omega^\alpha([X_i, X_j]) = 0$. But the latter equation holds for all $i,\ j$ if and only if each $[X_i, X_j]$ belongs to E (i.e., if E is involutive), while $d\omega^\alpha(X_i, X_j) = 0$ holds exactly when $d\omega^\alpha \in I(E)$. $\square$

Next we establish some equivalent conditions which all assure that a locally (finitely) generated ideal is a differential ideal.

Proposition 1.19 *Let I be an ideal of $\Lambda(\mathcal{M})$ locally generated by $m-k$ linearly independent 1-forms $\omega^{k+1},\ldots,\omega^m \in \Lambda^1(U)$. Furthermore let us denote by ω the $m-k$-form $\omega \equiv \omega^{k+1}\wedge\cdots\wedge\omega^m$, $\omega \in \Lambda^{m-k}(U)$. Then the following statements are equivalent:*

(i) I is an differential ideal

(ii) $d\omega^\beta = \sum_{\alpha=k+1}^m \omega^\beta{}_\alpha \wedge \omega^\alpha$ for some 1-forms $\omega^\beta{}_\alpha \in \Lambda^1(U)$, $\forall\ \beta \in \{k+1,\ldots,m\}$.

(iii) $\omega \wedge d\omega^\alpha = 0 \quad \forall\ \alpha \in \{k+1,\ldots,m\}$.

(iv) There exists $\theta \in \Lambda^1(U)$ such that $d\omega = \theta\wedge\omega$.

Proof: The equivalence of (i) and (ii) as well as the implication (i) $\Rightarrow$ (iii) follow immediately from the definitions. The same is true for the implication (ii) $\Rightarrow$ (iv). For the proof of (iv) $\Rightarrow$ (iii) note that the condition (iv) means that

$$\sum_{\alpha=k+1}^m (-1)^{\alpha-k+1} d\omega^\alpha \wedge \omega^{k+1}\wedge\cdots\wedge\widetilde{\omega^\alpha}\wedge\cdots\wedge\omega^m = \theta\wedge\omega^{k+1}\wedge\cdots\wedge\omega^m\,.$$

Multiplying this equation with ω^α the right hand side vanishes while on the left hand side only the term with $\widetilde{\omega^\alpha}$ survives and is proportional to $\omega\wedge d\omega^\alpha$, hence we obtain (iii). It remains to show that (iii) $\Rightarrow$ (ii): Again, let $\omega^1,\ldots,\omega^m$ be a basis of $\Lambda^1(U)$ such that $\omega^{k+1},\ldots,\omega^m$ generate I over U. Then

$$d\omega^i = \sum_{\ell<j} f^i_{\ell j}\omega^\ell\wedge\omega^j\,, \tag{1.51}$$

where $f^i_{\ell j}\in\mathcal{F}(U)$. But

$$0 = d\omega^\alpha\wedge\omega = \sum_{1\le\ell<j\le m} f^\alpha_{\ell j}\omega^\ell\wedge\omega^j\wedge\omega^{k+1}\wedge\cdots\wedge\omega^m\,.$$

Hence $f^\alpha_{\ell j}=0$ for $\alpha\in\{k+1,\ldots,m\}$ and $1\le\ell,j\le k$. Therefore the sum in (1.51) is of the form given in (ii). □

The preceding results can be expressed in the following version of the Frobenius theorem.

Theorem 1.6 (Frobenius' Theorem (second version)) *Let $\mathcal{M}$ be an m-dimensional manifold, E a k-dimensional distribution on $\mathcal{M}$ and $I(E)$ the associated ideal. The following statements are all equivalent:*

(i) E is integrable.

(ii) *E is involutive.*

(iii) *$I(E)$ is a differential ideal locally generated by $m-k$ linearly independent 1-forms $\omega^{k+1},\ldots,\omega^m \in \Lambda^1(U)$.*

(iv) *For every point in $\mathcal{M}$ there exists a neighborhood U and $\omega^{k+1},\ldots\omega^m$ generating $I(E)$ such that $d\omega^\beta = \sum_{\alpha=k+1}^m \omega^\beta{}_\alpha \wedge \omega^\alpha$ for some 1-forms $\omega^\beta{}_\alpha \in \Lambda^1(U)$, $\forall\ \beta \in \{k+1,\ldots,m\}$.*

(v) *$d\omega^\alpha \wedge \omega^{k+1} \wedge \cdots \wedge \omega^m = 0$ for $k+1 \le \alpha \le m$.*

(vi) *There exists a $\theta \in \Lambda^1(U)$ such that $d\omega = \theta \wedge \omega$ for $\omega = \omega^{k+1} \wedge \cdots \wedge \omega^m$.*

What remains to be shown here is that a involutive distribution is integrable, or, equivalently, that an arbitrary of the above points (ii) to (vi) implies (i).

1.7.1 Applications

Consider first a single timelike vector field X on a 4-dimensional Lorentz manifold $(\mathcal{M}, g)$. The orthogonal complement of $X^\perp(p)$ in every point $T_p(\mathcal{M})$ defines a 3-dimensional distribution E. Obviously, X is hypersurface orthogonal if and only if E is integrable. Since $X^\flat$ generates the ideal $I(E)$ for this case, the Frobenius theorem tells us that E is integrable if and only if

$$X^\flat \wedge dX^\flat = 0\,. \tag{1.52}$$

(We have used the equivalence of (i) and (v).) This shows that (1.52) is necessary and sufficient for X to be locally hypersurface orthogonal.

Exercise: Using (1.52), show that for $\mathcal{M} = \mathbb{R}^3$ a vector field X is hypersurface orthogonal if and only if $\mathrm{rot}X = 0$.

Our main application in GR will be the introduction of adapted coordinates if there are several Killing fields on $(\mathcal{M}, g)$, in particular, if there exist two commuting Killing fields. Let us, first introduce adapted coordinates for the more general situation of a k-dimensional distribution E. Let us consider a neighborhood U on which E is simply the span of k vector fields $\{X_1,\ldots,X_k\}$, i.e. at each point $p \in U$ the vectors $\{X_1(p),\ldots,X_k(p)\}$ form a basis of $E_p \subset T_p\mathcal{M}$. Let $\omega_{k+1},\ldots\omega_m$ denote the $m-k$ one forms which generate the ideal $I(E)$. In the remainder of this paragraph we use the following notation: Indices running from $1,\ 2,\ \ldots,\ m$ are denoted by Greek letters; for the first k numbers we use lower case Latin letters, while for indices with values $k+1,\ \ldots,\ m$ capital Latin letters will be used.

Assume that E is involutive and use adapted coordinates $\{x^\mu\}$ as in the first version of the Frobenius theorem. The (local) integral manifolds are then given by $\{x^A = \text{const}\}$. For the basis of vector fields X_a belonging to E we have

$$X_a = X_a^b \partial_b \, , \quad X_a^A = 0 \, . \tag{1.53}$$

Since $\omega^A(X_a) = 0$ we have

$$\omega^A = \omega^A{}_B dx^B \quad \omega^A{}_b = 0 \, . \tag{1.54}$$

Clearly, the matrices X_a^b and $\omega^A{}_B$ are non-singular.

Let us first mention the special case when the vector fields X_a commute. In this case also their flows, ϕ_s^a, commute and in the neighborhood of a point p_0 we can choose the coordinates $(s^1, \ldots, s^k)$ for the point

$$x = \phi^1_{s^1} \cdots \phi^k_{s^k}(p_0)$$

In these coordinates the vector fields X_a become simply partial derivatives,

$$X_a = \frac{\partial}{\partial s^a} \, . \tag{1.55}$$

We now assume that $\mathcal{M}$ is equipped with a (pseudo-)Riemannian metric g. We can then also introduce the 1-forms $\omega^a = X_a^\flat$. These generate the ideal of a distribution $E^\perp$. Let $E^\perp$ be spanned by vector fields X_A, such that

$$\omega^a(X_A) \equiv g(X_a, X_A) = 0 \qquad \text{hence} \quad E \perp E^\perp \, . \tag{1.56}$$

In what follows we assume that the restrictions $g|_E$ and $g|_{E^\perp}$ are non-singular (This is always the case for a Riemannian metric but not for a pseudo-Riemannian one!). In this case $E \cap E^\perp = \{0\}$ and

$$T_p\mathcal{M} = E_p \oplus E_p^\perp \, .$$

Hence the vector fields X_a, X_A form a basis of $T_p\mathcal{M}$ and of all $T_q\mathcal{M}$ in a neighborhood of x. Correspondingly

$$T_p^*\mathcal{M} = \text{span}\{\omega_p^a\} \oplus \text{span}\{\omega_p^A\} \, .$$

Here the span of several 1-forms is the vector space generated by them. Again this construction can be extended to a neighborhood of p. The vector fields X_A can be chosen such that $X_A^\flat = \omega^A$ since both sets of vectors span the same vector space $E^\perp$.

We then have constructed a basis of vector fields $\{X_\mu\}$ and a basis of 1-forms $\{\omega^\mu = X_\mu^\flat\}$ such that the forms $\{\omega^A\}$ generate the ideal $I(E)$ and the forms $\{\omega^a\}$ generate $I(E^\perp)$.

Let us now assume that E and $E^\perp$ are both involutive. We use coordinates $\{u^\mu\}$ which are adapted to E and also coordinates $\{v^\mu\}$ that are adapted to $E^\perp$. This means that the vector fields ∂_{u^a} span E while the vector fields ∂_{v^A}) span $E^\perp$. Then we have according to (1.53) and (1.54)

$$X_a u^A = 0 \qquad \omega^A = \omega^A{}_B du^B$$

$$X_A v^a = 0 \qquad \omega^a = \omega^a{}_b dv^b \,.$$

We now define the coordinates $\{x^\mu\}$ as

$$x^a = v^a \qquad x^A = u^A \,. \tag{1.57}$$

To show that these indeed form a good coordinate system we show that the dx^μ are linearly independent. Suppose there would be a linear relation

$$f_a dx^a + g_A dx^A = 0 \,, \quad \text{i.e.} \quad f_a dv^a + g_A du^A = 0 \,.$$

Applying this 1-form on X_c gives

$$f_a dv^a(X_c) = -g_A du^A(X_c) = g_A X_c u^A = 0 \,.$$

Since the dv^a are linearly independent it follows either $f_a = 0$ or $dv^a(X_c) = 0$ and since the dv^a span the ω^b also $\omega^b(X_c) = 0$. Hence $X_c \in E \cap E^\perp = \{0\}$. Hence we must request $f_a = 0$. Similarly by applying the identity to X_B we can imply $g_A = 0$.

The coordinates $\{x^\mu\}$ have the following properties

$$\text{(i)} \qquad X_a(x^A) = 0 \,, \tag{1.58}$$

$$\text{(ii)} \qquad g_{aA} = 0 \,. \tag{1.59}$$

The last property is obtained as follows: Set $\omega^a = \omega^a{}_\mu dx^\mu$. Then according to (1.56) $\omega^a{}_A = 0$. But $\omega^a = X_a^\flat$ hence

$$0 = \omega^a{}_A = X_a^b g_{bA} \,.$$

Since X_a^b is an invertible matrix this implies (ii). Summarizing we arrive at the following result:

Proposition 1.20 *Consider a (pseudo-)Riemannian manifold $\mathcal{M}$ of dimension m and an involutive distribution E of dimension k spanned locally by the vector fields $\{X_1, \ldots, X_k\}$. Consider the ideal $\mathcal{I}$ spanned by the 1-forms $\omega^i = X_i^\flat$. We assume also that $\mathcal{I}$ is differential. The distribution $E^\perp$ with $\mathcal{I} = I(E^\perp)$ is spanned by the vectors normal to E. We assume in addition, that $E \cap E^\perp = \{0\}$ (for Riemannian manifolds this is always true). Then we can introduce local coordinates $\{x^\mu\}$ such that*

$$X_a(x^A) = 0 \quad \textit{and} \quad g_{aA} = 0 \,. \tag{1.60}$$

Next we assume that the vector fields $\{X_1, \ldots, X_k\}$ are Killing fields of the metric g. Hence

$$0 = (L_{X_a} g)_{\mu\nu} = X_a^b g_{\mu\nu,b} + X_{a,\mu}^b g_{b\nu} + X_{a,\nu}^b g_{\mu b} \tag{1.61}$$

Applying this to $\mu\nu = AB$ and using (1.59) and (1.58) we obtain

$$g_{AB,b} = 0\,. \tag{1.62}$$

Hence the metric coefficients g_{AB} depend only on the x^C. Applying Eq. (1.61) on $\mu\nu = Ac$ we find

$$X_{a,A}^b = 0\,, \tag{1.63}$$

i.e. the components of the vector fields X_a depend only on the x^c. Summarizing, we have the useful result

Proposition 1.21 *If in addition to the assumptions of the Proposition 1.20 it is assumed that the X_a are Killing fields of the (pseudo-)Riemannian manifold $(\mathcal{M}, g)$, then there are local coordinates $\{x^\mu\}$ such that (in the notation introduced above)*

$$\begin{aligned} ds^2 &= g_{ab}(x^\mu)dx^a dx^b + g_{AB}(x^C)dx^A dx^B \\ X_a &= X_a^b(x^c)\partial_b\,. \end{aligned}$$

Note, as a result of this, that the metric $g_{ab}dx^a dx^b$ on the integral manifolds $\{x^A = \text{const}\}$ has k Killing fields X_a, hence these submanifolds are homogeneous spaces. Finally, we consider the important special case of commuting Killing fields, $[X_a, X_b] = 0$. We can then choose the adapted coordinates such that $X_a = \partial_a$, see (1.55). If we use this in (1.61) for $\mu\nu = cd$, we obtain $g_{cd,b} = 0$, hence also the metric components g_{ab} depend only on the x^C and on the integral submanifolds the metric is constant. Hence these are not only homogeneous spaces but simply flat space. The above equations then simplify to

$$\begin{aligned} ds^2 &= g_{ab}(x^C)dx^a dx^b + g_{AB}(x^C)dx^A dx^B \\ X_a &= \partial_a\,. \end{aligned}$$

We shall use this expressions as a staring point to derive the Kerr solution.

1.7.2 Proof of Frobenuis' theorem

We finally proof the first version of the theorem.

Since the theorem is local we can work in $\mathbb{R}^m$ and set $p = 0$. Moreover, we can assume that $E_0 \subset T_0\mathbb{R}^m$ is spanned by the basis vectors $\partial/\partial s^1, \ldots, \partial/\partial s^k$ at $p = 0$, where $(s^1, \ldots, s^m)$ are standard coordinates on $\mathbb{R}^m$. We denote by

$$\Pi \;:\; \mathbb{R}^m \to \mathbb{R}^k \;\; : (s^1, \ldots, s^m) \mapsto (s^1, \ldots, s^k)$$

the projection on the first k components of a point in $\mathbb{R}^m$. The corresponding tangent map $T\Pi$ maps E_0 isomorphically to $T_0\mathbb{R}^k$. By continuity $T\Pi$ is bijective in a neighborhood of $p = 0$. Hence near $p = 0$ we can choose unique vector fields $X_a(q) \in E_q$ such that

$$T\Pi X_a = \frac{\partial}{\partial s^a} \quad \forall \;\; 1 \leq a \leq k \; , \quad q \in U \subset \mathbb{R}^m \tag{1.64}$$

where U is a neighborhood of 0. The vector fields X_a are Π-related and so is their commutator,

$$T\Pi[X_a, X_b] = [T\Pi X_a, T\Pi X_b] = [\frac{\partial}{\partial s^a}, \frac{\partial}{\partial s^b}] = 0 \, . \tag{1.65}$$

Therefore we can introduce coordinates u^μ such that

$$X_a = \frac{\partial}{\partial u^a} \quad \forall \;\; 1 \leq a \leq k \, .$$

The sets $\{q \in U | u^{k+1} = a^{k+1}, \ldots, u^m = a^m\}$ are then obviously integral manifolds of E since their tangent space is spanned by the $\partial/\partial u^a$, $1 \leq a \leq k$. Conversely, if $\mathcal{N}$ is a connected integral manifold restricted to U, with the inclusion $\iota \; : \mathcal{N} \to U$, then we have for $u^\alpha \circ \iota$, $k < \alpha \leq m$, and $X(q) \in T_q\mathcal{N}$,

$$d(u^\alpha \circ \iota) X = X(u^\alpha \circ \iota) = T\iota X u^\alpha = 0$$

since $T\iota X \in E_q$ which is spanned by the ∂_a with $a \leq k$. This implies that $u^\alpha \circ \iota$ is constant on the connected manifold $\mathcal{N}$. □

Chapter 2

Cartan's formalism of GR

2.1 Cartan's structure equations

Let me first repeat the definitions of connection, torsion, curvature on a differentiable manifold $\mathcal{M}$ (assuming you have heard them previously).

Definition 2.1 (Connection, torsion, curvature)

- *A connection is a map* $\nabla \ : \ \chi(\mathcal{M}) \times \chi(\mathcal{M}) \to \chi(\mathcal{M}) : (X, Y) \mapsto \nabla_X Y$ *with the following properties*
  - ∇ *is* $\mathcal{F}(\mathcal{M})$*-linear in* X*, i.e.* $\nabla_{fX} Y = f \nabla_X Y$ *for* $f \in \mathcal{F}(\mathcal{M})$*.*
  - ∇ *is* $\mathbb{R}$*-linear in* Y*.*
  - *(Leibniz rule) For* $f \in \mathcal{F}(\mathcal{M})$*,* $\nabla_X(fY) = X(f)Y + f\nabla_X Y$*.*
- *The torsion of a connection* ∇ *is the following map,*

$$T : \ : \ \chi(\mathcal{M}) \times \chi(\mathcal{M}) \to \chi(\mathcal{M}) : (X, Y) \mapsto T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

- *The curvature of a connection* ∇ *is the following map,*

$$R \ : \ \chi(\mathcal{M}) \times \chi(\mathcal{M}) \times \chi(\mathcal{M}) \to \chi(\mathcal{M}) : (X, Y, W) \mapsto R(X, Y)W$$
$$R(X, Y)W = \nabla_X \nabla_Y W - \nabla_X \nabla_Y W - \nabla_{[X,Y]} W \,.$$

Note that both, the torsion and the curvature are $\mathcal{F}(\mathcal{M})$-linear in X and Y and the curvature is also $\mathcal{F}(\mathcal{M})$-linear in W. (Show this!)

To introduce the Cartan formalism, we now define 1-forms which fully describe the connection.

Definition 2.2 (Connection 1-forms) *Be ∇ an affine connection on $\mathcal{M}$ and be $(e_1, \dots, e_n)$ a basis of vector fields on an open set $\mathcal{U} \subset \mathcal{M}$. Be $(\theta^1, \dots, \theta^n)$ the dual basis of 1-forms determined by $\theta^i(e_j) = \delta^i_j$. We define the* **connection 1-forms** *$\omega^i{}_j \in \Lambda^1(\mathcal{U})$ by*

$$\nabla_X e_j = \omega^i{}_j(X) e_i. \tag{2.1}$$

We also define the Christoffel symbols with respect to the basis $\{e_i\}$ by

$$\nabla_{e_k} e_j = \Gamma^i_{kj} e_i = \omega^i{}_j(e_k) e_i. \tag{2.2}$$

With this we obtain

$$\omega^i{}_j = \Gamma^i_{kj} \theta^k. \tag{2.3}$$

Proposition 2.1 *For a vector field $X = X^i e_i$,*

$$\nabla X = e_i \otimes (dX^i + \omega^i{}_k X^k). \tag{2.4}$$

For a 1-form $\alpha = \alpha_i \theta^i$,

$$\nabla \alpha = \theta^i \otimes (d\alpha_i - \omega^k{}_i \alpha_k). \tag{2.5}$$

Proof: Note that for a vector field X, the covariant derivative ∇X is defined by $\nabla X(Y) = \nabla_Y X$. Then eq. (2.4) follows from (2.1) and the Leibniz rule. For (2.5), we use that ∇_X commutes with contractions:

$$0 = \nabla_X(\theta^i(e_j)) = (\nabla_X \theta^i)(e_j) + \theta^i(\nabla_X e_j).$$

Therefore

$$(\nabla_X \theta^i)(e_j) = -\omega^i{}_j(X),$$

so that

$$\nabla_X \theta^i = -\omega^i{}_j(X) \theta^j. \tag{2.6}$$

With this and the Leibniz rule, equation (2.5) follows. □

Definition 2.3 (torsion and curvature 2-forms) *Since the torsion $T(X, Y)$ and the curvature $R(X, Y)Z$ are anti-symmetric in X and Y, we can define* **torsion and curvature 2-forms** *Θ^i and $\Omega^i{}_j$ by*

$$T(X, Y) = \Theta^i(X, Y) e_i \tag{2.7}$$

$$R(X, Y) e_j = \Omega^i{}_j(X, Y) e_i. \tag{2.8}$$

Theorem 2.1 *The torsion and curvature 2-forms satisfy the* **structure equations of Cartan***:*

$$d\theta^i + \omega^i{}_j \wedge \theta^j = \Theta^i \tag{2.9}$$

$$d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j = \Omega^i{}_j \tag{2.10}$$

Proof: For (2.9):

$$\Theta^i(X,Y)e_i = \nabla_X Y - \nabla_Y X - [X,Y] = \nabla_X(\theta^i(Y)e_i) - \nabla_Y(\theta^i(X)e_i) - \theta^i([X,Y])e_i$$
$$= \left\{X(\theta^i(Y)) - Y(\theta^i(X)) - \theta^i([X,Y])\right\} e_i + \theta^i(Y)\omega^j_{\ i}(X)e_j - \theta^i(X)\omega^j_{\ i}(Y)e_j$$
$$= (d\theta^i + \omega^i_{\ l} \wedge \theta^l)(X,Y)e_i.$$

For the last equal sign we use (1.13) and the definition of $\wedge$.
And for (2.10):

$$\Omega^i_{\ j}(X,Y)e_i = \nabla_X\nabla_Y e_j - \nabla_Y\nabla_X e_j - \nabla_{[X,Y]}e_j$$
$$= \nabla_X(\omega^i_{\ j}(Y)e_i) - \nabla_Y(\omega^i_{\ j}(X)e_i) - \omega^i_{\ j}([X,Y])e_i$$
$$= \left\{X(\omega^i_{\ j}(Y)) - Y(\omega^i_{\ j}(X)) - \omega^i_{\ j}([X,Y])\right\} e_i + \{\omega^i_{\ j}(Y)\omega^k_{\ i}(X) - \omega^i_{\ j}(X)\omega^k_{\ i}(Y)\}e_k$$
$$= (d\omega^i_{\ j} + \omega^i_{\ l} \wedge \omega^l_{\ j})(X,Y)e_i \ .$$

□

Setting $R^i_{jkl} = \theta^i(R(e_k,e_l)e_j) = \Omega^i_{\ j}(e_k,e_l)$, we obtain

$$\Omega^i_{\ j} = \frac{1}{2}R^i_{jkl}\theta^k \wedge \theta^l. \tag{2.11}$$

Equivalently, with $T^i_{kl} = \theta^i(T(e_k,e_l))$ we have

$$\Theta^i = \frac{1}{2}T^i_{kl}\theta^k \wedge \theta^l \ . \tag{2.12}$$

Proposition 2.2 *A connection on a (pseudo-)Riemannian manifold $(\mathcal{M}, g)$ is metric if and only if*

$$dg_{ik} = \omega_{ik} + \omega_{ki} \tag{2.13}$$

where $\omega_{ik} := g_{il}\omega^l_{\ k}$; $g_{ij} = g(e_i,e_j)$.

Proof: By definition the connection ∇ is metric if $(\nabla_X g)_{ik} = X(g_{ik}) - g(\nabla_X e_i, e_k) - g(e_i, \nabla_X e_k) = 0$ for all vector fields X.Therefore, for a metric connection

$$dg_{ik}(X) \equiv X(g_{ik}) = g(\nabla_X e_i, e_k) + g(e_i, \nabla_X e_k) = g(\omega^j_{\ i}(X)e_j, e_k) + g(e_i, \omega^j_{\ k}(X)e_j)$$
$$= \omega^j_{\ i}(X)g_{jk} + \omega^j_{\ k}(X)g_{ij} = \omega_{ki}(X) + \omega_{ik}(X) \ .$$

□

For the **Riemannian connection** (or Levi-Civita connection) we therefore obtain the following equations:

$$\omega_{ij} + \omega_{ji} = dg_{ij} \tag{2.14}$$
$$d\theta^i + \omega^i_{\ j} \wedge \theta^j = 0 \tag{2.15}$$
$$d\omega^i_{\ j} + \omega^i_{\ k} \wedge \omega^k_{\ j} = \Omega^i_{\ j} = \frac{1}{2}R^i_{jkl}\theta^k \wedge \theta^l. \tag{2.16}$$

These are the Cartan structure equation for a Riemannian connection.

2.2 The formal solution of Cartan's structure equations for a Riemannian (or Levi-Civita) connection

Be $(e_i)_{i=1}^n$ and $(\theta^i)_{i=1}^n$ local dual bases of vector fields and 1-forms with $\theta^i(e_j) = \delta^i{}_j$, and $g_{ij} = g(e_i, e_j)$. For an orthonormal basis, $g_{ij} = \pm\delta_{ij}$.
We expand $d\theta^i$:

$$d\theta^i = -\frac{1}{2}C^i_{jl}\theta^j \wedge \theta^l \qquad C^i_{jl} = -C^i_{lj}\,. \tag{2.17}$$

(Note that in a coordinate basis, also called a holonomic basis, $\theta^i = dx^i$ the coefficients C^i_{jk} vanish.) The choice of the basis (θ^i) determines the C^i_{jk} and the metric components g_{ij} since $g = g_{ij}\theta^i\theta^j$ (here $\theta^i\theta^j = \frac{1}{2}(\theta^i \otimes \theta^j + \theta^j \otimes \theta^i)$). We now compute the connection 1-forms, $\omega^i{}_j$ and the curvature 2-forms $\Omega^i{}_j$ from the C^i_{jl} and the metric components g_{ij}. For a holonomic basis, i.e., a basis of the form $\theta^i = dx^i$, we have $C^i_{kl} \equiv 0$, while for an orthonormal basis $dg_{ij} = 0$.
With (2.3) and (2.15) (the first structure equation of Cartan) this yields

$$\left(-\frac{1}{2}C^i_{jl} + \Gamma^i_{jl}\right)\theta^j \wedge \theta^l = 0,$$

so that

$$\Gamma^i_{jl} - \Gamma^i_{lj} = C^i_{jl} \tag{2.18}$$

For a holonomic basis, the Γ^i_{jk} are symmetric.
We now define for an arbitrary basis

$$g_{ij,k} := e_k(g_{ij}),$$

so that $dg_{ij} = g_{ij,k}\theta^k$. Since $\omega_{ij} = g_{il}\Gamma^l_{kj}\theta^k$, (2.14) gives for an arbitrary basis

$$g_{il}\Gamma^l_{kj} + g_{jl}\Gamma^l_{ki} = g_{ij,k}.$$

(For a orthonormal basis, the $\Gamma_{ikj} := g_{il}\Gamma^l_{kj}$ are therefore antisymmetric in ij.)
With cyclic permutation we obtain

$$g_{ki,j} = g_{kl}\Gamma^l_{ji} + g_{il}\Gamma^l_{jk}$$

$$g_{jk,i} = g_{jl}\Gamma^l_{ik} + g_{kl}\Gamma^l_{ij}$$

With eq. (2.18) this leads to

$$(g_{ij,k} + g_{kj,i} - g_{ki,j}) = g_{kl}C^l_{ij} + g_{il}C^l_{kj} + g_{jl}(\Gamma^l_{ki} + \Gamma^l_{ik})$$

Multiplication with g^{mj} gives

$$\Gamma^m_{ki} + \Gamma^m_{ik} = g^{mj}(g_{ij,k} + g_{kj,i} - g_{ik,j}) - g^{mj}g_{kl}C^l_{ij} - g^{mj}g_{il}C^l_{kj}\ .$$

With (2.18) we find

$$\Gamma^m_{ki} = \frac{1}{2}g^{mj}(g_{jk,i} + g_{ji,k} - g_{ik,j}) + \frac{1}{2}(C^m_{ki} - g^{mj}g_{li}C^l_{kj} - g^{mj}g_{kl}C^l_{ij}). \qquad (2.19)$$

For a holonomic basis ($\theta^i = dx^i$), only the first part of (2.19) is non-vanishing and we find the well known standard expression for the Christoffel symbols.
For an orthonormal basis only the second part is non-vanishing and[1]

$$\Gamma^m_{ki} = \frac{1}{2}(C^m_{ki} - \varepsilon_m\varepsilon_i C^i_{km} - \varepsilon_m\varepsilon_k C^k_{im}) \ .$$

To solve the second structure eqn. (2.16), we proceed as follows: According to (2.3),

$$d\omega^i_{\ j} = d\Gamma^i_{kj} \wedge \theta^k + \Gamma^i_{kj} d\theta^k$$

$$d\Gamma^i_{kj} = e_l(\Gamma^i_{kj})\theta^l =: \Gamma^i_{kj,l}\theta^l$$

$$d\omega^i_{\ j} = \Gamma^i_{kj,l}\theta^l \wedge \theta^k - \frac{1}{2}\Gamma^i_{kj}C^k_{lm}\theta^l \wedge \theta^m$$

$$d\omega^i_{\ j} = \frac{1}{2}\left(\Gamma^i_{kj,l} - \Gamma^i_{lj,k} - \Gamma^i_{mj}C^m_{lk}\right)\theta^l \wedge \theta^k \ .$$

So that

$$\Omega^i_{\ j} = d\omega^i_{\ j} + \omega^i_{\ m} \wedge \omega^m_{\ j} = \left[\frac{1}{2}(\Gamma^i_{kj,l} - \Gamma^i_{lj,k} - \Gamma^i_{mj}C^m_{lk}) + \Gamma^i_{lm}\Gamma^m_{kj}\right]\theta^l \wedge \theta^k$$

$$= \frac{1}{2}R^i_{jlk}\theta^l \wedge \theta^k \ .$$

Hence

$$R^i_{jlk} = \Gamma^i_{kj,l} - \Gamma^i_{lj,k} - \Gamma^i_{mj}C^m_{lk} + \Gamma^i_{lm}\Gamma^m_{kj} - \Gamma^i_{km}\Gamma^m_{lj} \ . \qquad (2.20)$$

Proposition 2.3 2.3 The Bianchi identities

The torsion and curvature forms satisfy the Bianchi identities,

$$D\Theta^i \equiv d\Theta^i + \omega^i_l \wedge \Theta^l = \Omega^i_{\ j} \wedge \theta^j \ , \qquad (2.21)$$

$$D\Omega^i_{\ j} \equiv d\Omega^i_{\ j} + \omega^i_{\ l} \wedge \Omega^l_{\ j} - \omega^l_{\ j} \wedge \Omega^i_{\ l} = 0 \ . \qquad (2.22)$$

[1] $\varepsilon_m = g_{mm} = \pm 1$

Here D denotes the total exterior derivative. For an tensor valued p form with s covariant tensor indices and r contravariant indices we define the tensor values $p+1$ form

$$\begin{aligned} DS^{m_1\cdots m_r}_{i_1\cdots i_s} &= dS^{m_1\cdots m_r}_{i_1\cdots i_s} - S^{m_1\cdots m_r}_{j\cdots i_s}\wedge\omega^j{}_{i_1} - \cdots - S^{m_1\cdots m_r}_{i_1\cdots j}\wedge\omega^j{}_{i_s} \\ &\quad + S^{n\cdots m_r}_{i_1\cdots i_s}\wedge\omega^{m_1}{}_n + \cdots + S^{m_1\cdots n}_{i_1\cdots i_s}\wedge\omega^{m_r}{}_n\,. \end{aligned} \tag{2.23}$$

Proof: Of (2.21):

$$d\Theta^i + \omega^i_l\wedge\Theta^l = d(d\theta^i + \omega^i{}_j\wedge\theta^j) + \omega^i{}_j\wedge d\theta^j + \omega^i{}_l\wedge\omega^l{}_j\wedge\theta^j$$

$$= d\omega^i{}_j\wedge\theta^j + \omega^i{}_l\wedge\omega^l{}_j\wedge\theta^j = \Omega^i{}_j\wedge\theta^j\ .$$

Of (2.22):

$$d\Omega^i{}_j + \omega^i{}_l\wedge\Omega^l{}_j - \omega^l{}_j\wedge\Omega^i{}_l = d(d\omega^i{}_j + \omega^i{}_k\wedge\omega^k{}_j) + \omega^i{}_l\wedge(d\omega^l{}_j + \omega^l{}_k\wedge\omega^k{}_j) - \omega^l{}_j\wedge(d\omega^i{}_l + \omega^i{}_k\wedge\omega^k{}_l)$$

$$= d\omega^i{}_l\wedge\omega^l{}_j - \omega^i{}_l\wedge d\omega^l{}_j + \omega^i{}_l\wedge d\omega^l{}_j + \omega^i{}_l\wedge\omega^l{}_k\wedge\omega^k{}_j - \omega^l{}_j\wedge d\omega^i{}_l - \omega^l{}_j\wedge\omega^i{}_k\wedge\omega^k{}_l = 0\ .$$

□

Exercise: Show that in a holonomic basis, $e_i = \partial_i$, $\theta^i = dx^i$, the Bianchi identities (2.21) et (2.22) are equivalent to the Bianchi identities in a coordinate basis given by:

First Bianchi identity:

$$\sum_{\substack{cyclic\\ jkl}} R^i_{jkl} = \sum_{cyclic}\left[T^i_{mj}T^m_{kl} + T^i_{jk;l}\right]\ .$$

2nd Bianchi identity:

$$0 = \sum_{cyclic(mli)}\left[R^j_{kml;i} + R^j_{knm}T^n_{li}\right]\ .$$

The fact that with Cartan's formalism the Bianchi identities are nearly trivial shows how well this formalism is adapted to differential geometry.

For an arbitrary function we can define the m-form $*f = f\eta$. In terms of forms the integral of the function f is the integral of this m-form. The Einstein-Hilbert action is therefore

$$S_{EH} = \frac{1}{16\pi G}\int *R\,. \tag{2.24}$$

We now show the following

Proposition 2.4 *The Einstein Hilbert action*

$$\int *R \equiv \int R\eta \equiv \int R\sqrt{|g|}dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$$

can be given as $\int \Omega_{\mu\nu} \wedge *(\theta^\mu \wedge \theta^\nu)$*, where* ${\Omega^\mu}_\nu$ *is the curvature 2-form. More precisely,*

$$*R = \Omega_{\mu\nu} \wedge *(\theta^\mu \wedge \theta^\nu)\,. \tag{2.25}$$

Proof: We introduce

$$\eta^{\mu\nu} = *(\theta^\mu \wedge \theta^\nu) = \frac{1}{2}\eta_{\alpha\beta\sigma\rho}g^{\alpha\mu}g^{\beta\nu}\theta^\sigma \wedge \theta^\rho \tag{2.26}$$

and $\eta_{\alpha\beta} = \frac{1}{2}\eta_{\alpha\beta\sigma\rho}\theta^\sigma \wedge \theta^\rho$ so that

$$\eta^{\mu\nu} \wedge \Omega_{\mu\nu} = \eta_{\alpha\beta} \wedge \Omega^{\alpha\beta} = \frac{1}{2}\eta_{\alpha\beta} \wedge {R^{\alpha\beta}}_{\mu\nu}\theta^\mu \wedge \theta^\nu\,. \tag{2.27}$$

But

$$\eta_{\alpha\beta} \wedge \theta^\mu \wedge \theta^\nu = \frac{1}{2}\eta_{\alpha\beta\sigma\rho}\theta^\sigma \wedge \theta^\rho \wedge \theta^\mu \wedge \theta^\nu = (\delta^\mu_\alpha\delta^\nu_\beta - \delta^\nu_\alpha\delta^\mu_\beta)\eta \tag{2.28}$$

and

$$\eta^{\mu\nu} \wedge \Omega_{\mu\nu} = \frac{1}{2}(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\nu_\alpha\delta^\mu_\beta){R^{\alpha\beta}}_{\mu\nu}\eta = R\eta = *R\,. \tag{2.29}$$

□

Varying the action $\int \Omega_{\mu\nu} \wedge *(\theta^\mu \wedge \theta^\nu)$ one obtains Einstein's equation in the Cartan formalism.

2.4 Application: An exact gravitational plane wave solution

Proposition 2.5 *The metric*

$$ds^2 = -dt^2 + dr^2 - H(t-r, x^1, x^2)(dt-dr)^2 + \delta_{ab}dx^a dx^b\,, \tag{2.30}$$

$$\text{with} \qquad \Delta H \equiv \delta^{ab}\partial_a\partial_b H = 0 \quad a, b \in \{1, 2\} \tag{2.31}$$

is an exact solution of the vacuum Einstein equation.

This is an exact gravitational plane wave in the so called Brinkmann form [6].

Proof:
An orthonormal set of tetrads of this metric are

$$\theta^t = dt + \frac{1}{2}H(dt - dr) \tag{2.32}$$

$$\theta^r = dr + \frac{1}{2}H(dt - dr) \tag{2.33}$$

$$\theta^a = dx^a \qquad a \in \{1, 2\}\,. \tag{2.34}$$

It is easy to verify that

$$ds^2 = -(\theta^t)^2 + (\theta^r)^2 + \delta_{ab}\theta^a\theta^b\,. \tag{2.35}$$

As the metric coefficients are constant, $\omega_{\mu\nu} = -\omega_{\nu\mu}$. The Cartan connection 1-forms can then be inferred from $0 = \Theta^\nu = d\theta^\nu + \omega^\nu{}_\mu \wedge \theta^\mu$,

$$d\theta^t = \frac{1}{2}\partial_a H dx^a \wedge (dt - dr) = \frac{1}{2}\partial_a H \theta^a \wedge (\theta^t - \theta^r) = -\omega^t{}_\mu \wedge \theta^\mu \tag{2.36}$$

$$d\theta^r = \frac{1}{2}\partial_a H \theta^a \wedge (\theta^t - \theta^r) = -\omega^r{}_\mu \wedge \theta^\mu \tag{2.37}$$

$$d\theta^a = 0 = \omega^a{}_\mu \wedge \theta^\mu \qquad \Rightarrow \tag{2.38}$$

$$\omega^t{}_a = \frac{1}{2}\partial_a H(\theta^t - \theta^r) \;=\; \omega^r{}_a\,. \tag{2.39}$$

For the last equality sign we used the anti-symmetry.

From this we can easily calculate the curvature. Since all non-vanishing connection forms are proportional to $\theta^t - \theta^r$ there are no contributions of the form $\omega^\mu{}_\nu \wedge \omega^\nu{}_\lambda$ and we just have

$$\Omega^t{}_a = \Omega^r{}_a = d\omega^t{}_a = \frac{1}{2}\partial_a\partial_b H\theta_b \wedge (\theta^t - \theta^r) = \frac{1}{2}R^t{}_{a\mu\nu}\theta^\mu \wedge \theta^\nu = \frac{1}{2}R^r{}_{a\mu\nu}\theta^\mu \wedge \theta^\nu\,. \tag{2.40}$$

From this we can conclude

$$R^t{}_{atb} = -R^t{}_{arb} = R^r{}_{atb} = -R^r{}_{arb} = \frac{1}{2}\partial_a\partial_b H\,. \tag{2.41}$$

All other components of the Riemann tensor are either determined by symmetry or they vanish. For the Ricci tensor this yields

$$R_{ab} = R^t{}_{atb} + R^r{}_{arb} = 0 \tag{2.42}$$

$$R_{tt} = R^a{}_{tat} = -R^t{}_{ata} = -\frac{1}{2}\delta^{ab}\partial_a\partial_b H = R_{rr}\,. \tag{2.43}$$

Hence the Ricci tensor vanishes if and only if $\Delta H = 0$. □

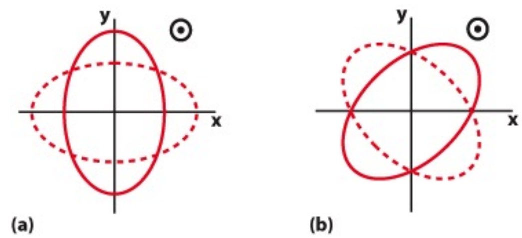

Figure 2.1: The two polarisation states of a gravitational plane wave in the plane normal to its propagation.

A simple example of such an exact gravitational wave solution is $H = \frac{1}{2}(x^2 - y^2)f(t-r)$ with

$$(H_{,ab}) = f(t-r)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (2.44)$$

This describes a gravitational wave which expands and contracts space along the x^1 and x^2 directions (see figure 2.1 a). Contrary, for $H = xyf(t-r)$,

$$(H_{,ab}) = f(t-r)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad (2.45)$$

which describes a gravitational wave expanding and contracting space in principle directions which are 45^o rotated wrt. x^1, x^2 (see figure 2.1 b).

Interestingly, this form of the metric, which of course is also a linearized gravitational wave on Minkowski spacetime, i.e. it satisfies the usual wave equation,

$$\left(-\partial_t^2 + \partial_r^2 + \partial_{x^1}^2 + \partial_{x^2}^2\right) H = 0$$

is an *exact* solution of Einstein's equation.

Exercice: Calculate the Ricci tensor of the metric (2.30) in the usual way, without using the Cartan formalism.

Exercice: Use the Cartan formalism to derive the Schwarzschild solution.
Hint: Make the following ansatz for the metric:

$$ds^2 = -e^{2a(r)}dt^2 + e^{2b(r)}dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)$$

and use the tetrads

$$\theta^0 = e^a dt \qquad (2.46)$$
$$\theta^1 = e^b dr \qquad (2.47)$$
$$\theta^2 = r d\vartheta \qquad (2.48)$$
$$\theta^3 = r\sin\vartheta\, d\varphi\,. \qquad (2.49)$$

Chapter 3

The 3+1 or ADM formalism of GR

(ADM stands for Arnowitt, Deser Misner.)
The 3+1 formulation of GR is important for theoretical developments (e.g. formulations of quantum gravity or the Hamiltonian formulation of GR) but also for numerical relativity. The 3+1 formalism is also very well adapted to cosmological perturbation theory.

Here we give an introduction to the topic with some applications. We assume

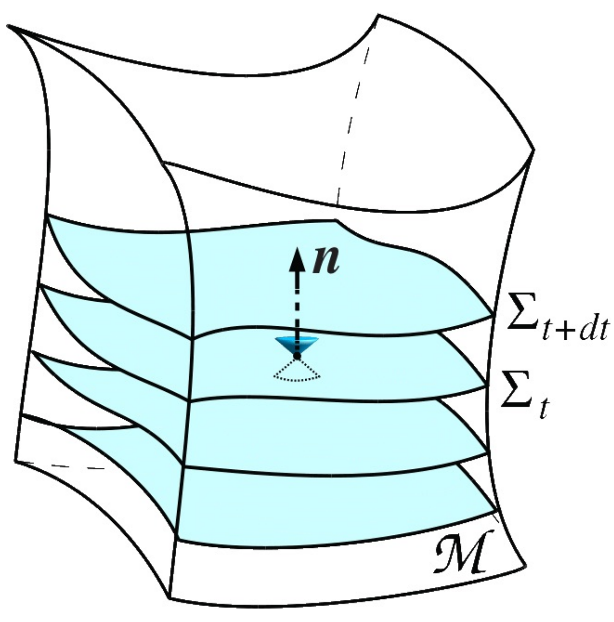

Figure 3.1: The slicing of spacetime into hypersurfaces of constant time.

that the spacetime $(\mathcal{M}, g)$ (a pseudo-Riemannian manifold with metric signature

$(-,+,+,+)$) admits a slicing by a 1-parameter family Σ_t of spacelike hypersurfaces, see Fig. 3.1 left panel. (A 3 dimensional submanifold of spacetime is called spacelike if at each point the normal to its tangent space is timelike.) More precisely, we assume the existence of a diffeomorphism

$$\phi : \mathcal{M} \to I \times \Sigma\,, \qquad I \subset \mathbb{R}\,, \tag{3.1}$$

where I is an interval, such that the manifolds $\Sigma_t = \phi^{-1}(\{t\} \times \Sigma)$ are spacelike and the curve $\phi^{-1}(I \times \{p\})$ is timelike for all points $p \in \Sigma$. These curves are called preferred timelike orbits. Their tangent vectors ∂_t define a vector field on $\mathcal{M}$. This can be decomposed into a part normal to Σ_t and a vector in $T\Sigma_t$ (see Fig. 3.2),

$$\partial_t = \alpha n + \bar{\beta}\,. \tag{3.2}$$

Here n denotes the future directed normal to Σ_t, with $n^2 \equiv n^\mu n^\nu g_{\mu\nu} = -1$, α

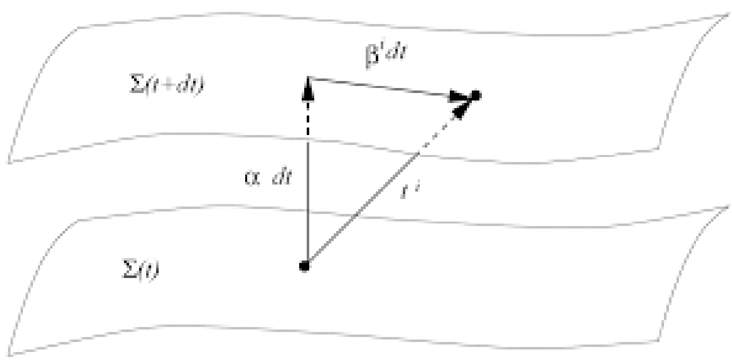

Figure 3.2: The decomposition of ∂_t into a component normal to Σ_t and a shift vector $\bar{\beta}$ in $T\Sigma_t$.

is the *lapse function* and $\bar{\beta}$ is called the shift vector. Here and in what follows we denote vectors on Σ_t with an overbar. A local coordinate system $\{x^i\}$ on Σ introduces natural (comoving) coordinates on $\mathcal{M}$ when giving the point $\phi^{-1}(t,p)$ the coordinates (t, x^i) where x^i are the coordinates of the point $p \in \Sigma$. In these coordinates $\bar{\beta} = \beta^i \partial_i$. The metric g induces a metric $\bar{g}$ on each Σ_t and we raise and lower indices of tensors on Σ_t (like e.g. the vector $\bar{\beta}$ with this metric. Using also $g(n, \partial_i) = 0$ we find

$$g(\partial_t, \partial_t) = -(\alpha^2 - \beta^i \beta_i) \;\text{ and }\quad g(\partial_t, \partial_i) = \beta_i\,. \tag{3.3}$$

In our comoving coordinates (t, x^i) the metric then takes the form

$$ds^2 \;=\; -(\alpha^2 - \beta^i\beta_i)dt^2 + 2\beta_i dx^i dt + \bar{g}_{ij} dx^i dx^j \tag{3.4}$$

$$\;=\; -\alpha^2 dt^2 + \bar{g}_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)\,. \tag{3.5}$$

As $\bar\beta$ is a vector on Σ, its components are raised and lowered with $\bar g_{ij}$. Eq (3.4) gives

$$(g_{\mu\nu}) = \begin{pmatrix} -\alpha^2 + \beta^i\beta_i & \beta_i \\ \beta_j & \bar g_{ij} \end{pmatrix} . \tag{3.6}$$

Exercise: Show that the inverse metric is given by

$$(g^{\mu\nu}) = \begin{pmatrix} -\alpha^{-2} & \alpha^{-2}\beta^i \\ \alpha^{-2}\beta^j & \bar g^{ij} - \alpha^{-2}\beta^i\beta^j \end{pmatrix} . \tag{3.7}$$

The form (3.5) of the metric shows that dt is orthogonal to the 1-forms $dx^i + \beta^i dt$ (in the sense that the corresponding vector fields $dt^\sharp$ and $(dx^i + \beta^i dt)^\sharp$ are orthogonal).

We call a tensor field S on $\mathcal{M}$ *tangential* (to the slicing Σ_t) if at each moment t it can be considered as a tensor field on Σ_t. In terms of comoving coordinates this means that S is of the form

$$\bar S = S^{i_1\cdots i_r}_{j_1\cdots j_s}\left(\partial_{i_1}\otimes\cdots\otimes\partial_{i_r}\otimes dx^{j_1}\otimes\cdots\otimes\ dx^{j_s}\right) . \tag{3.8}$$

In other words, at each moment t S 'lives' on $T\Sigma_t$ and $T^*\Sigma_t$. We shall denote such tensor fields by an over bar. Our first example of a tangential tensor field is the shift vector $\bar\beta$. Let us now consider a tangential p-form $\bar\omega$ on $\mathcal{M}$. In general its exterior derivative $d\bar\omega$ will no longer be tangential but of the form

$$d\bar\omega = \bar d\bar\omega + dt\wedge\partial_t\bar\omega\,, \tag{3.9}$$

where $\bar d$ denotes the exterior derivative in Σ_t. Both $\bar d\bar\omega$ and $\partial_t\bar\omega$ are tangential. In coordinates, $\partial_t\bar\omega = (\partial_t\omega_{i_1\cdots i_r})dx^{i_1}\wedge\cdots\wedge dx^{i_r}$.

Exercise: Show that for two tangential vector fields $[\bar X, \bar Y] = \overline{[\bar X, \bar Y]}$ and for a tangential vector field $\bar X$ on $T\Sigma_t$ and a tangential tensor field $\bar S$, the Lie derivative on $\mathcal{M}$ is given by the Lie derivative on Σ_t, $L_{\bar X}\bar S = \bar L_{\bar X}\bar S$. In other words, the commutator and the Lie derivative conserve the property of tangentiality (contrary to the exterior derivative).

We also introduce the projection map which projects an arbitrary vector in $T\mathcal{M}$ onto its part on $T\Sigma_t$,

$$h^\mu{}_\nu = \delta^\mu_\nu + n^\mu n_\nu\,. \tag{3.10}$$

Besides the dual basis $\{\partial_\mu\} = \{\partial_t, \partial_i\}$ and $\{dx^\mu\} = \{dt, dx^i\}$ adapted to the $3+1$ split we shall also use the dual basis

$$\{n, \partial_i\} \qquad \{\alpha dt, dx^i + \beta^i dt\}\,. \tag{3.11}$$

In order to use the Cartan formalism in our calculations it will be useful to replace the coordinate basis ∂_i by and orthornormal basis $\{\bar{e}_i\}$ on Σ_t. Denoting its dual basis on Σ_t by $\{\bar{\vartheta}^i\}$ we then have the following two pairs of duals bases which we shall use for what follows

$$\{e_0 \equiv n, \bar{e}_i\}\,, \qquad \{\theta^0 = \alpha dt, \theta^i = \bar{\vartheta}^i + \beta^i dt\} \quad \text{and} \tag{3.12}$$

$$\{\partial_t, \bar{e}_i\}\,, \qquad \{dt, \bar{\vartheta}^i\}\,. \tag{3.13}$$

Only the pair (3.12) is a dual pair of orthonormal bases on $T\mathcal{M}$ and $T^*\mathcal{M}$, while for the pair (3.13) the spatial part forms an orthonormal basis on Σ_t. We shall also often use

$$e_0 = n = \frac{1}{\alpha}(\partial_t - \bar{\beta})\,. \tag{3.14}$$

In terms of the orthonormal basis (3.12), the components of the projection operator becomes simply

$$h^\mu{}_\nu = \delta^\mu_i \delta^j_\nu \delta^i_j = \begin{cases} \delta^i_j & \text{if } \mu,\nu \text{ are spatial} \\ 0 & \text{if one index is } 0\,. \end{cases} \tag{3.15}$$

3.1 The formulas of Gauss and Weingarten

We now consider the first structure equation for the orthonormal frame (3.12),

$$d\theta^\mu + \omega^\mu_\nu \wedge \theta^\nu = 0\,. \tag{3.16}$$

Restricting this to $T\Sigma_t$ and using $\theta^0|_{T\Sigma_t} = 0$ we obtain ($\bar{\theta}^i \equiv \bar{\vartheta}^i$)

$$d\bar{\theta}^i + \omega^i{}_j \wedge \bar{\theta}^j \;=\; 0 \qquad \text{on } \; T\Sigma_t \tag{3.17}$$

$$-d\theta^0 = -\alpha_{,i}\theta^i \wedge dt = \omega^0{}_j \wedge \bar{\theta}^j \;=\; 0 \qquad \text{on } \; T\Sigma_t\,. \tag{3.18}$$

Here $\alpha_{,i} \equiv e_i(\alpha)$. Since the connections forms $\bar{\omega}^i{}_j$ on the hypersurface $T\Sigma_t$ also satisfy Eq. (3.17) and have the same symmetry properties we can conclude

$$\omega^i{}_j = \bar{\omega}^i{}_j \qquad \text{on } \; T\Sigma_t\,. \tag{3.19}$$

This has the following simple geometrical interpretation: For a fixed t, we denote by ∇ the Riemannian connection on $(\mathcal{M}, g)$ and by $\bar{\nabla}$ the one induced on $(\Sigma_t, \bar{g})$,

$$\bar{\nabla} = h\nabla\,, \qquad \bar{\nabla}_\mu = h^\nu_\mu \nabla_\nu \;\; \text{in coordinates.} \tag{3.20}$$

For a vector field $\bar{X}$ on $T\Sigma_t$, Eq. (3.19) is equivalent to

$$g(\nabla_{\bar{X}} e_j, e_i) = \omega_{ij}(\bar{X}) = \bar{\omega}_{ij}(\bar{X}) = \bar{g}(\bar{\nabla}_{\bar{X}} e_j, e_i) \tag{3.21}$$

This shows that for $\bar{X} \in \chi(\Sigma_t)$ and $Y \in \chi(\mathcal{M})$ with tangential projection $\bar{Y}$, i.e. $Y = \bar{Y} + y^0 e_0$, the tangential projection of $\nabla_{\bar{X}} Y$ on $T\Sigma_t$ is equal to $\bar{\nabla}_{\bar{X}}\bar{Y}$. Eq. (3.18) determines the component of $\nabla_{\bar{X}} Y$ normal to $T\Sigma_t$. Note also that $\omega^0{}_j = -\omega_{0j} = \omega_{j0} = \omega^j{}_0$. On $T\Sigma_t$ these 1-forms vanish, hence they satisfy the hypothesis of the following lemma:

Lemma 3.1 *(**Cartan**). If $\alpha^1, \ldots, \alpha^n$ are linearly independent 1-forms on a manifold $\mathcal{M}$ of dimension $n' \geq n$, and $\beta_1, \ldots, \beta_n$ are 1-forms on $\mathcal{M}$ satisfying*

$$\sum_{i=1}^{n} \alpha^i \wedge \beta_i = 0 \tag{3.22}$$

Then there are smooth functions f_{ij} on $\mathcal{M}$ such that

$$\beta_i = \sum_{j=1}^{n} f_{ij}\alpha^j \qquad \textit{and} \tag{3.23}$$

$$f_{ij} = f_{ji}\,. \tag{3.24}$$

Proof: In a neighborhood of any point we can choose $n' - n$ 1-forms $\alpha^{n+1}, \ldots, \alpha^{n'}$ such that they complete $\alpha^1, \ldots, \alpha^n$ to a basis of $T^*\mathcal{M}$. Then there are smooth functions f_{ij} ($1 \leq i \leq n$ and $1 \leq j \leq n'$) with

$$\beta_i = \sum_{j=1}^{n'} f_{ij}\alpha^j \tag{3.25}$$

The condition (3.22) then implies

$$\begin{aligned} 0 &= \sum_{i=1}^{n}\sum_{j=1}^{n'} f_{ij}\alpha^i \wedge \alpha^j \\ &= \sum_{1\leq i<j\leq n} (f_{ij} - f_{ji})\alpha^i \wedge \alpha^j + \sum_{i=1}^{n}\sum_{j>n} f_{ij}\alpha^i \wedge \alpha^j\,. \end{aligned} \tag{3.26}$$

Since the forms $\alpha_i \wedge \alpha_j$, $(i < j)$ are linearly independent 2-forms we conclude that $f_{ij} = 0$ for $j > n$ and $f_{ij} = f_{ji}$ for $j \leq n$. □

According to this lemma and Eq. (3.18) (the $\bar{\theta}^i$ play the role of the α^i and the $\omega^0{}_i$) the one of the β_i) there are functions K_{ij} on Σ_t such that

$$\omega^0{}_i = -K_{ij}\bar{\theta}^j \qquad \text{on} \;\; T\Sigma_t \qquad \text{with} \tag{3.27}$$

$$K_{ij} = K_{ji}\,. \tag{3.28}$$

The symmetric tensor field K_{ij} on Σ_t is called the extrinsic curvature of Σ_t or the second fundamental tensor (Often the expression 'second fundamental form' is used as a complement to the 'first fundamental form' which designs simply the metric $\bar{g}$. In this course we use the expression 'form' only to denote anti-symmetric co-variant tensor fields.). We also want to express the extrinsic curvature in the coordinate system (t, x^i). For this we first note that

$$K_{ij} = -\omega^0{}_i(e_j) = -\omega^i{}_0(e_j) = -(\nabla_j n)^i = -h^\alpha_j h^i_\beta (\nabla_\alpha n)^\beta \,. \tag{3.29}$$

For the last equal sign we used that in our orthonormal frame $h^k_j = \delta^k_j$ and $h^0_j = 0$. Expression (3.29) also holds in an arbitrary coordinate basis where K may have non-vanishing components also in direction ∂_t and where the index positions matter, so that in arbitrary coordinates or basis fields, where $h^\mu{}_\nu = \delta^\mu_\nu + n^\mu n_\nu$, we have

$$K_{\mu\nu} = -h^\alpha_\mu h_{\nu\beta}(\nabla_\alpha n)^\beta = -h^\alpha_\mu (\nabla_\alpha n)_\nu = -(n_{\nu;\mu} + n_\mu a_\nu)\,. \tag{3.30}$$

For the second equal sign we used that $n_\beta \nabla_\alpha n^\beta = 0$ which is a consequence of the normalisation condition, $n^2 = -1$. For the third equal sign we introduced the 'acceleration' of the normal field n,

$$a = \nabla_n n \,. \tag{3.31}$$

Hence (up to a sign) the extrinsic curvature is the projection of the covariant derivative of the normal n onto the hypersurface Σ_t. Since symmetry of a tensor field is invariant under coordinate transformation, also $K_{\mu\nu}$ is symmetric even though this is not evident from (3.30).

With this the 0-i first structure equations for our orthonormal basis e_μ become

$$g(\nabla_{e_i} e_j, n) = g(\nabla_{e_i} e_j, e_0) = -\omega^0{}_j(e_i) = K_{ji} = K_{ij} = g(\nabla_{e_j} e_i, n)\,. \tag{3.32}$$

The second fundamental tensor $K(\bar{X}, \bar{Y}) \equiv K_{ij} X^i Y^j$ on $T\Sigma_t$ determines the normal component of $\nabla_{\bar{X}} \bar{Y}$,

$$\nabla_{\bar{X}} \bar{Y} = \bar{\nabla}_{\bar{X}} \bar{Y} - K(\bar{X}, \bar{Y}) n \,. \tag{3.33}$$

Here $\bar{Y}, \bar{X} \in \chi(\mathcal{M})$ are tangent to $T\Sigma_t$ and $n = e_0$. Eq. (3.33) is called the **Gauss formula** and Eq. (3.32) are the **Weingarten equations**. We can write them for $\bar{X} = X^i e_i$ and $\bar{Y} = Y^i e_i$ as

$$K(\bar{X}, \bar{Y}) = g(n, \nabla_{\bar{X}} \bar{Y}) = g(n, \nabla_{\bar{Y}} \bar{X}) = -g(\nabla_{\bar{X}} n, \bar{Y}) = -g(\nabla_{\bar{Y}} n, \bar{x})\,. \tag{3.34}$$

The sign of the extrinsic curvature K is unfortunately not uniformly defined in the literature.

The map

$$K : T\Sigma_t \to T\Sigma_t : X^i e_i \mapsto K^i{}_j X^j e_i \tag{3.35}$$

is called the **Weingarten map** of the second fundamental tensor. Its eigenvalues are called the **principal curvatures**.

Execercise. Consider a torus imbedded in $\mathbb{R}^3$ with the metric induced from the flat metric on $\mathbb{R}^3$, i.e., $ds^2 = \delta_{ij}dx^i dx^j$. Determine the first and second fundamental tensor of the torus.

Execercise. Consider an ellipsoid with principle axes of lengths a, b and c imbedded in $\mathbb{R}^3$ with the induced metric. Determine the first and second fundamental tensor of the ellipsoid.
Hint: Such an ellipsoid is defined by the equation

$$f(x,y,z) = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1\,, \tag{3.36}$$

its normal is thus proportional to the gradient of f.

3.2 The Gauss and Codazzi-Mainardi equations

Before determining the parts of the connection forms $\omega^\mu{}_\nu$ which are not tangential to $T\Sigma_t$, we want to relate the curvature forms $\Omega^\mu{}_\nu$ on $\mathcal{M}$ on tangential vector fields to the ones on Σ_t and to the second fundamental tensor. We first consider the purely spatial components

$$\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j + \omega^i{}_0 \wedge \omega^0{}_j \tag{3.37}$$

We want to evaluate this curvature form first for vectors on $T\Sigma_t$. Since $\omega^i{}_j = \bar\omega^i_j$, the first two terms on $T\Sigma_t$ (where we can replace d by $\bar d$) simply give $\bar\Omega^i_j$. Using also (3.27) we find on $T\Sigma_t$

$$\Omega^i{}_j = \bar\Omega^i_j + K^i_l K_{jm}\bar\theta^l \wedge \bar\theta^m \qquad \text{on } \; T\Sigma_t\,. \tag{3.38}$$

For the 0-i components on $T\Sigma_t$ we find

$$\begin{aligned}
\Omega^0{}_j &= d\omega^0{}_j + \omega^0{}_k \wedge \omega^k{}_j && (3.39)\\
&= -d(K_{ji}\bar\theta^i) - K_{ki}\bar\theta^i \wedge \bar\omega^k_j \quad \text{on } \; T\Sigma_t\,, && (3.40)\\
&= -\bar d K_{ji} \wedge \bar\theta^i + K_{ji}\bar\omega^i_m \wedge \bar\theta^m - K_{ki}\bar\theta^i \wedge \bar\omega^k_j \quad \text{on } \; T\Sigma_t\,, && (3.41)\\
&= -\bar\nabla K_{ji} \wedge \bar\theta^i \quad \text{on } \; T\Sigma_t\,. && (3.42)
\end{aligned}$$

Here $\bar\nabla$ denotes the covariant derivative on $T\Sigma_t$. The formulas (3.38) and (3.41) are the famous **equations of Gauss and Codazzi–Mainardi** in terms of differential forms. We want to write them also in terms of the curvature tensor. For this we consider tangential vector $\bar{X}$ and $\bar{Y}$. The relation between the curvature form and the Riemann tensor gives

$$\begin{aligned} g(R(\bar{X},\bar{Y})e_j, e_i) &= \Omega_{ij}(\bar{X},\bar{Y}) \\ &= \bar{\Omega}_{ij}(\bar{X},\bar{Y}) + K^i_l K_{jm}(\bar{\theta}^l \wedge \bar{\theta}^m)(\bar{X},\bar{Y}) \\ &= \bar{g}(\bar{R}(\bar{X},\bar{Y})e_j, e_i) + K(e_i,\bar{X})K(e_j,\bar{Y}) - K(e_i,\bar{Y})K(e_j,\bar{X}) \,. \end{aligned} \tag{3.43}$$

For tangent vectors $\bar{X}$, $\bar{Y}$, $\bar{Z}$ and $\bar{W}$ we therefore obtain

$$g(R(\bar{X},\bar{Y})\bar{Z},\bar{W}) = \bar{g}(\bar{R}(\bar{X},\bar{Y})\bar{Z},\bar{W}) + K(\bar{W},\bar{X})K(\bar{Z},\bar{Y}) - K(\bar{W},\bar{Y})K(\bar{Z},\bar{X}) \,. \tag{3.44}$$

In components wrt the orthonormal spatial basis fields this reads

$$R^i{}_{jlm} = \bar{R}^i_{jlm} + K^i_l K_{jm} - K^i_m K_{jl} \,. \tag{3.45}$$

One calls this relation often **Gauss' Theorema Egregium**[1]. Due to the Minkowski signature the signs involving the extrinsic curvature K are different from the Riemannian case which is usually treated in the mathematics literature.

Similarly we obtain from (3.41)

$$\begin{aligned} g(R(\bar{X},\bar{Y})e_j, n) &= \Omega_{0j}(\bar{X},\bar{Y}) = (\bar{D}K_{ji} \wedge \bar{\theta}^i)(\bar{X},\bar{Y}) \\ &= \bar{\nabla}_k K_{ij}(\bar{\theta}^k \wedge \bar{\theta}^i)(\bar{X},\bar{Y}) = (X^k Y^i - X^i Y^k)\bar{\nabla}_k K_{ij} \\ &= (\bar{\nabla}_{\bar{X}} K)(\bar{Y}, e_j) - (\bar{\nabla}_{\bar{Y}} K)(\bar{X}, e_j) \,. \end{aligned} \tag{3.46}$$

Replacing e_j by an arbitrary tangent vector field $\bar{Z} = Z^j e_j$ this yields

$$g(R(\bar{X},\bar{Y})\bar{Z}, n) = (\bar{\nabla}_{\bar{X}} K)(\bar{Y},\bar{Z}) - (\bar{\nabla}_{\bar{Y}} K)(\bar{X},\bar{Z}) \,. \tag{3.47}$$

In components wrt the orthonormal basis fields, $e_0 \equiv n$, this reads

$$R^0{}_{jlm} = -[(\bar{\nabla}_l K)_{mj} - (\bar{\nabla}_m K)_{lj}] \,. \tag{3.48}$$

Exercice: Show that for a static metric, i.e. a metric of the form

$$ds^2 = -\alpha^2(x)dt^2 + \gamma_{ij}(x)dx^i dx^j \tag{3.49}$$

the extrinsic curvature vanishes. (We shall give a more mathematical definition of 'static' and 'stationary' in the next section.)

[1] For a n-dimensional surface imbedded in $n+1$ dimensional flat space, the combination $K^i_l K_{jm} - K^i_m K_{jl}$ depends only on the intrinsic curvature tensor but not on the imbedding.

3.3 The 3+1 form of Einstein's equations

To derive the $3+1$ split of Einstein's equations we also need the normal parts of the Riemann curvature, the components $R^0{}_{i0j}$. We determine these later in Section 3.3.2. First we want to explore what we can do already with the components of the Riemann tensor derived so far, namely $R^0{}_{iml}$ and $R^i{}_{jlm}$.

3.3.1 The time components of the Einstein and Ricci tensors

The basic equations of Gauss and Codazzi-Mainardi allow us to obtain interesting and useful expressions for the components $R_{0i} = G_{0i}$ and G_{00} of the Ricci and Einstein tensors. We first note that in full generality

$$i_{e_\alpha}\Omega^\alpha{}_\beta = \frac{1}{2}R^\alpha{}_{\beta\mu\nu}i_{e_\alpha}(\theta^\mu \wedge \theta^\nu) = R_{\beta\nu}\theta^\nu\,. \tag{3.50}$$

The components of the Ricci tensor are therefore given by

$$i_{e_\nu}(i_{e_\alpha}\Omega^\alpha{}_\beta) = \Omega^\alpha{}_\beta(e_\alpha, e_\nu) = R_{\beta\nu}\,. \tag{3.51}$$

Especially for the 0-i components relative to our adapted orthonormal basis we find

$$G_{0i} = R_{0i} = \Omega^j{}_0(e_j, e_i) = \bar\nabla_i K^j{}_j - \bar\nabla_j K^j{}_i\,. \tag{3.52}$$

Note that this gives the 0i components of the Einstein tensor wrt our orthonormal frame. We can also express this equations in coordinates using the $G_{0i} = n^\mu e_i^\nu G_{\mu\nu}$, where the latter $G_{\mu\nu}$ denotes the Einstein tensor in the coordinates (t, x^i). Using the projection operator for

$$\bar\nabla_i = h_i^\alpha \nabla_\alpha \tag{3.53}$$

and $K^j{}_j = K^\mu_\mu$ we obtain the following identity in coordinates

$$n^\mu G_{\mu i} = h_i^\alpha\left(\nabla_\alpha K^\nu{}_\nu - h_\nu^\beta \nabla_\beta K^\nu{}_\alpha\right)\,. \tag{3.54}$$

This equation is valid in arbitrary coordinates as long as the x^i are coordinates on Σ, in other words, the ∂_i are tangential.

Let us also consider, again in our orthonormal frame,

$$\begin{aligned} G_{00} &= R_{00} - \frac{1}{2}g_{00}R = \frac{1}{2}(R_{00} + R^i_i)\,. \\ R_{00} &= \Omega^j{}_0(e_j, e_0) \\ R^i_i &= \Omega^{ji}(e_j, e_i) + \Omega^{0i}(e_0, e_i)\,. \end{aligned}$$

Hence

$$2G_{00} = \Omega^{ji}(e_j, e_i) = \bar{R}^i_i + K^i_i K^j_j - K^i_j K^j_i \,, \tag{3.55}$$

$$G_{00} = \frac{1}{2}\left(\bar{R} + (\mathrm{tr}K)^2 - \mathrm{tr}(K^2)\right) = n^\mu n^\nu G_{\mu\nu} \,. \tag{3.56}$$

The last equal sign is valid in arbitrary coordinates while the first expression denotes G_{00} in our orthonormal frame.

Example 1: static and stationary spacetimes
Naively, a stationary spacetime is one where we can introduce coordinates such that the metric coefficients do not depend on time, $\partial_t g_{\mu\nu} = 0$. Here we want to translate this into a more geometric concept. For this we define

Definition 3.1 (stationary metric) *The metric of a Lorentz manifold* $(\mathcal{M}, g)$ *is called* **stationary** *if it admits a* **time-like** *Killing field,* b.

We now construct a local coordinate system in which g is time-independent. For this we consider in a neighborhood of a point $p_0 \in \mathcal{M}$ a three dimensional hypersurface Σ that is not tangent to b, i.e. $b(p) \notin T_p\Sigma$ for every p in a neighborhood of p_0.
Let $x^1(p)$, $x^2(p)$ and $x^3(p)$ be some coordinates on Σ and $\Phi_t := \Phi_{t,0}$ the flow of b. For the point $q = \Phi_t(p)$ we choose the coordinates $(t, x^1(p), x^2(p), x^3(p))$. With this construction $b = \partial_t = \partial_{x^0}$ and

$$L_b g = 0 \quad \text{is equivalent to} \quad g_{\mu\nu\,,0} = 0 \;.$$

It may happen that a Killing field is time-like in a certain domain, space-like in another one and light-like at the boundary (see black holes). In this sense the definition of stationarity given here is a local one.

Definition 3.2 (static metric) *A stationary metric is called* **static** *if the 1-form* $b^\flat$ *satisfies*

$$b^\flat \wedge db^\flat = 0 \;. \tag{3.57}$$

We want to show that in this case the surfaces Σ can be chosen such that b is normal to Σ and thus $g_{i0} = 0$.
(This is indeed a consequence of Frobenius' theorem: let us introduce the distribution $b(p)^\perp =: \{X_p \in T_p\mathcal{M} \mid g_p\left(X_p, b(p)\right) = 0\}$. If Eq. (3.57) is satisfied, the distribution $\{b(p)^\perp \mid p \in \mathcal{M}\}$ is locally integrable, but in this elementary case we show it without the use of Frobenius.). We first show that (3.57) implies that locally

$$b^\flat = \langle b, b\rangle df \tag{3.58}$$

for some function f. Here we use $\langle X,Y\rangle \equiv g(X,Y)$ interchangably. The surfaces $\{f = \text{const}\}$ are then normal to b and hence in coordinates (x^1, x^2, x^3) defined on $\{f = \text{const}\}$ together with $df = dt$ the metric has no components $g_{0i}dtdx^i$. We now prove the existence of the function f in (3.58). Since

$$\begin{aligned} 0 &= i_b(b^\flat \wedge db^\flat) = \langle b,b\rangle db^\flat - b^\flat \wedge i_b db^\flat \\ &= \langle b,b\rangle db^\flat - b^\flat \wedge (L_b b^\flat - d\langle b,b\rangle) \end{aligned}$$

But for an arbitrary vector field X

$$\left(L_b b^\flat\right)(X) = b(b^\flat(X)) - b^\flat([b,X]) = b\left(g(b,X)\right) - g(b,[b,X]) - g([b,b],X) = 0\,.$$

The last equation holds since b is a Killing field. With this we find

$$0 = \langle b,b\rangle db^\flat - d\langle b,b\rangle \wedge b^\flat = \langle b,b\rangle^2 d\left(\frac{b^\flat}{\langle b,b\rangle}\right)\,. \tag{3.59}$$

Hence $b^\flat/\langle b,b\rangle$ is closed and according to Poincaré's Lemma (3.58) holds locally. Note that we have used that b is a Killing field, otherwise the result would not hold. We therefore have

$$b^\flat = \langle b,b\rangle df =: \langle b,b\rangle dt$$

Conclusions: (for $df \equiv dt$):

- The flow Φ_t of b maps the hypersurfaces $t = \text{const}$ in an isometric way.
- An observer at rest propagates along integral curves of b.
- If there exists a time-like Killing field satisfying Eq. (3.57), there exists a preferred time t with $dt = \frac{b^\flat}{\langle b,b\rangle}$.
- For $\Sigma = \{t = \text{const}\}$, the Lagrangian coordinates introduced for the stationary case lead, in the static situation, to a metric of the form

$$\mathrm{d}s^2 = g_{00}(\mathbf{x})dt^2 + g_{ij}(\mathbf{x})dx^i dx^j = -\alpha^2(\mathbf{x})dt^2 + \bar{g}(\mathbf{x})\,. \tag{3.60}$$

On the hypersurface Σ we can introduce an orthonormal basis $\bar{e}_i$ of $\bar{g}$, which together with $e_0 \equiv n = \alpha^{-1}\partial_t = (\sqrt{-g_{00}(\mathbf{x})})^{-1}\partial_t$ form an orthonormal basis of spacetime with shift vector $\bar{\beta} = 0$. We denote the dual 1-forms by $\theta^0 = \alpha(\mathbf{x})dt$ and $\theta^i = \vartheta^i$. The first structure equation gives

$$\begin{aligned} d\theta^0 &= -\omega^0{}_i \wedge \theta^i = \alpha_{,i}\theta^0 \wedge \theta^i & (3.61)\\ d\theta^i &= -\omega^i{}_j \wedge \theta^i - \omega^i{}_0 \wedge \theta^0 = -\omega^i{}_j \wedge \theta^i\,. & (3.62) \end{aligned}$$

Here $\alpha_{,i} \equiv \bar{e}_i(\alpha)$ which in general is not just a partial deriovative wrt to some coodrinate x^i, since the $\bar{e}_i$ are orthonormal vector fields. The above eqns. imply ${\omega^0}_i = -\alpha_{,i}\theta^0$ and ${\omega^0}_i$ vanishes on $T\Sigma$ hence

$$K \equiv 0 \qquad \text{for a static spacetime.} \tag{3.63}$$

For the Einstein tensor we conclude $G_{0i} = 0$ and

$$G_{00} = \frac{1}{2}\bar{R} \qquad \text{for a static spacetime.} \tag{3.64}$$

Example: The Friedmann spacetime
We consider a spatially flat (for simplicity) Friedmann universe with metric

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \, . \tag{3.65}$$

We choose the orthonormal basis adapted to the constant time hypersurfaces

$$e_0 = \partial_t \, , \quad e_i = \frac{1}{a}\partial_i \, ; \qquad \theta^0 = dt \, , \theta^i = adx^i \tag{3.66}$$

The first structure equation yields

$$ddt = -{\omega^0}_i \wedge \theta^i = 0 \tag{3.67}$$

$$d\theta^i = -{\omega^i}_j \wedge \theta^j - {\omega^i}_0 \wedge dt = \frac{\dot{a}}{a} dt \wedge \theta^i \tag{3.68}$$

Together with the antisymmetry of $\omega_{\mu\nu}$ this implies $\omega_{ij} = 0$ and

$$-{\omega^0}_i = -{\omega^i}_0 = K_{ij}\theta^j = \frac{\dot{a}}{a}\delta_{ij}\theta^j \, . \tag{3.69}$$

In a Friedmann universe, the extrinsic curvature is diagonal and given by the expansion rate, $K_{ii} = \dot{a}/a = H$. For the components of the Einstein tensor this implies

$$G_{0i} = 0 \quad \text{and} \tag{3.70}$$

$$G_{00} = \frac{1}{2}(9H^2 - 3H^2) = 3H^2 \, . \tag{3.71}$$

3.3.2 The connections forms in the normal direction and other useful relations

In the previous section we have calculated ${\omega^i}_j(\bar{e}_k) = \bar{\omega}^i_j(\bar{e}_k)$ and ${\omega^0}_j(\bar{e}_k) = K_{jk}$. In terms of the orthonormal basis $\bar{\vartheta}^i$ on Σ_t we can write the extrinsic curvature, which is a symmetric covariant 2-tensor field on Σ_t also as

$$K = K_{ij}\bar{\theta}^i \otimes \bar{\theta}^j = \bar{K} = K_{ij}\bar{\vartheta}^i \otimes \bar{\vartheta}^j \, . \tag{3.72}$$

We also want to compute the connection forms in the direction $n = e_0$ normal to Σ_t. First we compute $\omega^0{}_i(e_0)$. The first structure equation yields

$$\begin{aligned} d\theta^0 &= d(\alpha dt) = \alpha_{,j}\vartheta^j \wedge dt = \frac{1}{\alpha}\alpha_{,j}\vartheta^j \wedge \theta^0 \\ &= \frac{1}{\alpha}\alpha_{,j}\theta^j \wedge \theta^0 \; = \; -\omega^0{}_j \wedge \theta^j \,. \end{aligned}$$

For the second line we made use of Eq. (3.12). Hence $\omega^0{}_i(e_0) = \alpha_{,i}/\alpha$ and with (3.27)

$$\omega^0{}_i = -K_{ij}\theta^j + \frac{1}{\alpha}\alpha_{,i}\theta^0 \,. \tag{3.73}$$

Finally we need to compute $\omega^i{}_j(e_0)$. The first structure equation gives

$$d\theta^i = -\omega^i{}_0 \wedge \theta^0 - \omega^i{}_j \wedge \theta^j = -(K^i_j\theta^0 + \omega^i{}_j) \wedge \theta^j$$

Hence

$$-i_{e_j} i_{e_0} d\theta^i = K^i_j + \omega^i{}_j(e_0) \,. \tag{3.74}$$

On the other hand

$$d\theta^i = d(\bar\vartheta^i + \beta^i dt) = \bar d\bar\vartheta^i + dt \wedge \partial_t\bar\vartheta^i + \bar d\beta^i \wedge dt \,.$$

With $e_0 = \alpha^{-1}(\partial_t - \beta^i\bar e_i)$ and $\bar d\bar\vartheta^i = -\bar\omega^i_j \wedge \bar\vartheta^j$ we obtain

$$\begin{aligned} i_{e_0} d\theta^i &= \frac{1}{\alpha}\left[i_{\bar\beta}\left(\bar\omega^i_j \wedge \bar\vartheta^j\right) + \partial_t\bar\vartheta^i - \bar d\beta^i + (\cdots)dt\right] \qquad \text{and} \\ i_{e_j}(i_{e_0} d\theta^i) &= \frac{1}{\alpha}\left[\left(\bar\omega^i_k \wedge \bar\vartheta^k\right)(\bar\beta, e_j) + \partial_t\bar\vartheta^i(e_j) - \bar d\beta^i(e_j)\right] \\ &= \frac{1}{\alpha}\left[\bar\omega^i_j(\bar\beta) - \beta^i_{|j} + \partial_t\bar\vartheta^i(e_j)\right] \,. \end{aligned} \tag{3.75}$$

Here the stroke in $\beta^i_{|j} = \beta^i_{,j} + \beta^k\omega^i{}_k(e_j)$ denotes the covariant derivative on $(\Sigma_t, \bar g)$. Let us define

$$\partial_t\bar\vartheta^i = c^i{}_j\bar\vartheta^j \,. \tag{3.76}$$

One easily derives from this that (exercise)

$$(\partial_t\bar g)_{ij} = c_{ij} + c_{ji} \,,$$

hence for an orthonormal basis on Σ_t the coefficients c_{ij} are antisymmetric. We then obtain from (3.75) and (3.74)

$$K_{ij} + \omega_{ij}(e_0) = -\frac{1}{\alpha}\left[\bar\omega_{ij}(\bar\beta) - \beta_{i|j} + c_{ij}\right] \,. \tag{3.77}$$

The symmetric and anti-symmetric parts of this relation determine K_{ij} and $\omega_{ij}(e_0)$:

$$\omega_{ij}(e_0) = -\frac{1}{\alpha}\bar{\omega}_{ij}(\bar{\beta}) + \frac{1}{2\alpha}\left[\beta_{i|j} - \beta_{j|i}\right] - \frac{1}{2\alpha}\left[c_{ij} - c_{ji}\right] \tag{3.78}$$

$$K_{ij} = \frac{1}{2\alpha}\left[\beta_{i|j} + \beta_{j|i} - (c_{ij} + c_{ji})\right] . \tag{3.79}$$

The last eqn. (3.79) provides a useful relation. Inserting (see prop. 1.15, $2\bar{\beta}_{i|j} = (\bar{L}_{\bar{\beta}}\bar{g})_{ij}) - (\bar{d}\bar{\beta})_{ij})$

$$(\bar{L}_{\bar{\beta}}\bar{g})_{ij} = \bar{\beta}_{i|j} + \bar{\beta}_{j|i} \tag{3.80}$$

we obtain

$$\bar{K} = -\frac{1}{2\alpha}\left[\partial_t \bar{g} - \bar{L}_{\bar{\beta}}\bar{g}\right] . \tag{3.81}$$

For the curvature we shall also need the explicit expression for $\bar{L}_{\bar{\beta}}\bar{K}$,

$$\bar{L}_{\bar{\beta}}\bar{K} = (\bar{L}_{\bar{\beta}}\bar{K}_{ij})\bar{\vartheta}^i \otimes \bar{\vartheta}^j + \bar{K}_{ij}(\bar{L}_{\bar{\beta}}\bar{\vartheta}^i) \otimes \bar{\vartheta}^j + \bar{K}_{ij}\bar{\vartheta}^i \otimes \bar{L}_{\bar{\beta}}\bar{\vartheta}^j$$

The Lie derivative of $\bar{\vartheta}^i$ on Σ_t is given by

$$\begin{aligned} \bar{L}_{\bar{\beta}}\bar{\vartheta}^i &= (\bar{d} \circ i_{\bar{\beta}} + i_{\bar{\beta}}\bar{d})\bar{\vartheta}^i = \bar{d}\beta^i - i_{\bar{\beta}}(\bar{\omega}^i_k \wedge \bar{\vartheta}^k) = \beta^i_{,k}\vartheta^k + \bar{\omega}^i{}_k\beta^k - \bar{\omega}^i_k(\bar{\beta})\bar{\vartheta}^k \\ &= (\beta^i{}_{|k} - \bar{\omega}^i{}_k(\bar{\beta}))\bar{\vartheta}^k . \end{aligned}$$

Inserting this above we find

$$(\bar{L}_{\bar{\beta}}\bar{K})_{ij} = \beta^k \bar{K}_{ij,k} + K_{ik}(\beta^k_{|j} - \bar{\omega}^k{}_j(\bar{\beta})) + K_{kj}(\beta^k_{|i} - \bar{\omega}^k{}_i(\bar{\beta})) . \tag{3.82}$$

We shall also need $\partial_t \bar{K}$. For this we use (3.76) so that

$$\partial_t \bar{K} = (\partial_t K_{ij} + K_{ik}c^k{}_j + K_{kj}c^k{}_i)\bar{\vartheta}^i \otimes \bar{\vartheta}^j . \tag{3.83}$$

Putting this together we have

$$(\partial_t \bar{K} - \bar{L}_{\bar{\beta}}\bar{K})_{ij} = (\partial_t - \bar{L}_{\bar{\beta}})K_{ij} + \left[K_{ik}\left(c^k{}_j - \beta^k_{|j} + \bar{\omega}^k{}_j(\bar{\beta})\right) + (i \leftrightarrow j)\right] . \tag{3.84}$$

Using (3.77) this becomes

$$(\partial_t \bar{K} - \bar{L}_{\bar{\beta}}\bar{K})_{ij} = (\partial_t - \bar{L}_{\bar{\beta}})K_{ij} - 2\alpha(\bar{K}^2)_{ij} - \alpha\left[K_{ik}\omega^k{}_j(e_0) + K_{kj}\omega^k{}_i(e_0)\right] . \tag{3.85}$$

This formula will be useful for the derivation of the spatial components of the Einstein tensor .

3.3.3 The spatial components of the Einstein and Ricci tensors

To derive also the spatial components of the Einstein tensor we need to determine the curvature forms also in the normal direction $n = e_0$. With the second structure equation and (3.73,3.77) this can be obtained as follows

$$\begin{aligned}\Omega^i{}_0 &= d\omega^i{}_0 + \omega^i{}_j \wedge \omega^j{}_0 \\ &= -d(K^i_j\theta^j) + d\left(\frac{\alpha^{|i}}{\alpha}\theta^0\right) + \omega^i{}_j \wedge \left(-K^j_k\theta^k + \frac{\alpha^{|j}}{\alpha}\theta^0\right) \qquad (3.86)\end{aligned}$$

For the first term we obtain

$$-dK^i_j \wedge \theta^j + K^i_j(\omega^j{}_k \wedge \theta^k + \omega^j{}_0 \wedge \theta^0) = -dK^i_j \wedge \theta^j + K^i_j(\omega^j{}_k \wedge \theta^k - K^j_k\theta^k \wedge \theta^0)\,.$$

We also need

$$\frac{\alpha^{|j}}{\alpha}\omega^i{}_j \wedge \theta^0 = \frac{\alpha^{|j}}{\alpha}\bar{\omega}^i_j(\bar{e}_k)\theta^k \wedge \theta^0\,,$$

and

$$\begin{aligned}d\left(\frac{\alpha^{|i}}{\alpha}\theta^0\right) &= d\left(\alpha^{,i}dt\right) = \alpha^{,i}_{,j}\theta^j \wedge dt \\ &= \frac{1}{\alpha}\alpha^{,i}_{,j}\theta^j \wedge \theta^0\,.\end{aligned}$$

We have used that $\theta^0 = \alpha dt$. Also note that for functions $\alpha_{,j} = e_j(\alpha) = \alpha_{|j}$.

The second and the last term in (3.86) therefore give together $\frac{1}{\alpha}\alpha^{|i}_{|j}\theta^j \wedge \theta^0$. In total we obtain for the curvature form

$$\Omega^i{}_0 = \frac{1}{\alpha}\alpha^{|i}_{|j}\theta^j \wedge \theta^0 - dK^i_j \wedge \theta^j + (K^i_j\omega^j{}_k - K^j_k\omega^i{}_j) \wedge \theta^k - K^i_jK^j_k\theta^k \wedge \theta^0\,. \qquad (3.87)$$

As a check one can restrict this to $T\Sigma_t$ to obtain the Codazzi-Mainardi eqn. (3.41).

The Ricci tensor is obtained from the curvature via (3.51)

$$R_{\mu\nu} = \Omega^\alpha{}_\mu(e_\alpha, e_\nu)\,. \qquad (3.88)$$

Using the fact that K_{ij} is symmetric and ω_{ij} is anti-symmetric we now obtain

$$\begin{aligned}R_{00} = \Omega^i{}_0(e_i, e_0) &= \frac{1}{\alpha}\bar{\Delta}\alpha + dK^i_i(e_0) - K^i_jK^j_i \qquad (3.89)\\ &= \frac{1}{\alpha}\bar{\Delta}\alpha - K^i_jK^j_i + \frac{3}{\alpha}(\partial_t - \bar{L}_{\bar{\beta}})H\,. \qquad (3.90)\end{aligned}$$

For the second equal sign we have introduced $3H = K^i_i$ which is often called the 'mean curvature'. Note that our definition of H differs by a factor 3 from the notation used in mathematics literature. We do this in order for H to denote the Hubble parameter in a Friedmann universe. We have also used

$$dK_{ij}(e_0) = e_0(K_{ij}) = \frac{1}{\alpha}(\partial_t - \bar{L}_{\bar{\beta}})K_{ij}\,. \tag{3.91}$$

Finally we want to determine

$$R_{ij} = \Omega^0{}_i(e_0, e_j) + \Omega^k{}_i(e_k, e_j)\,. \tag{3.92}$$

Eq. (3.38) gives the second term,

$$\Omega^k{}_i(e_k, e_j) = \bar{R}_{ij} + 3HK_{ij} - K_{ik}K^k_j\,. \tag{3.93}$$

For the first term we use (3.87)

$$\Omega^0{}_i(e_0, e_j) = \Omega^i{}_0(e_0, e_j) = -\frac{1}{\alpha}\alpha_{|ij} - dK_{ij}(e_0) + K_{ik}(\omega^k{}_j(e_0) + K^k_j) - \omega^i{}_k(e_0)K^k_j\,. \tag{3.94}$$

Using again (3.91) and adding both contributions we can write R_{ij} as

$$R_{ij} = \bar{R}_{ij} + 3HK_{ij} - \frac{1}{\alpha}\alpha_{|ij} - \frac{1}{\alpha}(\partial_t - \bar{L}_{\bar{\beta}})K_{ij} + K_{ik}\omega^k{}_j(e_0) + K^k_j\omega_{ki}(e_0)\,.$$

For the last three terms we now use (3.85) which finally leads to

$$R_{ij} = \bar{R}_{ij} + 3HK_{ij} - 2(\bar{K}^2)_{ij} - \frac{1}{\alpha}(\partial_t\bar{K} - \bar{L}_{\bar{\beta}}\bar{K})_{ij} - \frac{1}{\alpha}\alpha_{|ij}\,. \tag{3.95}$$

For the Riemann scalar we obtain

$$R = R^i_i - R_{00} = \bar{R} + (3H)^2 + \mathrm{tr}(\bar{K}^2) - \frac{6}{\alpha}(\partial_t - \bar{L}_{\bar{\beta}})H - \frac{2}{\alpha}\Delta\alpha\,. \tag{3.96}$$

Note that $\mathrm{tr}\left(\partial_t\bar{K} - \bar{L}_{\bar{\beta}}\bar{K}\right) = (\partial_t - \bar{L}_{\bar{\beta}})H - 2\alpha\mathrm{tr}(\bar{K}^2)$. This is best seen from (3.85).

With this we finally also obtain the ij components of the Einstein tensor,

$$\begin{aligned} G_{ij} &= R_{ij} - \frac{1}{2}R\delta_{ij} \\ &= \bar{R}_{ij} + 3HK_{ij} - 2(\bar{K}^2)_{ij} - \frac{1}{\alpha}(\partial_t\bar{K} - \bar{L}_{\bar{\beta}}\bar{K})_{ij} - \frac{1}{\alpha}\alpha_{|ij} \\ &\quad -\frac{1}{2}\left(\bar{R} + (3H)^2 + \mathrm{tr}(\bar{K}^2) - \frac{6}{\alpha}(\partial_t - \bar{L}_{\bar{\beta}})H - \frac{2}{\alpha}\Delta\alpha\right)\delta_{ij}\,. \end{aligned} \tag{3.97}$$

Note that the only second time derivative in these equations is $\partial_t\bar{K}$ (and $3\partial_t H = \partial_t\mathrm{tr}\bar{K}$). Hence the (ij) Einstein equations determine the time evolution of $\bar{K}$

(assuming that the matter equations can be solved and T_{ij} can be expressed in terms of the metric and its first derivatives). The time evolution of the metric is then obtained via (3.81),

$$\partial_t \bar{g} = \bar{L}_{\bar{\beta}} \bar{g} - 2\alpha \bar{K} \,. \tag{3.98}$$

The equations (3.98) and $G_{ij} = 8\pi G T_{ij}$ where $\partial_t \bar{K}$ is isolated in G_{ij} as given in (3.97) are called the ADM (Arnowitt, Deser, Misner) evolution equations. This system of equations has traditionally been used in numerical relativity. It turned out, however not to be ideal. Especially, the ADM evolution equations do not satisfy any known hyperbolicity criterion. This situation led to the development of alternative schemes; among them several which are explicitly hyperbolic. [2]. There are, however relativly simple ways to rewrite these equation (e.g. by separating out the conformal degree of freedom of $\bar{g}$) which lead to explicitely hyperbolic equations which are numerically stable, see [4].

Exercise: Consider a static spherically symmetric metric of the form

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right) \,. \tag{3.99}$$

Use the 3+1 formalism for this metric to compute the Einstein tensor. What properties does the energy momentum tensor need to have in order to satisfy these equations? Discuss 3 examples:

- The vacuum (Schwarzschild).
- A cosmological constant (Kottler or Schwarzschild de Sitter).
- An electric point charge (Reissner-Nordstrom).

Show that solutions are 'additive' in the sense that if the functions f and g solve the Einstein equations with the above metric and energy momentum tensor T^f and T^g then $f+g$ solves Einstein's equations for $T = T^f + T^g$.

3.3.4 Gaussian normal coordinates

We may interpret Eqs. (3.98) and (3.97) as evolution equations for the metric $\bar{g}$ and for the extrinsic curvature $\bar{K}$. Together with the constraints (3.56, 3.52) which have to be satisfied on Σ_t for all t.

To simplify the problem one may specify up to four coordinate conditions to fix the Σ_t slicing and the spatial coordinates x^i on Σ. An especially simple local coordinate system are so called Gaussian normal coordinates which are defined by

$$\alpha = 1, \quad \beta^i = 0 \quad \text{Gaussian normal coordinates} \,. \tag{3.100}$$

We now want to show that such coordinates always exist locally. For this we consider a hypersurface Σ_0 with timelike normal n. In the neighborhood U of a point $p \in \Sigma_0$ we construct the timelike geodesic $c(t,q)$ tangent to n with $c(0,q) = q$. We choose coordinates x^1, x^2, x^3 on U and consider the points $x^1 = x^1_{(0)} + s, x^2, x^3$ with s small enough so that these coordinates label points in U. We now consider the two parameter family of points $c(t,s) = c(t, x^1_{(0)} + s, x^2, x^3)$. In the coordinates (t, x^1, x^2, x^3) we have $n = \partial_t$ and $\partial_{x^1} = \partial_s$ since $[\partial_t, \partial_s] = 0$ also $[n, c^*\partial_s] = 0$. Identifying $\partial_s \equiv c^*\partial_s$ we therefore have

$$0 = [n, \partial_s] = \nabla_n \partial_s - \nabla_{\partial_s} n \tag{3.101}$$

Since $g(n,n) \equiv \langle n, n\rangle = -1$ clearly for the times slicing $\Sigma_t = \{c(t,q) | q \in \Sigma_0 \cap U\}$ we have $\alpha = 1$. We now want to show that also $\beta = 0$, i.e. $n = \partial_t$ is normal on Σ_t. For $t = 0$ this is so by construction. For $t > 0$ we have

$$t = -\int_0^t \langle n, n\rangle dt \tag{3.102}$$

independent of the position $q \in U$. Therefore

$$\begin{aligned} 0 &= \frac{1}{2}\nabla_{\partial_s}\int_0^t \langle n, n\rangle dt' = \int_0^t \langle \nabla_{\partial_s} n, n\rangle dt' = \int_0^t \langle \nabla_n \partial_s, n\rangle dt' \\ &= \int_0^t \nabla_n \langle \partial_s, n\rangle dt' = \int_0^t \frac{d}{dt'}\langle \partial_s, n\rangle dt' = \langle \partial_s, n\rangle(t) - \langle \partial_s, n\rangle(0)\,, \end{aligned}$$

where we have used $\nabla_n n = 0$ for the second line. Since n is normal to Σ at $t = 0$ and ∂_s is tangent, the second term vanishes and we obtain $\langle \partial_s, n\rangle(t) = 0$. Hence the coordinate x^1 remains normal to n in the entire neighbourhood of values t for which the geodesics are well defined. The same argument applies to x^2 and x^3 and we find that in the chosen coordinates the metric takes the form

$$ds^2 = -dt^2 + g_{ij}(t, \mathbf{x}) dx^i dx^j\,. \tag{3.103}$$

Hence $\alpha = 1$ and $\boldsymbol{\beta} = 0$ as requested.

For the Friedmann metric, cosmic time specifies Gaussian normal coordinates, see Eq. (3.65).

Exercise: Write the Einstein equations in Gaussian normal coordinates.

Exercise: Another often used coordinate system are 'maximal slicing' coordinates which satisfy $\mathrm{tr}\bar{K} = 3H = 0$. In this coordinate system the 00 equation yields

$$\frac{1}{\alpha}\bar{\Delta}\alpha - \mathrm{tr}(\bar{K}^2) = 16\pi G T_{00}\,.$$

Find maximal slicing coordinates for a spatially flat Friedmann universe. (In these coordinates the Universe is not expanding.)

3.4 The Hamiltonian formulation of GR

In their 4-dimensional coordinate invariant form the Einstein equations are not adapted to a Hamiltonian treatment. Contrary to a Lagrangian, a Hamiltonian density requires a time coordinate with respect to which we define 'evolution in time'. This is already evident by the definition of a Hamiltonian as

$$\mathcal{H}(\phi,\pi_\phi) = \pi_\phi \frac{\partial \mathcal{L}}{\partial \dot\phi} - \mathcal{L}(\phi,\dot\phi)\,, \qquad \pi_\phi \equiv \frac{\partial \mathcal{L}}{\partial \dot\phi}\,. \tag{3.104}$$

Here ϕ denotes one or several fields and we assume $\mathcal{L}$ to depend on the fields, their spatial derivatives and their first time derivatives denoted by $\dot\phi$.

The 0μ Einstein equations are

$$2G_{00} = \bar R + (3H)^2 - \mathrm{tr}(\bar K^2) = 16\pi G T_{00} \qquad \text{(energy constraint)} \tag{3.105}$$

$$G_{0i} = 3\bar H_{|i} - \bar K^j_{i|j} = 8\pi G T_{0j} \qquad \text{(momentum constraint)}\,. \tag{3.106}$$

As already discussed above, these equations contain only first time derivatives of the metric. Hence they are constraints and not evolution equations.

In order to find a Hamiltonian we start with the usual gravitational action

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{\det g}R\,. \tag{3.107}$$

We use (3.96)

$$R = \bar R + (3H)^2 + \mathrm{tr}(\bar K^2) - \frac{6}{\alpha}(\partial_t - \bar L_{\bar\beta})H - \frac{2}{\alpha}\Delta\alpha\,.$$

We now show that the last two terms can be replaced by a divergence, namely

$$\frac{3}{\alpha}(\partial_t - \bar L_{\bar\beta})H - \frac{2}{\alpha}\Delta\alpha = \nabla_\mu(3Hn^\mu + a^\mu) + (3H)^2\,. \tag{3.108}$$

Using $K_{\mu\nu} = -\nabla_\mu n_\nu - n_\mu a_\nu$ with $\nabla_n n = a$ and $n = \alpha^{-1}(\partial_t - \beta)$, we obtain $3H = K^\mu_\mu = -\nabla^\mu n_\mu$ so that

$$\nabla_\mu(3Hn^\mu + a^\mu) = \frac{3}{\alpha}(\partial_t - \bar L_{\bar\beta})H - (3H)^2 + \nabla_\mu a^\mu\,.$$

Hence our claim (3.108) follows if $\nabla_\mu a^\mu = \bar\Delta\alpha/\alpha$. To show this we first note that $a = \nabla_{e_0}e_0 = \omega^i{}_0(e_0)\bar e_i = \alpha^{,i}/\alpha \bar e_i$. In other words, if α is independent of the spatial coordinates then $a = 0$ and n is a geodesic. Now

$$a^\mu_{;\mu} = a^\mu_{,\mu} + a^\nu\omega^\mu{}_\nu(e_\mu) = a^i_{|i} + a^i\omega^0{}_i(e_0) = a^i_{|i} + a^i\alpha^{,i}/\alpha\,.$$

For the last equal sign we used (3.73) for $\omega^0{}_i(e_0)$. With $a^i = \alpha^{,i}/\alpha$ we have $a^i_{|i} = \bar{\Delta}\alpha/\alpha - (\alpha^{,i}/\alpha)^2$. Combining this with the previous expressions yields

$$a^\mu_{;\mu} = \frac{\bar{\Delta}\alpha}{\alpha}\,, \tag{3.109}$$

and hence implies (3.108). With this we can write ($3H = K^\mu_\mu$)

$$R = \bar{R} + K_{ij}K^{ij} - (3H)^2 - 2\nabla_\mu(3Hn^\mu + a^\mu) = L_{ADM} - 2\nabla_\mu(3Hn^\mu + a^\mu)\,. \tag{3.110}$$

We this the action becomes

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{\det g}L_{ADM} - 2\nabla_\mu(3Hn^\mu + a^\mu) = S_{ADM} + S_{\text{surface}}\,, \tag{3.111}$$

$$L_{ADM} = \bar{R} + K_{ij}K^{ij} - (3H)^2\,. \tag{3.112}$$

The second term is a surface term which does not contribute to the equations of motion. The first term is called the ADM (Arnowitt, Deser, Misner) Lagrangian.

In order to study this action we use that the determinant of our metric (3.4) is $\det g = \alpha^2 \det \bar{g}$ (**exercise**) so that we can write

$$S_{ADM} = \frac{1}{16\pi G}\int dt \int d^4x\alpha\sqrt{\det\bar{g}}\left(\bar{R} + K_{\mu\nu}K^{\mu\nu} - (3H)^2\right)\,. \tag{3.113}$$

This action is correct in an arbitrary coordinate system and we shall now consider it wrt the coordinates (t, x^i). In the rest of this section, tensorial components are considered wrt the dual bases ∂_t, ∂_i and dt, dx^i.

Exercise: Show that for the metric

$$(g_{\mu\nu}) = \left(\begin{array}{c|c} -(\alpha^2 - \beta^2) & \beta_i \\ \hline \beta_j & \bar{g}_{ji} \end{array}\right) \tag{3.114}$$

the determinant is given by $\det g = -\alpha^2 \det \bar{g}$. Here $\beta^2 = \beta_i\beta_j\bar{g}^{ij}$.

We now consider Eq (3.113) as our Lagrangian for a field theory of the fields α, β^i and $\bar{g}_{ij}$. From the expression for $K_{\mu\nu}$ (see eq. (3.79)) we see that no time derivatives of α and β^i enter in $\sqrt{\det g}L_{ADM}$. Therefore the equations from varying α and β^i will give constraint equations and these variables are not dynamical. However, $c_{ij} + c_{ji} = \partial_t\bar{g}_{ij}$ appears in $K_{\mu\nu}$ and therefore the 6 variables $\bar{g}_{ij}$ are dynamical.

To derive the Hamiltonian we introduce their momenta

$$\pi^{ij} = \frac{\partial}{\partial\dot{\bar{g}}_{ij}}\left(\sqrt{\det g}L_{ADM}\right)\,. \tag{3.115}$$

Since L_{ADM} depends on $\dot{\bar{g}}_{ij}$ only through K_{ij} we have

$$\pi^{ij} = \frac{\partial K_{mn}}{\partial \dot{\bar{g}}_{ij}} \left(\sqrt{\det g} \frac{\partial}{\partial K_{mn}} L_{ADM} \right) . \tag{3.116}$$

But with (3.81) we get

$$\begin{aligned} \frac{\partial K_{mn}}{\partial \dot{\bar{g}}_{ij}} &= -\frac{1}{2\alpha}\delta^i_m \delta^j_n \qquad \text{and} \\ \frac{\partial}{\partial K_{mn}} L_{ADM} &= \frac{\partial}{\partial K_{mn}}[(\bar{g}^{il}\bar{g}^{jk} - \bar{g}^{ij}\bar{g}^{lk})K_{ij}K_{lk}] \\ &= (\delta^m_i \delta^n_j K_{lk} + \delta^m_l \delta^n_k K_{ij})[\bar{g}^{il}\bar{g}^{jk} - \bar{g}^{ij}\bar{g}^{lk}] \qquad \text{so that} \\ \pi^{ij} &= -\sqrt{\det \bar{g}}\left[K^{ij} - \bar{g}^{ij}3H\right] . \end{aligned}$$

This leads to the following expression for the Hamiltonian density,

$$\mathcal{H}_{ADM} = \left(\pi^{ij}\dot{\bar{g}}_{ij} - L_{ADM}\right) \tag{3.117}$$

$$= \sqrt{\det \bar{g}}\left[\left(K^{ij} - \bar{g}^{ij}3H\right)\left(2\alpha K_{ij} - \beta_{i|j} - \beta_{j|i}\right) - \alpha\left(\bar{R} + K_{lm}K^{lm} - (3H)^2\right)\right] \tag{3.118}$$

$$= \sqrt{\det \bar{g}}\left[\alpha(K_{lm}K^{lm} - (3H)^2 - \bar{R}) - 2[(K^{ij} - 3H\bar{g}^{ij})\beta_i]_{|j} + 2(K^{ij} - 3H\bar{g}^{ij})_{|j}\beta_i\right] . \tag{3.119}$$

Ignoring the total (spatial) divergence term we can write this Hamiltonian as

$$H_{ADM}(t) = \int_{\Sigma_t} d^3x\sqrt{\bar{g}}\left[\alpha(K_{ij}K^{ij} - (3H)^2 - \bar{R}) + 2(K^{ij} - 3H\bar{g}^{ij})_{|j}\beta_i\right] . \tag{3.120}$$

This is the ADM Hamiltonian. In this expression K_{ij} is to be understood as a function of the canonical variables $\bar{g}_{ij}$ and π^{ij}. More precisely

$$\sqrt{\det \bar{g}}K^{ij} = -\left(\pi^{ij} - \frac{1}{2}\pi^m_m\bar{g}^{ij}\right) . \tag{3.121}$$

To obtain the evolution equations we can now use the canonical equations,

$$\dot{\bar{g}}_{ij} = \frac{\partial}{\partial \pi^{ij}}\mathcal{H}_{ADM} \tag{3.122}$$

$$\dot{\pi}^{ij} = -\frac{\partial}{\partial \bar{g}_{ij}}\mathcal{H}_{ADM} . \tag{3.123}$$

It is an **exercise** to show that these equations are equivalent to (3.98) and the $G_{ij} = 0$ vacuum Einstein equation. To obtain also the constraint equations we must vary the Lagrangian. Here we vary $L_H = \mathcal{H} - \pi^{ij}\dot{\bar{g}}_{ij}$, i.e.

$$16\pi G S_H = \int_{t_1}^{t_2} dt \left[\int_{\Sigma_t} d^3x \pi^{ij} \dot{\bar{g}}_{ij} - \mathcal{H}_{ADM} \right] . \tag{3.124}$$

The dynamical variables are now $(\bar{g}_{ij}, \beta^i, \alpha)$ and the momenta π^{ij}. A somewhat lengthy calculation gives

$$\delta H_{ADM} = \int_{\Sigma_t} \left(\Pi^{ij} \delta \bar{g}_{ij} + Q_{ij} \delta \pi^{ij} - C \delta\alpha - 2 B_i \delta\beta^i \right) d^3x \,, \tag{3.125}$$

where the two constraint functions are

$$C = \bar{R} + (3H)^3 - K^{ij}K_{ij} \qquad (= 2G_{00} \text{ wrt the orthonormal frame}) \tag{3.126}$$

$$B_i = -\left(K_i^j - 3H\delta_i^j \right)_{|j} \qquad (= G_{0i} \text{ wrt the orthonormal frame}). \tag{3.127}$$

The two other derivatives give

$$Q_{ij} = 2\frac{\alpha}{\sqrt{\det \bar{g}}} \left(\pi_{ij} - \frac{1}{2}\pi^m_m \bar{g}_{ij} \right) + \beta_{i|j} + \beta_{j|i} \tag{3.128}$$

$$\begin{aligned} \Pi^{ij} = {} & \alpha\sqrt{\det\bar{g}}G^{ij} - \frac{1}{2}\frac{\alpha}{\sqrt{\det\bar{g}}} \left(\pi^{mn}\pi_{mn} - \frac{1}{2}\pi^m_m\pi^n_n \right) \bar{g}^{ij} \\ & + 2\frac{\alpha}{\sqrt{\det\bar{g}}} \left(\pi^i_m \pi^{mj} - \frac{1}{2}\pi^m_m \pi^{ij} \right) - \sqrt{\det\bar{g}} \left(\alpha^{|ij} - \bar{g}^{ij}\beta^m_{|m} \right) \\ & - \sqrt{\det\bar{g}} \left((\sqrt{\det\bar{g}})^{-1} \pi^{ij}\beta^m \right)_{|m} + \pi^{mi}\beta^j_{|m} + \pi^{mj}\beta^i_{|m} \,. \end{aligned} \tag{3.129}$$

Note that the first term G^{ij} here is the expression for the spatial components of the Einstein tensor in coordinates, not wrt. the orthonormal frame, but in coordinates! We leave the derivation as an exercise for rainy Easter days...

The variation of the full action therefore gives $Q_{ij} - \dot{\bar{g}}_{ij}$ and $\Pi_{ij} + \dot{\pi}_{ij}$. The **vacuum** Einstein equations therefore are

$$B_i = 0 \,, \qquad C = 0 \qquad \text{and} \tag{3.130}$$

$$\dot{\bar{g}}_{ij} = Q_{ij} \,, \qquad \dot{\pi}_{ij} = -\Pi_{ij} \,. \tag{3.131}$$

The $\dot{\bar{g}}_{ij}$ equation is clearly equivalent to (3.98). A somewhat lengthy but straight forward algebra shows that the $\dot{\pi}_{ij}$ equation is equivalent to $G^{ij} = 0$ in the orthonormal frame. The fastest way to see this is to express G^{ij} in coordinates in terms of $G^{\mu\nu}$ wrt the orthonormal frame which we have determined in the previous section. Considering the cumbersome expression for Π^{ij} it becomes clear that Gaussian normal coordinates where this expression reduces tho the first three terms are very useful.

Exercise: Superspace and the Wheeler-DeWitt equation
We consider Gaussian normal coordinates, $\alpha = \beta^i = 0$ so that the ADM Hamiltonian becomes

$$(16\pi G)H_{ADM} = \int_{\Sigma_t}(K_{\mu\nu}K^{\mu\nu} - (3H)^2 - \bar{R})\sqrt{\bar{g}}d^3x \equiv \int d^3x\mathcal{H}_{ADM}\,. \qquad (3.132)$$

a) Show that the vacuum Einstein equations imply $\mathcal{H}_{ADM} = 0$.

b) Derive the following equivalent expressions for $\mathcal{H}_{ADM}$

$$\mathcal{H}_{ADM} = \frac{1}{\sqrt{\bar{g}}}\left(\pi^{km}\bar{g}_{ki}\bar{g}_{mj}\pi^{ij} - \frac{1}{2}\bar{g}_{ki}\bar{g}_{mj}\pi^{ki}\pi^{mj}\right) - \sqrt{\bar{g}}\bar{R} \qquad (3.133)$$

$$= \frac{1}{2}G_{ijkm}\pi^{ij}\pi^{km} + V(\bar{g}_{ij}) \qquad (3.134)$$

$$= \frac{1}{2}G^{ijkm}\pi_{ij}\pi_{km} + V(\bar{g}_{ij})\,, \qquad (3.135)$$

where $V = -\sqrt{\bar{g}}\bar{R}$ and

$$G_{ijkm} = \frac{1}{\sqrt{\bar{g}}}\left(\bar{g}_{ik}\bar{g}_{jm} + \bar{g}_{im}\bar{g}_{jk} - \bar{g}_{ij}\bar{g}_{km}\right) \qquad (3.136)$$

$$G^{ijkm} = \frac{\sqrt{\bar{g}}}{4}\left(\bar{g}^{ik}\bar{g}^{jm} + \bar{g}^{im}\bar{g}^{jk} - 2\bar{g}^{ij}\bar{g}^{km}\right)\,. \qquad (3.137)$$

Note that in (3.135) the indices in π_{ij} are not lowered with the 3-metric $\bar{g}_{ij}$ but with G_{ijkm} so that now $\pi_{ij} \equiv G_{ijkm}\pi^{km}$.
Show also that $G_{ijkm}G^{lnkm} = (\delta^l_i\delta^n_j + \delta^l_j\delta^n_i)/2$.
The space of 3-metrics $\bar{g}_{ij}(x)$ is called 'superspace' and G_{ijkm} can be considered as a metric on this superspace. It is called the Wheeler-DeWitt metric.

c) Consider the space of diagonal metrics and call $\bar{g}_{AA} = h_A$ so that $G_{ijkm}\pi^{ij}\pi^{km}$ becomes $G_{AB}\pi^A\pi^B$. One can now introduce a line element on this 'superspace' $dL^2 = G_{AB}dh^Adh^B$, where the summation and integrations d^3x are understood. Show that the 3-metrics which satisfy Einstein's vacuum equations satisfy a 'geodesic equation' on mini superspace with an extra force term coming from the potential V. Derive this equation.

d) Quantum cosmological models are usually based on 'mini-superspace' where one additionally assumes that the metric is homogeneous, h_A does not depend on x. The wave function on superspace, $\Psi[\bar{g}]$ satisfies $\mathcal{H}_{ADM}\Psi = 0$. Derive this equation in mini-superspace where it leads to a system of equations for

$h_A(t)$.
The equation

$$\mathcal{H}_{ADM}\Psi[\bar{g}_{ij}] = 0 \tag{3.138}$$

is called the Wheeler-DeWitt equation.

3.5 ADM energy and momentum of isolated systems in asymptotically flat spacetimes

As is well known, energy and momentum of the gravitational field are in general not well defined and the *covariant* 'conservation' equation, $T^{\mu\nu}_{;\mu} = 0$ is not a conservation equation, i.e., in general there is no integral over a spatial hypersurface which is time independent. Nevertheless, it is possible to define (several) so called 'pseudo-tensor(s)' of the gravitational field, $\tau_{\mu\nu}$ which, together with the energy momentum 3-form $*T^\alpha \equiv *(T^\alpha_\mu \theta^\mu)$ satisfy

$$d[\sqrt{-g}(*T^\alpha + *\tau^\alpha)] = 0\,. \tag{3.139}$$

So that for any domain D in spacetime Stokes' theorem implies

$$0 = \int_D d(\sqrt{-g}(*T^\alpha + *t^\alpha)) = \int_{\partial D} \sqrt{-g}(*T^\alpha + *\tau^\alpha)\,. \tag{3.140}$$

In an asymptotically flat spacetime we can choose the domain, bounded by some initial hypersurface Σ_{t_1} and a final hypersurface Σ_{t_2}, very large such that we may neglect T^α and τ^α at the border $\partial\Sigma_t \times \{t\}$. In this case we obtain

$$P^\alpha(t_1) \equiv \int_{\Sigma_{t_1}} \sqrt{-g}(*T^\alpha + *\tau^\alpha) = \int_{\Sigma_{t_2}} \sqrt{-g}(*T^\alpha + *\tau^\alpha) \equiv P^\alpha(t_2)\,. \tag{3.141}$$

Hence in this situation

$$P^\alpha(t) = \int_{\Sigma_t} \sqrt{-g}(*T^\alpha + *\tau^\alpha) \tag{3.142}$$

is a conserved 4-momentum. (If we work wrt an orthonormal frame the pre-factor $\sqrt{-g}$ can be omitted.)

We now show that the Landau-Lifshitz energy momentum pseudo-tensor is such that (3.139) is satisfied , where

$$*\tau^\alpha_{LL} \equiv -\frac{1}{16\pi G}\eta^{\alpha\beta\gamma\delta}\left(\omega_{\sigma\beta}\wedge\omega^\sigma{}_\gamma\wedge\theta_\delta - \omega_{\beta\gamma}\wedge\omega_{\sigma\delta}\wedge\theta^\sigma\right)\,. \tag{3.143}$$

Here $\eta^{\alpha\beta\gamma\delta} = -\epsilon_{\alpha\beta\gamma\delta}/\sqrt{-g}$ is the totally anti-symetric tensor with indices raised.

To see this, we introduce the 1-form $\eta^{\alpha\beta\gamma} \equiv \eta^{\alpha\beta\gamma\delta}\theta_\delta$. It is easy to verify that (exercise!)

$$\eta^{\alpha\beta\gamma} = *(\theta^\alpha \wedge \theta^\beta \wedge \theta^\gamma) \tag{3.144}$$

With this, Einstein's equation can be written in the form (exercise!)

$$-\frac{1}{2}\Omega_{\beta\gamma} \wedge \eta^{\beta\gamma\alpha} = 8\pi G * T^\alpha \,. \tag{3.145}$$

From the second structure equation ($\Omega^\beta{}_\gamma = d\omega^\beta{}_\gamma + \omega^\beta{}_\sigma \wedge \omega^\sigma{}_\gamma$) and $dg_{\mu\nu} = \omega_{\mu\nu} + \omega_{\nu\mu}$ we conclude

$$\Omega_{\beta\gamma} = d\omega_{\beta\gamma} - \omega_{\sigma\beta} \wedge \omega^\sigma{}_\gamma \,, \tag{3.146}$$

hence (3.145) is equivalent to

$$-\frac{1}{2}\eta^{\alpha\beta\gamma\delta}\theta_\delta \wedge (d\omega_{\beta\gamma} - \omega_{\sigma\beta} \wedge \omega^\sigma{}_\gamma) = 8\pi G * T^\alpha \,. \tag{3.147}$$

On the first term we perform an 'integration by parts' writing

$$\theta_\delta \wedge d\omega_{\beta\gamma} = -d(\theta_\delta \wedge \omega_{\beta\gamma}) + d\theta_\delta \wedge \omega_{\beta\gamma} \,.$$

With the first structure equation and the above expression for $dg_{\mu\nu}$ we have

$$d\theta_\delta = d(g_{\delta\mu}\theta^\mu) = dg_{\delta\mu} \wedge \theta^\mu + g_{\delta\mu} d\theta^\mu = \omega_{\mu\delta} \wedge \theta^\mu \,.$$

Inserting this above together with (3.143) yields

$$-\frac{1}{2}\eta^{\alpha\beta\gamma\delta} d\left(\omega_{\beta\gamma} \wedge \theta_\delta\right) = 8\pi G(*T^\alpha + *\tau^\alpha_{LL}) \,. \tag{3.148}$$

We now multiply this expression with $\sqrt{-g}$ using that $\sqrt{-g}\eta^{\alpha\beta\gamma\delta} = \epsilon_{\alpha\beta\gamma\delta}$ is constant we obtain

$$-d\left(\sqrt{-g}\eta^{\alpha\beta\gamma\delta}\omega_{\beta\gamma} \wedge \theta_\delta\right) = -d\left(\sqrt{-g}\omega_{\beta\gamma} \wedge \eta^{\alpha\beta\gamma}\right) = 16\pi G\sqrt{-g}(*T^\alpha + *\tau^\alpha_{LL}) \,, \tag{3.149}$$

which implies the conservation law

$$d\left(\sqrt{-g}(*T^\alpha + *\tau^\alpha_{LL}\right) = 0 \,. \tag{3.150}$$

To define also a conserved angular momentum it is important to notice that in a coordinate basis $\tau^{\alpha\beta}_{LL}$ is symmetric.

Exercise: Show that $\tau^{\alpha\beta}_{LL}$ defined by $\tau^\alpha_{LL} = \tau^{\alpha\beta}_{LL}\theta_\beta$ is symmetric in a coordinate basis. To do so, show first that this is equivalent to

$$*\tau^\alpha_{LL} \wedge dx^\mu = *\tau^\mu_{LL} \wedge dx^\alpha \,.$$

Then show this latter equation (this is a lengthy derivation!).

It is important to note that with the connection forms also $\tau_{LL}^{\alpha\beta}$ does not transform like a tensor. At a given point p we can always choose coordinates such that $\omega_{\alpha\beta}(p) = 0$ which implies that $\tau_{LL}^{\alpha\beta}(p) = 0$ in these coordinates. Nevertheless, its integral over spatial hypersurfaces is meaningful in asymptotically flat spacetimes.

Denoting the total energy momentum pseudo-tensor by

$$\mathcal{T}^{\alpha\beta} = T^{\alpha\beta} + \tau_{LL}^{\alpha\beta} \tag{3.151}$$

we now introduce the pseudo-angular momentum density in a coordinate basis (x^μ)

$$*M^{\alpha\beta} = x^\alpha \ *\mathcal{T}^\beta - x^\beta \ *\mathcal{T}^\alpha \,. \tag{3.152}$$

With $d(\sqrt{-g} \ *\mathcal{T}^\alpha) = 0$ we now have

$$d(\sqrt{-g} * M^{\alpha\beta}) = \sqrt{-g}(dx^\alpha \wedge *\mathcal{T}^\beta - dx^\beta \wedge *\mathcal{T}^\alpha) = 0 \,, \tag{3.153}$$

where we have used the symmetry of $\mathcal{T}^{\alpha\beta}$ for the last equal sign.

The conserved integrals

$$P^\mu = \int_{\Sigma_t} \sqrt{-g} * \mathcal{T}^\mu \qquad \text{and} \tag{3.154}$$

$$J^{\mu\nu} = \int_{\Sigma_t} \sqrt{-g} * M^{\mu\nu} \tag{3.155}$$

are called the ADM energy (mass) P^0, the ADM momentum P^i, and the ADM angular momentum J^{ij} of the isolated system.

Integrating (3.149) over a large spatial domain $D \subset \Sigma_t$ we obtain with Stokes

$$16\pi G \int_D \sqrt{-g} * \mathcal{T}^\alpha = -\int_{\partial D} \sqrt{-g}\omega_{\beta\gamma} \wedge \eta^{\alpha\beta\gamma} \,. \tag{3.156}$$

Extending D to all of space ∂D becomes a sphere at infinity which we denote by $\mathbb{S}_\infty$ and we obtain

$$P^\mu = \frac{-1}{16\pi G} \int_{\mathbb{S}_\infty} \sqrt{-g}\omega_{\beta\gamma} \wedge \eta^{\alpha\beta\gamma} \,. \tag{3.157}$$

Also the angular momentum can be written as a flux integral. Setting

$$h^\alpha \equiv -\sqrt{-g}\omega_{\beta\gamma} \wedge \eta^{\alpha\beta\gamma}$$

we can write, again using (3.149)

$$16\pi G\sqrt{-g} * M^{\mu\nu} = x^\mu dh^\nu - x^\nu dh^\mu \tag{3.158}$$

$$= d\left(x^\mu h^\nu - x^\nu h^\mu\right) - \left(dx^\mu \wedge h^\nu - dx^\nu \wedge h^\mu\right) . \tag{3.159}$$

With a somewhat lengthy calculations (see [16] for details) one finds that also the second term is a differential,

$$dx^\mu \wedge h^\nu - dx^\nu \wedge h^\mu = -d\left(\sqrt{-g}\eta^{\mu\nu}\right) , \tag{3.160}$$

where we have introduced

$$\eta^{\mu\nu} = \eta^{\mu\nu}{}_{\alpha\beta} dx^\alpha \wedge dx^\beta \, . \tag{3.161}$$

If we insert this expression for $*M^{\mu\nu}$ in (3.155) we find

$$J^{\mu\nu} = \frac{1}{16\pi G}\int_{\mathbb{S}_\infty}\left(x^\mu h^\nu - x^\nu h^\mu + \sqrt{-g}\eta^{\mu\nu}\right) . \tag{3.162}$$

Exercise: Compute P^μ and $J^{\mu\nu}$ for the Schwarzschild metric.

To make contact with the literature, e.g. Ref. [9] where $\tau_{LL}^{\alpha\beta}$ is computed explicitly in coordinates one may solve the following exercise.
Exercise: Show that

$$dh^\mu = \frac{1}{\sqrt{-g}} H^{\mu\alpha\nu\beta}_{,\alpha\beta}\eta^\nu \, , \tag{3.163}$$

where

$$\eta^\nu = \eta^\nu{}_{\alpha\beta\gamma}\theta^\alpha \wedge \theta^\beta \wedge \theta^\gamma \qquad \text{and} \tag{3.164}$$

$$H^{\mu\alpha\nu\beta} = \tilde{g}^{\mu\nu}\tilde{g}^{\alpha\beta} - \tilde{g}^{\alpha\nu}\tilde{g}^{\mu\beta} \qquad \tilde{g}^{\mu\nu} \equiv \sqrt{-g} g^{\mu\nu} \, . \tag{3.165}$$

The tensor $H^{\mu\alpha\nu\beta}$ is sometimes called the Landau-Lifshitz super potential. This result implies

$$-gG^{\mu\nu} = \frac{1}{2} H^{\mu\alpha\nu\beta}_{,\alpha\beta} + 8\pi G g \tau_{LL}^{\mu\nu} \, . \tag{3.166}$$

(The solution to this rather lengthy exercise can be found in [16].)

Finally we want to derive asymptotic expressions for P^μ for a mtetric which has the following decay properties: We request that outside a compact region, the spatial metric can be written as

$$g_{ij} = \delta_{ij} + h_{ij} \tag{3.167}$$

in standard $\mathbb{R}^3$ cartesian coordinates where

$$h_{ij} \sim \mathcal{O}(1/r), \qquad h_{ij,m} \sim \mathcal{O}(1/r^2) \qquad h_{ij,mn} \sim \mathcal{O}(1/r^3)$$

and the extrinsic curvature decays like

$$K_{ij} \sim \mathcal{O}(1/r^2) \qquad K_{ij,m} \sim \mathcal{O}(1/r^3) \, .$$

To lowest order in $1/r$ we then have

$$\begin{aligned} \omega_{\alpha\beta} &= g_{\alpha\gamma}\Gamma^{\gamma}_{\beta\nu}dx^{\nu} = \frac{1}{2}\left(g_{\beta\alpha,\nu} + g_{\nu\alpha,\beta} - g_{\nu\beta,\alpha}\right)dx^{\nu} + \cdots && (3.168) \\ \eta^{\mu}_{\alpha\beta} &= \epsilon^{\mu}_{\alpha\beta\gamma}dx^{\gamma} + \cdots && (3.169) \end{aligned}$$

so that

$$\begin{aligned} P^0 &= \frac{1}{16\pi G}\epsilon^{0ij}{}_k \int_{\mathbb{S}_\infty} g_{jm,i}dx^m \wedge dx^k && (3.170) \\ &= \frac{-1}{16\pi G}\epsilon_{ijk} \int_{\mathbb{S}_\infty} g_{jm,i}dx^m \wedge dx^k && (3.171) \\ &= \frac{1}{16\pi G}\int_{\mathbb{S}_\infty} (g_{ml,m} - g_{mm,l})N^l ds\,. && (3.172) \end{aligned}$$

For the last equality we have used $dx^m \wedge dx^k = \epsilon^{lmk}N_l ds$ where we have introduced the unit normal to the sphere N^l and the surface element of the sphere, $ds = r^2 d\Omega$. For the second equality we have used $1 = \epsilon_{0123} = -\epsilon^{012}{}_3 = \epsilon_{123}$. As $g_{ij,k} \sim 1/r^2$ and $ds \propto r^2$, this integral is independent of r and the limit $r \to \infty$ can be taken. Furthermore, all higher order terms in $1/r$ do not contribute in this limit. Eq. (3.172) is our final expression for the ADM mass of an isolated system.

Similarly one derives for the ADM momentum, using $\omega_{0j}|_{\Sigma_t} = K_{ji}dx^i$

$$\begin{aligned} P^n &= \frac{-1}{16\pi G}\epsilon^{n\alpha\beta}{}_k \int_{\mathbb{S}_\infty} \omega_{\alpha\beta} \wedge dx^k && (3.173) \\ &= \frac{-2}{16\pi G}\epsilon^{n0j}{}_k \int_{\mathbb{S}_\infty} \omega_{0j} \wedge dx^k && (3.174) \\ &= \frac{-1}{8\pi G}\epsilon_{njk} \int_{\mathbb{S}_\infty} K_{ji}dx^i \wedge dx^k && (3.175) \\ &= \frac{1}{8\pi G}\int_{\mathbb{S}_\infty} (K_{nj} - \delta_{nj}K^i_i)N^j ds\,. && (3.176) \end{aligned}$$

Eq. (3.176) is our final expression for the ADM momentum of an isolated system. The integrals are to be understood as limits $r \to \infty$ of integrals over a large sphere of radius r.

Exercise: Derive a corresponding expression for the ADM angular momentum for a stationary spacetime. In the stationary case one can require that the metric asymptotically takes the form

$$g_{00} = -\left(1 - \frac{2m}{r} + \frac{2m^2}{r^2}\right) + \mathcal{O}(r^{-3}) \qquad (3.177)$$

$$g_{0i} = -2\epsilon_{ijk}\frac{S^j x^k}{r^3} + \mathcal{O}(r^{-3}) \qquad (3.178)$$

$$g_{ij} = -\left(1+\frac{2m}{r}\right)\delta_{ij} + \mathcal{O}(r^{-2})\,. \qquad (3.179)$$

3.6 Static and stationary spacetimes

We have seen that a stationary spacetime, i.e., a metric with a timelike Killing field b, $L_b g = 0$, is one where we can introduce coordinates such that the metric coefficients do not depend on time, $\partial_t g_{\mu\nu} = 0$. If the metric is static, i.e., $b^\flat \wedge db^\flat = 0$, we can even find coordiates such that the shift vector vanishes and g takes the form

$$\mathrm{d}s^2 = g_{00}(\mathbf{x})dt^2 + g_{ij}(\mathbf{x})dx^i dx^j = -\alpha^2(\mathbf{x})dt^2 + \bar{g}(\mathbf{x})\,. \qquad (3.180)$$

On the hypersurface Σ we can then introduce an orthonormal basis $\bar{e}_i$ of $\bar{g}$, which together with $e_0 \equiv n = \alpha^{-1}\partial_t = (\sqrt{-g_{00}(\mathbf{x})})^{-1}\partial_t$ form an orthonormal basis of spacetime with shift vector $\vec{\beta} = 0$. We denote the dual 1-forms by $\theta^0 = \alpha(\mathbf{x})dt$ and $\theta^i = \vartheta^i$. The first structure equation gives

$$d\theta^0 = -\omega^0{}_i \wedge \theta^i = \alpha_{,i}\theta^0 \wedge \theta^i \qquad (3.181)$$

$$d\theta^i = -\omega^i{}_j \wedge \theta^i - \omega^i{}_0 \wedge \theta^0 = -\omega^i{}_j \wedge \theta^i\,. \qquad (3.182)$$

Here $\alpha_{,i} \equiv \bar{e}_i(\alpha)$ which in general is not just a partial derivative wrt to some coordinate x^i, since the $\bar{e}_i$ are orthonormal vector fields. The above eqns. imply $\omega^0{}_i = -\alpha_{,i}\theta^0$ and $\omega^0{}_i$ vanishes on $T\Sigma$ hence

$$K \equiv 0 \qquad \text{for a static spacetime.} \qquad (3.183)$$

For the Einstein tensor we conclude $G_{0i} = 0$ and

$$G_{00} = \frac{1}{2}\bar{R} \qquad \text{for a static spacetime.} \qquad (3.184)$$

3.6.1 The Komar formula

For stationary asymptotically flat spacetimes there is the following interesting formula to obtain the total mass of the system,

$$M = -\frac{1}{8\pi G}\int_{\mathbb{S}_\infty} *db^\flat\,. \qquad (3.185)$$

Let us prove this formula using Eqs. (3.177-3.179) for the asymptotic form of the metric. At large r,

$$\begin{aligned} b^\flat &= (\partial_t)^\flat = g_{0\mu}dx^\mu = -\left(1-\frac{2m}{r}\right)dt + \mathcal{O}(1/r^2) \\ db^\flat &= -\frac{2m}{r^3}x^i dx^i \wedge dt + \mathcal{O}(1/r^3) \\ *\,db^\flat &= -\frac{2m}{r^3}x^i \epsilon_{ijk} dx^j \wedge dx^k + \mathcal{O}(1/r^3)\,. \end{aligned}$$

Inserting this above we find

$$-\frac{1}{8\pi G}\int_{\mathbb{S}_\infty} *db^\flat = \frac{2m}{8\pi G}\int_{\mathbb{S}_\infty} r^{-3}x^i\epsilon_{ijk}dx^j\wedge dx^k = \frac{m}{4\pi G}\int_{\mathbb{S}_\infty} r^{-3}x^i N^i r^2 d\Omega = \frac{m}{G} = M\,. \tag{3.186}$$

For stationary, asymptotically flat metrics therefore the ADM mass and the Komar mass agree. If spacetime is not stationary, the Komar mass is not well defined (it is in general not a conserved quantity) but the ADM mass is.

Chapter 4

Black holes

4.1 Axi-symmetric, stationary spacetimes

A spacetime $(\mathcal{M}, g)$ is called axisymmetric if it admits the group $SO(2)$ as an isometry group with closed spacelike orbits. In the following we shall also request that spacetime be asymptotically flat. A spacetime $(\mathcal{M}, g)$ is axisymmetric and stationary if the group $\mathbb{R} \times SO(2)$ acts isometrically and the Killing field belonging to time translations, $\mathbb{R}$, is at least asymptotically timelike. We denote the two Killing fields belonging to the $\mathbb{R}$ and $SO(2)$ symmetry by b and m respectively. As the group action on $\mathcal{M}$ is commutative, also the generators commute [1],

$$[b, m] = 0\,. \tag{4.1}$$

The orbits of the 2-dimensional symmetry group $\mathbb{R} \times SO(2)$ form 2-dimensional sub-manifolds, $\Sigma \subset \mathcal{M}$ whose tangent space is spanned by b and m. The collection of these tangent spaces therefore forms an involutive (integrable) 2-dimensional distribution E. We also consider the orthogonal 2-dimensional distribution, $E^\perp$ such that $T_p\mathcal{M} = E_p \oplus E_p^\perp$, $p \in \mathcal{M}$. For vectors $X \in E^\perp$ we have $b^\flat(X) = \langle b, X\rangle = 0$ and equivalently $m^\flat(X) = 0$. Hence $b^\flat$ and $m^\flat$ generate the ideal $I(E^\perp)$. Frobenius' theorem implies that also $E^\perp$ is involutive if and only if this ideal is differential. But this is equivalent to the Frobenius conditions

$$b^\flat \wedge m^\flat \wedge db^\flat = 0 = b^\flat \wedge m^\flat \wedge dm^\flat\,. \tag{4.2}$$

Below we shall show that Einstein's vacuum equations imply (4.2). If this condition is satisfied we call $(\mathcal{M}, g)$ circular. As we have shown in Section 1.7 in this case we can find coordinates adapted to E and $E^\perp$ such that

$$b = \partial_t\,, \qquad m = \partial_\phi \qquad (t = x^0, \varphi = x^1) \tag{4.3}$$

[1] Show as an exercise that two vector fields X and Y commute if and only if their respective flows Φ_s^X and Φ_t^Y commute, i.e. $\Phi_s^X \circ \Phi_t^Y(p) = \Phi_t^Y \circ \Phi_s^X(p)$

$$ \begin{aligned} {}^{(4)}g &= \sigma_{ab}(x^i)dx^a dx^b + g_{ij}(x^k)dx^i dx^j \quad \text{where} \quad a,b \in \{0,1\} \\ \text{and} \quad & i,j,k \in \{2,3\}\,. \end{aligned} \tag{4.4} $$

In summary, a stationary and axisymmetric asymptotically flat vacuum spacetime $(\mathcal{M}, {}^{(4)}g)$ is circular and can be described locally as

$$ \mathcal{M} = \Sigma \times \Gamma\,, \qquad {}^{(4)}g = \sigma + g\,. \tag{4.5} $$

Here Σ is diffeomorphic to $\mathbb{R} \times SO(2)$ and the metric coefficients in the adapted coordinates ($x^0 = t,\ x^1 = \varphi$) depend only on the coordinates of Γ. (Σ, σ) is a 2-dimensional Lorentz manifold while (Γ, g) is a two dimensional Riemannian manifold orthogonal to Σ. The two Killing fields b and m are tangent to Σ and orthogonal to Γ. We shall choose the indices a, b, c to denote coordinates on Σ and i, j, k to denote coordinates on Γ.

To show (4.2) we introduce the 1-form $R(m) = R_{\mu\nu}m^\mu dx^\nu$ which is called the Ricci form of m. We shall now show that (4.2) is equivalent to the 'Ricci cirularity condition',

$$ b^\flat \wedge m^\flat \wedge R(b) = 0 = b^\flat \wedge m^\flat \wedge R(m)\,. \tag{4.6} $$

In vacuum these Ricci forms of course vanish which then proves (4.2).

To show this equivalence we first introduce the 'twist 1-forms' belonging to b and m,

$$ \omega_b = \frac{1}{2} * (b^\flat \wedge db^\flat) \qquad \omega_m = \frac{1}{2} * (m^\flat \wedge dm^\flat)\,. \tag{4.7} $$

The first 4-form in (4.2) is proportional to $m^\flat \wedge *\omega_b = \langle m, \omega_b^\sharp \rangle \eta$ where η is the volume form on $\mathcal{M}, g$ (see exercises of Chapter 1). Here we have introduced the scalar product $\langle \cdot, \cdot \rangle$ for arbitrary tensor fields of equal rank. For two tensor fields, U and V of rank s we simply set

$$ \langle U, V \rangle = U_{i_1 \cdots i_s} V^{i_1, \cdots, i_s} = U^{i_1 \cdots i_s} V_{i_1, \cdots, i_s}\,. \tag{4.8} $$

Hence the Frobenius conditions are equivalent to

$$ \langle m, \omega_b^\sharp \rangle = \langle b, \omega_m^\sharp \rangle = 0\,. \tag{4.9} $$

But since m is a Killing field commuting with b, $L_m g = 0$, $L_m b = 0$ and therefore $L_m \omega_b = 0$ (Remember that L commutes with d hence from $L_m b = 0$ we follow that $L_m db = dL_m b = 0$.). With the Cartan identity, $L_m = d \circ i_m + i_m \circ d$ we obtain

$$ d\langle m, \omega_b^\sharp \rangle = d i_m \omega_b = -i_m d\omega_b\,. $$

Below we shall show that

$$ d\omega_b = *(b^\flat \wedge R(b)) \tag{4.10} $$

so that this gives

$$d\langle m, \omega_b^\sharp \rangle = -i_m * (b^\flat \wedge R(b)) \tag{4.11}$$

and equivalently with m and b interchanged,

$$d\langle b, \omega_m^\sharp \rangle = -i_b * (m^\flat \wedge R(m))\,. \tag{4.12}$$

This shows that the Frobenius condition implies the Ricci circularity condition since in complete generality for a $p-$form α and a vector field X we have (**Exercise**)

$$i_X * \alpha = * \left(\alpha \wedge X^\flat\right)\ . \tag{4.13}$$

Conversely, the Ricci circularity condition implies $d\langle m, \omega_b^\sharp\rangle = d\langle b, \omega_m^\sharp\rangle = 0$, hence these scalar products are constant. But as our spacetime is asymptotically flat there are fixpoints under $SO(2)$ as there are in flat space. Hence there are points where $m = 0$ and consequently the above scalar products must vanish, hence (4.2) is valid if (4.10) holds.

4.1.1 Derivation of Eq. (4.10)

We first show that for an arbitrary Killing field k

$$d * dk^\flat = 2 * R(k)\,. \tag{4.14}$$

To see this we use the Ricci identity,

$$k_{\sigma;\rho\mu} - k_{\sigma;\mu\rho} = R^\lambda{}_{\sigma\rho\mu} k_\lambda \tag{4.15}$$

The Killing equation, $k_{\sigma;\rho} + k_{\rho;\sigma} = 0$ implies $k^\rho_{;\rho} = 0$. Contracting σ and μ above therefore yields

$$k_{\sigma;\rho}{}^{;\sigma} = -k_{\rho;\sigma}{}^{;\sigma} = R_{\lambda\rho}k^\lambda = R(k)_\rho\,. \tag{4.16}$$

But it is easy to see that for an aribtrary p-form α in coordinates $(\delta\alpha)_{k_1\cdots k_{p-1}} = \alpha_{jk_1\cdots k_{p-1}}{}^{;j}$. Applying this to the 2-form $dk^\flat = (k_{\rho;\sigma} - k_{\sigma;\rho})dx^\sigma \wedge dx^\rho)$ we obtain

$$\delta dk^\flat = -(*d*)dk^\flat = -2R(k)\,. \tag{4.17}$$

We used that in 4 dimensions with $\mathrm{sgn}(g) = -1$ we have $\delta = -(*d*)$. Taking the $*$ on both sides and using $** = \mathrm{id}$ for 3-forms we obtain (4.14).

We now show also the following identity for a Killing field k and a p-form α:

$$\delta(k^\flat \wedge \alpha) = -k^\flat \wedge \delta\alpha + L_k\alpha\,. \tag{4.18}$$

For this we use

$$\begin{aligned}\delta(k^\flat \wedge \alpha) &= (-1)^p (*)^{-1} d * (k^\flat \wedge \alpha) = (*)^{-1} d * (\alpha \wedge k^\flat) \\ &= (*)^{-1} d(i_k * \alpha) = (*)^{-1} (L_k - i_k d) * \alpha \\ &= L_k \alpha - (*)^{-1} i_k d * \alpha \,.\end{aligned}$$

Exercise: Show that for an arbitrary vector field X and p-form α we have $*(\alpha \wedge X^\flat) = i_X * \alpha$.

For the last term above we use

$$i_k d * \alpha = i_k (*)^{-1} * d * \alpha = (-1)^{p+1} i_k * \delta\alpha = *(k^\flat \wedge \delta\alpha)\,.$$

Inserting this above we have (4.18).

With this we can now show (4.10) for $\omega_k = \frac{1}{2} * (k^\flat \wedge dk^\flat)$. We write

$$d\omega_k = \frac{1}{2} d * (k^\flat \wedge dk^\flat) = \frac{1}{2} * \delta(k^\flat \wedge dk^\flat) \tag{4.19}$$

and apply (4.18) on $\alpha = dk^\flat$. Since $L_k dk^\flat = dL_k k^\flat = [k,k]^\flat = 0$ this yields

$$d\omega_k = -\frac{1}{2} * (k^\flat \wedge \delta dk^\flat) = *(k^\flat \wedge R(k))\,. \tag{4.20}$$

4.2 Elements of the derivation of the Kerr solution

We now go on to compute the Riemann tensor from the metric

$${}^{(4)}g = \sigma_{ab}(x^i) dx^a dx^b + g_{ij}(x^k) dx^i dx^j = \sigma_{ab}\theta^a \otimes \theta^b + g_{ij}\theta^i \otimes \theta^j \tag{4.21}$$

where $\theta^0 = dt$ and $\theta^1 = d\phi$ and $\theta^i = dx^i$. We also introduce

$$V = -\langle b, b\rangle = -\sigma_{tt}\,, \quad W = \langle b, m\rangle = \sigma_{t\varphi}\,, \quad X = \langle m, m\rangle = \sigma_{\varphi\varphi} \ \text{ and } \tag{4.22}$$

$$A = \frac{W}{X}\,, \quad S = \sqrt{-\sigma} = \sqrt{VX + W^2}\,. \tag{4.23}$$

We need the connection forms relative to the coordinate basis $dx^\mu = \theta^\mu$. The first structure equations together with $\omega_{\mu\nu} + \omega_{\nu\mu} = dg_{\mu\nu}$ yield

$$\omega_{ai} + \omega_{ia} = 0 \tag{4.24}$$

$$\omega_{ab} + \omega_{ba} = d\sigma_{ab} = \sigma_{ab,i}\theta^i \tag{4.25}$$

$$\omega_{ij} + \omega_{ji} = dg_{ij} \tag{4.26}$$

$$\omega^i{}_j \wedge \theta^j + \omega^i{}_a \wedge \theta^a = 0 \tag{4.27}$$

$$\omega^a{}_b \wedge \theta^b + \omega^a{}_i \wedge \theta^i = 0 \tag{4.28}$$

It is easy to check that the following Ansatz satisfies the equations:

$$\begin{aligned} \omega^i{}_j &= \text{connection forms of } (\Gamma, g) && (4.29)\\ \omega_{ab} &= \frac{1}{2} d\sigma_{ab} && (4.30)\\ \omega_{ia} &= -\omega_{ai} = -\frac{1}{2}\sigma_{ab,i}\theta^b\,. && (4.31) \end{aligned}$$

It is now straight forward to compute the curvature forms. We concentrate on the terms which enter the Ricci tensor,

$$^{(4)}R_{\mu\nu} = \Omega^\alpha{}_\mu(e_\alpha, e_\nu) \qquad e_\nu \equiv \partial_\nu\,. \tag{4.32}$$

To determine $^{(4)}R_{ij} = \Omega^a{}_i(e_a, e_j) + \Omega^l{}_i(e_l, e_j)$ we use the second structure equation,

$$\begin{aligned} \Omega^i{}_j &= d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j + \omega^i{}_a \wedge \omega^a{}_j && (4.33)\\ &= {}^{(g)}\Omega^i{}_j + \propto \theta^a \wedge \theta^b && (4.34) \end{aligned}$$

From this it follows that

$$^{(4)}R^k{}_{imj} = {}^{(g)}R^k{}_{imj} \quad \text{and} \quad {}^{(4)}R^k{}_{ikj} = {}^{(g)}R^k{}_{ikj} \tag{4.35}$$

For $^{(4)}R^a{}_{iaj}$ we use

$$\begin{aligned} \Omega^a{}_j &= d\omega^a{}_j + \omega^a{}_k \wedge \omega^k{}_j + \omega^a{}_b \wedge \omega^b{}_j && (4.36)\\ &= d\left(\frac{1}{2}\sigma^{ac}\sigma_{cb,j}\theta^b\right) + \frac{1}{2}\sigma^{ac}\sigma_{cb,k}\theta^b \wedge \omega^k{}_j + \omega^a{}_b \wedge \omega^b{}_j && (4.37)\\ &= \left(\frac{1}{2}\sigma^{ac}\sigma_{cb,j}\right)_{,i}\theta^i \wedge \theta^b + \frac{1}{2}\sigma^{ac}\sigma_{cb,k}\theta^b \wedge \omega^k{}_j + \omega^a{}_b \wedge \omega^b{}_j && (4.38)\\ &= {}^{(g)}\nabla_i\left(\frac{1}{2}\sigma^{ac}\sigma_{cb,j}\right)\theta^i \wedge \theta^b + \omega^a{}_b \wedge \omega^b{}_j\,. && (4.39) \end{aligned}$$

Denoting the covariant derivative on (Γ, g) by a stroke, $_|$, we find with this

$$\begin{aligned} \Omega^a{}_j(e_a, e_i) &= -\frac{1}{2}\left(\sigma^{ac}\sigma_{ac,j}\right)_{|i} - \omega^a{}_b(e_i) \wedge \omega^b{}_j(e_a) && (4.40)\\ &= -\left(\frac{S_{,j}}{S}\right)_{|i} - \frac{1}{4}\sigma^{ac}\sigma_{cb,i}\sigma^{bd}\sigma_{da,j} && (4.41)\\ &= -\left(\frac{S_{,j}}{S}\right)_{|i} + \frac{1}{4}\sigma^{ad}_{,i}\sigma_{da,j} && (4.42)\\ &= -\frac{S_{|ij}}{S} + \frac{S_{,j}S_{,i}}{S^2} + \frac{1}{4}\sigma^{ab}_{,i}\sigma_{ab,j}\,. && (4.43) \end{aligned}$$

Adding the two contributions we find

$$ {}^{(4)}R_{ij} = {}^{(g)} R_{ij} - \frac{S_{|ij}}{S} + \frac{S_{,j}S_{,i}}{S^2} + \frac{1}{4}\sigma^{ab}_{,i}\sigma_{ab,j}\,. \tag{4.44} $$

This can be expressed in terms of the functions S, X, W and V. A short calculation gives

$$ {}^{(4)}R_{ij} = {}^{(g)} R_{ij} - \frac{S_{|ij}}{S} + \frac{1}{4S^2}\left[V_{,i}X_{,j} + X_{,i}V_{,j} + 2W_{,i}W_{,j}\right]. \tag{4.45} $$

Let us go on to determine $R_{ab} = \Omega^c{}_a(e_c, e_b) + \Omega^i{}_a(e_i, e_b)$. Using the second structure equation we find

$$ \Omega_{ai} = \frac{1}{2}\sigma_{ab|ij}\theta^j \wedge \theta^b - \frac{1}{4}\sigma_{ac,j}\sigma^{cd}\sigma_{bd,i}\theta^j \wedge \theta^b\,. \tag{4.46} $$

Hence

$$ {}^{(4)}R_{aibj} = -\frac{1}{2}\sigma_{ab|ij} + \frac{1}{4}\sigma_{ac,j}\sigma^{cd}\sigma_{bd,i} \tag{4.47} $$

$$ {}^{(4)}R^i{}_{aib} = -\frac{1}{2}\sigma_{ab|}{}^{,i}{}_{i} + \frac{1}{4}\sigma_{ac,i}\sigma^{cd}\sigma_{bd}{}^{,i}\,. \tag{4.48} $$

We still need

$$ \Omega^c{}_a(e_c, e_b) = d\omega^c{}_a(e_c, e_b) + (\omega^c{}_i \wedge \omega^i{}_a)(e_c, e_a) \tag{4.49} $$

$$ = (\omega^c{}_i \wedge \omega^i{}_a)(e_c, e_a) \tag{4.50} $$

$$ = \frac{1}{4}\sigma^{cd}\left(\sigma_{db,i}\sigma^{,i}_{ac} - \sigma_{cd,i}\sigma^{,i}_{ab}\right) \tag{4.51} $$

The second term equals $-S_{,i}\sigma^{,i}_{ab}/2S$. And using $(\sigma^{cd}\sigma_{db})_{,i} = 0$ the first term becomes

$$ -\frac{1}{4}\sigma^{cd}_{,i}\sigma_{bd}\sigma_{ac}{}^{,i} = -\frac{1}{4}\sigma_{ad}\sigma^{cd}_{,i}\sigma_{bc}{}^{,i} $$

so that

$$ {}^{(4)}R^c{}_{acb} = -\frac{S_{,i}}{2S}\sigma_{ab}{}'^{i} - \frac{1}{4}\sigma_{ad}\sigma^{cd}_{,i}\sigma_{bc}{}^{,i}\,. \tag{4.52} $$

Adding (4.52) and (4.48) gives

$$ {}^{(4)}R_{ab} = -\frac{1}{2S}\sigma_{ad}\left(S\sigma^{cd}\sigma_{cb}{}^{,i}\right)_{|i}\,. \tag{4.53} $$

In particular the partial trace R^a_a is

$$ {}^{(4)}R^a_a = -\frac{1}{2S}\left(S\sigma^{ca}\sigma_{ca,i}\right)^{|i} = -\frac{1}{S}\left(S\frac{1}{S}S_{,i}\right)^{|i} = -\frac{1}{S}\,{}^{(g)}\Delta S\,. \tag{4.54} $$

Hence the vacuum equations imply that the function $S = \sqrt{-\det\sigma}$ is harmonic.

A somewhat lengthy exercise shows that the mixed components vanish, ${}^{(4)}R_{ai} = 0$.

It is now quite easy to check that the following 'Ansatz' satisfies the vacuum field equations, ${}^{(4)}R_{\mu\nu} = 0$:

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \frac{4Mar\sin^2\vartheta}{\rho^2}dtd\varphi + \left(r^2 + a^2 + \frac{2Ma^2r\sin^2\vartheta}{\rho^2}\right)\sin^2\vartheta d\varphi^2$$
$$+\frac{\rho^2}{\Delta}dr^2 + \rho^2 d\vartheta^2 \,, \qquad (4.55)$$

where

$$\rho^2(r,\vartheta) \equiv r^2 + a^2\cos^2\vartheta \,, \quad \Delta(r) \equiv r^2 - 2Mr + a^2 \,. \qquad (4.56)$$

The first line of (4.55) is the metric $\sigma_{ab}dx^a dx^b$ while the second line is $g_{ij}dx^i dx^j$. Eq. (4.55) gives the famous Kerr metric in so called Boyer-Lindquist coordinates.

Exercise:
Use the expressions (4.45) and (4.53) to compute the Ricci tensor for the Kerr solution (4.55) and show that it vanishes. (This requires still a considerable amount of algebra. It might be useful to do it with a GR package of Mathematica.)

4.3 Some properties of the Kerr solution

A direct calculation of the Kretschman scalar $\mathcal{K} \equiv R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ shows that $\mathcal{K}$ diverges when

$$\rho^2 = r^2 + a^2\cos^2\vartheta \to 0 \,. \qquad (4.57)$$

Hence $(r = 0, \vartheta = \pi/2)$ is a true singularity of the Kerr spacetime.

The two parameters M and a represent the mass and the angular momentum. It is easy to see that for $a = 0$, hence $\rho = r$ this reduces to the Schwarzschild metric. In the limit $M \to 0$ the metric reduces to the form

$$ds^2 = -dt^2 + \frac{\rho^2}{r^2 + a^2}dr^2 + \rho^2 d\vartheta^2 + (r^2 + a^2)\sin^2\vartheta d\varphi^2 \,. \qquad (4.58)$$

This is nothing but flat space expressed in spheroidal coordinates. Performing the coordinate transformation

$$x = \sqrt{r^2 + a^2}\sin\vartheta\cos\varphi \,, \quad y = \sqrt{r^2 + a^2}\sin\vartheta\sin\varphi \,, \quad z = r\cos\vartheta \,, \qquad (4.59)$$

reduces (4.58) to the Minkowski metric. This actually indicates that in the $\vartheta = \pi/2$ plane $r = 0$ corresponds to a ring of radius a and hence r is not the radial coordinate we are used to, which rather corresponds to $R^2 = r^2 + a^2 \sin^2\vartheta$. This also indicates that the singularity $(r = 0, \vartheta = \pi/2)$ is actually a ring (not a point) in space.

To interpret the parameter a it is useful to write the Kerr metric in the following equivalent form:

$$\begin{aligned} ds^2 &= -(1-\frac{2Mr}{\rho^2})dt^2 - \frac{4Mar\sin^2\vartheta}{\rho^2}dtd\varphi + \frac{\Sigma^2\sin^2\vartheta}{\rho^2}d\varphi^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\vartheta^2 && (4.60) \\ &= -\frac{\rho^2\Delta}{\Sigma^2}dt^2 + \frac{\Sigma^2\sin^2\vartheta}{\rho^2}\left(d\varphi - \frac{2Mra}{\Sigma^2}dt\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\vartheta^2\,, && (4.61) \end{aligned}$$

where

$$\Sigma^2 = (r^2+a^2)^2 - a^2\Delta\sin^2\vartheta\,. \tag{4.62}$$

The form (4.61) represents the metric in a form that suggests a rotating object. In the limit $r \to \infty$ one has $\Sigma^2 \simeq r^4$, $\rho^2 \sim \Delta \sim r^2$ and the metric reduces to the weak field at large distance from a rotating body (see eq. (5.213) of [16]), which is given by

$$\begin{aligned} g_{00} &= -\left(1 - \frac{2M}{r} + 2\frac{M^2}{r^2}\right) + \mathcal{O}(r^{-3})\,, && (4.63) \\ g_{i0} &= 2\epsilon_{ijk}\frac{J^k x^j}{r^3} + \mathcal{O}(r^{-3})\,, && (4.64) \\ g_{00} &= \left(1 + \frac{2M}{r}\right) + \mathcal{O}(r^{-2})\,, && (4.65) \end{aligned}$$

where M/G is the mass and $\vec{J}/G$ is the angular momentum of the body. Comparison of these two expressions, using $\vec{e}_r \times \vec{e}_z = -\sin\vartheta\vec{e}_\varphi$ and $(\vec{e}_\varphi)_i dx^i = r\sin\vartheta d\varphi$, gives

$$g_{0i}dx^i dt = \frac{2J}{r}\sin^2\vartheta d\varphi dt \qquad \text{for } \vec{J} = J\vec{e}_z\,, \qquad \text{hence} \tag{4.66}$$

$$\vec{J} = Ma\vec{e}_z\,. \tag{4.67}$$

From the Schwarzschild case we know that M is also the Komar mass,

$$M/G = -\frac{1}{8\pi G}\int_{\mathbb{S}^\infty} *db^\flat\,. \tag{4.68}$$

The angular momentum $J = aM$ can also be obtained as a Komar integral,

$$J/G = \frac{1}{16\pi G}\int_{\mathbb{S}^\infty} *dm^\flat\,. \tag{4.69}$$

To see this we expand

$$\begin{aligned} m^\flat &= g_{\varphi\mu}dx^\mu \simeq -\frac{2aM}{r}\sin^2\vartheta dt + r^2\sin^2\vartheta d\varphi && (4.70) \\ dm^\flat &\simeq \frac{2aM}{r^2}\sin^2\vartheta dr\wedge dt + 2r\sin^2\vartheta dr\wedge d\varphi + \cdots && (4.71) \end{aligned}$$

The terms not written out will not contribute to the hodge dual $*dm^\flat$ in the integral over $\mathbb{S}$. We work in the (up to order $1/r^2$) orthonormal basis

$$\theta^0 = \left(1 - \frac{2M}{r}\right) dt, \quad \theta^1 = dr, \quad \theta^2 = r d\vartheta \tag{4.72}$$

$$\theta^3 = r \sin\vartheta \left(d\varphi - \frac{2aM}{r^3} dt\right) . \tag{4.73}$$

With this

$$dm^\flat \simeq \frac{2aM}{r^2} \sin^2\vartheta \theta^1 \wedge \theta^0 + 4\frac{aM}{r^2} \sin^2\vartheta \theta^1 \wedge \theta^0 + \cdots \tag{4.74}$$

$$= \frac{6aM}{r^2} \sin^2\vartheta \theta^1 \wedge \theta^0 + \cdots \tag{4.75}$$

$$* \, dm^\flat = \frac{6aM}{r^2} \sin^2\vartheta \theta^2 \wedge \theta^3 + \cdots \tag{4.76}$$

In the integral (4.69) this yields

$$J/G = \frac{6aM}{16\pi G} \int_{\mathbb{S}\infty} \sin^3\vartheta d\vartheta \wedge d\varphi = aM/G \, . \tag{4.77}$$

4.4 Properties of the Kerr-Newman family of solutions

In this section we slightly generalize our solution to allow also for charged back holes. This yields the so called Kerr Newman family of black hole solutions which are given in terms of 3 parametes M, a and Q. Setting now

$$\Delta = r^2 - 2Mr + a^2 + Q^2 \tag{4.78}$$

and as before

$$\rho^2 = r^2 + a^2 \cos^2\vartheta \, , \qquad \Sigma^2 = (r^2 + a^2)^2 - a^2 \Delta \sin^2\vartheta \, , \tag{4.79}$$

the metric is given by

$$g_{rr} = \frac{\rho^2}{\Delta} \, , \quad g_{\vartheta\vartheta} = \rho^2 \, , \quad g_{\varphi\varphi} = \frac{\Sigma^2}{\rho^2} \sin^2\vartheta \, , \tag{4.80}$$

$$g_{tt} = -1 + \frac{2Mr - Q^2}{\rho^2} \, , \quad g_{t\varphi} = -a\frac{2Mr - Q^2}{\rho^2} \sin^2\vartheta \, . \tag{4.81}$$

We orient the angular momentum such that $a \geq 0$ and also request $M \geq 0$ and $Q \geq 0$. The family contains the following special cases

- $Q = a = 0$: Schwarzschild solution,
- $a = 0$: Reissner-Nordstrøm solution,
- $Q = 0$: Kerr solution.

We can write the metric in standard $3+1$ form

$$ds^2 = -\alpha^2 dt^2 + g_{\varphi\varphi}(d\varphi + \beta^\varphi dt)^2 + g_{rr}dr^2 + g_{\vartheta\vartheta}d\vartheta^2\,, \tag{4.82}$$

where only the φ-component of the shift vector is non zero. A brief calculation yields

$$\alpha^2 = \frac{-1}{g_{\varphi\varphi}}(g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2) = \frac{\rho^2}{\Sigma^2}\Delta\,, \tag{4.83}$$

$$\beta^\varphi = \frac{g_{t\varphi}}{g_{\varphi\varphi}} = -a\frac{2Mr - Q^2}{\Sigma^2} \tag{4.84}$$

The electromagnetic field of the Kerr-Newman solution is

$$F = \frac{Q}{\rho^4}\Big[(r^2 - a^2\cos^2\vartheta)dr \wedge (dt - a\sin^2\vartheta d\varphi) + 2ar\cos\vartheta\sin\vartheta d\vartheta \wedge ((r^2+a^2)d\varphi - adt)\Big]\,. \tag{4.85}$$

Exercise: Verify that the homogeneous Maxwell equations are satisfied, $dF = 0$.

Denoting the current by the 1-form $j = j_\mu dx^\mu$, the inhomogeneous Maxwell equation give

$$\delta F = 4\pi j\,, \qquad 0 = \delta\delta F = *d*j\,, \text{ hence } \ d*j = j^\mu_{;\mu}\eta = 0\,, \tag{4.86}$$

which implies that the integral of $*j$ over a 3d domain $\mathcal{D}$ is conserved. A short calculation shows

$$Q = \int_{\mathcal{D}} *j = \frac{1}{4\pi}\int_{\mathcal{D}} d*F = \frac{1}{4\pi}\int_{\partial\mathcal{D}} *F = \frac{1}{4\pi}\int_{\mathbb{S}^\infty} *F\,. \tag{4.87}$$

For the last equal sign we have chosen $\mathcal{D}$ to be all of 3d space.

Exercise: Calculate $*d*F$ to determine j. At arbitrary r this is a lengthy calculation.

At large distance, $r \gg M$, $r \gg a$ and $r \gg Q$ one finds in terms of the orthornormal triad ($\bar{e}_{\hat{r}} = \partial_r$, $\bar{e}_{\hat{\vartheta}} = r^{-1}\partial_\vartheta$, $\bar{e}_{\hat{\varphi}} = (r\sin\vartheta)^{-1}\partial_\varphi$

$$E_{\hat{r}} = F_{rt} = \frac{Q}{r^2} + \mathcal{O}(r^{-3})\,, \quad E_{\hat{\vartheta}} = \frac{F_{\vartheta t}}{r} = \mathcal{O}(r^{-4})\,, \quad E_{\hat{\varphi}} \propto F_{\varphi t} = 0 \tag{4.88}$$

$$B_{\hat{r}} = \frac{F_{\vartheta\varphi}}{r^2\sin\vartheta} = \frac{2Qa}{r^3}\cos\vartheta + \mathcal{O}(r^{-4})\,, \tag{4.89}$$

$$B_{\hat{\vartheta}} = \frac{F_{\varphi r}}{r\sin\vartheta} = \frac{2Qa}{r^3}\sin\vartheta + \mathcal{O}(r^{-4})\,, \quad B_{\hat{\varphi}} \propto F_{\vartheta r} = 0\,. \tag{4.90}$$

The electric field is a Coulomb field with charge Q. The magnetic field is the field of a magnetic dipole in $\bar{e}_z$-direction with dipole moment

$$\mu = Qa = \frac{Q}{M}J \equiv g\frac{Q}{2M}J \quad \text{with} \quad g = 2\,. \tag{4.91}$$

Surprisingly, a classical Kerr-Newman black hole has a magnetic g-factor $g = 2$ like a Dirac electron!

4.4.1 Static limit and stationary observers

We consider an observer moving on a world line with constant r and ϑ and uniform angular velocity such that she sees an unchanging spacetime, i.e. a stationary observer. Her angular velocity as measured from an observer at rest at infinity is

$$\omega = \frac{d\varphi}{dt} = \frac{\dot\varphi}{\dot t} = \frac{u^\varphi}{u^t}\,, \tag{4.92}$$

where u^μ is the 4-velocity of the observer. We suppose ω to be constant (i.e. independent of time and of φ). As both, ∂_t and ∂_φ are Killing fields, u is then proportional to a timelike Killing field,

$$u = u^t\left(\partial_t + \omega\partial_\varphi\right) = \frac{b+\omega m}{|b+\omega m|}\,, \tag{4.93}$$

where

$$|b+\omega m| = \sqrt{-\langle b+\omega m, b+\omega m\rangle}\,.$$

Since u must be timelike,

$$g_{tt} + 2\omega g_{\varphi t} + \omega^2 g_{\varphi\varphi} < 0\,. \tag{4.94}$$

When the lefthand side of (4.94) vanishes u becomes lightlike. This happens when

$$\omega = \frac{-g_{\varphi t} \pm \sqrt{g_{\varphi t}^2 - g_{tt}g_{\varphi\varphi}}}{g_{\varphi\varphi}}\,. \tag{4.95}$$

Setting

$$\Omega = -\frac{g_{\varphi t}}{g_{\varphi\varphi}} = -\frac{\langle b,m\rangle}{\langle m,m\rangle} = a\frac{2Mr-Q^2}{\Sigma^2}\,, \tag{4.96}$$

we find that $\omega_{\min} < \omega < \omega_{\max}$ with

$$\omega_{\min} = \Omega - \sqrt{\Omega^2 - \frac{g_{tt}}{g_{\varphi\varphi}}} \tag{4.97}$$

$$\omega_{\max} = \Omega + \sqrt{\Omega^2 - \frac{g_{tt}}{g_{\varphi\varphi}}}\,. \tag{4.98}$$

For an interpretation of Ω consider stationary observers that are not rotating wrt radially infalling test particles. Since the angular momentum of such test particles vanishes, we have for these special stationary observers, so-called Bardeen observers, $\langle u, m\rangle = 0$, hence $\langle b+\omega m, m\rangle = 0$, i.e., $\omega = \Omega$. The velocity field of the Bardeen observer is proportional to $\xi = b + \Omega m$. Note that

$$\langle \xi, \xi\rangle = -\frac{\rho^2 \Delta}{\Sigma}\,. \tag{4.99}$$

Obviously $\omega_{\min} = 0$ if and only if $g_{tt} = 0$, that is $\langle b, b\rangle = 0$ which is equivalent to $\rho^2 + Q^2 - 2Mr = 0$, hence

$$r = r_\pm(\vartheta) = M \pm \sqrt{M^2 - Q^2 - a^2 \cos^2 \vartheta}\,. \tag{4.100}$$

We shall always assume that $M^2 > Q^2 + a^2$ since otherwise the metric has a naked singularity as we shall see below (an exception is the case $M = Q = 0$ where, as we have already seen, the spacetime is flat).

The surfaces $\{r = r_\pm(\vartheta)\}$ are called the outer $(+)$ and inner $(-)$ ergosurface.

An observer is said to be static (relative to the 'fixed stars') if $\omega = 0$, so that u is proportional to b. Static observers can exist only outside the static limit, i.e. for $r \geq r_+(\vartheta)$ (or for $r \leq r_-(\vartheta)$) where $\omega_{\min} \leq 0$.

At the static limit, defined by the surface $r = r_+(\vartheta)$, b becomes lightlike. An observer would have to move at the speed of light in order to remain at rest with respect to the fixed stars. But if the observer rotates, $\omega > 0$, her 4-velocity need not become lightlike at the static limit. This already indicates that the static limit is not a horizon.

The redshift which an asymptotic observer measures for light emitted from a source 'at rest' ($u \propto b$) outside the static limit is

$$\frac{\nu_e}{\nu_o} = \sqrt{\frac{\langle b, b\rangle_0}{\langle b, b\rangle_e}} \simeq \sqrt{\frac{-1}{\langle b, b\rangle_e}}\,. \tag{4.101}$$

This expression diverges at the static limit. Note that for a Schwarzschild black hole the static limit coincides with the horizon.

Between the outer and the inner static limit, $r_+(\vartheta) > r > r_-(\vartheta)$, $\omega_{\min} > 0$ hence an observer must rotate. Inside the inner static limit, $r < r_-(\vartheta)$, $\omega_{\min}$ becomes again negative such that $\omega = 0$ is possible. However, such observers are not visible from far away and their interpretation is not straight forward.

4.4.2 The Killing horizon and the Ergosphere

For $\Omega^2 = g_{tt}/g_{\varphi\varphi} \equiv \langle b,b\rangle/\langle m,m\rangle$ we have $\omega_{\rm min} = \omega_{\rm max} = \Omega$. On this surface there is only one possible angular velocity for a stationary observer and it is the one of the Bardeen observer. On this critical surface we have

$$g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2 = 0 \qquad \text{or} \qquad \Delta = r^2 - 2Mr + a^2 + Q^2 = 0\,. \tag{4.102}$$

This equation is satisfied on the two hypersurfaces given by

$$S_{H\pm} = \{r = r_{H\pm}\} \quad \text{with} \quad r_{H\pm} = M \pm \sqrt{M^2 - a^2 - Q^2}\,. \tag{4.103}$$

Here again, we need $M^2 \geq a^2 + Q^2$ for r_H to exist. If this condition is not satisfied, $\Delta > 0$ for all r and the singularity at $\rho = 0$ can be seen from far away, i.e. it is 'naked' (except if $M = Q = 0$).

We shall see that the surfaces $S_{H_\pm}$ are Killing horizons. They are denoted the outer $(+)$ and the inner $(-)$ horizon of the Kerr black hole. For an outside observer (i.e. for all astrophysical observations) only the outer horizon is really relevant and we shall concentrate on it in this discussion and we call it S_H for simplicity. But all can be repeated for the inner horizon.

On the hypersurface S_H, Ω becomes

$$\Omega_H = -\frac{g_{\varphi t}(r_H,\vartheta)}{g_{\varphi\varphi}(r_H,\vartheta)} = a\frac{2Mr_H - Q^2}{(r_H^2 + a^2)^2} = \frac{a}{r_H^2 + a^2}\,. \tag{4.104}$$

This is a very remarkable result: The black hole rotates like a rigid body on S_H. We have obtained this fact from the explicit formulas. In the Appendix we give a general argument why Ω_H has to be constant. Because of this, the vector field

$$\ell = b + \Omega_H m \tag{4.105}$$

is a Killing field. On the horizon this Killing field coïncides with ξ and is light like, $\langle \ell,\ell\rangle|_{r=r_H} = 0$. Furthermore, the flow of ℓ leaves the horizon invariant. Indeed, the horizon can be defined as $\langle \ell,\ell\rangle = 0$ and since ℓ is a Killing field

$$L_\ell\langle \ell,\ell\rangle = (L_\ell g)(\ell,\ell) - 2\langle[\ell,\ell],\ell\rangle = 0\,. \tag{4.106}$$

For the Kerr solution we have seen that the following theorem holds. In the Appendix we shall show that this is true for a generic axisymmetric, stationary vacuum spacetime (This is a first step in the direction of the uniqueness theorem for the Kerr-Newmann family of axisymmetric, stationary vacuum solutions which we shall not demonstrate in this course.).

Theorem 4.1 Weak Rigidity Theorem
Let $(\mathcal{M}, g)$ be a circular spacetime with commuting Killing fields b, m and set $\xi = b + \Omega m$ with $\Omega = -\langle b, m\rangle/\langle m, m\rangle$. Then Ω is constant on the hypersurface on which ξ is lightlike, $S_\xi = \{\langle \xi, \xi\rangle = 0\}$, which we assume to exist. Moreover, S_ξ is a null hypersurface, invariant under the isometry group $\mathbb{R} \times SO(2)$.

The surface S_ξ is a Killing horizon in the sense of the following definition:

Definition 4.1 Killing horizon
Let k be a Killing field and H_k the set of points where k is null, and not identically vanishing. A connected component of this set which is a null hypersurface, and any union of such null surfaces is called a Killing horizon (generated by k).

Obviously $S_\xi \equiv H_\ell \equiv S_H$ and on S_H the Killing field ℓ agrees with ξ. Any hypersurface $\{\langle \ell, \ell\rangle = \text{constant}\}$ is left invariant under the action of the symmetry group $\mathbb{R} \times SO(2)$. In particular the Killing field ℓ is a tangent null vector of $S_H = H_\ell$.

We emphasize that the notion of a Killing horizon – in contrast to the one of the event horizon – is of a local nature. It is natural to expect that the event horizon of a stationary black hole is a Killing horizon. That this is indeed true has been proven only relatively recently as part of a corrected version of the so-called "strong rigidity theorem". In addition, this theorem states that k is either the stationary Killing field (non-rotating black hole) or spacetime is axisymmetric (rotating black hole). For a circular spacetimes the event horizon agrees with the Killing horizon H_ℓ. This is Theorem 4.2. in [5].

Since a Killing horizon is a null hypersurface, the tangent space at each point is orthogonal to a null vector, and therefore does not contain timelike vectors (see Exercise at the end of this section). Moreover, the set of null vectors is one-dimensional and spanned by any normal vector (same Exercise). In other words, a Killing horizon is tangent to the light cone at each point. Therefore, crossing is possible in only one direction. In the situation above ℓ is as a null vector tangent to S_H but also proportional to the normal vector field on S_H, i.e.,

$$d\langle \ell, \ell\rangle = -2\tilde{\kappa}\ell^\flat \,. \tag{4.107}$$

The proportionality factor $\tilde{\kappa}$ is called the **surface gravity** of the Killing horizon. It plays an important role for black hole thermodynamics. Since ℓ is a Killing field Eq. (4.107) implies (see prop. 1.15)

$$d\langle \ell, \ell\rangle = di_\ell \ell^\flat = -i_\ell d\ell^\flat = 2i_\ell \nabla \ell^b = -2\nabla_\ell \ell^\flat = -2\tilde{\kappa}\ell^\flat, \tag{4.108}$$

hence

$$\nabla_\ell \ell = \tilde{\kappa}\ell \,. \tag{4.109}$$

Equation (4.109) tells us that the integral curves of ℓ are non–affinely parameterized geodesics. (**Exercise:** Show that by an appropriate re-parametrization these integral curves satisfy the standard geodesic equation.) These null geodesics generate the Killing horizon.

With some tricks (see [16]) one can show that on the horizon, $r = r_H$,

$$\kappa(r_H) = G\tilde{\kappa} = \frac{G(r_H - M)}{r_H^2 + a^2} = \frac{G(r_H - M)}{2Mr_H^2 - Q^2} \,. \tag{4.110}$$

The static limit (outer ergosurface) is timelike, except at the poles, $\vartheta = 0$ or π which are critical points of m, where it agrees with the horizon which is null and a normal vector to the static hypersurface, being orthogonal to the null vector b, cannot be timelike. Therefore one can pass through the outer ergosurface in both directions, in contrast to the Killing horizon. The region between the outer static limit and the outer horizon is the so-called outer **ergosphere** (for reasons which will be clarified below). The region between the inner static limit and the inner horizon is the so-called inner **ergosphere**. The static limit and the horizon come together at the poles, see Fig. 4.1). Inside the ergosphere, b is spacelike. This implies that inside the ergosphere nothing can prevent on observer from rotating about the black hole, any timelike velocity has $\omega > 0$. The ergosphere disappears when $a \to 0$.

Note that the inner ergosurface lies inside the inner horizon. Like the outer ergosurface it touches is at $\vartheta = 0$ and π. At $\vartheta = \pi/2$ it touches the ring singularity $r = 0, \vartheta = \pi/2$, i.e. $x^2 + y^2 = a^2$. The ergosurfaces are given by $g_{tt} = 0$ such that b becomes light like. The horizon is given by $g^{rr} = \Delta/\rho = 0$ so that the radial gradient becomes lightlike, $g^{\mu\nu}\partial_\mu r\, \partial_\nu r = 0$, which is equivalent to the condition that the surface $\{r = r_H\}$ is lightlike.

We add some remarks about Killing horizons for static spacetimes, for which the stationary Killing field satisfies the Frobenius condition $\omega_k = k \wedge dk = 0$ (vanishing twist). Setting $N \equiv \langle k, k\rangle$ we have the following general identity for Killing fields with vanishing twist (see proof below):

$$N\langle dk^\flat, dk^\flat\rangle = \langle dN, dN, \rangle \,. \tag{4.111}$$

For such a Killing field k we can therefore conclude that dN is null on the surface $S_k = \{N = \langle k, k\rangle = 0\}$. In other words, for the non-degenerate case, $dN \neq 0$ the hypersurface S_k is the Killing horizon H_k. In [5] (Theorem 4.1) it is shown that the event horizon of a (non-degenerate) static black hole coincides with H_k.

To show Eq. (4.111) we consider that $N = i_k k^\flat$ hence for a Killing field with vanishing twist

$$Ndk^\flat + k^\flat \wedge dN \;=\; (i_k k^\flat)dk^\flat + k^\flat \wedge d(i_k k^\flat)$$

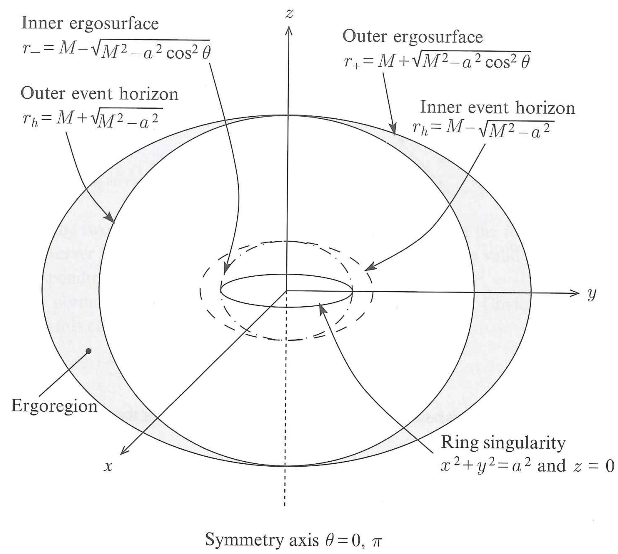

Figure 4.1: Cross section through the axis of rotation of a Kerr solution ($Q = 0$ in the depicted case). The two ergosurfaces and the two horizons are indicated together with the ergo regions (ergospheres) and the ring singularity (Figure from [13]). Here the coordinates drawn are $x = \sqrt{r^2 + a^2}\sin\vartheta\cos\varphi$, $y = \sqrt{r^2 + a^2}\sin\vartheta\sin\varphi$, $z = r\cos\vartheta$.

$$\begin{aligned} &= (i_k k^\flat) dk^\flat - k^\flat \wedge (i_k dk^\flat) \\ &= i_k(k^\flat \wedge dk^\flat) = 0 \qquad \text{hence} \\ dk^\flat &= -\frac{1}{N} k^\flat \wedge dN\,. \end{aligned} \tag{4.112}$$

For the fourth equal sign we used the vanishing twist condition. But

$$\langle k^\flat \wedge dN, k^\flat \wedge dN \rangle = \langle k, k \rangle \langle dN, dN \rangle - \langle k^\flat, dN, \rangle^2 = \langle k, k \rangle \langle dN, dN, \rangle\,. \tag{4.113}$$

Here we have used that for a Killing field $0 = L_k \langle k, k \rangle = i_k dN = \langle k^\flat, dN \rangle = 0$. Taking the scalar product of (4.112) with itself therefore gives Eq. (4.111).

Exercise Be V an $n+1$-dimensional Minkowski vector space with inner product denoted by $\langle x, y \rangle$ for $x, y \in V$. Show that the following facts hold:

i) Two timelike vectors are never orthogonal.
ii) A timelike vector is never orthogonal to a null vector.
iii) Two null vectors are orthogonal if and only if they are linearly dependent.
iv) The orthogonal complement of a null vector is an n-dimensional subspace of V in which the inner product is positive semi-definite and degenerate with rank $n-1$.

4.4.3 Coordinate singularities and true singularities

At the horizon, $\Delta = 0$, the Kerr metric expressed in terms of the Boyer- Lindquist coordinates appears singular. However, as in the case of the Schwarzschild solution, this is merely a coordinate singularity. This can be seen by transforming to the so-called Kerr coordinates. These new coordinates are generalizations of the Eddington-Finkelstein coordinates for spherically symmetric black holes and are defined by

$$d\tilde{v} = dt + \frac{r^2+a^2}{\Delta}dr \qquad (4.114)$$

$$d\tilde{\varphi} = d\varphi + \frac{a}{\Delta}dr\,. \qquad (4.115)$$

The exterior differential of the rhs vanishes for both definitions, since Δ is a function or r only, hence $\tilde{v}$ and $\tilde{\varphi}$ are fine coordinates. The Kerr metric can be written in terms of the new coordinates $(\tilde{v}, r, \vartheta, \tilde{\varphi})$ as follows

$$ds^2 = {}^{(4)}g_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{2Mr - Q^2}{\rho^2}\right)d\tilde{v}^2 + 2drd\tilde{v} + \rho^2 d\vartheta^2$$
$$+\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\vartheta}{\rho^2}\sin^2\vartheta d\tilde{\varphi}^2 - 2a\sin^2\vartheta d\tilde{\varphi}dr - 2a\frac{2Mr-Q^2}{\rho^2}d\tilde{\varphi}d\tilde{v}\,. \qquad (4.116)$$

This expression is regular at the horizon ($\Delta = 0$). Instead of $\tilde{v}$ one often also uses $\tilde{t} = \tilde{v} - r$. In this coordinates the Killing fields are

$$b = (\partial_{\tilde{t}})_{r,\vartheta,\tilde{\varphi}}\,, \qquad m = (\partial_{\tilde{\varphi}})_{\tilde{t},r,\vartheta}\,, \qquad (4.117)$$

where the variables inicated as subscripts are the ones to be kept constant.

The Kerr-Newman metric has a true singularity which lies inside the horizon for $M^2 > a^2 + Q^2$.

Visualizing the spacetime geometry is made easier by considering the structure of the light cones. We examine this more closely in the equatorial plane, $\vartheta = \pi/2$. as indicated in Fig. 4.2. Each point in this plane represents an integral curve of the Killing field b. The wave fronts of light signals which have formed shortly after being emitted from the marked points are also shown in Fig. 4.2. We note the following facts:

a) Since b is timelike outside the outer ergosurface, the points of emission are inside the wave fronts.

b) At the outer ergosurface b becomes lightlike, and the point of emission lies thus on the wave front.

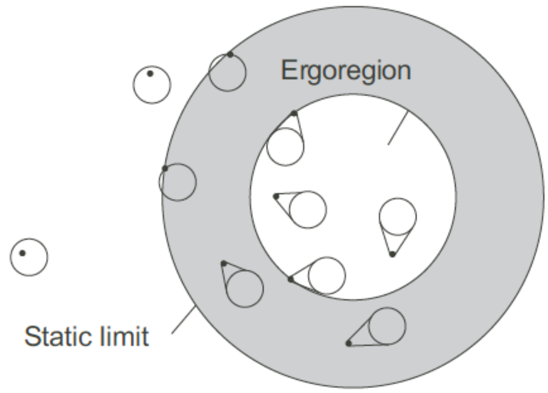

Figure 4.2: We show the lightcones for the Kerr metric in the equatorial plane, $\vartheta = \pi/2$ where the distance between the horizon and the static limit, and therefore the ergosphere is maximal and the inner ergosphere coïncides with the singularity $r = 0$. Here, the singularity at $r = 0$ is represented as a point in the center. (Figure from [16])

c) Inside the ergosphere b is spacelike and hence the emitting points are outside the wave fronts.

d) For $r = r_H$, the Killing field b is still spacelike, furthermore, the wave fronts arising from a point of emission on this surface lie entirely inside the surface, except for touching points, because $\{r = r_H\}$ (i.e. $\{\Delta = 0\}$) is a null surface. This demonstrates that this surface is indeed an event horizon. This lightlike hypersurface is invariant with respect to the flow of b and m (prove this!).

4.5 The Penrose process and black hole thermodynamics

In this section we want to show that it is in principle possible to extract energy out of a Kerr black hole. We shall study only Kerr black holes and set $Q = 0$ in this section. This is not essential but simplifies the discussion somewhat. As we shall see, energy extraction is possible as long as the angular momentum a is non-vanishing. Physically the process is the following: We let a particle with energy E fall behind the ergosurface of a Kerr black hole. There it disintegrates into 2 particles with energies E_1 and E_2 such that $E = E_1 + E_2$, and with momenta that are such that e.g. E_1 is ejected out of the ergosphere and E_2 falls into the black hole. We now show that it can happen that $E_2 < 0$ so that $E_1 > E$.

For this we first note that for a particle moving along a geodesic with 4-momentum

$p = mu$, the energy $E = -\langle p, b\rangle = -m(g_{tt}u^t + g_{t\varphi}u^\varphi) = -p_t$ is conserved. In a Kerr spacetime g_{tt} can change sign and hence this conserved energy can be negative. This exactly what happens inside the ergosphere. However, far away this energy the energy of a particle must necessarily be positive. Therefore, a particle entering the ergosphere from far away necessarily has $E > 0$, however, a particle originating within the ergosphere may well have $E < 0$.

Remark : As a side let us show the conservation of energy. More precisely we show that for each Killing field of a metric, there exists a conserved quantity along its geodesics. More precisely, if k is a Killing field and u a geodesic then $\langle u, k\rangle$ is conserved along u. To see this we use that

$$\nabla_u\langle u, k\rangle = \langle u, \nabla_u k\rangle = \langle u, \nabla_k u\rangle + \langle u, [u, k]\rangle\,.$$

For the last equal sign we used $\nabla_u k - \nabla_k u = [u, k]$. Since $\langle u, u\rangle$ is constant, $\langle u, \nabla_u k\rangle = \nabla_k\langle u, u\rangle/2 = 0$. Furthermore, $[u, k] = -L_k u$. Using now that k is a Killing field we have

$$0 = (L_k g)(u, u) = L_k(\langle u, u\rangle) - 2\langle u, L_k u\rangle\,.$$

As $\langle u, u\rangle \equiv -1$, the first term on the rhs vanishes identically, hence also the second term vanishes and

$$\nabla_u\langle u, k\rangle = 0\,.$$

To study in more detail when we may have $E_2 < 0$ we write the Kerr metric as

$$ds^2 = -e^{2\nu}dt^2 + e^{2\psi}(d\varphi - \omega dt)^2 + e^{2\mu_1}dr^2 + e^{2\mu_2}d\vartheta^2\,. \tag{4.118}$$

The inverse metric is

$$\begin{aligned} g^{tt} &= -e^{-2\nu}\,, \qquad g^{t\varphi} = -\omega e^{-2\nu}\,, \qquad g^{\varphi\varphi} = e^{-2\psi} - \omega^2 e^{-2\nu}\,, \\ g^{rr} &= e^{-2\mu_1}\,, \qquad g^{\vartheta\vartheta} = e^{-2\mu_2}\,. \end{aligned} \tag{4.119}$$

We now consider a particle of mass m with 4-momentum p such that

$$-m^2 = p^2 = -e^{-2\nu}p_t^2 - 2\omega e^{-2\nu}p_t p_\varphi + \left(e^{-2\psi} - \omega^2 e^{-2\nu}\right)(p_\varphi)^2 + e^{-2\mu_1}p_r^2 + e^{-2\mu_2}(p_\vartheta)^2\,. \tag{4.120}$$

Solving for the conserved energy $E = -p_t$ yields

$$E = \omega p_\varphi + e^\nu\left[e^{-2\psi}(p_\varphi)^2 + e^{-2\mu_1}p_r^2 + e^{-2\mu_2}(p_\vartheta)^2 + m^2\right]^{1/2}\,. \tag{4.121}$$

The sign on the square root must be positive in order to allow $E \to m$ for a particle at rest at infinity. We choose $a > 0$, hence $\omega > 0$. If we want $E < 0$ we need $p_\varphi < 0$ and

$$e^\nu\left[e^{-2\psi}(p_\varphi)^2 + e^{-2\mu_1}p_r^2 + e^{-2\mu_2}(p_\vartheta)^2 + m^2\right]^{1/2} < -\omega p_\varphi\,.$$

This is best achieved if we choose the square root as small as possible, hence by setting $p_r = p_\vartheta = 0$. In the limiting case of a highly relativistic particle, $m \ll |p_\varphi|$ this boils down to

$$e^{2(\nu-\psi)} < \omega^2 \,, \tag{4.122}$$

which is equivalent to $g_{tt} > 0$. Hence this can be achieved inside the ergosphere.

Let us now consider the amount of negative energy, $-E_2 = \langle p_2, b\rangle$ and angular momentum $L_2 = \langle p_2, m\rangle$ the particle can carry into the black hole. For this we make use of the fact that also $\langle p_2, \ell\rangle$ is conserved since also ℓ is a Killing field. Since ℓ is timelike in the ergosphere this quantity must be negative (see ex. on p88), hence

$$0 > \langle p_2, \ell\rangle = -E_2 + \Omega_H L_2 \,, \tag{4.123}$$

which yields the bound $L_2 < E_2/\Omega_H$. Especially since $E_2 < 0$ also $L_2 < 0$.

When the particle falls in the horizon, the conservation of the Komar integral requests that $\delta J = L_2$ and $\delta M = E_2$. Hence both, the mass and the angular momentum of the black hole decrease. The bound (4.123) translates to

$$\delta M > \Omega_H \delta J = \frac{a}{r_H^2 + a^2}\delta J \,. \tag{4.124}$$

This result can be expressed in a more suggestive form making contact with 'black hole thermodynamics'. For this we calculate the area of the Kerr horizon. On the horizon the metric is

$$ds^2 = (r_H^2 + a^2\cos^2\vartheta)d\vartheta^2 + \frac{(r_H^2+a^2)^2\sin^2\vartheta}{(r_H^2 + a^2\cos^2\vartheta)}d\varphi^2 = \alpha_{pq}d\theta^p d\theta^q \,. \tag{4.125}$$

The surface of the horizon is therefore

$$A = \int \sqrt{\det\alpha}\, d\vartheta d\varphi = \int (r_H^2 + a^2)\sin\vartheta d\vartheta d\varphi = 4\pi(r_H^2 + a^2) = 4\pi\frac{a}{\Omega_H} \,. \tag{4.126}$$

$r_H = M + \sqrt{M^2 - a^2}$. We now introduce the so called 'irreducible mass' as

$$M_{\text{irr}}^2 = \frac{1}{2}\left(M^2 + \sqrt{M^4 - J^2}\right) = \frac{A}{16\pi}\,, \qquad J = aM\,, \qquad M^2 = M_{\text{irr}}^2 + \left(\frac{J}{2M_{\text{irr}}}\right)^2 , \tag{4.127}$$

which shows that the square of total energy/mass of the black hole can be split into an 'irreducible part' and a term that can be interpreted as rotational energy. Note also that for a Schwarzschild black hole $M = M_{\text{irr}}$ which also follows from $M^2 = A/(16\pi)$ which is the area formula for Schwarzschild black holes.

Form (4.127) we can infer

$$\delta M_{\text{irr}} = \frac{a}{4M_{\text{irr}}\sqrt{M^2-a^2}}\left(\Omega_H^{-1}\delta M - \delta J\right) \,. \tag{4.128}$$

Hence (4.124) implies

$$\delta M_{\rm irr} > 0\,, \qquad \delta A > 0\,. \tag{4.129}$$

The area of the horizon increases during the Penrose process. This result is actually more general than our derivation: no classical physical process involving black holes can decrease their total horizon area!

To make contact with black hole thermodynamics we write the bound on the change in the area in a different form

$$\delta A = 8\pi \frac{a}{\Omega_H \sqrt{M^2 - a^2}} \left(\delta M - \Omega_H \delta J\right)\,, \tag{4.130}$$

or

$$\delta M = \frac{\kappa}{8\pi G}\delta A + \Omega_H \delta J\,, \tag{4.131}$$

where we have inserted the surface gravity for $Q = 0$,

$$\kappa = \kappa(r_H) = \frac{G\sqrt{M^2 - a^2}}{2M(M + \sqrt{M^2 - a^2})}\,, \tag{4.132}$$

see Eq. (4.110). In this form Eq. (4.131) has exactly the form of the second law of thermodynamics where $A/(4G)$ plays the role of the entropy (which is compatible with $\delta A > 0$) and $\kappa/(2\pi G)$ is acting as the 'temperature'. The term $\Omega_H \delta J/G$ is the 'work' exerted from the black hole and $\delta M/G$ is the change of energy.

Exercise:

- Show that in units where $c = k_{\rm Boltzmann} = 1$, the quantity $\hbar\kappa/G$ has the units of a temperature. Calculate the temperature $\hbar\kappa/(2\pi G)$ for a solar mass Schwarzschild black hole in units of Kelvin.

- Show that in the same units $\hbar^{-1}A/4G$ is dimensionless. Determine also this dimensionless entropy and compare it to the maximal entropy of photons with total energy M/G enclosed in a box of size $2r_H$.

- Consider a Kerr black hole with $a \sim M/2$ and $M \sim M_\odot$. Compare the different terms in Eq. (4.131).

Appendix A

The weak rigidity Theorem

Theorem A.1 Weak Rigidity Theorem
Let $(\mathcal{M}, g)$ be a circular spacetime with commuting Killing fields b, m and set $\xi = b + \Omega m$ with $\Omega = -\langle b, m\rangle / \langle m, m\rangle$. Then Ω is constant on the hypersurface on which ξ is lightlike, $H = \{\langle \xi, \xi\rangle = 0\}$, which we assume to exist. Moreover, H is a null hypersurface, i.e. H is a Killing horizon, invariant under the isometry group $\mathbb{R} \times SO(2)$.

Proof: Clearly H is invariant under the isometry group since all the elements used in its definition, b, m and the metric are. We first establish that H is a null hypersurface. For this we introduce

$$\sigma = -\frac{1}{2}\langle b^\flat \wedge m^\flat, b^\flat \wedge m^\flat\rangle = VX + W^2 = -\det \sigma_{ab}\,, \tag{A.1}$$

where $V = -\sigma_{tt} = -\langle b, b\rangle$, $X = \sigma_{\varphi\varphi} = \langle m, m\rangle$ and $W = \sigma_{\varphi t} = \langle m, b\rangle$. Note that the hypersurface H is defined by $\sigma = 0$. Intuitively it is clear that the hypersurfaces $\{\sigma = \text{const.}\}$ are invariant under the symmetry group $\mathbb{R} \times SO(2)$ since the length of both b and m are invariant. Formally this follows from

$$L_b W = (L_b g)(b, m) - g(L_b b, m) - g(b, L_b m) = 0\,, \tag{A.2}$$

and in the same way $L_b X = L_b V = L_m W = L_m X = L_m V = 0$. Hence the gradient $d\sigma^\sharp$ is perpendicular to b and m, in other words, $d\sigma^\sharp \in E^\perp$, where E denotes the tangent space of the surface H. Below we show that on H we also have $d\sigma^\sharp \in E$. This implies that $d\sigma^\sharp\big|_H \in E \cap E^\perp$ is null. This intersection is 1–dimensional on H and $d\sigma^\sharp$ must therefore be proportional to the null field ξ. Hence the normal $d\sigma^\sharp$ is null on H which proves that H is lightlike.

To show that $d\sigma^\sharp \in E$ we now calculate it in detail.

$$d\sigma = 2W dW + X dV + V dX\,. \tag{A.3}$$

We use

$$dW = di_b m^\flat = -i_b dm^\flat\,, \qquad W = -i_m db^\flat\,, \tag{A.4}$$

$$dX = -i_m dm^\flat\,, \qquad dV = i_b db^\flat\,, \tag{A.5}$$

so that

$$d\sigma = -W(i_b dm^\flat + i_m db^\flat) + X i_b db^\flat - V i_m dm^\flat\,. \tag{A.6}$$

We now multiply this equation from the right with $b^\flat \wedge m^\flat$, and use the Frobenius condition (4.2) to convert e.g. $i_b dm^\flat \wedge (b^\flat \wedge m^\flat) = dm^\flat \wedge (i_b b^\flat m^\flat - b^\flat i_b m^\flat) = dm^\flat \wedge (-Vm^\flat - Wb^\flat)$. Collecting the terms in the resulting equation we arrive at

$$d\sigma \wedge (b^\flat \wedge m^\flat) = \sigma d(b^\flat \wedge m^\flat)\,. \tag{A.7}$$

Especially on H this implies $d\sigma \wedge (b^\flat \wedge m^\flat) = 0$. We also use here that the group action is such that b and m are linearly independent (span a 2-dimensional tangent space). Then $d\sigma^\sharp$ must be a linear combination of b and m which is null, hence it must be proportional to ξ.

Next we want to show that $\Omega|_H \equiv \Omega_H$ is constant. This follows with a similar argument. We consider the gradient $d\Omega^\sharp$. With the same considerations as for $d\sigma^\sharp$ it follows that $d\Omega^\sharp$ is perpendicular the b and m and hence $d\Omega^\sharp \in E^\perp$. On the other hand,

$$d\Omega = -d\frac{\langle b, m\rangle}{\langle m, m\rangle} = -\frac{1}{X} dW + \Omega dX) = \frac{1}{X} i_\xi m^\flat\,. \tag{A.8}$$

The same amnipulations which led to (A.7) now give

$$d\Omega \wedge (b^\flat \wedge m^\flat) = -\frac{1}{X}\langle \xi, \xi\rangle m^\flat \wedge dm^\flat\,, \tag{A.9}$$

where, as above, we have used $i_\xi \xi^\flat = \langle \xi, \xi\rangle$ and $i_\xi m^\flat = \langle \xi, m\rangle = 0$. But on H the right hand side of (A.9) vanishes hence also $d\Omega^\sharp$ is a linear combination of b and m, which implies that it is in E. Like for $d\sigma^\sharp$, this implies $d\Omega^\sharp \in E^\perp \cap E$ hence $d\Omega^\sharp \propto \xi$ so that H is a null–surface on which the function Ω is constant. □

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